Approximation Complexity of Additive Random Fields

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Abstract

Let $X(t), t \in [0,1]^d$ be an additive random field. We investigate the complexity of finite rank approximation

$$X(t, \omega) \approx \sum_{k=1}^{n} \xi_k(\omega)\varphi_k(t).$$

The results obtained in asymptotic setting $d \to \infty$, as suggested H.Woźniakowski, provide quantitative version of dimension curse phenomenon: we show that the number of terms in the series needed to obtain a given relative approximation error depends on d exponentially and find the explosion coefficients.

Key words: approximation complexity, dimension curse, Gaussian processes, linear approximation error, random fields, tractability

1 Introduction

Let $X(t) = \sum_{k=1}^{\infty} \xi_k \varphi_k(t), t \in T$, be a random function represented via random variables ξ_k and the deterministic real functions φ_k. Let $X_n(t) = \sum_{k=1}^{n} \xi_k \varphi_k(t)$ be the approximation to X of rank n. How large should be n in order to make approximation error small enough? Provided a functional norm $\|\cdot\|$ is given on the sample paths' space, the question can be stated in the average and in the probabilistic setting. Namely, find

$$n^{avg}(\varepsilon) := \inf \left\{ n : \mathbb{E}||X - X_n||^2 \leq \varepsilon^2 \right\},$$

or

$$n^{pr}(\varepsilon, \delta) := \inf \{n : \mathbb{P}\{||X - X_n|| \geq \varepsilon\} \leq \delta\}.$$

In this work we mostly consider the additive random fields X of tensor product-type with $T \subset \mathbb{R}^d$. The word "additive" means that X can be represented as a sum of terms depending only of appropriate groups of coordinates.

In the first part of the article we investigate the problem for fixed X, T, and d.

In the second part we consider sequences of related tensor product-type fields $X^{(d)}(t), t \in T^{(d)} \subset \mathbb{R}^d$, with $d \to \infty$ and study the influence of dimension parameter d. It turns out that the rank n that is necessary to obtain a relative error ε increases exponentially in d for any fixed ε. The explosion coefficient admits a simple explicit expression and does not depend on ε. Interestingly, the phenomenon of exponential explosion does not depend on the smoothness properties of the underlying fields. Recall that exponential explosion of the difficulty in approximation problems that include dimension parameter is well known as "dimensionality curse" or "intractability", see e.g. [11], For more recent results on tractability and intractability, see [12]. On ideological level, we were much inspired by this work.

Throughout the article, we use the following notation: for integers let $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{N}^* = \{1, 2, \dots\}$. We write $a_n \sim b_n$ iff $\lim_n a_n/b_n = 1$.

The article is organized as follows. In Section 2 we specify the class of random fields to work with and introduce the necessary notation. After recalling some basic known approximation results in Section 3, we handle a given additive field in Section 4 while Section 5 is devoted to the asymptotic setting: we deal with a series of random fields with parameter dimension $d \to \infty$. Finally, in Section 6 we give some extensions of our results to more general class of random field.

2 Additive random fields

s:fields

In this article we investigate additive random fields. The simplest example of additive field is given by

$$X(t) = \sum_{l=1}^{d} X_l(t_l), \qquad t \in \mathbb{R}^d,$$

where X_l are independent copies of a one-parametric process. The behavior of X was studied in [2] and in some other works. During last years, the additive fields of higher orders also attracted the interest of researchers. In this general case, the additive d-parametric random field is a sum of i.i.d. fields, each depending on a smaller number of parameters. To give a precise definition, we need some notation. Let us fix $d, b \in \mathbb{N}$ such that $d \geq b \geq 1$, and let $T_d = [0,1]^d$, $T_b = [0,1]^b$. We denote by D and D_b the following sets of indexes:

$$D = \{1, \cdots, d\}, \quad D_b = \{A \subset D \,, |A| = b\}\,.$$

For each $A = \{a_1, \cdots, a_b\} \in D_b$, we define the projection $\Pi_A : T_d \to T_b$ by $\Pi_A(t) = (t_{a_1}, \cdots, t_{a_b})\,.$

We consider the following process defined for every $t \in T_d$ by

$$X(t) = \sum_{A \in D_b} X_A\left(\Pi_A(t)\right),$$

where X_A are i.i.d. copies of a b-parametric random field and call X an additive random field of order b.

The additive structure becomes especially important if the order b is much smaller than time dimension d. Since in this article we are mainly interested in the role of dimension, we are going to discuss the families of additive random fields with varying d and b. In this setting, it is quite natural to assume that X is actually generated by a *one-parametric process* via taking *tensor degrees*.

Recall the notion of tensor product for second order random fields. Given two centered fields $\{Y_1(t_1)\}_{t_1 \in T_{d_1}}$ and $\{Y_2(t_2)\}_{t_2 \in T_{d_2}}$ with covariances $\mathcal{K}_1(\cdot,\cdot)$ and $\mathcal{K}_2(\cdot,\cdot)$, respectively, we define their tensor product $\{(Y_1 \otimes Y_2)(t)\}_{t \in T_{d_1+d_2}}$ as a centered second order random field with covariance

$$\mathcal{K}\left((t_1,t_2),(t_1',t_2')\right) := \mathcal{K}_1(t_1,t_1')\mathcal{K}_2(t_2,t_2').$$

The definitions of multiple tensor products $\bigotimes_{j=1}^{b} Y_j$ and tensor degrees $Y^{\otimes b}$ are now straightforward.

Let now $\{Y(u)\}_{u \in [0,1]}$ be a given second order one-parametric process expanded with respect to an orthonormal basis $(\varphi_i)_{i \in \mathbb{N}} \in L_2([0,1])$:

$$Y(u) = \sum_{i=0}^{\infty} \lambda(i)\varphi_i(u)\xi_i\,,$$

where $\lambda(i) \geq 0$, $\sum_{i=0}^{\infty} \lambda(i)^2 < \infty$ and $(\xi_i)_{i\in\mathbb{N}}$ are non-correlated random variables, $E(\xi_i) = 0$ and $\mathrm{Var}(\xi_i) = 1$. For any integer $b \geq 1$ the b-th tensor degree of Y writes as

$$X(t) := Y^{\otimes b}(t) = \sum_{k\in\mathbb{N}^b} \prod_{l=1}^{b} \lambda(k_l)\varphi_{k_l}(t_l)\xi_k \,, \qquad t \in T_b,$$

where the variables $(\xi_k)_{k\in\mathbb{N}^b}$ are non-correlated, $E(\xi_k) = 0$ and $\mathrm{Var}(\xi_k) = 1$.

Now the d-parametric additive random field of order b generated by Y has a form

$$\begin{aligned} X_{d,b}(t) &= \sum_{A\in D_b} \sum_{k\in\mathbb{N}^b} \left(\prod_{l=1}^{b} \lambda(k_l)\varphi_{k_l}\left([\Pi_A(t)]_l\right) \right) \xi_k^A \\ &= \sum_{A\in D_b} \sum_{k\in\mathbb{N}^A} \left(\prod_{a\in A} \lambda(k_a) \prod_{a\in A} \varphi_{k_a}(t_a) \right) \xi_k^A \,. \end{aligned} \tag{2.1}$$

If k_Y denotes the covariance function of Y, i.e. $\mathrm{Cov}(Y(u), Y(u')) = k_Y(u, u')$ we easily see that

$$\mathrm{Cov}(X_{d,b}(t), X_{d,b}(t')) = \sum_{A\in D_b} \prod_{a\in A} k_Y(t_a, t'_a) \,.$$

For the rest of this section, we make the following assumption that substantially simplifies the calculations:

zero **Assumption 2.1**

$$\forall u \in [0,1] \,, \ \varphi_0(u) = 1 \,.$$

This assumption leads to the principal results in a more direct way. However, we will show later in Section 6 that it can be dropped, at least sometimes. We are, of course, not the first to notice the advantages of this assumption. See for example a recent work [4] where important random fields satisfying this property are handled.

Under Assumption 2.1 we have the following lemma.

lem1 **Lemma 2.2** *Let $k, k' \in \mathbb{N}^d$ and $A, A' \subset D$. If Assumption 2.1 is valid, then the functions $\psi(t) = \prod_{a\in A} \varphi_{k_a}(t_a)$ and $\psi'(t) = \prod_{a\in A'} \varphi_{k'_a}(t_a)$ are either identical or orthogonal in $L_2(T_d)$.*

Proof: We are interested in the scalar product

$$(\psi,\psi') = \int_{T_d} \psi(t)\psi'(t)\mathrm{d}\lambda_d(t)\,, \tag{2.2}$$

where λ_d is the Lebesgue measure on T_d. We can represent this integral as a product of three factors:

$$(\psi,\psi') = \Pi_1\Pi_2\Pi_3$$

where

$$\begin{aligned}
\Pi_1 &= \prod_{a\in A\cap A'} \int_{[0,1]} \varphi_{k_a}(t_a)\varphi_{k'_a}(t_a)\mathrm{d}t_a\,,\\
\Pi_2 &= \prod_{a\in A\setminus A'} \int_{[0,1]} \varphi_{k_a}(t_a)\mathrm{d}t_a\,,\\
\Pi_3 &= \prod_{a\in A'\setminus A} \int_{[0,1]} \varphi_{k'_a}(t_a)\mathrm{d}t_a\,.
\end{aligned}$$

In the first factor Π_1, whenever $k_a \neq k'_a$, the integral is null, since the functions (φ_i) are orthogonal. In the second factor Π_2, if $k_a \neq 0$, it follows from the orthogonality of the family (φ_i) and Assumption 2.1 that the integral $\int_{[0,1]} \varphi_{k_a}(t_a)\mathrm{d}t_a$ is null. The same argument applies to the third factor Π_3.

We see that $\Pi_1\Pi_2\Pi_3$ does not vanish only if $k_a = k'_a$ for $a \in A\cap A'$ and $k_a = 0$, $k'_a = 0$ elsewhere, and the assertion follows. $\square$

Notice that in expression (2.1) different sets A can generate the same product $\prod_{a\in A}\varphi_{k_a}(t_a)$. Therefore, it is more convenient to write $X_{d,b}$ in a different way:

$$X_{d,b}(t) = \sum_{h=0}^{b} \sum_{\substack{C\subset D\\ |C|=h}} \sum_{k\in(\mathbb{N}^*)^C} \prod_{a\in C} \varphi_{k_a}(t_a) \prod_{a\in C}\lambda(k_a) \sum_{\substack{F\subset(D\setminus C)\\ |F|=b-h}} \lambda(0)^{b-h}\xi_{\overline{k}}^{C\cup F}\,, \tag{2.3}$$

proces2

where $\overline{k}\subset \mathbb{N}^{C\cup F}$ is made of k by adding zeros. We can simplify this expression to

$$X_{d,b}(t) = \sum_{h=0}^{b} \sum_{\substack{C\subset D\\ |C|=h}} \left[\sum_{k\in(\mathbb{N}^*)^C} \prod_{a\in C} \varphi_{k_a}(t_a) \prod_{a\in C}\lambda(k_a)\right]\eta^C\,, \tag{2.4}$$

proces3

where $(\eta^C)_{C\in D}$ are non-correlated centered random variables of variance

$$\mathrm{Var}(\eta^C) = C_{d-h}^{b-h} \lambda(0)^{2(b-h)} .$$

Expression (2.4) is convenient to handle, since by Lemma 2.2 all terms in the right hand side are orthogonal in $L_2(T_d)$.

The spectrum of the covariance operator of $X_{d,b}$ can be described as follows. For every fixed $h \in \{0, \cdots, b\}$, and every $k \in (\mathbb{N}^*)^h$ take the eigenvalue $C_{d-h}^{b-h} \left[\prod_{l=1}^{h} \lambda(k_l)\right]^2 \lambda(0)^{2(b-h)}$ of multiplicity C_d^h coming from C_d^h different choices of subset $C \subset D$. It is more convenient to us not to identify the equal eigenvalues generated by permutations of (k_l).

3 Approximation of simple tensor products

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In this section we recall some results of the paper [8] on approximation of tensor product random fields. In terms of Section 2 the setting of [8] corresponds to the "elementary" case $b = d$ which does not contain additivity effect. The facts known about this case will be the starting point of our study. According to (2.1), for $b = d$ let $X(t) = X_{d,b}(t)$ be a random field given by

$$X(t) := \sum_{k\in\mathbb{N}^d} \prod_{l=1}^{d} \lambda(k_l)\xi_k \prod_{l=1}^{d} \varphi_{k_l}(t_l) , \qquad t \in [0,1]^d,$$

where (φ_i) is an orthonormal system in $L_2[0,1]$ and ξ_k are non-correlated random variables.

3.1 Fixed dimension

Assume that d is fixed and that the assumptions

$$\Lambda := \sum_{i=1}^{\infty} \lambda(i)^2 < \infty, \tag{3.1}$$

$$\lambda(i) \sim \mu i^{-r} (\log i)^q \tag{3.2}$$ rq

are satisfied with $\mu > 0, r > 1/2, q \neq -r$. We approximate X by a finite sum X_n corresponding to the n maximal eigenvalues. Recall that approximation cardinality $n^{avg}(\varepsilon)$ is defined as

$$n^{avg}(\varepsilon) = \inf\{n ;\ E\|X - X_n\|_{L_2(T_d)}^2 \leq \varepsilon^2\} .$$

Then we have the following theorem:

thm_LT1 **Theorem 3.1** ([8]) *Under assumption* (3.2) *it is true that*

$$n_d^{avg}(\varepsilon) \sim \left(\frac{B_d}{\sqrt{2}(r-1/2)^{r\beta+1/2}} \frac{|\log \varepsilon|^{r\beta}}{\varepsilon} \right)^{1/(r-1/2)} , \qquad (3.3)$$ asym_LT1

where for $\alpha = q/r$

$$B_d = \mu^d \Pi_d^r, \quad \beta = (d-1) + d\alpha \qquad \text{if } \alpha > -1, \qquad (3.4)$$

$$B_d = \mu d^r S^{(d-1)r}, \quad \beta = \alpha \qquad \text{if } \alpha < -1, \qquad (3.5)$$

$$S = \sum_{i=1}^{\infty} \lambda(i)^{1/r}, \quad \Pi_d = \frac{\Gamma(\alpha+1)^d}{\Gamma(d(\alpha+1))} .$$

3.2 Increasing dimension

Suppose $d \to \infty$ and assume that

$$\sum_{i=1}^{\infty} |\log \lambda(i)|^2 \lambda(i)^2 < \infty . \qquad (3.6)$$ Mfinite

The total size of the field X is characterized by

$$E\|X\|^2 = \Lambda^d .$$

As above, define the cardinality associated to the *relative* error

$$\tilde{n}_d^{avg}(\varepsilon) = \inf\{n\,;\; E\|X - X_n\|^2_{L_2(T_d)} \le \varepsilon^2 \Lambda^2\} .$$

Then we have

thm_LT2 **Theorem 3.2** ([8]) *Under assumption* (3.6) *it is true that*

$$\lim_{d\to\infty} \frac{\log \tilde{n}_d^{avg}(\varepsilon)}{d} = \log A , \qquad (3.7)$$ asym_LT2

where $A = \Lambda e^{2M}$ *and* $M = -\sum_{i=1}^{\infty} \log \lambda(i) \frac{\lambda(i)^2}{\Lambda}$.

We stress that no regularity assumption like (3.2) is required.

4 Approximation in fixed dimension

s:fixed

In this section, we fix d and b and consider the quality of approximation to an additive field $X_{d,b}$ by means of the processes of rank n, as $n \to \infty$. Namely, we approximate $X_{d,b}$ with the finite sum X_n from (2.4) corresponding to n maximal eigenvalues of covariance operator. As a measure of approximation, we use

$$n^{avg}(\varepsilon, d, b) = \inf\{n\,;\ E\|X_{d,b} - X_n\|^2_{L_2(T_d)} \leq \varepsilon^2\}\,.$$

Analogously to (3.2), we will consider here the practically important case described by the following

valpropre **Assumption 4.1** *$\lambda(i) \sim \mu\, i^{-r}(\log i)^q\,, \quad i \to \infty$, for some $\mu > 0$, $r > 1/2$, and $q \neq -r$.*

We write $\alpha = q/r$. For any $h \in \{1, \cdots, b\}$ and $k \in (\mathbb{N}^*)^h$, let us write

$$\lambda_k^2 = \prod_{l=1}^{h} \lambda^2(k_l)\,,$$

and $(\overline{\lambda}^2_{n,h}\,, n \in \mathbb{N})$ for the decreasing rearrangement of the array $(\lambda_k^2)\,, k \in (\mathbb{N}^*)^h$. We know from [8] that

$$\overline{\lambda}^2_{n,h} \;\sim\; B_h^2 n^{-2r}(\log n)^{2r\beta}\,, \quad n \to \infty, \tag{4.1}$$

where

- $\alpha > -1 : \begin{cases} B_h = \mu^h \left(\frac{\Gamma(\alpha+1)^h}{\Gamma(h(\alpha+1))}\right)^r \\ \beta = (h-1) + h\alpha \end{cases}$
- $\alpha < -1 : \begin{cases} B_h = \mu h^r \left[\sum_{i\geq 1} \lambda(i)^{1/r}\right]^{(h-1)r} \\ \beta = \alpha \end{cases}$

Note that equivalent results can be found e.g. in Csáki [3], Li [7] Papageorgiou and Wasilkowski [9] (for $q = 0$) and especially in Karol', Nikitin, and Nazarov [5] for even more general case than we need here.

prop1 **Proposition 4.2** *Under Assumptions* 2.1 *and* 4.1 *we have:*

a) If $\alpha > -1$, then

$$n^{avg}(\varepsilon, d, b) \sim [C_d^b]^{\frac{2r}{2r-1}} \left(\frac{B_b}{\sqrt{2}(r-1/2)^{r\beta+1/2}}\frac{|\log\varepsilon|^{r\beta}}{\varepsilon}\right)^{(r-1/2)^{-1}}\,, \quad \varepsilon \to 0. \tag{4.2}$$

prop11

b) If $\alpha < -1$, then

$$n^{avg}(\varepsilon, d) \sim \left(\frac{\sqrt{Q}}{\sqrt{2}(r-1/2)^{r\alpha+1/2}} \frac{|\log \varepsilon|^{r\alpha}}{\varepsilon} \right)^{(r-1/2)^{-1}} , \quad \varepsilon \to 0, \qquad (4.3) \quad \boxed{\texttt{prop12}}$$

where

$$Q = \left(\sum_{h=1}^{b} C(h)^{\frac{1}{2r}} \right)^{2r} \quad \text{and} \quad C(h) = C_{d-h}^{b-h} [C_d^h]^{2r} \lambda(0)^{2(b-h)} B_h^2. \qquad (4.4) \quad \boxed{\texttt{BCi}}$$

Proof:
If $\alpha > -1$, then β depends on h, hence, in asymptotic setting, the only relevant eigenvalues are those corresponding to the maximal β, i.e. the asymptotics is determined by the array of eigenvalues corresponding to $h = b$. In this case, $\lambda(0)$ does not appear in the array and we have

$$\sum_{m>n} \overline{\lambda}_{m,b}^2 \sim B_b^2 (2r-1)^{-1} n^{1-2r} (\log n)^{2r\beta} , \qquad (4.5) \quad \boxed{\texttt{ass\_err}}$$

where $\beta = (b-1) + b\alpha$. We look for

$$n^{avg}(\varepsilon, d, b) = C_d^b \cdot \inf\{n\, ;\ C_d^b \sum_{m>n} \overline{\lambda}_{m,b}^2 \le \varepsilon^2\}$$

and the result follows from (4.5).

If $\alpha < -1$, then β does not depend on h. Therefore, the eigenvalues $\overline{\lambda}_{n,h}^2$ have the same order of decay for all h. For a given $h \in \{1, \cdots, b\}$, we have to consider eigenvalues $C_{d-h}^{b-h} \overline{\lambda}_{n,h}^2 \lambda(0)^{2(b-h)}$ of multiplicity C_d^h. We include, say, n_h maximal terms in approximating process of rank n. The contribution of the non-included terms to approximation error for this given h is

$$C_d^h \cdot \sum_{m > \frac{n_h}{C_d^h}} C_{d-h}^{b-h} \overline{\lambda}_{m,h}^2 \lambda(0)^{2(b-h)} \sim C(h)(2r-1)^{-1} n_h^{1-2r} (\log n_h)^{2r\beta} , \quad n_h \to \infty ,$$

where $C(h)$ is defined in (4.4).

Let us denote by f the function defined on $[1, \infty[^b$ by

$$f(x_1, \cdots, x_b) = \sum_{h=1}^{b} C(h)(2r-1)^{-1} x_h^{1-2r} (\log x_h)^{2r\beta} .$$

We want to minimize f under the constraint $x_1 + \cdots + x_b = n$. This leads to the optimal values $n_1, \cdots, n_b$. We derive

$$n_h \sim n \cdot \frac{C(h)^{\frac{1}{2r}}}{\sum_{j=1}^{b} C(j)^{\frac{1}{2r}}}$$

which gives

$$f(n_1, \cdots, n_b) \sim Q\,(2r-1)^{-1} n^{1-2r} (\log n)^{2r\beta}\,,$$

where Q is defined in (4.4). The result of Proposition 4.2 b) is now immediate. □

s:increasing

5 Approximation in increasing dimension

We study the approximation of $X_{d,b}$ by a finite sum X_n when the dimension d is increasing. We still work under Assumption 4.1 and consider two basic different situations:
a) the case where the additivity order b is fixed,
b) the case where b goes to infinity and the positive limit $\lim_{d\to\infty} b/d$ exists.
In order to deal with relative errors, we compute the total size of the additive process:

$$\begin{aligned} E\|X_{d,b}\|^2_{L_2(T_d)} &= \sum_{h=0}^{b} C_d^h \sum_{k\in(\mathbb{N}^*)^h} \prod_{k=1}^{h} \lambda(k_l)^2 C_{d-h}^{b-h} \lambda(0)^{2(b-h)} \\ &= C_d^b \sum_{h=0}^{b} C_b^h \tilde{\Lambda}^h \lambda(0)^{2(b-h)} \\ &= C_d^b (\lambda(0)^2 + \tilde{\Lambda})^b = C_d^b \Lambda^b\,. \end{aligned} \tag{5.1}$$

where $\tilde{\Lambda} = \sum_{i=1}^{\infty} \lambda(i)^2$ and $\Lambda = \sum_{i=0}^{\infty} \lambda(i)^2$ We want to evaluate the relative average approximation complexity:

$$\tilde{n}^{avg}(\varepsilon, b, d) = \inf\{n\,;\ E\|X_{d,b} - X_n\|^2_{L_2(T_d)} \le C_d^b \Lambda^b \varepsilon^2\}\,. \tag{5.2}$$

For both cases a) and b), the idea is to compare (in terms of cardinality) the contribution of each array for a fixed h.

5.1 Case b fixed

We have the following approximation.

prop_bfixed **Proposition 5.1** *Let Assumptions* 2.1 *and* 4.1 *hold. When b is fixed and $d \to \infty$,*

$$\tilde{n}^{avg}(\varepsilon, b, d) \sim \frac{d^b}{b!} \Lambda^{-b/(2r-1)} \; n_b^{avg}(\varepsilon)\,,$$

and the asymptotics of $n_b^{avg}(\varepsilon)$ is given in (3.3).

Proof: Recall that the spectrum of covariance operator of additive process of order b can be obtained as follows. To any fixed $h = 1, \cdots, b$ associate an array of eigenvalues

$$\{\lambda_k^2 = \prod_{l=1}^{h} \lambda(k_l)^2\,, k \in (\mathbb{N}^*)^h\}$$

to which two operations are applied:

a) every eigenvalue λ_k^2 is multiplied by $C_{d-h}^{b-h} \lambda(0)^{2(b-h)}$,

b) the array is taken with multiplicity C_d^h.

If we forget about all arrays except for the last one corresponding to $h = b$, then Theorem 3.1 provides the required lower bound

$$\tilde{n}^{avg}(\varepsilon, b, d) \geq C_d^b \; n_b^{avg}(\varepsilon \Lambda^{b/2}) \sim \frac{d^b}{b!} \; \Lambda^{-b/(2r-1)} \; n_b^{avg}(\varepsilon),$$

as $d \to \infty$.

Now we give an approximation construction providing the upper bound for $\tilde{n}^{avg}(\varepsilon, b, d)$. Fix a small δ and include in approximation part $C_d^b \, n_b^{avg}(\varepsilon \Lambda^{b/2})$ terms from the last array ($b = h$) and $C_d^h \; n_h^{avg}(L_{b,h} \delta \varepsilon \Lambda^{b/2})$ terms from every array with $1 \leq h \leq b-1$, where

$$L_{b,h}^2 := [C_b^h]^{-1} = \frac{C_d^b}{C_d^h \; C_{d-h}^{b-h}} \; .$$

The squared approximation error for each $h \leq b-1$ can be evaluated by

$$C_d^h \cdot L_{b,h}^2 \delta^2 \varepsilon^2 \Lambda^b \cdot C_{d-h}^{b-h} \lambda(0)^{2(b-h)} = \delta^2 \varepsilon^2 C_d^b \Lambda^b \cdot \lambda(0)^{2(b-h)},$$

hence the total squared error is bounded by $\delta^2 \varepsilon^2 C_d^b \Lambda^b \sum_{h=1}^{b-1} \lambda(0)^{2(b-h)}$. Taking into account the error in the last array, we see that the total squared relative error of our procedure does not exceed $\left(1+\delta^2\sum_{h=1}^{b-1}\lambda(0)^{2(b-h)}\right)\varepsilon^2$ which can be made arbitrary close to ε as $\delta\to 0$.

Finally, let us evaluate the number of terms in approximation part. For each $h\le b-1$ there exist constants $c_i(b,h,\delta)$ such that

$$\begin{aligned} C_d^h \;\; n_h^{avg}(L_{b,h}\;\delta\varepsilon\Lambda^{b/2}) &\le C_d^h\; c_1(b,h,\delta)\; n_h^{avg}(\varepsilon\Lambda^{b/2}) \\ &\le \frac{d^h}{h!}\; c_2(b,h,\delta)\; n_b^{avg}(\varepsilon\Lambda^{b/2}) \\ &\le d^{b-1}\; c_3(b,h,\delta)\; n_b^{avg}(\varepsilon\Lambda^{b/2}). \end{aligned}$$

Hence the total number of terms in approximation part is bounded by

$$\left(\frac{d^b}{b!}+c_3(b,h,\delta)\,d^{b-1}\right)\; n_b^{avg}(\varepsilon\Lambda^{b/2})\sim\frac{d^b}{b!}\;\Lambda^{-b/(2r-1)}\; n_b^{avg}(\varepsilon),\qquad d\to\infty,$$

as required. $\square$

5.2 Case $b\to\infty$

In this case it is insightful to look at the relative weight of each array of eigenvalues for $h=1,\cdots,b$. Let us fix h and compute the weight of the array, that is the sum of the eigenvalues, taking into account multiplication and multiplicity (see the beginning of the proof of Proposition 5.1).

$$W_h := C_d^h\; C_{d-h}^{b-h}\;\lambda(0)^{2(b-h)}\sum_{k\in(\mathbb{N}^*)^h}\prod_{l=1}^h\lambda(k_l)^2 = C_d^b\; C_b^h\lambda(0)^{2(b-h)}\tilde\Lambda^h\,,$$

and the relative weight

$$\frac{W_h}{E\|X_{d,b}\|^2_{L_2(T_d)}} = C_b^h(1-p)^{b-h}p^h\,. \qquad (5.3) \quad \boxed{\texttt{Bern}}$$

where

$$p=\frac{\tilde\Lambda}{\Lambda}\,.$$

Recall the notation $M=\sum_{i=0}^\infty(-\log\lambda(i))\dfrac{\lambda(i)^2}{\tilde\Lambda}$, $A=e^{2M}\Lambda$.

We see from (5.3) that the distribution of the relative weights is the Binomial distribution $\mathcal{B}(b,p)$. If $b\to\infty$, the main contribution is given by the arrays with h such that $h/b\sim p$. This observation yields the following result:

prop_bincr **Proposition 5.2** *Assume $b, d \to \infty$ with $b/d \to \beta \in [0,1]$ and let Assumptions* 2.1 *and* (3.6) *be valid. Then*

$$\lim_{d\to\infty} \frac{\log \tilde{n}^{avg}(\varepsilon, b, d)}{d} = \log V, \tag{5.4}$$

where

$$V = (1-\beta p)^{-1+\beta p}\beta^{-\beta p}(1-p)^{(1-p)\beta}A^{\beta}\,.$$

Proof: We first give an appropriate approximation procedure. Let $H = H(d,p) = \{h \in \mathbb{N} : \, pd - d^{1/3} \le h \le pd + d^{1/3}\}$. We include in the error term all arrays with $h \notin H$ and include in approximation part $C_d^h\, \tilde{n}_h^{avg}(\varepsilon)$ terms for any $h \in H$. According to (5.3) the total squared error is

$$\sum_{h\notin H} W_h + \varepsilon^2 \sum_{h\in H} W_h \le \left(\mathcal{B}(b,p)(\mathbb{N}\backslash H) + \varepsilon^2\right) C_d^b \Lambda^b.$$

By Moivre-Laplace central limit theorem, $\mathcal{B}(b,p)(\mathbb{N}\backslash H) \to 0$, as $d \to \infty$. Hence the relative error of our procedure is at most $\varepsilon + o(1)$.

Now we will evaluate the number N of terms included in the approximation part. Recall that

$$N = \sum_{h\in H} C_d^h\, \tilde{n}_h^{avg}(\varepsilon).$$

Under our assumptions on b/d and by the choice of H Stirling formula yields, uniformly over $h \in H$,

$$\lim_{d\to\infty} (C_d^h)^{1/d} = \lim_{d\to\infty} (\frac{d-h}{d})^{-(d-h)/d} (\frac{d}{h})^{h/d} = (1-\beta p)^{\beta p - 1}(\beta p)^{-\beta p}\,. \tag{5.5}$$ stir

Moreover, Theorem 3.2 yields

$$\tilde{n}_h^{avg}(\varepsilon)^{1/h} \to \tilde{A}\,, \quad h \to \infty$$

where $\tilde{A} = e^{2\tilde{M}}\tilde{\Lambda}$ and $\tilde{M} = \sum_{i=1}^{\infty}(-\log\lambda(i))\frac{\lambda(i)^2}{\tilde{\Lambda}}$. It follows that, uniformly over $h \in H$,

$$\lim_{d\to\infty} n_h^{avg}(\varepsilon)^{1/d} = \lim_{d\to\infty} n_h^{avg}(\varepsilon)^{\frac{1}{h}\cdot\frac{h}{b}\cdot\frac{b}{d}} = \tilde{A}^{p\beta}.$$

We obtain

$$\lim_{d\to\infty} (C_d^h n_h^{avg}(\varepsilon))^{1/d} = (1-\beta p)^{-1+\beta p}(\beta p)^{-\beta p}\tilde{A}^{\beta p}.$$

Coming back to the constants associated to the full sequence of eigenvalues, we obtain

$$\tilde{M} = \frac{\Lambda}{\tilde{\Lambda}}\, M + \log\lambda(0)\cdot\frac{\lambda(0)^2}{\tilde{\Lambda}} = \frac{M}{p} + \log\lambda(0)(\frac{1}{p}-1),$$

hence

$$\begin{aligned}\tilde{A}^p &= e^{2\tilde{M}p}\tilde{\Lambda}^p = e^{2M+2(1-p)\log\lambda(0)}\ (p\Lambda)^p \\ &= (e^{2M}\Lambda)\ p^p\Lambda^{p-1}[\lambda(0)^2]^{1-p} = A\ p^p\Lambda^{p-1}[\Lambda(1-p)]^{1-p} \\ &= Ap^p(1-p)^{1-p}\,,\end{aligned}$$

therefore

$$\lim_{d\to\infty}(C_d^h n_h^{avg}(\varepsilon))^{1/d} = (1-\beta p)^{-1+\beta p}\beta^{-\beta p}(1-p)^{(1-p)\beta}A^\beta \qquad (5.6)$$ final

as required. Finally, notice that the size of H grows polynomially and does not influence the logarithmic limit. Therefore, our approximation procedure has all required properties.

We will now provide a lower bound for approximation cardinality. Let the set H be as above and let d be so large that (by Moivre-Laplace theorem)

$$\sum_{h\notin H} W_h \le \frac{1}{2}\ C_d^b\Lambda^d.$$

Fix $\varepsilon > 0$ and let X_n be an n-term approximation of $X_{d,b}$ such that

$$E\|X_{d,b} - X_n\|^2_{L_2(T_d)} \le \varepsilon^2\, C_d^b\Lambda^d.$$

Write the expansion $X_{d,b} := \sum_{h=0}^b X_{d,b}^{(h)}$, as done in (2.4). Similarly, we can expand $X_n := \sum_{h=0}^b X_n^{(h)}$. In view of orthogonality we have

$$\begin{aligned}\varepsilon^2\, C_d^b\Lambda^d &\ge E\|X_{d,b} - X_n\|^2_{L_2(T_d)} \\ &= \sum_{h=0}^b E\|X_{d,b}^{(h)} - X_n^{(h)}\|^2_{L_2(T_d)} \\ &\ge \sum_{h\in H} E\|X_{d,b}^{(h)} - X_n^{(h)}\|^2_{L_2(T_d)}. \qquad (5.7)\end{aligned}$$

On the other hand,

$$\sum_{h\in H} E\|X_{d,b}^{(h)}\|_{L_2(T_d)}^2 = \sum_{h\in H} W_h \geq \frac{1}{2}\, C_d^b\, \Lambda^d. \qquad (5.8) \quad \boxed{\texttt{low2}}$$

By comparing (5.7) and (5.8) we see that for some $h \in H$

$$E\|X_{d,b}^{(h)} - X_n^{(h)}\|_{L_2(T_d)}^2 \leq 2\varepsilon^2 E\|X_{d,b}^{(h)}\|_{L_2(T_d)}^2.$$

This means that the relative approximation error for $X_{d,b}^{(h)}$ is small. Recall that the spectral structure of $X_{d,b}^{(h)}$ differs only by multiplication and multiplicity from the field X considered in Section 3 if we put there $b = d = h$. Multiplication of eigenvalues does not influence *relative* approximation error. Multiplicity of eigenvalues should be taken into account when we compute approximation cardinality. We see that

$$n \geq C_d^h\ \tilde{n}_h^{avg}(\sqrt{2}\varepsilon).$$

By using Theorem 3.2 and (5.5) we get

$$\begin{aligned} \lim_{d\to\infty} (\tilde{n}^{avg}(\varepsilon, b, d))^{1/d} &\geq \liminf_{d\to\infty} \inf_{h\in H} (C_d^h)^{1/d}\ (n_h^{avg}(\sqrt{2}\varepsilon))^{\frac{1}{h}\cdot\frac{h}{b}\cdot\frac{b}{d}} \\ &= (1-\beta p)^{-1+\beta p}\beta^{-\beta p}(1-p)^{(1-p)\beta} A^\beta = V, \end{aligned}$$

as required. $\square$

Comments : Let us describe more precisely what happens in some special cases:

• If $\beta = 1$, we get

$$(1-p)^{-1+p}(1-p)^{1-p}A^1 = A\,.$$

This essentially corresponds to the case considered in Section 3.

• It is surprising to note that for $\beta < 1$, the result depends on $p = \tilde{\Lambda}/\Lambda$. We can examine some special values of p:

► if $p = 0$, there is only one eigenvalue, hence $A = 1$, and $V = 1$. There is no exponential explosion.

► if $p = 1$, then $\lambda(0) = 0$ and $V = (1-\beta)^{\beta-1}\beta^{-\beta}A^\beta$.

• If $\beta = 0$, then $V = 1$. There is no exponential explosion, and this includes the case b fixed and $d \to \infty$.

We can prove the following more precise statement:

prop_binc0 **Proposition 5.3** *Assume $b, d \to \infty$ with $b/d \to 0$ and let Assumptions* 2.1 *and* 3.6 *be valid. Then*

$$\lim_{d\to\infty} \frac{\log \tilde{n}^{avg}(\varepsilon, b, d)}{b \log(d/b)} = p. \tag{5.9}$$

Proof: Since the idea is the same as in the previous statement, we omit the details. Now Stirling formula yields

$$\frac{\log C_d^h}{h} = \frac{1}{2h}[\log\left(\frac{d}{d-h}\right) - \log(2\pi h)] + \frac{d-h}{h}\log\left(\frac{d}{d-h}\right) + \log\frac{d}{h} + o(1). \tag{5.10}$$

beta00

We have to compare different terms in expression above. Since $d/b \to \infty$, $b, h \to \infty$ and $b/h \to p$, it is clear that

$$\begin{aligned} \frac{1}{2h}\log\left(\frac{d}{d-h}\right) &= o(1), \\ \frac{1}{2h}\log(2\pi h) &= o(1), \\ \frac{d-h}{h}\log\left(\frac{d}{d-h}\right) &= O(1). \end{aligned}$$

Hence the main term is $\log(d/h)$ and

$$\frac{\log C_d^h}{h} \sim \log(d/h) \sim \log(d/b).$$

Recall that for any $\varepsilon > 0$ it is true that $\log \tilde{n}_h^{avg}(\varepsilon) \sim \tilde{A}h$. Hence,

$$\log\left(C_d^h \cdot \tilde{n}_h^{avg}(\varepsilon)\right) \sim h\log(d/b) \sim p\, b\ \log(d/b),$$

and the proof is complete along the same lines as above. □

6 Some extensions

s:general

6.1 Approximation arguments based on ℓ-numbers

We now briefly remind some precise arguments for elimination of negligible parts from expansions. Let X be a centered Gaussian vector in a Banach space L. The ℓ-numbers $\ell_n(X)$ are defined by

$$\ell_n(X)^2 = \inf\left\{\mathbb{E}\|X - \sum_{j=1}^{n}\varphi_j\xi_j\|^2, \varphi_j \in L, \xi_j \sim \mathcal{N}(0,1)\right\}. \tag{6.1}$$

ln

It is clear from (6.1) that for any vectors X_1 and X_2 and any $n, m \in \mathbb{N}$

$$\ell_{n+m}(X_1 + X_2) \leq \ell_n(X_1) + \ell_m(X_2).$$

By the same argument,

$$\ell_{n+m}(X_1) = \ell_{n+m}((X_1 + X_2) - X_2) \leq \ell_n(X_1 + X_2) + \ell_m(X_2).$$

It follows that

$$\ell_{n+m}(X_1) - \ell_m(X_2) \leq \ell_n(X_1 + X_2) \leq \ell_{n-m}(X_1) + \ell_m(X_2). \qquad (6.2)$$ bilat

Hence the following is true.

x1x2 **Lemma 6.1** *Let (a_n) be a regularly varying sequence. Assume that random vectors X_1, X_2 satisfy $\ell_n(X_1) \sim a_n$ and $\ell_n(X_2) = o(a_n)$. Then $\ell_n(X_1 + X_2) \sim a_n$.*

Proof: Let us fix $\delta > 0$ and set $m = m(n) = [\delta n]$. Then (6.2) yields

$$\ell_n(X_1 + X_2) \leq \ell_{n-[\delta n]}(X_1) + \ell_{[\delta n]}(X_2).$$

We have

$$\begin{aligned} & \limsup_{n\to\infty} \frac{\ell_n(X_1 + X_2)}{a_n} \\ \leq\ & \limsup_{n\to\infty} \frac{\ell_{n-[\delta n]}(X_1)}{a_{n-[\delta n]}} \cdot \frac{a_{n-[\delta n]}}{a_n} + \limsup_{n\to\infty} \frac{\ell_{[\delta n]}(X_2)}{a_{[\delta n]}} \cdot \frac{a_{[\delta n]}}{a_n} \\ \leq\ & 1 \cdot (1-\delta)^\alpha + 0 \cdot \delta^\alpha, \end{aligned}$$

where α is the non-positive regularity index of (a_n). By letting $\delta \to 0$ we obtain

$$\limsup_{n\to\infty} \frac{\ell_n(X_1 + X_2)}{a_n} \leq 1.$$

The lower bound follows along the same lines. $\square$

We stress that no independence or any other condition is assumed on X_1, X_2 in this lemma.

While the definition of ℓ-numbers applies to any Banach space, in the Hilbert space case they are particularly easy to handle. Namely, if

$$X = \sum_{j=1}^{\infty} \lambda_j \varphi_j \xi_j,$$

where (φ_j) is an orthonormal system in L, (ξ_j) i.i.d. standard normal and λ_j a non-increasing positive sequence, then (see [1], [6] or [10], p.51)

$$\ell_n(X)^2 = \mathbb{E}\Big\| \sum_{j=n+1}^{\infty} \lambda_j \varphi_j \xi_j \Big\|^2 = \sum_{j=n+1}^{\infty} \lambda_j^2 \, .$$

We observe that $\ell_n(X)$ is just the inverse sequence to $n^{avg}(\varepsilon)$ for X. Therefore, we can restate Lemma 6.1 as follows.

x1x2a **Lemma 6.2** *Let g be a regularly varying function defined in a neighborhood of zero. Assume that random vectors X_1, X_2 satisfy*

$$n^{avg}(X_1;\varepsilon) \sim g(\varepsilon) \quad \textit{and} \quad n^{avg}(X_2;\varepsilon) = o\left(g(\varepsilon)\right) \quad \textit{as } \varepsilon \to 0.$$

Then $n^{avg}(X_1 + X_2;\varepsilon) \sim g(\varepsilon)$.

6.2 Approximation without Assumption 2.1

In this subsection we explain how to get rid of restrictive Assumption 2.1. Let $Y(u), u \in [0,1]$ be arbitrary centered second order process. Let denote $I := \int_0^1 Y(u)\mathrm{d}u$, $\sigma^2 = \mathbb{E}[I^2]$, and $\kappa(u) := cov\left(Y(u), I\right)/\sigma^2$. We split Y in two non-correlated parts; one of them is degenerate (has rank one), while another satisfies Assumption 2.1. Namely, let

$$Y(u) = Y_0(u) + \hat{\mathcal{Y}}(u) := [Y - \kappa(u)I] + \kappa(u)I. \qquad (6.3)$$ 0hat

Indeed, $\hat{\mathcal{Y}}$ has rank one and for all $u_0, u \in [0,1]$ we have

$$\begin{aligned} cov(Y_0(u_0), \hat{\mathcal{Y}}(u)) &= \mathbb{E}\left[(Y(u_0) - \kappa(u_0)I)\,\kappa(u)I\right] \\ &= \kappa(u)\left[cov(Y(u_0), I) - \kappa(u_0)\sigma^2\right] = 0 \end{aligned}$$

and

$$\begin{aligned} \int_0^1 Y_0(u)du &= I - \sigma^{-2}\int_0^1 cov(Y(u), I)\mathrm{d}u \cdot I \\ &= \left[1 - \sigma^{-2} cov\left(\int_0^1 Y(u)\mathrm{d}u, I\right)\right] \cdot I = 0. \end{aligned}$$

It follows from the latter identity that Y_0 satisfies Assumption 1.1 with $\lambda(0) = 0$. The parts of the decomposition (6.3) are *not* orthogonal in $L_2[0,1]$. The same is true for multi-parametric expansions based on (6.3).

Now we recall some elementary algebra of tensor products. Given a finite sequence of fields $\{Y_j(t)\}_{t\in T_{d_j}}, 1 \le j \le b$, each of them being decomposed in two non-correlated parts $Y_j = Y_{j0} + Y_{j1}$, we have

$$\bigotimes_{j=1}^{b} Y_j = \sum_{i\in\{0,1\}^b} \bigotimes_{j=1}^{b} Y_{ji_j}$$

where the terms of the right hand side are pairwise non-correlated. This formula is obvious if we look at the respective covariances.

For tensor degrees of a one-parametric process $Y = Y_0 + Y_1$ the above formula yields

$$\begin{aligned} Y^{\otimes b} &= \sum_{i\in\{0,1\}^b} \bigotimes_{j=1}^{b} Y_{i_j} \\ &= \sum_{A\subset\{1,\dots,b\}} Y_0^{\otimes|A|}(\Pi_A(\cdot)) \otimes Y_1^{\otimes(b-|A|)}(\Pi_{A^c}(\cdot)). \end{aligned}$$

Applying this to (6.3), we obtain

$$Y^{\otimes b} = \sum_{A\subset\{1,\dots,b\}} Y_0^{\otimes|A|}(\Pi_A(\cdot)) \otimes \hat{\mathcal{Y}}^{\otimes(b-|A|)}(\Pi_{A^c}(\cdot)) := \sum_{A\subset\{1,\dots,b\}} Z_A.$$

Now let us consider the approximation properties of each term in this expansion.

Assume that Assumption 4.1 is verified and let $\alpha = q/r > -1$.

Let us fix A and let $h = |A|$. Since $\hat{\mathcal{Y}}$ has rank one, the same is true for $\hat{\mathcal{Y}}^{\otimes(b-h)}$. Therefore, the second factor does not influence approximation properties. On the other hand, since Y_0 differs from Y only by a process of rank one, it inherits from Y the validity of Assumption 4.1 by Weil lemma. Now we consider separately the main term corresponding to $A = \{1,\dots,b\}$ and all other terms (with $h < b$). Indeed, under $h < b$ Theorem 3.1 yields

$$\begin{aligned} n^{avg}(Z_A,\varepsilon) &= O\left(\left(\frac{|\log\varepsilon|^{r(h-1)+h\alpha}}{\varepsilon}\right)^{(r-1/2)^{-1}}\right) \\ &= o\left(\left(\frac{|\log\varepsilon|^{r(b-1)+b\alpha}}{\varepsilon}\right)^{(r-1/2)^{-1}}\right). \end{aligned}$$

The main term has a larger cardinality asymptotics of the just mentioned order.

Let us now consider the additive processes. We can write (2.1) as

$$X_{d,b}(t) = \sum_{A\subset D_b} Y_A^{\otimes b}\left([\Pi_A(t)]\right)$$

where $Y_A^{\otimes b}$ are non-correlated copies of $Y^{\otimes b}$, and introduce its main part generated by Y_0 as

$$X_{d,b}^0(t) = \sum_{A\subset D_b} Y_{0,A}^{\otimes b}\left([\Pi_A(t)]\right),$$

where $Y_{0,A}^{\otimes b}$ are non-correlated copies of $Y_0^{\otimes b}$. Since Y_0 satisfies Assumption 2.1, Proposition 4.2 applies and we get the asymptotics (4.2) for average cardinalities of $X_{d,b}^0$. On the other hand, the difference between $X_{d,b}^0$ and $X_{d,b}$ is a finite sum of the fields with lower order of average cardinalities. Therefore, by Lemma 6.2 for $X_{d,b}$ we get the same result (4.2) as for $X_{d,b}^0$. We get the following:

Corollary 6.3 *If $\alpha = q/r > -1$ in Assumption* 4.1*, Proposition* 5.1 *is true without Assumption* 2.1.

Our arguments do not apply to the case $\alpha < -1$, where the secondary terms bring the contribution of the same order as the main term.

It would be very interesting to understand what happens with the results about additive process with *variable* b in absence of Assumption 2.1. Recall that the eigenvalue $\lambda(0)^2$ directly related to this assumption explicitly appears in the answer via parameter $p = 1 - \frac{\lambda(0)^2}{\Lambda}$. Therefore, we can not expect that the results like Proposition 5.2 will be the same in absence of Assumption 2.1.

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