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Julien Brémont

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# ENTROPY AND MAXIMIZING MEASURES OF GENERIC CONTINUOUS FUNCTIONS

Julien Brémont

Université Paris 12, novembre 2007

## Abstract

In the natural context of ergodic optimization, we provide a short proof of the assertion that the maximizing measure of a generic continuous function has zero entropy.

## Abstract

Dans le cadre usuel de l'étude des mesures maximisantes, nous donnons une preuve courte du fait que la mesure maximisante d'une fonction continue générique est d'entropie nulle.

## 1 Introduction

Let  $(X, T)$  be a topological dynamical system, where  $X$  is a compact metric space with a continuous transformation  $T : X \rightarrow X$ . Introduce the set  $\mathcal{M}_T$  of Borel  $T$ -invariant probability measures on  $X$ , endowed with the compact and metrizable weak-\* topology. We assume that measures supported by a periodic orbit are dense in  $\mathcal{M}_T$  and that the map  $\mu \mapsto h(\mu)$  is upper-semi-continuous (usc) on  $\mathcal{M}_T$ . These assumptions are for instance verified if  $(X, T)$  satisfies expansiveness and specification (cf Denker-Grillenberger-Sigmund [5]).

Fixing a continuous  $f : X \rightarrow \mathbb{R}$ , “ergodic optimization” (see Jenkinson [6] and references therein) is concerned with the following variational problem :

$$\beta(f) = \sup \{ \mu(f) \mid \mu \in \mathcal{M}_T \} \text{ and } \text{Max}(f) = \{ \mu \in \mathcal{M}_T \mid \beta(f) = \mu(f) \},$$

where  $\mu(f)$  is for  $\int f d\mu$ . The aim is to describe the set  $\text{Max}(f)$  of *maximizing measures* for  $f$ , which is always a non-empty compact and convex subset of  $\mathcal{M}_T$ . Notice also that any measure in the ergodic decomposition of a maximizing measure is a maximizing measure. We consider here genericity results in functional spaces. Recall that a set is *residual* if it contains a dense  $G_\delta$ -set. A property defining a residual set is *generic*. An element in a residual set is declared *generic*.

The regularity of  $f$  plays a crucial role. In a Hölder or Lipschitz functional space, the Conze-Guivarc'h-Mañé lemma (see [6] for instance) gives a characterization of the maximizing measures via their support. The analysis is fairly delicate and difficult conjectures about periodic measures remain open (cf [6], [3] and references therein). The analysis of the case of the space  $C(X)$  of real-valued continuous functions on  $X$  (endowed with the supremum norm) is completely different. The Conze-Guivarc'h-Mañé lemma is not valid any more, but duality arguments are available. Bousch and Jenkinson [1, 2] showed that for a generic  $f$  in  $C(X)$  the situation is somehow pathological.

### **Theorem 1.1** (*Bousch-Jenkinson*)

*A generic function in  $C(X)$  has a unique maximizing measure and it has full support.*

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In a recent article on a closely related problem, Jenkinson and Morris [7] considered the entropy of “Lyapunov maximizing measures” for  $C^1$ -expanding maps of the Circle. Certainly, their method allows to complete the picture in the following way, in some sense restricting the “pathology” :

**Theorem 1.2** (*Jenkinson-Morris*)

*The maximizing measure of a generic function in  $C(X)$  has zero entropy.*

The purpose of this note is to give a short and rather elegant proof of the latter result. Let us also mention that in the particular case of a symbolic setup (such as the shift  $T$  on some product space  $X = \{0, \dots, m-1\}^{\mathbb{Z}}$ ), theorem (1.2) can be proved elementarily using the density in  $C(X)$  of locally constant functions, cf Conze-Guivarc’h [4].

## 2 Proof of theorem 1.2

Define a non-negative map  $\varphi : f \mapsto \sup_{\mu \in \text{Max}(f)} h(\mu)$  on  $C(X)$ . Let us check that it is usc. Indeed, since  $\mu \mapsto h(\mu)$  is usc and  $\text{Max}(f)$  is compact,  $\varphi(f) = h(\mu_f)$  for some  $\mu_f \in \text{Max}(f)$ . If now  $f_n \rightarrow f$ , then up to extraction  $\mu_{f_n}$  weakly converges to some  $\mu$  in  $\text{Max}(f)$  and thus  $\varphi(f) \geq h(\mu) \geq \limsup h(\mu_{f_n})$ . This proves the assertion.

As a result,  $\varphi$  is continuous on a residual set  $R$ . We will show that  $\varphi$  in restriction to  $R$  is equal to zero. This latter fact will be a corollary from the following claim, of independent interest.

**Proposition 2.1** *On a dense set  $D$  in  $C(X)$ , the maximizing measure is unique and supported by a periodic orbit.*

Assuming this result, let  $f \in R$  and  $f_n \rightarrow f$  with  $f_n$  in  $D$ . Since  $\varphi(f_n) = 0$  and  $\varphi$  is continuous at  $f$ , we get  $\varphi(f) = 0$ . Thus  $\varphi|_R$  is zero, as announced. This gives theorem (1.2).

To prove the latter proposition, first notice that it is enough to show that densely in  $C(X)$  there is a periodic maximizing measure. Indeed, if  $g$  has a maximizing measure  $\mu$  supported by some periodic orbit  $\text{Orb}(x_0)$ , introduce for  $\eta_0 > 0$  the map  $\eta(x) = -\eta_0 \text{dist}(x, \text{Orb}(x_0))$ ,  $\forall x \in X$ . Then for  $\nu \in \mathcal{M}_T$ , one has  $\nu(g + \eta) = \nu(g) + \nu(\eta)$  and  $\nu(g) \leq \beta(g)$  and  $\nu(\eta) \leq 0$ , with both equalities simultaneously if and only if  $\nu = \mu$ . We therefore obtain  $\text{Max}(g + \eta) = \{\mu\}$  and this gives the result since  $\|\eta\|_\infty \rightarrow 0$  as  $\eta_0 \rightarrow 0$ .

To conclude, take any  $f$  and a measure  $\mu \in \mathcal{M}_T$  supported by a periodic orbit with small  $\beta(f) - \mu(f)$ . By the next proposition,  $f$  can be perturbed into  $g$  with a maximizing measure  $\nu$  such that  $\mu$  and  $\nu$  are not mutually singular (taking  $\varepsilon = 1/2$  in the statement of the proposition). As  $\mu$  is ergodic, it appears in the ergodic decomposition of  $\nu$  and thus  $\mu \in \text{Max}(g)$ . □

The next proposition comes from the classical proof of the Bishop-Phelps theorem. It is adapted from a preliminary version of Pollicott and Sharp [8].

**Proposition 2.2** *Let  $f \in C(X)$  and  $\mu \in \mathcal{M}_T$ . Write  $\beta(f) - \mu(f) = \varepsilon\delta$ , with  $\varepsilon \geq 0, \delta \geq 0$ . Then there exist  $g \in C(X)$  and  $\nu \in \text{Max}(g)$  such that  $\|f - g\|_\infty \leq \delta$  and  $\|\mu - \nu\|_{C(X)} \leq \varepsilon$ .*

*Proof of the proposition :* From homogeneity and the fact that  $\text{Max}(g) = \text{Max}(\lambda g)$  for  $\lambda > 0$ , it is enough to suppose that  $\delta = 1$ . Clearly we can also assume that  $\varepsilon > 0$ . Define  $\Phi(u) = \beta(u) - \mu(u)$  on  $C(X)$  and let, for  $v \in C(X)$  :

$$A(v) = \{u \in C(X) \mid \Phi(u) \leq \Phi(v) - \varepsilon\|v - u\|_\infty\}.$$

By the triangular inequality, observe that  $A(u) \subset A(v)$  if  $u \in A(v)$ . Let now  $f_0 = f$ , with  $\Phi(f_0) = \varepsilon$ , and for  $n \geq 0$ , choose  $f_{n+1} \in A(f_n)$  such that  $\Phi(f_{n+1}) \leq 2^{-n-1}\varepsilon + \inf\{\Phi(u) \mid u \in A(f_n)\}$ . Then  $(A(f_n))$  is decreasing and one has for  $n \geq 0$  and any  $u \in A(f_n)$  :

$$\Phi(f_n) - \varepsilon 2^{-n} \leq \Phi(u) \leq \Phi(f_n) - \varepsilon\|f_n - u\|_\infty.$$

As a result  $\|f_n - u\|_\infty \leq 2^{-n}$ . Thus  $(f_n)$  is a Cauchy sequence converging to some  $g$  and  $\text{diam}(A(f_n)) \leq 2^{-n+1}$ . Therefore  $\|f - g\|_\infty \leq 1$  and  $A(g) = \{g\}$ .

By this last property of  $g$ , the open convex set  $\{(u, y) \in C(X) \times \mathbb{R} \mid y < \Phi(g) - \varepsilon\|g - u\|_\infty\}$  and the convex set  $\{(u, y) \in C(X) \times \mathbb{R} \mid y \geq \Phi(u)\}$  are disjoint. From the Hahn-Banach separation theorem (cf Ruelle [9], Appendix A.3.3 (a)), there is a linear form  $L(u, y) = y - \tilde{\mu}(u)$ , with a signed Borel measure  $\tilde{\mu}$ , and  $t \in \mathbb{R}$  such that for all  $u \in C(X)$  :

$$\Phi(g) - \varepsilon\|g - u\|_\infty - \tilde{\mu}(u) \leq t \leq \Phi(u) - \tilde{\mu}(u).$$

Taking  $u = g$  gives  $t = \Phi(g) - \tilde{\mu}(g)$ . Thus for all  $u \in C(X)$ , we have  $\Phi(g) - \tilde{\mu}(g - u) \leq \Phi(u)$  and  $\tilde{\mu}(g - u) \leq \varepsilon\|g - u\|_\infty$ , which can be rewritten as  $\beta(g) + (\mu + \tilde{\mu})(u) \leq \beta(g + u)$  and  $|\tilde{\mu}(u)| \leq \varepsilon\|u\|_\infty$ .

Consequently and by definition,  $\nu = \mu + \tilde{\mu}$  is a tangent functional for  $\beta$  at  $g$  (cf Ruelle [9], Appendix A.3.6). As detailed in the next lemma, it is thus a maximizing measure for  $g$ . □

**Lemma 2.3** *Let  $f \in C(X)$  and a signed Borel measure  $\nu$  be such that  $\beta(f) + \nu(g) \leq \beta(f + g)$ , for all  $g \in C(X)$ . Then  $\nu$  is an invariant probability measure and it belongs to  $\text{Max}(f)$ .*

*Proof of the lemma :* Let  $g \geq 0$  in  $C(X)$ . Since  $\beta(f) \geq \beta(f - g)$ , we get  $\nu(g) \geq \beta(f) - \beta(f - g) \geq 0$ . Thus  $\nu$  is positive. Also for any real constant  $a$ , we have  $\beta(f + a) = \beta(f) + a$ , giving  $\nu(a) \leq a$ . Therefore  $\nu(1) = 1$  and  $\nu$  is a probability measure.

Let  $g \in C(X)$ . Since  $\beta(f + g - g \circ T) = \beta(f)$ , we have  $\nu(g - g \circ T) \leq 0$ . Taking  $-g$ , we get equality. Thus  $\nu$  is  $T$ -invariant. Next, as  $\beta(0) = 0$ , when taking  $g = -f$  we obtain  $\beta(f) - \nu(f) \leq 0$ . This shows that  $\nu \in \text{Max}(f)$  and concludes the proof of the lemma. □

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LABORATOIRE D'ANALYSE ET DE MATHÉMATIQUES APPLIQUÉES, UNIVERSITÉ PARIS XII, FACULTÉ DES SCIENCES ET TECHNOLOGIES, 61 AVENUE DU GÉNÉRAL DE GAULLE, 94010 CRÉTEIL CEDEX, FRANCE  
*E-mail address* : [bremont@univ-paris12.fr](mailto:bremont@univ-paris12.fr)