Medium frequency linear vibrations of anisotropic elastic structures

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MEDIUM FREQUENCY LINEAR VIBRATIONS OF ANISOTROPIC ELASTIC STRUCTURES

by

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SUMMARY

A new numerical method is proposed for studying medium frequency linear vibrations of anisotropic viscous elastic structures, based on an energy principle and without using the base of the vibration eigenmodes of associated conservative system. The values generated are all deterministic. These can be used to characterize the vibratory state by frequency band, to study the response to a medium frequency deterministic or random steady excitation, to determine for a fixed frequency band the space distributions of the excitation forces that, when applied to the entire structure or to a given part of the structure, produce the maximum vibrations on the entire structure or an a specified part. The method also facilitates the study of the spatial propagation of vibrations in the elastic medium and makes it possible to compute the modal density.

I – INTRODUCTION

Theoretically, the linear vibrations of an elastic, viscous anisotropic structure occupying a bounded domain in space, slightly damped, can be studied without difficulty if we know explicitly the spectrum $\{\omega_j\}, j \in \mathbb{N}$ of eigenfrequencies for the associated undamped system, and the corresponding modal basis $\{\varphi_j\}, j \in \mathbb{N}$.

In practice, for anisotropic elastic structures of any given *a priori* geometry, the modal basis $\{\varphi_j\}$ is not explicitly known and must be calculated numerically. We are then led to consider several cases :

1) In the low frequency range, linear vibrations of structures can conventionally be solved :

M1 – By direct numerical time integration of the equations.

M2 – By a numerical time integration of the uncoupled equations in the truncated modal basis $\{\varphi_1, \varphi_2, \ldots, \varphi_m\}$. This is the modal synthesis in the time domain.

M3 – By calculating the frequency response function T_ω, which can be

(*a*) carried out using the truncated modal basis $\{\varphi_1, \varphi_2, \ldots, \varphi_m\}$:

$$T_\omega \simeq \sum_{j=1}^{m} h_j(\omega)\,\varphi_j \otimes \varphi_j ;$$

(*b*) or carried out directly calculating, for each value of ω considered :

$$T_\omega = (-\omega^2 M + i\omega C + K)^{-1},$$

where M, C and K designate the mass, damping and stiffness operators, respectively.

2) We are interested here in the medium frequency range, which we define as follows :

– the excitation frequencies are not high enough to be able to use asymptotic methods *a priori*.

– the excitation frequencies are not low enough for the response to be of the modal type *a priori*, but bring into the answer a very large number of high order vibration eigenmodes (the eigenmodes here being ordered by increasing eigenfrequencies) while the modal density can be high *a priori* (one mode per hertz, for example, in the vicinity of 3 000 Hz).

Let us note first of all that the structure must be discretized finely into finite elements in this medium frequency range. This leads us to reason using discretized systems with a large number of degrees of freedom.

If applied to medium frequencies, the methods described above for low frequencies would lead to the following adaptations in use :

– As the frequencies are high, 3 000 or 4 000 Hz for instance, method M1 requires a very small integration time step, not to mention the numerical damping problems that may crop up in certain cases.

– Methods M2 and M3(*a*) call for the calculation of the vibration eigenmodes up to a high order. As the discretized system has a great many degrees of freedom, it seems difficult to determine the necessary truncated modal base directly and with enough precision to separate the modes, considering the current state of numerical precision and knowledge of the algorithms used to find the eigenvalues and eigenvectors. One can then consider, for example, dynamic substructuring techniques to determine the eigenmodes.

Method M3(*b*) involves finding the solution to a complex linear system containing a large number of equations, with the reduction of the matrix,

$$-\omega^2 [M] + i\omega [C] + [K]$$

carried out for each ω. If the calculation is to be carried out only for a few values of ω, this method is then very effective in the medium frequency range insofar as, for the calculated values of ω, there is no numerical conditioning problem. This is true when the excitation is on one or several rays. On the other hand, if the excitation is on a band that is not narrow, as is often the case for both deterministic and stationary random excitations in the medium frequency domain, numerous values of ω must then be considered *a priori* for dynamic identification. The cost of this method may then appear prohibitive. Finally, as far as vibration prediction problems are concerned, since the excitation is not at a fixed frequency but must cover a relatively wide band (e.g. 2 000 to 3 000 Hz), the method M3(*b*) should normally lead to the consideration of very many values of ω, because since the elastic medium is bounded, anisotropic and of indeterminate geometry, nothing is known beforehand of the variations in the operator T_ω when ω varies throughout the domain of frequencies studied. We can obtain modal behaviors as in low frequency, locally or not, as well as global dynamic behaviors (grouping by eigenmode packages when the modal density is large enough) or quasi-static behaviors. All of these behaviors may appear simultaneously at certain frequencies in the band considered, depending on the part of the structure studied or the component of vibrations observed. For example, for a radially excited slender cylindrical shell a modal behavior with respect to the axial displacement can be obtained at certain frequencies simultaneously with a global radial behavior calling for the superposition of a modal package. Under these conditions, it seems difficult to assume a slow variation of $\omega \mapsto T_\omega$ in the medium frequency band studied before any calculations are made. This excludes, *a priori*, a calculation involving only a few values of ω.

3) We are proposing here a numerical method of solution for the medium frequencies, based on the following assumptions :

– For the medium frequency range, the vibratory state of the system is characterized by frequency band and no longer by discrete frequency values.

– The frequency response function is replaced by a calculated frequency response function, calculated by frequency band.

– The response at time t of an observation of the vibrating system is replaced by the average response over the band, that is the square root of the response energy over the band.

– The idea of "appropriation" of an eigenmode to its eigenfrequency, an idea which is used in low frequency, is replaced by the search for extremum vibratory states per frequency band.

– The idea of modal density per frequency band is retained. This can of course be calculated without calculating the spectrum $\{\omega_j\}, j \in \mathbb{N}$ of eigenfrequencies of the associated undampened system. Nonetheless, the modal density does not appear in this theory and can thus be considered as a "by-product" of the calculations.

The quantities we are going to construct are all deterministic and can be used :

– to identify the medium frequency dynamics of a viscous, anisotropic linear elastic medium of any given shape ;

– to study the response to any deterministic excitation in the medium frequency domain, or to a stationary random excitation with the spectral power measurement concentrated in a medium frequency band ;

– to study the space propagation of the vibrations in the elastic medium ;

– to determine, for a fixed frequency band, the space distributions of the excitation forces which, when applied to the whole structure or to a given part of the structure, produce extremum vibratory states throughout the elastic system or in a specified part of the same system ;

– possibly to calculate the modal density.

II – VIBRATION EQUATIONS AND ASSUMPTIONS

We study the linear vibrations of an elastic anisotropic body that, in its reference configuration, occupies an open bounded domain Ω of $\mathbb{R}^3$, with boundary $\partial\Omega$ supposed C^0 and C^1 by parts.

We call $x=(x_1, x_2, x_3)$ a point of $\mathbb{R}^3, \partial_t$ (resp. ∂_j) the partial derivative with respect to t (resp. x_j).

The common convention of summation on repeated dummy indices is used.

Let $u=(u_1, u_2, u_3)$ be the field of displacement of the elastic body. On the part Γ_u of $\partial\Omega$, measured positive, we set $u|_{\Gamma_u}=0, \forall t$.

The variational formulation of this elastodynamic problem involves a classical introduction of the following complex Hilbert spaces : $H=\{u;\, u_j \in L^2(\Omega)\}$ associated with the scalar product :

$$(u, v)_H = \int_\Omega u_j(x)\overline{v_j(x)}\,dx,$$

and

$$V=\{u;\, u_j \in H^1(\Omega);\, u_j|_{\Gamma_u}=0\}.$$

We have :

$$V \subset H = H' \subset V'.$$

II.1 – MASS, DAMPING AND STIFFNESS OPERATORS.

Let M (resp. C), the linear mass (damping resp.) operator, be real continuous, symmetric and positive definite on H. We then have :

$$\forall u, v \in H, \quad (M u, v)_H = \int_\Omega \rho(x) u_j(x)\overline{v_j(x)}\,dx, \quad (1)$$

where ρ is the density of the elastic medium, which verifies the hypothesis :

$$\forall x \in \Omega, \quad 0<\rho_1 \leqq \rho(x) \leqq \rho_2 < +\infty, \quad (2)$$

where ρ_1 and ρ_2 are two fixed real constants, strictly positive and finite.

Let K be the linear stiffness operator such that :

$$\left.\begin{aligned} &\forall u, v \in V, \\ &\langle K u, v\rangle_{V',V} = \int_\Omega a_{ijkh}(x)\varepsilon_{kh}(u)\varepsilon_{ij}(\overline{v})\,dx, \end{aligned}\right\} \quad (3)$$

where $\varepsilon_{ij}(u)=(\partial_j u_i+\partial_i u_j)/2$. The strain tensor, the elasticity constants $a_{ijkh}(x)$ verify the usual properties of symmetry and positiveness.

Under these conditions the linear stiffness operator K is continuous, real, symmetric from V to V', and we have :

$$\forall u \in V, \quad \langle K u, v\rangle_{V',V} \geqq \mu \|u\|_V^2, \quad (4)$$

where μ is a real constant, fixed strictly positive.

II.2 – SYSTEM FREQUENCY RESPONSE OPERATORS.

Let $i=\sqrt{-1}$. Considering our assumptions, [6,9] show that $\forall \omega \in \mathbb{R}$, the linear operator $-\omega^2 M + i\omega C + K$ in H having a domain $\mathrm{Dom}\, K=\{u \in V,\, K u \in H\}$ allows as an inverse the compact linear operator T_ω from H to H :

$$T_\omega = (-\omega^2 M + i\omega C + K)^{-1}. \quad (5)$$

The operator T_ω is called frequency response operator relative to the displacement field u. It is also shown that the series of the squares of the eigenvalue moduli is convergent, and is called a Hilbert Schmidt operator.

II.3 – DEFINITION OF THE CLASS OF EXCITATION FORCES.

As we indicated in the introduction, we are going to study medium frequency vibrations in a frequency band, and no longer at discrete frequency values. In all that follows, the "frequency ω" indicates an angular frequency.

Then let B_n be a medium frequency band with a central frequency $n\,\Delta\omega$, n being an integer and $\Delta\omega$

being the bandwidth. The band B_n is the compact interval of $\mathbb{R}$:

$$B_n=\left[\left(n-\frac{1}{2}\right)\Delta\omega,\left(n+\frac{1}{2}\right)\Delta\omega\right]. \qquad (6)$$

To characterize the medium frequency vibrations of the elastic medium, we consider the excitation class comprising the functions $F_n\ (t, x)$ such that :

$$F_n=f_n\otimes\psi, \qquad (7)$$

where :

$f_n\ (t)$ is the time component of the excitation at complex values such that $f_n\in L^2(\mathbb{R})$, and such that its Fourier transform $\hat{f}_n=\mathscr{F}(f_n)$ have the compact support B_n ;

$\psi\ (x)$ is the space component of the excitation. We assume here that the surface forces applied to $\Gamma_s=\partial\Omega\setminus\Gamma_u$ and the volume forces applied in Ω are such that the space component of the excitation can be represented by a given element ψ in H ; (9)

Consequently, $F_n=f_n\otimes\psi\in L^2(\mathbb{R}, H)$. (10)

II.4 – PROPERTY OF THE FUNCTIONS F_n.

(a) The partial Fourier transform $\hat{F}_n$ with respect to t of $F_n\in L^2(\mathbb{R}, H)$ is the element of $L^2(\mathbb{R}, H)$ which is written :

$$\hat{F}_n=\hat{f}_n\otimes\psi. \qquad (11)$$

Consequently, $\omega\mapsto\hat{F}_n(\omega)$, at values in H, is compact support B_n.

(b) Let $f_0\ (t)$ be the function such that :

$$f_0(t)=f_n(t)\exp(-in\,\Delta\omega\, t), \qquad \forall\, t\in\mathbb{R}. \qquad (12)$$

Then $f_0\in L^2(\mathbb{R})$ and its Fourier transform support is

$$\operatorname{supp}\hat{f}_0=B_0=\left[-\frac{\Delta\omega}{2},\frac{\Delta\omega}{2}\right].$$

as

$$\hat{f}_n(\omega)=\int_{\mathbb{R}} e^{-i\omega t} f_n(t)\,dt$$

$$=\int_{\mathbb{R}} e^{-i(\omega-n\Delta\omega)t} f_0(t)\,dt=\hat{f}_0(\omega-n\,\Delta\omega),$$

and since $\operatorname{supp}\hat{f}_n=B_n$, we have $\operatorname{supp}\hat{f}_0=B_0$.

(c) For a given excitation $F_n=f_n\otimes\psi$ the energy is defined as the energy of the signal F_n ; that is

$$e_n(\psi)=\int_{\mathbb{R}}\|F_n\|_H^2\,dt. \qquad (13)$$

Using Plancherel's formula we obtain :

$$e_n(\psi)=\|\psi\|_H^2\int_{\mathbb{R}}|f_n(t)|^2\,dt \qquad (14)$$

$$=\frac{1}{2\pi}\|\psi\|_H^2\int_{B_n}|\hat{f}_n(\omega)|^2\,d\omega.$$

(d) Let us give an example of the function f_n.

For all $n\in\mathbb{Z}$, let $\omega\mapsto p_n(\omega)$ be the function of $L^2(\mathbb{R})$ such that

$$\left.\begin{aligned} p_n(\omega)&=1 \qquad \text{si} \quad \omega\in B_n;\\ p_n(\omega)&=0 \qquad \text{si} \quad \omega\notin B_n.\end{aligned}\right\} \qquad (15)$$

Then taking

$$f_0(t)=\frac{1}{\pi t}\sin\left(t\frac{\Delta\omega}{2}\right), \qquad (16)$$

we have

$$\left.\begin{aligned} &\hat{f}_0(\omega)=p_0(\omega);\quad \hat{f}_n(\omega)=p_n(\omega);\\ &\int_{B_n}|\hat{f}_n(\omega)|^2\,d\omega=\Delta\omega.\end{aligned}\right\} \qquad (17)$$

II.5 – EQUATION OF THE MEDIUM FREQUENCY VIBRATIONS.

For a given $F_n=f_n\otimes\psi$ belonging to the class defined in para. II.3, we are interested in the medium frequency vibration u in the band B_n defined by its partial Fourier transform $\hat{u}$ with respect to t, such that

$$\omega\mapsto\hat{u}(\omega)=\hat{f}_n(\omega)(T_\omega\psi), \qquad (18)$$

where T_ω is the frequency response operator defined by (5).

As $\hat{f}_n$ has compact support B_n, equation (18) shows that the Fourier transform $\omega\mapsto\hat{u}(\omega)$ also has compact support B_n.

The vibration u defined by (18) is interpreted as the forced solution associated with $f_n\otimes\psi$ of :

$$M\,\partial_t^2 u+C\,\partial_t u+K\,u=f_n\otimes\Psi, \qquad (19)$$

where $u\in L^2(\mathbb{R}, V)$ and $\partial_t u\in L^2(\mathbb{R}, H)$.

II.6 – MECHANICAL SYSTEM OBSERVATION OPERATOR.

When the displacement field $u=(u_1,\ u_2,\ u_3)$ is known, the vibratory state of the mechanical system subjected to excitation F_n in the band B_n is completely determined.

We can nonetheless consider one or more components of the field u at a fixed point of $\bar{\Omega}$, or a component of a strain tensor or the tensor of stresses at one point, etc.

An observation operator Q is thus introduced, operating only on the space variable x. When applied to the displacement field u, this gives the observation defined by Q and denoted by σ.

We suppose that the observation σ at any fixed instant t takes its values in a complex finite or infinite Hilbert space W with the scalar product denoted by $(v, w)_W$ and that the operator Q is *real*, linear, continuous from H to W :

$$\sigma=Q\,u. \qquad (20)$$

For example, to obtain the value of the displacement field at a given point x_0 in $\overline{\Omega}$, we must take $W=\mathbb{C}^3$ and $Q=\delta_{x_0}$, where δ_{x_0} designates the measure of Dirac at the point x_0. In the same way, the jth component of the displacement field at x_0 is obtained by taking $W=\mathbb{C}$ and $Q=\delta_{x_0}\circ\Pi_j$, where Π_j is the jth canonical projection from $\mathbb{C}^3$ on $\mathbb{C}$. We then have $\sigma(t)=(\delta_{x_0}\circ\Pi_j)(u)=u_j(t, x_0)$.

In the same way, if we want the component $\varepsilon_{ij}(u)$ of the strain tensor at the point x_0 of Ω, we take $W=\mathbb{C}$ and $\sigma=Qu=(\varepsilon_{ij}(u))_{x=x_0}$, etc.

Note that, for $W=H$, and Q is the identical application of H, $\sigma=u$; that is, the observation is the entire field of displacement.

III – ENERGETIC CHARACTERISTICS OF THE VIBRATIONS IN THE BAND B_n.

In this paragraph we will introduce some energy quantities that will be used to characterize the vibratory state of the mechanical system subject to an excitation $f_n\otimes\psi$ in the band B_n.

III.1 – CHARACTERIZATION OF THE VIBRATORY INTENSITY OF AN OBSERVATION IN THE BAND B_n.

Let $\sigma=Qu$ be an observation of the mechanical system.

The vibratory intensity of the response σ due to the given excitation $F_n=f_n\otimes\psi$ of energy $e_n(\psi)$ will be characterized by the square root of the scalar :

$$E_Q^n(\psi)=\int_{\mathbb{R}}\|\sigma\|_W^2\,dt. \tag{21}$$

This scalar represents the total energy of the signal $\sigma\in L^2(\mathbb{R}, W)$. By applying Plancherel's theorem we can write the following, since $\omega\mapsto\hat{u}(\omega)$ has support B_n, and considering (18) :

$$\begin{aligned}E_Q^n(\psi)&=\frac{1}{2\pi}\int_{B_n}\|Q\,\hat{u}\|_W^2\,d\omega\\&=\frac{1}{2\pi}\int_{B_n}|\hat{f}_n(\omega)|^2\,\|QT_\omega(\psi)\|_W^2\,d\omega.\end{aligned} \tag{22}$$

In the same way, if we consider the partial derivative $\partial_t\sigma$ with respect to t of the observation σ, the vibratory intensity of the response $\partial_t\sigma=Q\,\partial_t u$ due to the excitation F_n will be characterized by the square root of the scalar :

$$E_Q^{n\cdot}(\psi)=\int_{\mathbb{R}}\|\partial_t\sigma\|_W^2\,dt=\frac{1}{2\pi}\int_{B_n}\omega^2\,\|Q\hat{u}\|_W^2\,d\omega \tag{23}$$

because the Fourier transform of $\partial_t u$ has support B_n.

III.2 – CHARACTERIZATION OF THE SPACE PROPAGATION OF THE VIBRATIONS IN THE BAND B_n.

Because the vibratory intensity of the observation σ is characterized by the scalars (or their square root) $E_Q^n(\psi)$, given the excitation $F_n=f_n\otimes\psi$ in the band B_n, the spatial propagation of the vibrations is obtained by studying the function of the type $x\mapsto E_{\delta_x}^n(\psi)$, defined on Ω.

In effect, for $Q=\delta_x$ and $W=\mathbb{C}^3$, σ represents the value of the displacement field at the point x and $E_{\delta_x}^n(\psi)$ represents the intensity of the vibration at this same point x. Considering (21) and (22), we have

$$\begin{aligned}x\mapsto E_{\delta_x}^n(\psi)&=\int_{\mathbb{R}}\|u(t, x)\|_{\mathbb{C}^3}^2\,dt\\&=\frac{1}{2\pi}\int_{B_n}\|\hat{u}(\omega, x)\|_{\mathbb{C}^3}^2\,d\omega.\end{aligned} \tag{24}$$

It should be noted that the function (23) defines the deflection of the elastic medium displacement for the vibratory state caused by the excitation F_n in the band B_n. If we desire information on the phase-shift between any two points x_1 and x_2 in the elastic medium, this is obtained by calculating correlations of the type

$$\begin{aligned}E_{\delta_{x_1},\delta_{x_2}}^n(\psi)&=\int_{\mathbb{R}}(u(t, x_1), u(t, x_2))_{\mathbb{C}^3}\,dt\\&=\frac{1}{2\pi}\int_{B_n}(\hat{u}(\omega, x_1), \hat{u}(\omega, x_2))_{\mathbb{C}^3}\,d\omega.\end{aligned} \tag{25}$$

Finally, the spatial propagation can be studied with respect to any given subspace of $\mathbb{C}^3$. For example, the spatial propagation of the component u_j of the vibrations is obtained by studying the function

$$\begin{aligned}x\mapsto E_{\delta_x\circ\pi_j}^n(\psi)&=\int_{\mathbb{R}}|u_j(t, x)|^2\,dt\\&=\frac{1}{2\pi}\int_{B_n}|\hat{u}_j(\omega, x)|^2\,d\omega.\end{aligned} \tag{26}$$

III.3 – PROBABILISTIC INTERPRETATION OF ENERGY CHARACTERISTICS.

The characteristics of the type (21) were defined for a deterministic excitation of the type $F_n=f_n\otimes\psi$. In this paragraph, we will give the probabilistic interpretation of these characteristics when the excitation is a stationary random process. It should be noted that the results obtained below go farther than the simple probabilistic interpretation, as they make it possible to reduce the second order statistical calculation of medium frequency stationary random vibrations to the calculation of medium frequency deterministic vibrations, which we will solve explicitly below in para. V and the following.

We thus let $(\Omega, \mathcal{T}, P)$ be a probability space $\mathbb{E}$ designating the mathematical expectation. We then assume that the excitation F_n can be expressed as

$$F_n = \zeta_n \otimes \psi, \tag{27}$$

where the space component is always deterministic and is represented by a given element ψ in H, and where the time component is a stochastic process $\{\zeta_n(t),\ t \in \mathbb{R}\}$ defined on $(\Omega,\ \mathcal{T},\ P)$, indexed on $\mathbb{R}$, with values in $L^2(\Omega)$), centered, stationary, of second order, continuous (in quadratic means). We also suppose that its spectral measure μ_{ζ_n} allows a density S_{ζ_n} with respect to $d\omega$ which has support B_n :

$$\mu_{\zeta_n}(\omega) = S_{\zeta_n}(\omega)\, d\omega. \tag{28}$$

For all $\tau \in \mathbb{R}$, the autocorrelation function of the process $\zeta_n(t)$ is written as :

$$\begin{aligned} R_{\zeta_n}(\tau) &= \mathbb{E}(\zeta_n(t+\tau)\overline{\zeta_n(t)}) \\ &= \int_{\mathbb{R}} e^{i\omega\tau} \mu_{\zeta_n}(\omega) = \int_{Bn} e^{i\omega\tau} S_{\zeta_n}(\omega)\, d\omega. \end{aligned} \tag{29}$$

Considering the general properties of S_{ζ_n}, and that $\operatorname{supp} S_{\zeta_n} = B_n$, we know that the function $\hat{f}_n \in L^2(\mathbb{R})$ exists with compact support B_n such that :

$$S_{\zeta_n}(\omega) = \frac{1}{2\pi} |\hat{f}_n(\omega)|^2, \qquad \omega \in \mathbb{R}. \tag{30}$$

Under these conditions the inverse Fourier transform of $\hat{f}_n$, denoted f_n is a function belonging to $L^2(\mathbb{R})$.

As the process $\zeta_n(t)$ is stationary and centered, for any t fixed in $\mathbb{R}$, the variance V_{ζ_n} of the random variable $\zeta_n(t)$ is equal to the power of the process $\{\zeta_n(t),\ t \in \mathbb{R}\}$:

$$V_{\zeta_n} = \mathbb{E}(|\zeta_n(t)|^2) = \mu_{\zeta_n}(\mathbb{R}) = \frac{1}{2\pi}\int_{B_n} |\hat{f}_n(\omega)|^2\, d\omega. \tag{31}$$

The process $\{F_n(t),\ t \in \mathbb{R}\}$ with values in H, defined by (27), is centered, stationary, of second order, continuous and its spectral measure allows a density $S_{F_n}(\omega)$ with respect to $d\omega$ such that, for ω fixed in $\mathbb{R}$, $S_{F_n}(\omega)$ is the linear operator of H expressed as :

$$S_{F_n}(\omega) = S_{\zeta_n}(\omega)(\psi \otimes \overline{\psi}). \tag{32}$$

The total power of the process $F_n(t)$ is written :

$$e_{F_n}(\psi) = \mathbb{E}(\| F_n \|_H^2) = \| \psi \|_H^2\, \mathbb{E}(|\zeta_n(t)|^2),$$

or, using (31) :

$$e_{F_n}(\psi) = \frac{1}{2\pi} \| \psi \|_H^2 \int_{B_n} |\hat{f}_n(\omega)|^2\, d\omega. \tag{33}$$

We note that the right hand member of (33) is identical to (14).

Thus the process $\{U(t), t \in \mathbb{R}\}$ with values in H, resulting from the filtering of the process $\{F_n(t),\ t \in \mathbb{R}\}$ by the frequency response operator convolution filter T_ω defined by (5) is stationary, centered, of second order, continuous and its spectral measure admits the linear operator $S_U(\omega)$ of H as the density with respect to $d\omega$, written :

$$S_U(\omega) = T_\omega S_{F_n}(\omega) T_\omega^*, \qquad \omega \in \mathbb{R}, \tag{34}$$

where T_ω^* designates the adjoint operator of T_ω in H.

The spectral density of an observation process $\{\sigma(t),\ t \in \mathbb{R}\}$ with values in W, defined by $\sigma = QU$ (cf II.6) is expressed :

$$S_\sigma(\omega) = Q S_U(\omega)\, {}^tQ, \tag{35}$$

where ${}^tQ$ is the linear operator on W to H, transposed from the real operator Q.

The total power of the process $\{\sigma(t),\ t \in \mathbb{R}\}$ is thus expressed as :

$$\mathbb{E}(\| \sigma \|_W^2) = \int_{\omega \in \mathbb{R}} (\operatorname{tr} S_\sigma(\omega))\, d\omega. \tag{36}$$

and as $\operatorname{supp} S_{\zeta_n} = B_n$, considering relations (35), (34), (32) and (30), we have :

$$\begin{aligned} \mathbb{E}(\| \sigma \|_W^2) &= \int_{\omega \in B_n} (\operatorname{tr}\{ Q T_\omega (\psi \otimes \overline{\psi}) T_\omega^*\, {}^tQ \})\, S_{\zeta_n}(\omega)\, d\omega \\ &= \int_{\omega \in B_n} (Q T_\omega(\psi),\ Q T_\omega(\psi))_W\, S_{\zeta_n}(\omega)\, d\omega. \end{aligned}$$

whence :

$$\mathbb{E}(\| \sigma \|_W^2) = \frac{1}{2\pi}\int_{\omega \in B_n} |\hat{f}_n(\omega)|^2\, \| Q T_\omega(\Psi) \|_W^2\, d\omega. \tag{37}$$

Comparing (37) with (22), we see that :

$$\mathbb{E}(\| \sigma \|_W^2) = E_Q^n(\psi). \tag{38}$$

Consequently, the energy characteristic (22) of an observation of the system, constructed from the deterministic excitation :

$$F_n = f_n \otimes \psi, \qquad \psi \in H,\ \ f_n \in L^2(\mathbb{R}),\ \ \operatorname{supp} \hat{f}_n = B_n,$$

can be interpreted as the total power of the same observation for a random, centered, stationary excitation $F_n = \zeta_n \otimes \psi$, with the spectral density $S_{\zeta_n}(\omega)$ of the process $\{\zeta_n(t),\ t \in \mathbb{R}\}$ being expressed :

$$S_{\zeta_n}(\omega) = \frac{1}{2\pi} |\hat{f}_n(\omega)|^2, \qquad \omega \in \mathbb{R}.$$

III.4 – COMMENTS ON THE ENERGY CHARACTERISTICS.

Let us note that the energy characteristics we have introduced to study the intensity of the vibrations in the medium frequency domain are relative to a fixed frequency band B_n. In practice, it is better not to make the bands B_n too large, to avoid calculating characteristics that are too general and also to ensure that they remain linked in some way to the frequency aspect.

Consequently, if we want to study the medium frequency dynamics over a rather large medium frequency interval I, we will cut this interval into a certain number of bands B_n such that $I = \cup_n B_n$, $\cap_n B_n = \emptyset$, and we will study the problem in each band B_n.

Note that it is always possible to apply the case of any given deterministic excitation $F \in L^2(\mathbb{R}, H)$, with supp $\hat{F} = I$, to several elementary cases :

$$F_n = f_n \otimes \psi, \quad \psi \in H, \quad f_n \in L^2(\mathbb{R}), \quad \text{supp}\, \hat{f}_n = B_n.$$

The remark remains valid in the case of a stationary random excitation which has a spectral measure with support on I, considering the results in para. III.3.

Aside from what we have just discussed, it will also be shown when we develop the numerical construction of the solution that it is advantageous not to take the bands B_n too wide, to avoid weighing down the cost of the numerical calculations, even if it means cutting the original band into several bands that are small enough.

IV – EXTREMUM VIBRATORY STATE IN A BAND B_n.

IV.1 – EXPLANATION OF THE PROBLEM.

In the domain of low frequency vibrations, for a conservative linear elastic medium it is conventionally said that the system "resonates" at a harmonic excitation $F = \exp(i\omega_0 t) \otimes \psi_0$, if ω_0 is a vibration eigenfrequency and if the spatial component ψ_0 is the eigenmode associated with ω_0. If the mechanical system is slightly damped one would still say that there is resonance at the same frequency (in fact the resonant frequency of the slightly dissipative system is a little different from ω_0).

The problem consisting of exciting this same system with a force $F = \exp(i\omega_0 t) \otimes \psi$, where ω_0 is still an eigenfrequency, and finding ψ in H to obtain the resonance, is referred to as a problem of appropriating the eigenmode of frequency ω_0. In the previously described case, we would have to take $\psi = \psi_0$.

For the study of vibrations in the medium frequency domain, we no longer use discrete values of the frequency, but rather frequency bands B_n. The quantities used to characterize the intensity of the vibration are given in para. III. They express the total signal energy of an observation in a given band B_n.

Thus, for a fixed medium frequency band B_n, the problem corresponding to the appropriation in the low frequency domain is: knowing that the excitation applied is of the type $F_n = f_n \otimes \psi$, with f_n a fixed function in $L^2(\mathbb{R})$ such that supp $\hat{f}_n = B_n$, do particular elements ψ exist in H that lead to extremum vibratory states in the B_n band. with the energy $e_n(\psi)$ of F_n as defined by (14) of course being fixed ? This is the problem of finding the space distributions of excitation forces producing "resonances" or "extremum vibratory states" in the band B_n. We will see that these extremum vibratory states exist and that the most interesting of these is of course the one corresponding to the vibration maximum that can be obtained in the band B_n, when ψ describes H, if the limitation that $e_n(\psi)$ is fixed.

It is of great interest in mechanics, to find these states, to predict the vibrations in the medium frequency domain.

By knowing this information, the maximum intensity of the vibrations in a given zone of the structure can be determined when the source of excitation is not spatially fixed but can, *a priori*, be placed anywhere within a certain area of a structure (we then look for the extremum ψ in a subspace H_1 of H, or anywhere in the structure (the extremum ψ are then looked for in H). In the same direction, we can also determine the best spatial location of excitation sources to maximize or minimize the vibrations throughout the structure or in a specific area of the structure.

These various aspects will lead us to introduce an operator which, in finding its eigenvalues and eigenvectors, will make it possible to answer the questions asked.

IV.2 – SESQUILINEAR FORM OF THE SIGNAL ENERGY IN THE BAND B_n AND THE ASSOCIATED OPERATOR.

Let Q be an observation operator of H to W, having the properties defined in the para. II.6.

$$F_{n,j} = f_n \otimes \psi^{(j)}, \qquad j \in \{1, 2\},$$

two excitations belonging to the class defined in the para. II.3 :

$$\psi^{(j)} \in H, \quad f_n \in L^2(\mathbb{R}), \qquad \text{supp}\, \hat{f}_n = B_n.$$

Let $u^{(j)}$ be the vibration due to the excitation $F_{n,j}$, which is such that (*cf* (18)) : $\hat{u}^{(j)}(\omega) = \hat{f}_n(\omega)(T_\omega \psi^{(j)})$. Let $\sigma^{(j)} = Q u^{(j)}$ be the value of the observation associated with the vibration $u^{(j)}$. Then the Fourier transform with repect to t of $\sigma^{(j)}$ is expressed by :

$$\hat{\sigma}^{(j)}(\omega) = \hat{f}_n(\omega) \quad (Q T_\omega \psi^{(j)}), \qquad \omega \in \mathbb{R}, \quad j \in \{1, 2\}. \tag{39}$$

We let :

$$E_Q^n(\psi^{(1)}, \psi^{(2)}) = \int_{\mathbb{R}} (\sigma^{(1)}, \sigma^{(2)})_W \, dt. \tag{40}$$

Using Plancherel's theorem, the relation (39) and the transposition, we get :

$$\begin{aligned}
E_Q^n(\psi^{(1)}, \psi^{(2)}) &= \frac{1}{2\pi} \int_{\omega \in B_n} (\hat{\sigma}^{(1)}, \hat{\sigma}^{(2)})_W \, d\omega \\
&= \frac{1}{2\pi} \int_{\omega \in B_n} |\hat{f}_n(\omega)|^2 (Q T_\omega \psi^{(1)}, Q T_\omega \psi^{(2)})_W \, d\omega \\
&= \frac{1}{2\pi} \int_{\omega \in B_n} |\hat{f}_n(\omega)|^2 (T_\omega^{*}\, {}^t Q Q T_\omega \psi^{(1)}, \psi^{(2)})_H \, d\omega \\
&= \left(\left\{ \frac{1}{2\pi} \int_{\omega \in B_n} |\hat{f}_n(\omega)|^2 \, T_\omega^{*}\, {}^t Q Q T_\omega \, d\omega \right\} \psi^{(1)}, \psi^{(2)} \right)_H,
\end{aligned}$$

where T_ω^* is the adjoint operator of T_ω in H and ${}^tQ$ is the real linear operator on W to H, transposed from the operator Q.

We conclude that :

$$\left.\begin{array}{c}\forall\, \psi^{(1)} \text{ and } \psi^{(2)} \in H, \\ E_Q^n(\psi^{(1)}, \psi^{(2)}) = (\mathscr{E}_Q^n \psi^{(1)}, \psi^{(2)})_H,\end{array}\right\} \tag{41}$$

where we let :

$$\mathscr{E}_Q^n = \frac{1}{2\pi}\int_{\omega\in B_n} |\hat{f}_n(\omega)|^2\, T_\omega^*\, {}^tQQT_\omega\, d\omega. \tag{42}$$

The relation (41) shows that $\psi^{(1)}, \psi^{(2)} \mapsto E_Q^n(\psi^{(1)}, \psi^{(2)})$ is a sesquilinear form on $H \times H$ which we will call a "sesquilinear form of the signal energy over band B_n", $\mathscr{E}_Q^n$ the linear operator in H being called the associated "energy operator of the signal". We will study the properties of the operator $\mathscr{E}_Q^n$ in the para. IV.3.

Comments (43)

(a) Let us note that for $\psi^{(1)} = \psi^{(2)} = \psi \in H$, we have $\sigma^{(1)} = \sigma^{(2)} = \sigma$, and thus :

$$E_Q^n(\psi, \psi) = (\mathscr{E}_Q^n \psi, \psi)_H = \int_{\mathbb{R}} \|\sigma\|_W^2\, dt = \frac{1}{2\pi}\int_{B_n} \|\hat{\sigma}\|_W^2\, d\omega \tag{44}$$

is nothing other than the total energy of the signal over the band B_n of the observation σ defined by (21), due to the excitation $F_n = f_n \otimes \psi$.

(b) By taking $W = H$ and $Q = \mathrm{Id}$, where Id designates the identity operator of H, we have

$$E_{\mathrm{Id}}^n(\psi, \psi) = (\mathscr{E}_{\mathrm{Id}}^n \psi, \psi)_H = \int_{\mathbb{R}} \|u\|_H^2\, dt = \frac{1}{2\pi}\int_{B_n} \|\hat{u}\|_H^2\, d\omega, \tag{45}$$

i.e. $E_{\mathrm{Id}}^n(\psi, \psi)$ represents the total energy of the signal for the entire structure over the band B_n due to the excitation $F_n = f_n \otimes \psi$. Thus the larger $E_{\mathrm{Id}}^n(\psi, \psi)$ the larger the vibrations in the set Ω of the elastic medium. The operator $\mathscr{E}_{\mathrm{Id}}^n$ thus makes it possible to determine the extremum vibratory states for the entire elastic medium, for the band B_n.

(c) If we are concerned only with the energy of the signal u over the band B_n in the part $\Omega_1 \subset \Omega$ of Ω, we take $W = H$, and for Q we take the operator of multiplication by the indicator function $\mathbf{1}_{\Omega_1}$ of the set Ω_1, which is such that $\mathbf{1}_{\Omega_1}(x) = 0$ if $x \in \Omega \setminus \Omega_1$, $\mathbf{1}_{\Omega_1}(x) = 1$ if $x \in \Omega_1$. With these conditions, we have :

$$E_{\mathbf{1}_{\Omega_1}}^n(\psi, \psi) = (\mathscr{E}_{\mathbf{1}_{\Omega_1}}^n \psi, \psi)_H = \int_{\mathbb{R}}\left(\int_{\Omega_1} \|u(t,x)\|_{\mathbb{C}^3}^2\, dx\right) dt. \tag{46}$$

The scalar $E_{\mathbf{1}_{\Omega_1}}^n(\psi, \psi)$ clearly represents the energy of the signal in the part Ω_1 of the structure over the band B_n due to the excitation $F_n = f_n \otimes \psi$. Thus the greater $E_{\mathbf{1}_{\Omega_1}}^n(\psi, \psi)$ the greater the vibrations will be in the Ω region of the structure. The operator $\mathscr{E}_{\mathbf{1}_{\Omega_1}}^n$ will thus make it possible to determine the extremum vibratory states in the Ω_1 part of the elastic medium.

IV.3 – PROPERTIES OF THE ENERGY OPERATOR OF THE SIGNAL AND DETERMINATION OF THE EXTREMUM VIBRATORY STATES.

Considering the assumptions, and the properties of the operators T_ω and Q, [6 and 9] show that the operator $\mathscr{E}_Q^n$ defined by (42) is a linear operator, self adjoint ($\mathscr{E}_Q^{n*} = \mathscr{E}_Q^n$), positive, compact in H. Consequently, the spectrum of eigenvalues of the operator $\mathscr{E}_Q^n$ is discrete and is a sequence of positive real numbers $\lambda_1 \geq \lambda_2 \geq \lambda_3 \to \ldots \to 0$, each eigenvalue having a finite multiplicity and being an isolated point, perhaps with the exception of 0. There exist associated eigenvectors $\psi_1, \psi_2, \psi_3, \ldots$, forming an orthonormal basis of H. It is also shown that $\sum_{j=1}^{\infty} \lambda_j < +\infty$ (we then say that $\mathscr{E}_Q^n$ is nuclear). In what follows, we will denote by $\lambda^{(j)}$ the jth eigenvalue, counted once only if it is repetitive, such that $\lambda^{(1)} > \lambda^{(2)} > \lambda^{(3)} \to \ldots \to 0$, and we will denote by $X^{(j)}$ the eigen subspace of H associated with the eigenvalue $\lambda^{(j)}$.

(a) *Finding the extremum vibratory states.*

For a given excitation energy $e_n(\psi)$ (*cf* [14]) and considering the properties of the operator $\mathscr{E}_Q^n$, the Hermitian form on H associated with (41), i.e.

$$\psi \to E_Q^n(\psi, \psi) = (\mathscr{E}_Q^n \psi, \psi)_H, \tag{47}$$

goes through extrema when ψ describes H, and reaches its extrema when ψ belongs to the eigen-subspaces $X^{(j)}$ of H.

We are thus led to introduce the following definition: with the constant

$$a_n = (2\pi)^{-1}\int_{B_n} |\hat{f}_n(\omega)|^2\, d\omega \tag{48}$$

being fixed, the jth extremum vibratory state in the band B_n will be obtained for a spatial force distribution ψ such that :

$$\|\psi\|_H = 1, \qquad \psi \in X^{(j)}. \tag{49}$$

Under these conditions, the total energy of the response over the band B_n for the jth extremum vibratory state is expressed :

$$E_Q^n(\psi, \psi) = (\mathscr{E}_Q^n \psi, \psi)_H = \lambda^{(j)}. \tag{50}$$

Of course the energy level $\lambda^{(j)}$ of the jth vibratory state depends on the normalization of ψ, since we have defined $\|\psi\|_H = 1$ (49). If the normalization is changed, that is if we take $\varphi = b\,\psi$ while $b \in \mathbb{C}$, ψ still verifies (49), the jth extremum vibratory state will still be reached but the response will have an energy level $|b|^2 \lambda^{(j)}$.

Let us check the equality (50).

Let q be the necessarily finite dimension of the eigen-subspace $X^{(j)}$, and let $\psi_1^{(j)}, \psi_2^{(j)}, \ldots, \psi_q^{(j)}$ be the eigenvectors of $\mathscr{E}_Q^n$ associated with $\lambda^{(j)}$. They form a basis of $X^{(j)}$ and are orthonormal in H. Any ψ verifying (49) is expressed :

$$\left.\begin{aligned} \psi = \sum_{p=1}^{q} \mu_p \psi_p^{(j)}, \qquad \mu_p \in \mathbb{C}, \\ \sum_{p=1}^{q} |\mu_p|^2 = 1. \end{aligned}\right\} \tag{51}$$

Substituting ψ given by (51) in the first equation (50), we get :

$$E_Q^n(\psi, \psi) = (\mathscr{E}_Q^n \psi, \psi)_H$$

$$= \sum_p \sum_{p'} \mu_p \overline{\mu_{p'}} (\mathscr{E}_Q^n \psi_p^{(j)}, \psi_{p'}^{(j)})_H$$

$$= \lambda^{(j)} \sum_p \sum_{p'} \mu_p \overline{\mu_{p'}} (\psi_p^{(j)}, \psi_{p'}^{(j)})_H = \lambda^{(j)} \sum_p |\mu_p|^2 = \lambda^{(j)}.$$

(b) *Maximum vibratory state in the band B_n.*

As $\lambda^{(1)}$ is the dominant eigenvalue, the upper bound of $\psi \mapsto E_Q^n(\psi, \psi)$ when ψ describes H, with the constraints a_n fixed and $\|\psi\|_H = 1$, is reached for $\psi \in X^{(1)}$; that is, it belongs to the dominant eigensubspace. Consequently, the maximum vibratory state that can be obtained in the band B_n with the constraints a_n fixed (*cf* (48)) and $\|\psi\|_H = 1$, is reached for a spatial distribution of force ψ belonging to the dominant eigensubspace $X^{(1)}$ associated with the dominant eigenvalue $\lambda^{(1)}$ of $\mathscr{E}_Q^n$, the total energy of the response then being $E_Q^n(\psi, \psi) = \lambda^{(1)}$.

(c) *Extremum vibratory states relative to a subspace H_1 of H.*

As we indicated in para. IV.1, it is often of interest to find the extremum vibratory states in the band B_n when the spatial component of the excitation ψ does not describe all of H, but only a vector subspace H_1 of H. This situation corresponds, for example, to the case where the forces are spatially restricted to a certain part of the structure.

Using a logic similar to that used in para. (a) above shows that the extremum vibratory states relative to the subspace H_1 are obtained by calculating the eigensubspaces of the operator $(\mathscr{E}_Q^n)_1$ which is the restriction to H_1 of the operator $\mathscr{E}_Q^n$ in H.

For example if H_1 is a vector subspace of H, of finite dimension m, associated with the scalar product induced by that of H, and denoting by $(\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_m)$ an orthonormal basis for H_1, the matrix of the operator $(\mathscr{E}_Q^n)_1$ relative to the basis $\{\mathbf{b}_p\}$ will be a Hermitian, positive square $(m \times m)$ matrix and will have the following elements :

$$[(\mathscr{E}_Q^n)_1]_{pp'} = (\mathscr{E}_Q^n \mathbf{b}_{p'}, \mathbf{b}_p)_H = E_Q^n(\mathbf{b}_{p'}, \mathbf{b}_p), \qquad p, p' \in \{1, 2, \ldots, m\}. \tag{52}$$

The jth extremum vibratory state relative to H_1 will be obtained for the ψ of the form $\psi = \sum_{p=1}^{m} \mu_p \mathbf{b}_p$, with $\mu = (\mu_1, \mu_2, \mu_3, \ldots, \mu_m)$ belonging to the eigensubspace associated with the eigenvalue $\lambda^{(j)}$ of the matrix $[(\mathscr{E}_Q^n)_1]$ and verifying the normalization condition $\sum_{p=1}^{m} |\mu_p|^2 = 1$.

In particular, the finding of the maximum vibratory state in the Ω_1 part of the structure when ψ describes the subspace H_1 defined above, consists of calculating the dominant eigenvalue and the eigensubspace associated with the positive Hermitian matrix of elements :

$$[(\mathscr{E}_{1_{\Omega_1}}^n)_1]_{pp'} = (\mathscr{E}_{1_{\Omega_1}}^n \mathbf{b}_{p'}, \mathbf{b}_p)_H = E_{1_{\Omega_1}}^n(\mathbf{b}_{p'}, \mathbf{b}_p), \tag{53}$$

where 1_{Ω_1} has the meaning given in the comment (43 c).

V – OPERATOR OF TOTAL KINETIC ENERGY OVER THE BAND B_n AND EXPRESSION FOR THE MODAL DENSITY

In this paragraph, we introduce a certain operator $\mathscr{E}_c^n$ in H such that, for any ψ in H and given f_n :

$$E_c^n(\psi, \psi) = (\mathscr{E}_c^n \psi, \psi)_H, \tag{54}$$

represents the total kinetic energy of the response of the elastic medium to the excitation $f_n \otimes \psi$ in the band B_n.

This operator is entirely equivalent to the operator $\mathscr{E}_Q^n$ defined by (42) insofar as concerns the determination of the extremum vibratory states. Thus we shall not belabor this aspect. On the other hand, this operator will make it possible to calculate the modal density N_n over the band B_n, i.e. $N_n \Delta\omega$ represents the number of eigenmodes of vibration of the associated conservative system that there are in the band B_n.

V.1 – SESQUILINEAR FORM OF THE TOTAL KINETIC ENERGY OVER THE BAND B_n, AND THE ASSOCIATED OPERATOR.

We will use the notations and assumptions given at the start of para. IV.2. The sesquilinear form of the total kinetic energy on $H \times H$ for the band B_n is defined by :

$$E_c^n(\psi^{(1)}, \psi^{(2)}) = \frac{1}{2} \int_{\mathbb{R}} (M\, \partial_t u^{(1)}, \partial_t u^{(2)})_H \, dt, \tag{55}$$

$u^{(j)}$ being the vibration due to the excitation

$$F_{n,j} = f_n \otimes \psi^{(j)}.$$

The relation (55) shows that the total kinetic energy over B_n of the response of the elastic medium to excitation $f_n \otimes \psi$ is expressed :

$$E_c^n(\psi, \psi) = \frac{1}{2} \int_{\mathbb{R}} (M\, \partial_t u, \partial_t u)_H \, dt, \tag{56}$$

u being the vibration due to the excitation $f_n \otimes \psi$, defined by (18).

Using Plancherel's theorem, the relation (18) and the transposition, the relation (55) allows one to write :

$$E_c^n(\psi^{(1)}, \psi^{(2)}) = \frac{1}{4\pi}\int_{\omega\in B_n} \omega^2 (M\,\hat{u}^{(1)}, \hat{u}^{(2)})_H\, d\omega$$

$$= \frac{1}{4\pi}\int_{\omega\in B_n} \omega^2 |\hat{f}_n(\omega)|^2 (MT_\omega \psi^{(1)}, T_\omega \psi^{(2)})_H\, d\omega$$

$$= \frac{1}{4\pi}\int_{\omega\in B_n} \omega^2 |\hat{f}_n(\omega)|^2 (T_\omega^* MT_\omega \psi^{(1)}, \psi^{(2)})_H\, d\omega$$

$$= \left(\left\{\frac{1}{4\pi}\int_{\omega\in B_n} \omega^2 |\hat{f}_n(\omega)|^2\, T_\omega^* MT_\omega\, d\omega\right\}\psi^{(1)}, \psi^{(2)}\right)_H.$$

We conclude that :

$$\left.\begin{array}{c}\forall\, \psi^{(1)},\quad \psi^{(2)}\in H,\\ E_c^n(\psi^{(1)}, \psi^{(2)}) = (\mathscr{E}_c^n \psi^{(1)},\quad \psi^{(2)})_H,\end{array}\right\} \qquad (57)$$

where we let :

$$\mathscr{E}_c^n = \frac{1}{4\pi}\int_{\omega\in B_n} \omega^2 |\hat{f}_n(\omega)|^2\, T_\omega^* MT_\omega\, d\omega. \qquad (58)$$

We call $\mathscr{E}_c^n$ the total kinetic energy operator over B_n.

Considering the assumptions and properties of the operators T_ω and M in H, we show as for the operator $\mathscr{E}_Q^n$ that the operator $\mathscr{E}_c^n$ defined by (58) is a linear, self-adjoint, positive, compact operator in H, and that the series $\sum_{j=1}^{+\infty} \nu_j$ of its eigenvalues is convergent ($\mathscr{E}_c^n$ is nuclear). For such an operator we can define its trace, the trace operator being denoted by tr.

V.2 – MODAL DENSITY RELATIVE TO THE BAND B_n.

Here we set up a relation that will make it possible to calculate numerically, subsequently, the modal density, denoted by N_n, taken as constant by construction in the band B_n, and such that the number of eigenmodes of vibration in the associated undampened elastic system in the band B_n (bandwidth $\Delta\omega$) be equal to $N_n\,\Delta\omega$.

Let $\mathscr{E}_{c,\xi}^n$ be the total kinetic energy operator (58) constructed with :

$$C = 2\xi n \Delta\omega\, M, \qquad \xi\in\mathbb{R}^+, \qquad (59)$$

$$f_n(t) = (\pi t)^{-1}\sin(t\Delta\omega/2)\exp(in\Delta\omega\, t) \qquad (60)$$

Thus ξ represents the average damping rate of the structure over the band B_n, and f_n is the function defined in (II.4 d).

With these conditions, we have

$$a_n \alpha_n < N_n < b_n \alpha_n, \qquad (61)$$

where :

$$a_n = \frac{n}{n+(1/2)}, \qquad b_n = \frac{n}{n-(1/2)}, \qquad (62)$$

$$\alpha_n = 16\, n \lim_{\xi\to 0_+} \xi\, \mathrm{tr}(M\,\mathscr{E}_{c,\xi}^n). \qquad (63)$$

Numerically, for $n \gg 1$, $a_n \simeq b_n \simeq 1$, and for a given $\xi \ll 1$, we have :

$$N_n \simeq 8\pi n\xi\left(\mathrm{Arctg}\left(\frac{1}{n\xi}\right) + n\xi\,\mathrm{Log}\left(\frac{n\xi}{\sqrt{1+n^2\xi^2}}\right)\right)^{-1} \mathrm{tr}(M\,\mathscr{E}_{c,\xi}^n). \qquad (64)$$

The proof for this is given in the appendix.

VI – CONSTRUCTION OF THE SOLUTION

VI.1 – CALCULATIONS NEEDED TO IMPLEMENT THE ENERGY METHOD.

The energy characteristics (*cf* para III) are obtained by calculating numbers such as $E_Q^n(\psi)$, given by (21) in the deterministic case, and by (38) in the random case.

In the same way, the extremum vibratory states are obtained by studying the eigenvalues problem of the operator $\mathscr{E}_Q^n$ defined by (42). However (41) defines this operator in the sesquilinear form $E_Q^n(\psi^{(1)}, \psi^{(2)})$ given by (40).

Considering the comment (43 a), we have

$$E_Q^n(\psi) = E_Q^n(\psi, \psi).$$

Consequently, all of these quantities will be determined as we can calculate $E_Q^n(\psi^{(1)}, \psi^{(2)})$, defined by (41) with any given $\psi^{(1)}$ and $\psi^{(2)}$ in H.

We will thus attempt to calculate this quantity numerically, using an appropriate numerical method (*cf* para I).

The method that we are going to develop will make it possible to calculate the frequency response operator T_ω, also sometimes called the "generalized Green function".

Finally, let us note that the calculation of the modal density [*cf* (61) to (64)] requires the determination of the operator $\mathscr{E}_c^n$, which is also defined in sesquilinear form $E_c^n(\psi^{(1)}, \psi^{(2)})$ [*cf* (57)], which is of the same type as $E_Q^n(\psi^{(1)}, \psi^{(2)})$. We will thus also give the numerical method for calculating E_c^n.

VI.2 – LOW FREQUENCY EQUATION ASSOCIATED WITH THE MEDIUM FREQUENCY PROBLEM.

Let f_n be a function belonging to the excitation class defined in para. II.3, and let f_0 be the associated function such that :

$$f_0(t) = f_n(t)\exp(-in\Delta\omega\, t). \qquad (65)$$

then, according to (II.4 b) :

$$f_0 \in L^2(\mathbb{R}) \text{ and } \mathrm{supp}\,\hat{f}_0 = B_n = \left[-\frac{\Delta\omega}{2}, \frac{\Delta\omega}{2}\right]$$

is a low frequency signal associated with the medium frequency signal f_n relative to the band B_n. We have the following proposition.

Proposition : (66)

The medium frequency vibration u, such that $\hat{u} = B_n$, defined by :

$$\hat{u}(\omega) = \hat{f}_n(\omega)(T_\omega \psi), \tag{67}$$

and interpreted as the forced solution associated with the medium frequency excitation $f_n \otimes \psi$ over the band B_n of :

$$M\,\partial_t^2 u + C\,\partial_t u + K\,u = f_n \otimes \psi, \tag{68}$$

is expressed :

$$u(t) = u_0(t)\exp(in\,\Delta\omega\, t), \tag{69}$$

where, u_0 is the forced solution of :

$$\begin{aligned} &M\,\partial_t^2 u_0 + (C + 2\,in\,\Delta\omega\, M)\,\partial_t u_0 \\ &\qquad + (K + in\,\Delta\omega\, C - (n\,\Delta\omega)^2\, M)\,u_0 = f_0 \otimes \psi \end{aligned} \tag{70}$$

associated with the low frequency excitation $f_0 \otimes \psi$ and defined by its Fourier transform :

$$\hat{u}_0(\omega) = \hat{f}_0(\omega)(T_{\omega + n\Delta\omega}\,\psi), \tag{71}$$

which has the compact support :

$$B_0 = \left[-\frac{\Delta\omega}{2}, \frac{\Delta\omega}{2}\right].$$

Comments and Proof of the Proposition (66) (72)

(a) By substituting (69) in (68), and taking (65) into account, (70) is obtained directly because :

$$\partial_t u(t) = (\partial_t u_0(t) + in\,\Delta\omega\, u_0(t))\exp(in\,\Delta\omega\, t), \tag{73}$$

$$\begin{aligned} \partial_t^2 u(t) = (&\partial_t^2 u_0(t) + 2\,in\,\Delta\omega\,\partial_t u_0(t) \\ &- (n\,\Delta\omega)^2\, u_0(t))\exp(in\,\Delta\omega\, t). \end{aligned} \tag{74}$$

(b) From the relation (69) we obtain

$$u_0(t) = u(t)\exp(-in\,\Delta\omega\, t). \tag{75}$$

Therefore $\hat{u}_0(\omega) = \hat{u}(\omega + n\,\Delta\omega)$. In the same way, considering (65) and the results of (II.4 b), we have $\hat{f}_n(\omega + n\,\Delta\omega) = \hat{f}_0(\omega)$.

We conclude (71) considering (67). Since supp $\hat{f}_0 = B_0$, the relation (71) shows that supp $\hat{u}_0 = B_0$.

Consequently, u_0 defined by (71) is exactly the low frequency vibration on B_0, the forced solution of (70) associated with the medium frequency vibration u over B_n, by the relation (69).

(c) The existence and uniqueness of the forced solution u_0 of (70) results directly from (71).

(d) The low frequency equation (70) will be solved numerically using an appropriate step by step integration method. The comparison of relations (67) and (71) shows that any implicit numerical scheme that is unconditionally stable when applied to the equations (68) will also be unconditionally stable when applied to the equation (70).

VI.3 – ACTUAL CONSTRUCTION OF THE MEDIUM FREQUENCY SOLUTION.

We let :

$$\omega_L = \frac{\Delta\omega}{2}, \qquad \tau_L = \frac{\pi}{\omega_L} = \frac{2\pi}{\Delta\omega} \tag{76}$$

For any $m \in \mathbb{Z}$, we define the family of functions φ_m on $\mathbb{R}$ with values in $\mathbb{R}$ such that :

$$t \longmapsto \varphi_m(t) = \frac{\sin\omega_L(t - m\,\tau_L)}{\sqrt{\pi\omega_L}\,(t - m\,\tau_L)}. \tag{77}$$

With these notations we have the following proposition :

Proposition (78)

The medium frequency vibration u defined by (67), the forced solution of the equation :

$$M\,\partial_t^2 u + C\,\partial_t u + K\,u = f_n \otimes \psi, \tag{79}$$

is written for $t \in \mathbb{R}$:

$$u(t) = \tau_L^{1/2}\, e^{in\Delta\omega t} \sum_{m\in\mathbb{Z}} u_0(m\,\tau_L)\,\varphi_m(t), \tag{80}$$

where u_0 is the forced solution of (70) defined by (71).

The Fourier transform of $u(t)$ with respect to t, which has compact support B_n is written :

$$\hat{u}(\omega) = \tau_L\, p_n(\omega) \sum_{m\in\mathbb{Z}} u_0(m\,\tau_L)\, e^{-im\tau_L\omega}, \tag{81}$$

where p_n is defined by (15). In the same way for $t \in \mathbb{R}$, we have :

$$\partial_t u(t) = \tau_L^{1/2}\, e^{in\Delta\omega t} \sum_{m\in\mathbb{Z}} z(m\,\tau_L)\,\varphi_m(t), \tag{82}$$

where

$$z(t) = \partial_t u_0(t) + in\,\Delta\omega\, u_0(t). \tag{83}$$

Comments (84)

(a) The series on the right sides of equations (80), (81) and (82) are convergent in L^2.

(b) The relationship (81) determines operator T_ω (considering (67)) as a function of the low frequency sampled time solution u_0 of the equation (70), and thus gives a constructive numerical method for obtaining the generalized Green function.

(c) The results of proposition (78) will make it possible to calculate the numbers $E_c^n(\psi^{(1)}, \psi^{(2)})$ and $E_Q^n(\psi^{(1)}, \psi^{(2)})$, directly, that is to determine all the quantities of interest (*cf* VI.1).

Proof of the proposition (78).

The functions φ_m defined by (77) are in $L^2(\mathbb{R})$, and $\{\varphi_m\}$, $m \in \mathbb{Z}$ is an orthonormal system of $L^2(\mathbb{R})$, but not complete. It is thus not an orthonormal basis of $L^2(\mathbb{R})$.

On the other hand, $\{\varphi_m\}$, $m \in \mathbb{Z}$ is an orthonormal basis of the vectorial subspace of $L^2(\mathbb{R})$ defined by $\{f \in L^2(\mathbb{R}) \mid \operatorname{supp} \hat{f} = B_0\}$, for the scalar product and the norm induced by $L^2(\mathbb{R})$.

As $u_0 \in L^2(\mathbb{R}, V)$ and $\operatorname{supp} \hat{u}_0 = B_0$, u_0 can be expanded on the basis $\{\varphi_m\}$, which is nothing other than the expression of the sampling theorem, and gives :

$$u_0(t) = \sum_{m \in \mathbb{Z}} u_0(m\tau_L)\, \tau_L^{1/2}\, \varphi_m(t). \tag{85}$$

Considering (69), we conclude (80). The relationship (81) is obtained directly by taking the Fourier transform of (80).

A similar train of thought leads us from (73) to the relation (82).

Proposition (86)

Let $u_0^{(1)}$ and $u_0^{(2)}$ be the respective forced solutions of (70) for the excitations $f_0 \otimes \psi^{(1)}$ and $f_0 \otimes \psi^{(2)}$, for given $\psi^{(1)}$ and $\psi^{(2)}$ in H, with f_0 given by (65). Then :

(*a*) The value at the point $\{\psi^{(1)}; \psi^{(2)}\}$ of $H \times H$ of the sesquilinear form of the signal energy in the medium frequency band B_n defined by (40) is expressed :

$$E_Q^n(\psi^{(1)}, \psi^{(2)}) = \tau_L \sum_{m \in \mathbb{Z}} (Q\, u_0^{(1)}(m\tau_L),\, Q\, u_0^{(2)}(m\tau_L))_W, \tag{87}$$

where Q is the observation operator from H to W, defined in para. II.6.

(*b*) The value at the point $\{\psi^{(1)}; \psi^{(2)}\}$ of $H \times H$ of the sesquilinear form of the total kinetic energy in the medium frequency band B_n defined by (55) is expressed by :

$$E_c^n(\psi^{(1)}, \psi^{(2)}) = \frac{1}{2}\tau_L \sum_{m \in \mathbb{Z}} (M\, z^{(1)}(m\tau_L),\, z^{(2)}(m\tau_L))_H, \tag{88}$$

where

$$\left.\begin{aligned} & j \in \{1, 2\}, \\ & z^{(j)}(t) = \partial_t u_0^{(j)}(t) + in\,\Delta\omega\, u_0^{(j)}(t). \end{aligned}\right\} \tag{89}$$

Comments (90)

Considering what was said in para. VI.1, the relations (87) and (88) show that we can calculate very simply all of the quantities characterizing the medium frequency dynamics in the band B_n simply by knowing, at times $\{m\tau_L, m \in \mathbb{Z}\}$, the forced solution $u_0^{(j)}$ and its partial derivative $\partial_t u_0^{(j)}$), $j \in \{1, 2\}$ for the equation :

$$\begin{aligned} & M\, \partial_t^2 u_0^{(j)} + (C + 2\,in\,\Delta\omega\, M)\, \partial_t u_0^{(j)} \\ & \quad + (K + in\,\Delta\omega\, C - (n\,\Delta\omega)^2 M)\, u_0^{(j)} = f_0 \otimes \psi^{(j)}. \end{aligned} \tag{91}$$

Moreover, the equation (91) is a second-order differential equation with respect to t, linear, in the low-frequency domain :

$$\left(\operatorname{supp} \hat{f}_0 = \operatorname{supp} \hat{u}_0^{(j)} = B_0 = \left[-\frac{\Delta\omega}{2}, \frac{\Delta\omega}{2}\right]\right).$$

This equation can thus be solved easily by a direct numerical time integration method with large time steps, since the highest frequency component is $\Delta\omega/2$, while a direct numerical integration of the equation (68) would lead to very small time steps since the highest frequency component is then $(n + 1/2)\,\Delta\omega$. This generally leads to insurmountable calculation problems (*cf* Introduction, para. I).

We will discuss the numerical problems in para. VII.

Proof of the proposition (86).

Considering (40) and (80), we have :

$$\begin{aligned} E_Q^n(\psi^{(1)}, \psi^{(2)}) &= \int_{\mathbb{R}} (\sigma^{(1)}, \sigma^{(2)})_W\, dt \\ &= \int_{\mathbb{R}} (Q\, u^{(1)}, Q\, u^{(2)})_W\, dt \\ &= \tau_L \sum_{m \in \mathbb{Z}} \sum_{m' \in \mathbb{Z}} (Q\, u_0^{(1)}(m\tau_L), \\ &\qquad Q\, u_0^{(2)}(m'\tau_L))_W \int_{\mathbb{R}} \varphi_m(t)\, \varphi_{m'}(t)\, dt. \end{aligned}$$

But since $\{\varphi_m\}$ is orthonormal in $L^2(\mathbb{R})$:

$$\int_{\mathbb{R}} \varphi_m(t)\, \varphi_{m'}(t)\, dt = \delta_{mm'},$$

whence the result (87).

In the same way, the relations (55) and (82) yield :

$$\begin{aligned} E_c^n(\psi^{(1)}, \psi^{(2)}) &= \frac{1}{2} \int_{\mathbb{R}} (M\, \partial_t u^{(1)}, \partial_t u^{(2)})_H\, dt \\ &= \frac{1}{2}\tau_L \sum_{m \in \mathbb{Z}} \sum_{m' \in \mathbb{Z}} (M\, z^{(1)}(m\tau_L), \\ &\qquad z^{(2)}(m'\tau_L))_H \int_{\mathbb{R}} \varphi_m(t)\, \varphi_{m'}(t)\, dt, \end{aligned}$$

whence the relation (88).

VII – NUMERICAL ANALYSIS

VII.1 – EXPLANATION OF THE PROBLEM.

Let us first examine the various numerical problems encountered in solving the problem posed, i.e. mainly the calculation of (87) and (88). We will then study the various points in detail.

(a) *Time aspect*

– As the time integration of the equation (70) connot be carried out numerically over all of $\mathbb{R}$, it will be carried out over a bounded time interval (t_I, t_S), where $t_I < 0$ and $t_S > 0$.

This introduces a systematic error because it does not take into account the ernergy contained in the interval $(-\infty, t_I)$ and $(t_S, +\infty)$. We will see that it is possible to reduce this error as much as we want, and that in practice we obtain a very good precision when $t_S - t_I$ is "small", i.e. by integrating over a time interval demanding reasonable calculation costs.

– To solve the equation (70), a step-by-step integration algorithm must be used that is unconditionally stable and produces the values $u_0(t)$ and $\partial_t u_0(t)$ directly for sampled values of t, $m\tau_L$, $m \in \mathbb{Z}$.

Furthermore, the truncation need on the sums in $m \in \mathbb{Z}$ appearing in the relations (87) and (88) is directly linked to the choice of t_I and t_S.

(b) *Spatial aspect*

Let us note that all of the approximation procedures for time integration of the equation (70) can be studied by keeping the operators M, C and K, without introducing their finite dimension approximations.

Nonetheless, as the mechanical structures considered can be anisotropic and of any geometry, only finite dimension approximations of these operators can be obtained to represent them. Naturally, to approach the operators M, C and K, we will use the usual method of finite elements. For the medium frequency domain and possibly even for the high frequency domain, the only problem with using this method is selecting the fineness of the grid in the various parts of the structure.

VII.2 – SPATIAL DISCRETIZING OF THE EQUATIONS BY THE FINITE ELEMENT METHOD.

The forced solution u constructed by (69) and (70) is strictly the same as the forced solution for (68). This is what proposition (66) states.

Moreover, the direct solution of the equation (68) by the finite element method, for the excitation $f_n \otimes \psi$ in the band B_n would require a structure grid compatible with the frequency components of the band B_n. Let us recall that the method of finite elements consists of introducing a subspace V_d of space V, of finite dimension d, with d representing the number of degrees of freedom corresponding to the finite element selected model.

The approximate solution, denoted by u_d of u is then expressed as :

$$u_d(t, M) = \sum_{j=1}^{d} U_j(t)\,\mathbf{b}_j(M),$$

where $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_d\}$ is a basis of V_d. The problems of convergence are studied exhaustively in the literature and we shall not touch upon them here. As we are involved with a classic case of elastodynamics in an anisotropic linear elastic medium, there are no difficulties in addition to the usual theory arising from the context of the medium frequency domain.

Thus let $U(t)$ be the $(d \times 1)$ column matrix on $\mathbb{C}^d$ of the nodal unknowns $U_j(t)$, and let P be the $(d \times 1)$ column matrix on $\mathbb{C}^d$ of the spatial component of nodal forces equivalent to ψ.

Let $[M]$, $[C]$ and $[K]$ be the matrices of the corresponding approximations of the operators M, C and K. These are the mass, damping and stiffness matrices, respectively. They are $(d \times d)$ square, real, symmetric and positive definite, considering the assumptions concerning the operators M, C and K.

If we denote the derivative of any quantity x with respect to t by $\dot{x}$, the approximation of the equation (68) by the finite element method gives

$$[M]\ddot{U}(t) + [C]\dot{U}(t) + [K]\,U(t) = f_n(t)\,P, \qquad \forall\, t \in \mathbb{R}. \tag{92}$$

The proposition (66) means that the forced solution $U(t)$ of (92) relative to the medium frequency excitation $f_n(t).P$, over band B_n, is expressed as :

$$U(t) = U_0(t)\exp(in\,\Delta\omega\, t), \tag{93}$$

where $U_0(t)$ is the $(d \times 1)$ column matrix on $\mathbb{C}^d$ such that $t \mapsto U_0(t)$ is the forced solution of :

$$[M]\ddot{U}_0(t) + [\mathscr{C}_n]\dot{U}_0(t) + [\mathscr{K}_n]\,U_0(t) = f_0(t)\,P, \qquad \forall\, t \in \mathbb{R}, \tag{94}$$

where we let :

$$[\mathscr{C}_n] = [C] + 2\,in\,\Delta\omega\,[M], \tag{95}$$

$$[\mathscr{K}_n] = [K] + in\,\Delta\omega\,[C] - (n\,\Delta\omega)^2\,[M], \tag{96}$$

and where $f_0(t)$ is the function associated with $f_n(t)$ by (65).

Note that the matrices $[\mathscr{C}_n]$ and $[\mathscr{K}_n]$ are $(d \times d)$ square, complex, symmetric (but not Hermitian) and that the initial band structure of the matrices $[M]$, $[C]$ and $[K]$ is kept for $[\mathscr{C}_n]$ and $[\mathscr{K}_n]$. Let $\{\eta_1, \eta_2, \ldots\}$ be an orthonormal basis of the observation space W, introduced in para. II.6. If W is of finite dimension p, we designate by $[Q]$ the $(p \times d)$ matrix of the approximation of the observation operator Q from V_d to W. If W is infinite, we still designate by $[Q]$ the $(p \times d)$ matrix of the approximation of the operator Q from V_d to W_p, where W_p is the finite subspace p of W generated by $\{\eta_1, \eta_2, \ldots, \eta_p\}$. Considering the properties of Q (cf para. II.6), $[Q]$ is a real, $(p \times d)$ matrix.

To abridge the notation that follows, if A and B are two complex column matrices of the same length, we let :

$$\langle A, B \rangle = {}^t\!AB, \tag{97}$$

where ${}^t\!A$ is the row matrix transposed from A, and we will identify an element of A in $\mathbb{C}^d$ with its matrix (again denoted by A) on the cononical basis of $\mathbb{C}^d$.

Considering (80), (81), (82) and (83), the desired solution to the equation (92) is such that :

$$U(t) = \tau_L^{1/2}\, e^{in\Delta\omega t} \sum_{m \in \mathbb{Z}} U_0(m\tau_L)\,\varphi_m(t), \tag{98}$$

$$\hat{U}(\omega) = \tau_L\, p_n(\omega) \sum_{m \in \mathbb{Z}} U_0(m\tau_L)\, e^{-im\tau_L\omega}, \tag{99}$$

$$\dot{U}(t)=\tau_L^{1/2}\, e^{in\,\Delta\omega\, t} \sum_{m\in Z} Z(m\,\tau_L)\,\varphi_m(t), \qquad (100)$$

$$\hat{\dot{U}}(\omega)=\tau_L\, p_n(\omega) \sum_{m\in Z} Z(m\,\tau_L)\, e^{-im\tau_L\omega'} \qquad (101)$$

with $Z(t)$ being the $(d \times 1)$ column matrix such that :

$$Z(t)=\dot{U}_0(t)+in\;\Delta\omega\; U_0(t). \qquad (102)$$

Remarks

1) On the numerical level, the relations that make it possible to calculate the energy characteristics depending on the displacement velocity field show that is of no interest to write $\dot{U}(\omega)=i\,\omega\,\hat{U}(\omega)$, but that using (101) is.

2) The relations (99) and (101) directly yield the frequency response operator matrices (still called generalized Green functions) relative to the displacement field and to the displacement velocity field.

The proposition (86) shows that the sesquilinear forms of the signal energy and of the total kinetic energy defined on $\mathbb{C}^d \times \mathbb{C}^d$ are expressed for any $P^{(1)}$ and $P^{(2)}$ in $\mathbb{C}^d$ by :

$$E_Q^n(P^{(1)}, P^{(2)}) = \tau_L \sum_{m\in Z} \langle [Q]\, U_0^{(1)}(m\,\tau_L),\; [Q]\,\overline{U_0^{(2)}(m\,\tau_L)} \rangle, \quad (103)$$

$$E_c^n(P^{(1)}, P^{(2)}) = \frac{1}{2}\tau_L \sum_{m\in Z} \langle [M]\, Z^{(1)}(m\,\tau_L),\; \overline{Z^{(2)}(m\,\tau_L)} \rangle, \quad (104)$$

with, for $j\in\{1,2\}$:

$$Z^{(j)}(t)=\dot{U}_0^{(j)}(t)+in\;\Delta\omega\; U_0^{(j)}(t), \qquad (105)$$

and where $U_0^{(j)}(t)$ is the forced solution of the equation (94) for excitation $f_0(t)\,P^{(j)}$

Remarks :

1) The relation (103) can be used to calculate the energy characteristics (21) and (22), to study the spatial propagation of the vibrations (24), (25) and (26), for a deterministic or stationary random excitation, to determine extremum vibratory states (*cf* para. IV).

2) The relation (104) can be used to calculate the modal density (*cf* para. V). In effect, the relation (64) requires the calculation of $\mathrm{tr}(M\,\mathscr{E}^n_{c,\xi})$, the approximation of which is expressed as $\mathrm{tr}([M][\mathscr{E}^n_{c,\xi}])$, the Hermitian $(d\times d)$ matrix $[\mathscr{E}^n_{c,\xi}]$ having the following elements, as per (54) :

$$\left.\begin{array}{l}[\mathscr{E}^n_{c,\xi}]_{p'p}=E_c^n(e_p, e_{p'}),\\ p, p' \in \{1, 2, \ldots, d\},\end{array}\right\} \qquad (106)$$

where $\{e_1, e_2, \ldots, e_p\}$ is the canonical basis of $\mathbb{C}^d$.

Let us note that, using an additional approximation, we can avoid having to construct the entire matrix $[\mathscr{E}^n_{c,\xi}]$.

In effect, as the matrix $[M]$ is real, symmetric, positive definite, it can be expressed as :

$$[M]=\sum_{j=1}^{d} \mu_i\,\Lambda_j\,{}^t\Lambda_j, \qquad (107)$$

where $\mu_1 \geqq \mu_2 \geqq \mu_3 \geqq \ldots \geqq \mu_d > 0$ are the eigenvalues of $[M]$ and $\Lambda_1, \Lambda_2, \ldots, \Lambda_d$, are the $(d\times 1)$ column matrices of the associated eigenvectors, which are such that $\langle \Lambda_j, \Lambda_{j'} \rangle = \delta_{jj'}$. We then have :

$$\begin{aligned}\mathrm{tr}([M][\mathscr{E}^n_{c,\xi}]) &= \sum_{j=1}^{d} \mu_j\,\mathrm{tr}(\Lambda_j\,{}^t\Lambda_j[\mathscr{E}^n_{c,\xi}])\\ &= \sum_{j=1}^{d} \mu_j \langle [\mathscr{E}^n_{c,\xi}]\,\Lambda_j, \Lambda_j\rangle\\ &= \sum_{j=1}^{d} \mu_j\, E_c^n(\Lambda_j, \Lambda_j). \quad (108)\end{aligned}$$

When d is large, solving the entire eigen value problem of $[M]$, which is necessary if we are to use (108), is more costly than the direct calculation of (106). However, if the mass matrix $[M]$ is such that the q first eigenvalues $\mu_1, \mu_2, \ldots, \mu_d$, with $q \ll d$ are dominant with respect to $\mu_{q+1}, \ldots, \mu_d$, we can then approach (108) using :

$$\mathrm{tr}([M][\mathscr{E}^n_{c,\xi}]) \simeq \sum_{j=1}^{q} \mu_j\, E_c^n(\Lambda_j, \Lambda_j). \qquad (109)$$

The approximate value (109) is then advantageous because it requires only the calculation of the first eigenvalues and eigenvectors of $[M]$, then the calculation of (104) for a small number of spatial force distribution.

$$P^{(1)}=P^{(2)}=\Lambda_j, \qquad j\in\{1, 2, \ldots, q\}.$$

VII.3 – TIME INTEGRATION OF THE ASSOCIATED LOW FREQUENCY EQUATION.

The results of the para. VII.2 show that all of the quantities are expressed by solving the low frequency equation (94). We will study in this paragraph the numerical aspects related to the time integration of (94).

This equation is standard and we can thus use any unconditionally stable step-by-step integration algorithm (Newmark method, θ-Wilson method, etc.).

(a) *Selection of the integration step.*

We retain the notations (76). We have seen that $\mathrm{supp}\,\hat{f}_0 = \mathrm{supp}\; U_0 = B_0 = [-\Delta\omega/2, \Delta\omega/2]$. This is why the forced solution of (94) that we are looking for is called the associated low frequency solution to the medium frequency problem in band B_n.

As the highest angular frequency existing in the signals f_0 and U_0 is $\omega_L = \Delta\omega/2$, the smallest period is $2\,\pi/\omega_L = 2\,\tau_L$. According to the Shannon theorem, the integration step t must be less than $2\,\tau_L/2 = \tau_L$. Now to use formulas of the type (103), (104), $U_0(t)$ and $\dot{U}_0(t)$ must be known at instants $m\,\tau_L$, $m\in\mathbb{Z}$. Consequently, the integration step, denoted by Δt, will be such that $\Delta t = \tau_L / m_T$, where m_T is a positive integer greater than 1. The choice of m_T partially conditions the precision that will be obtained in the solution and depends only on the numerical method of integration used.

(b) *Choice of the initial instant of integration t_I.*

The time $t_I < 0$ at which integration will start will be taken at $t_I = - m_I \tau_L$, where m_I is a positive integer. The initial instant t_I is thus defined by m_I. To optimize the cost of numerical calculation, this must be as small as possible, with two constraints : not to truncate the excitation energy and not to introduce a numerical transient disturbing the solution that is being looked for.

The selection of m_I is thus related to the asymptotic behavior of the function f_0 for $t \to -\infty$. But since $f_0 \in L^2(\mathbb{R})$, we know that for any fixed positive ϵ as small as we want, $\exists\, m_I$ such that

$$\int_{-\infty}^{-m_I \tau_L} | f_0(t) |^2 \, dt < \varepsilon .$$

We can thus make the part of the energy truncated by the choice of a finite initial instant t_I as small as we wish. For example, for f_0 defined by (16)

$$\int_{-\infty}^{-m_I \tau_L} | f_0(t) |^2 \, dt \leqq \frac{1}{\pi^2} \int_{-\infty}^{-m_I \tau_L} \frac{dt}{t^2} = \frac{\Delta\omega}{2\pi^3 m_I},$$

and since by (17) :

$$\int_{-\infty}^{+\infty} | f_0(t) |^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} | \hat{f}_0(\omega) |^2 \, d\omega = \frac{\Delta\omega}{2\pi},$$

the relative error introduced is less than $(\pi^2 m_I)^{-1}$, which gives 0.03 for $m_I = 3$, for example.

Furthermore, the energy of the transient created by the algorithm can be made negligible by taking zero initial conditions

$$U_0(-m_I \tau_L) = \dot{U}_0(-m_I \tau_L) = 0,$$

and m_I large enough so that the system can adapt during the first instants of integration, with the excitation energy transferred to the system remaining small during these instants.

(c) *Choice of the final instant t_S.*

The final instant of integration, designated t_S, will be taken as $t_S = m_S \tau_L$, with m_S a positive integer. As $U_0 \in L^2(\mathbb{R}, \mathbb{C}^d)$, again, with $\forall\, \varepsilon$ fixed as small as desired, finite m_S exists such that :

$$\int_{m_S \tau_L}^{+\infty} \langle U_0(t), \overline{U_0(t)} \rangle \, dt < \varepsilon .$$

Thus we can make the energy not taken into account in the interval $(t_S, +\infty)$ as small as we want.

The choice of m_S is directly related to the dynamics of the system governed by the structural damping. As the total energy introduced is known before integration is started and as at each instant $t > t_I$, the total energy dissipated during the time interval $t - t_I$ can be determined, the energy budget makes it possible to determine m_S automatically to obtain a given precision beforehand in the solution.

(d) *Remark.*

The relations (98) to (101), (103) and (104) use a sum at every instant $m\tau_L$, $m \in \mathbb{Z}$. The energy considerations discussed in points (*b*) and (*c*) above which define the initial instant $t_I = - m_I \tau_L$ and the final instant $t_S = m_S \tau_L$, directly give the truncations to be carried out on the sums $\sum_{m \in \mathbb{Z}}$. The $\sum_{m \in \mathbb{Z}}$ need only be replaced by $\sum_{m=-m_I}^{m_S}$.

(e) *Numerical method of step-by-step integration.*

Here we present a numerical method based on the implicit second central difference, unconditionally stable and equivalent to the Newmark method [2], but developed in a nonconventional form due to [1]. This method that we have tested numerically for our dynamics problem in the medium frequency domain gives very good results with a minimum of operations.

We thus have to solve the following problem :

$$\left.\begin{aligned} [M]\,\ddot{U}_0(t) + [\mathscr{C}_n]\,\dot{U}_0(t) & \\ + [\mathscr{K}_n]\,U_0(t) = f_0(t)\,P, & \qquad t \in (t_I, t_S), \\ U_0(t_I) = \dot{U}_0(t_I) = 0 & \end{aligned}\right\} \tag{110}$$

where $[\mathscr{C}_n]$ and $[\mathscr{K}_n]$ are the matrices defined by (95) and (96).

Let $t_1, t_2, \ldots, t_m, \ldots$ be the finite set of the interval (t_I, t_S) such that $t_1 = t_I$ and $\forall\, m,\ t_{m+1} - t_m = \Delta t$. At the instant "1/2" of the interval $[t_m, t_{m+1}]$, denoted by :

$$t_{1/2} = \frac{t_m + t_{m+1}}{2} = t_m + \frac{\Delta t}{2}, \tag{111}$$

we let

$$U_m = U_0(t_m); \qquad V_m = \dot{U}_0(t_m). \tag{112}$$

At the instant $t_{1/2}$, the implicit central difference method leads us to define :

$$U_{1/2} = \frac{U_m + U_{m+1}}{2} \tag{113}$$

$$\dot{U}_{1/2} = \frac{U_{m+1} - U_m}{\Delta t} = \frac{V_{m+1} + V_m}{2} \tag{114}$$

$$\ddot{U}_{1/2} = \frac{V_{m+1} - V_m}{\Delta t}. \tag{115}$$

From the second equation (114) we conclude :

$$V_{m+1} = 2\left(\frac{U_{m+1} - U_m}{\Delta t} \right) - V_m. \tag{116}$$

(115) can also be expressed :

$$\ddot{U}_{1/2}=\left(\frac{V_{m+1}+V_m}{2}\right)\frac{2}{\Delta t}-2\frac{V_m}{\Delta t},$$

or, using the second equation (114) :

$$\ddot{U}_{1/2}=\frac{2}{(\Delta t)^2}U_{m+1}-\frac{2}{(\Delta t)^2}U_m-\frac{2}{\Delta t}V_m. \quad (117)$$

By writing the equilibrium at the instant $t_{1/2}$, (110) yields :

$$[M]\ddot{U}_{1/2}+[\mathscr{C}_n]\dot{U}_{1/2}+[\mathscr{K}_n]U_{1/2}=f_0(t_{1/2})P, \quad (118)$$

and by substituting the relations (117), (114) and (113) in (118) we get :

$$\left(\frac{2}{(\Delta t)^2}[M]+\frac{1}{\Delta t}[\mathscr{C}_n]+\frac{1}{2}[\mathscr{K}_n]\right)U_{m+1}$$

$$=\left(\frac{2}{(\Delta t)^2}[M]+\frac{1}{\Delta t}[\mathscr{C}_n]-\frac{1}{2}[\mathscr{K}_n]\right)U_m$$

$$+\frac{2}{\Delta t}[M]V_m+f_0\left(t_m+\frac{\Delta t}{2}\right)P. \quad (119)$$

By replacing the expressions (95) and (96) of $[\mathscr{C}_n]$ and $[\mathscr{K}_n]$, in (119), considering (116), we get the system of equations of the numerical integration method :

$$[A_n]U_{m+1}=[B_n]U_m+\frac{2}{\Delta t}[M]V_m$$

$$+f_0\left(t_m+\frac{\Delta t}{2}\right)P, \qquad m\geqq 1, \quad (120)$$

$$V_{m+1}=\frac{2}{\Delta t}(U_{m+1}-U_m)-V_m, \quad (121)$$

where, for the initial instant $t_I=t_1$, $U_1=V_1=0$, and where we let :

$$[A_n]=\left(\frac{2}{(\Delta t)^2}+\frac{2in\Delta\omega}{\Delta t}-\frac{(n\Delta\omega)^2}{2}\right)[M]$$

$$+\left(\frac{1}{\Delta t}+in\frac{\Delta\omega}{2}\right)[C]+\frac{1}{2}[K], \quad (122)$$

$$[B_n]=\left(\frac{2}{(\Delta t)^2}+\frac{2in\Delta\omega}{\Delta t}+\frac{(n\Delta\omega)^2}{2}\right)[M]$$

$$+\left(\frac{1}{\Delta t}-in\frac{\Delta\omega}{2}\right)[C]-\frac{1}{2}[K] \quad (123)$$

Remark 1 : Note that $[A_n]$ and $[B_n]$ are both $(d\times d)$ square, complex, symmetric matrices and they have the same band structure as the matrices $[M]$, $[C]$ and $[K]$.

Remark 2 : For a fixed medium frequency band B_n, the solution of (120) for all values of m and for values of P that must be considered require only one triangularization of the band matrix $[A_n]$.

Remark 3 : In the special case where the structural damping is introduced by the mean damping rate ξ in the band B_n, we have $[C]=2\xi n\Delta\omega[M]$. [*cf* (59) and (A.3)] and the relationships (122) and (123) are expressed :

$$[A_n]=\left(\frac{2}{(\Delta t)^2}+\frac{2n\Delta\omega}{\Delta t}(i+\xi)\right.$$

$$\left.+(n\Delta\omega)^2\left(i\xi-\frac{1}{2}\right)\right)[M]+\frac{1}{2}[K], \quad (124)$$

$$[B_n]=\left(\frac{2}{(\Delta t)^2}+\frac{2n\Delta\omega}{\Delta t}(i+\xi)\right.$$

$$\left.-(n\Delta\omega)^2\left(i\xi-\frac{1}{2}\right)\right)[M]-\frac{1}{2}[K]. \quad (125)$$

VIII – EXAMPLE

Here we present the example of a plane, homogeneous rectangular plate, isotropic in bending.

We have voluntarily selected this very simple case so as to be able to :

– compare the numerical results obtained with the exact theoretical solution known ;

– study the numerical convergences ; in effect, we known that there is a convergence mathematically, but we want to see if the needed calculations remain reasonable.

To do this, we developed a specific finite element semi-analytical computer program for this type of plate, to construct its mass, damping and stiffness matrix.

VIII.1 – GEOMETRICAL AND MECHANICAL DATA.

Relative to an orthonormal $oxyz$ system, we consider a plane rectangular plate of dimensions $L_x=6.0$ and $L_y=0.5$, the central plane of which is the oxy plane. The coordinates of the four corners of the plate are (0, 0, 0), $(L_x, 0, 0)$, $(L_x, L_y, 0)$ and $(0, L_y, 0)$.

The plate is homogeneous, isotropic of constant thickness $e=0.001$, density $\rho=7\,850$, Poisson constant $\nu=0.3$ and Young's modulus $E=2\times 10^{11}$. We want to know the bending, with the displacement of a point (x, y) in the oz direction in the central plane being denoted by $w(x,y)$. We hypothesize that the plate is simply resting on its edges.

VIII.2 – FINITE ELEMENTS CONSIDERED.

For such a plate, we know how to construct the exact solution explicitly. To test the proposed method in the medium frequency domain, we modelize the plate with finite semi-analytic elements, discretizing in the ox direction and integrating analytically along oy.

The finite element used is constructed by expanding in a direct sum the displacement field space $w(x, y)$ into a subspace characterized by an integer $N > 1$ such that the displacement field in the subspace N is expressed by :

$$w_N(x, y) = u(x) \sin\left(\frac{N\pi y}{L_y}\right).$$

Having chosen a grid in the ox direction, the finite semi-analytical element is constructed for each subspace N by taking cubic interpolation functions in x for the function $u(x)$ and integrating analytically over y.

We thus obtain for each subspace N a mass matrix $[M]$ and a stiffness matrix $[K]$. In the present example, we generated the damping matrix relative to a medium frequency band B_n by the relationship

$$[C] = 2\xi n \Delta\omega [M].$$

The scalar $\xi > 0$ then represents the mean critical damping rate in the band B_n.

By knowing these three matrices the vibratory state of the plate can be determined for medium frequency excitations in the band B_n, the spatial component of the excitation being in the considered subspace N, i.e. of the form $P \sin\left(\frac{N\pi y}{L_y}\right)$, with P being a column matrix of constants.

For the numerical results given below, we took a grid with 91 nodes at a constant mesh in the ox direction, which gives 90 finite semi-analytical elements and 180 degrees of freedom (considering the boundary conditions).

In fact, for the medium frequency bands B_n studied this grid is a bit too dense ; but we did not want to introduce systematic errors of approximation due to the finite element method, considering the comparisons with the exact theoretical solution, because we are attempting to test the convergences on the other parameters of the numerical model we are proposing (choice of parameter values for m_I, m_S, m_T, Δt, etc.).

VIII.3 – NUMERICAL RESULTS OBTAINED.

A. *Numerical convergence.*

For a fixed band B_n, we study the convergences of the solutions as a function of the parameters m_I, m_S, m_T defined in paragraph VII.3. For the time component of the excitation, we use the function $f_n(t)$ defined in II.4d. For the space component of the excitation relative to a subspace N, we apply the nodal force $1 \times \sin\left(\frac{N\pi y}{L_y}\right)$ along the degree of freedom z at node 46, which is the middle of the plate, and 0 on all of the other degrees of freedom.

The quantity observed to test the numerical convergence is the value of $u(x)$ at node 46, denoted by u_{46}, with the complete displacement field for this node being given by :

$$w_N(x_{46}, y) = u_{46} \sin\left(\frac{N\pi y}{L_y}\right), \qquad y \in (0, L_y).$$

For this observation u_{46}, we calculate its energy characteristic given by (21), which we designate by the simplified notation E_u.

Table I summarizes the calculations processed and indicates the number of the figures 1 to 6 graphing the numerical results obtained by the method explained in this article.

The various figures 1 to 6 show that the convergence is rapid and that the solution is obtained with good precision and after relatively few steps in the calculation.

The number of steps is $(m_I + m_S) \times m_T$. In the cases covered, with some 30 steps, we obtain a solution having more than sufficient precision for practical purposes. It should be noted that the ordinate scale is very much expanded.

TABLE I

Subspace N	Band B_n			Mean damping rate ξ	Theoretical number of eigenmodes in the band B_n and in the subspace N	Theoretical value of E_u (m)	Figure no.
	Center frequency Hz	Bandwidth $\Delta\omega$ (Hz)	Value of n				
5	260.	26.	10	0,001	12	$0{,}1667 \times 10^{-4}$	1
5	260.	26.	10	0,01	12	$0{,}8110 \times 10^{-5}$	2
15	2180,1	50,7	43	0,001	26	$0{,}1683 \times 10^{-5}$	3 and 4
15	2180,1	50,7	43	0,01	26	$0,\ 849 \times 10^{-6}$	5 and 6

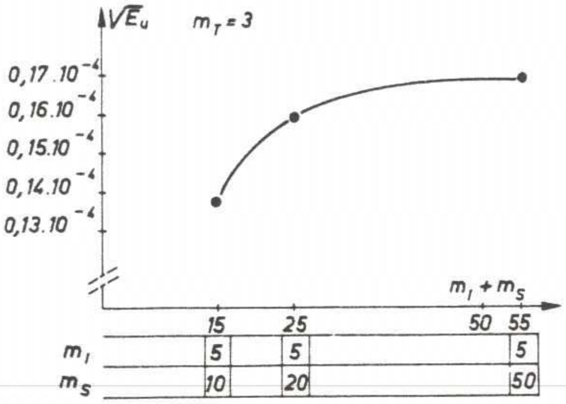

Fig. 1.

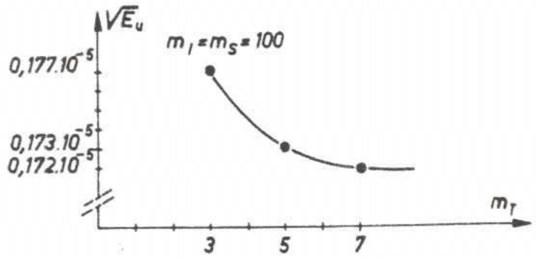

Fig. 4.

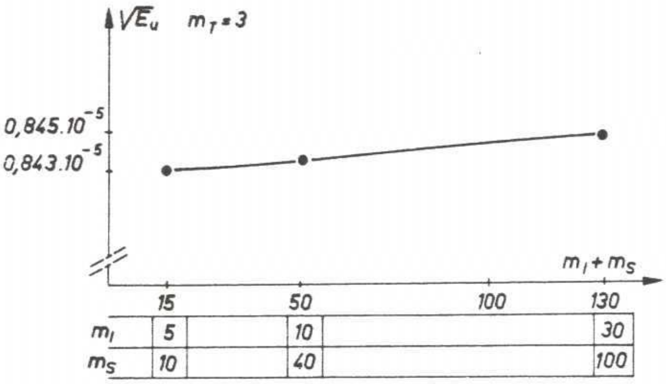

Fig. 2.

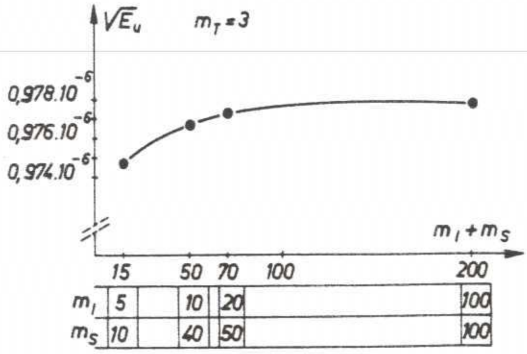

Fig. 5.

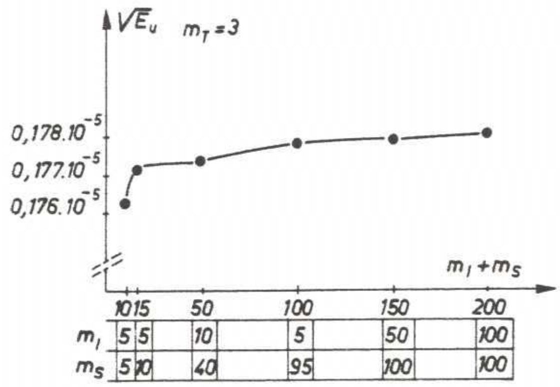

Fig. 3.

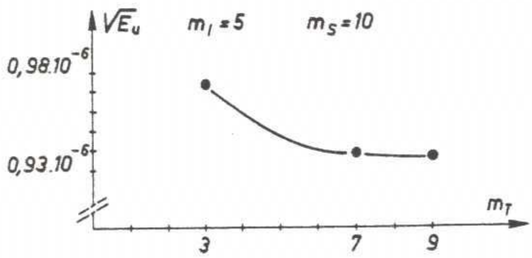

Fig. 6.

B. *Level and spatial propagation of vibrations.*

Table 2 summarizes the computations graphed in the figures 7 through 14.

For all of the calculations, the time component of the excitation f_n is the function defined in II.4d. For the space component of the excitation relative to a subspace

TABLE II

Subspace N	Band B_n			Mean damping rate ξ	Theoretical number of eigenmode in the band B_n and in the subspace N	m_l	m_S	m_T	Figure no.
	Center frequency (Hz)	Bandwidth $\Delta\omega$ (Hz)	Value of n						
15	2180,1	50,7	43	0,001	26	5	10	3	7
15	2180,1	50,7	43	0,01	26	5	10	3	8
15	2180,1	50,7	43	0,001	26	10	40	3	9
15	2180,1	50,7	43	0,01	26	10	40	3	10
5	260.	26.	10	0,001	12	5	20	3	11
5	260.	26.	10	0,01	12	5	20	3	12
5	260.	26.	10	0,001	12	5	20	3	13
5	260.	26.	10	0,01	12	5	20	3	14

N, the nodal force $1 \times \sin\left(\frac{N \pi y}{L_y}\right)$ is applied along the degree of freedom z at the node indicated on each figure by a vertical arrow next to the $P = 1$ symbol, and 0 is applied to all of the other degrees of freedom. Each figure (7 to 14) represents the graph of the function :

$$x \mapsto \sqrt{E_u(x)} \, (\max_{x \in (0,\, L_x)} \sqrt{E_u(x)})^{-1}, \qquad x \in (0,\, L_x),$$

where $E_u(x)$ is the energy characteristic of the observation $u(x)$ given by (21). This function thus makes it possible to calculate the intensity of the vibrations, and to study their spatial propagation, at all points in the plate. Each figure shows the solution obtained from the exact theory and the numerical solution obtained by means of the theory discussed here. Note that the prediction obtained may be qualified as very good.

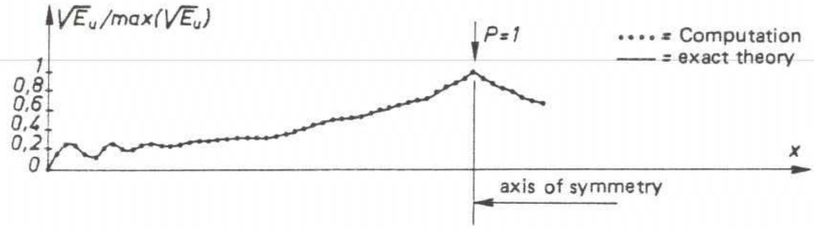

Fig. 7.

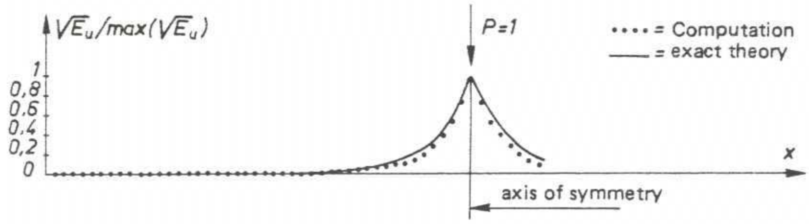

Fig. 8.

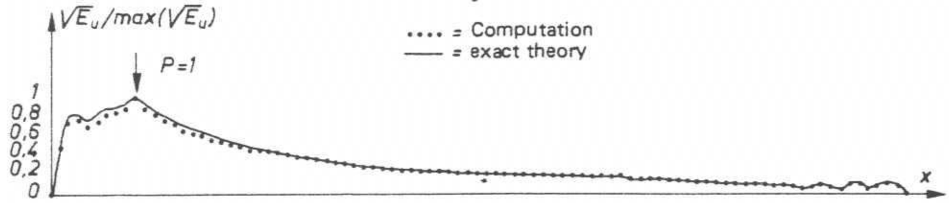

Fig. 9

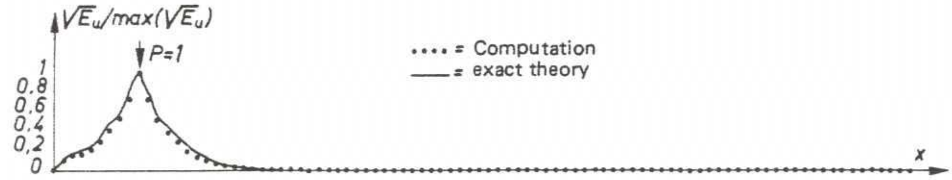

Fig. 10.

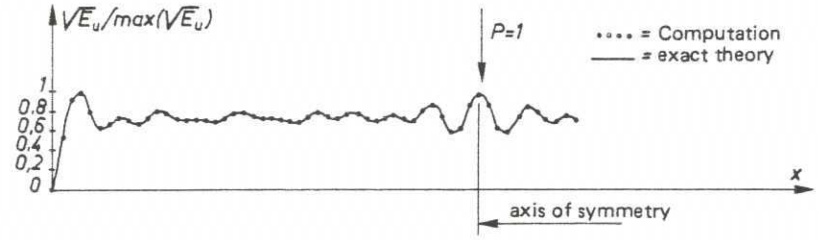

Fig. 11.

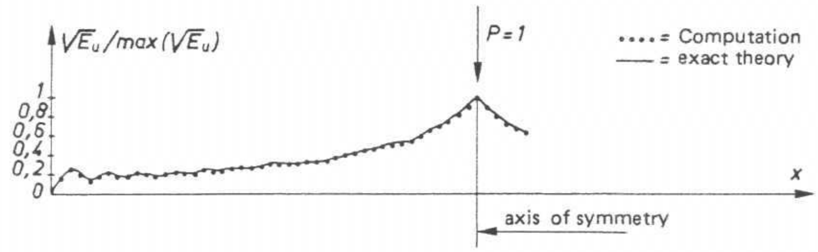

Fig. 12.

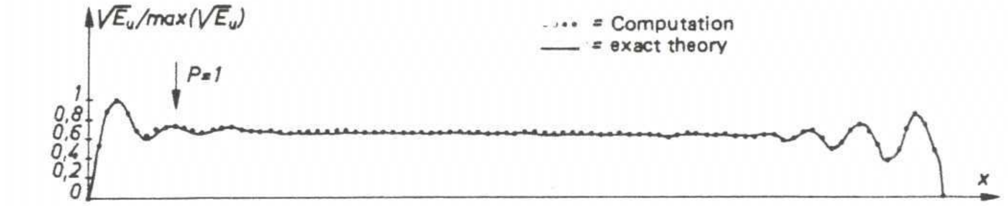

Fig. 13.

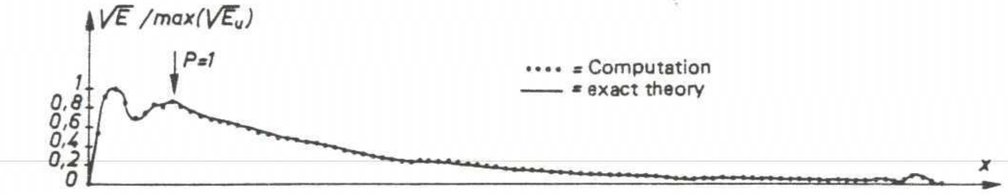

Fig. 14.

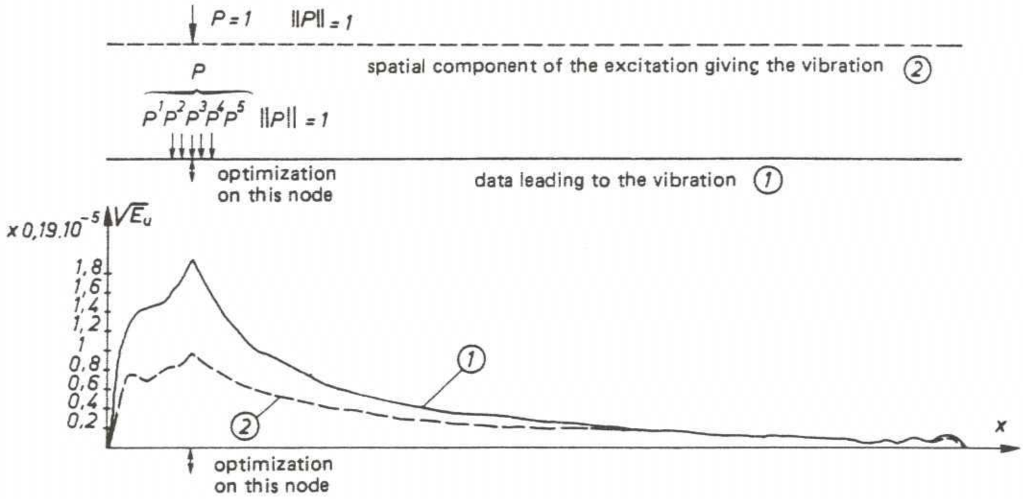

Fig. 15.

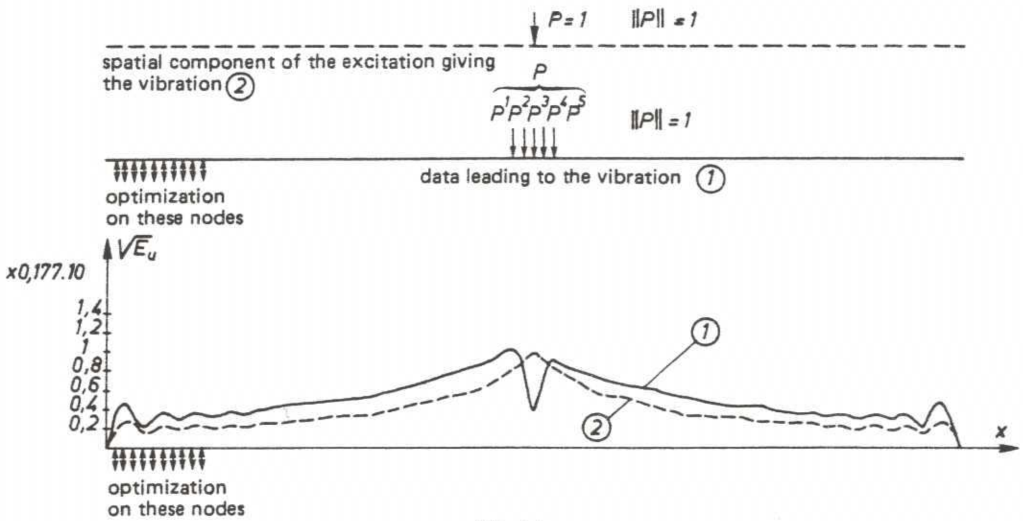

Fig. 16.

C. *Maximum vibratory states.*

The figures 15 and 16 illustrate the implementation of the developments in paragraph IV, to determine the maximum vibratory states.

The hypothesis of calculation for these two figures are as follows :

The subspace considered is $N = 15$. The center frequency of the band B_n is 2 180.1 Hz, with a bandwidth $\Delta\omega$ of 50.7 Hz, which corresponds to $n = 43$. In this band, the theoretical number of eigenmodes (and in the subspace $N = 15$) is 26. The mean critical damping rate in the band is taken to be $\xi = 0.001$. The calculation parameters are $m_I = 5$, $m_S = 10$ and $m_T = 3$. The time component of the excitation f_n is the function defined in II.4d.

Each curve in the figures 15 and 16 is a graph of the function $x \mapsto \sqrt{E_u(x)}$, $x \in (0, L_x)$, where $E_u(x)$ is the energy characteristic of the observation $u(x)$ given by (21) and corresponding to the following spatial excitations :

(a) The curves referenced (2) correspond to the vibration obtained for an excitation the spatial distribution of which is a nodal force $1 \times \sin\left(\frac{N \pi y}{L_y}\right)$ all along the degree of freedom z of the node indicated in the figures by a vertical arrow next to the symbol $P = 1$, and 0 for all of the other degrees of freedom.

(b) The curves referenced (1) correspond to the maximum vibration, determined by the theory explained in paragraph IV, that there can be at a point in the plate (Fig. 15) or in an area of the plate (Fig. 16), indicated in the figures by double arrows and the note "optimization on these nodes", when the spatial component of the excitation $P \sin\left(\frac{N \pi y}{L_y}\right)$ of the subspace N, represented by $P \in \mathbb{C}^{180}$, describes the subspace $\mathbb{R}^5$ of $\mathbb{C}^{180}$, defined by the degrees of freedom indicated in the two figures by the vertical arrows under to the symbol $P^{(1)}, P^{(2)}, P^{(3)}, P^{(4)}, P^{(5)}$. These scalars $P^{(j)}$ represent the components of the forces along the considered degrees of freedom, with the normalization conditions

$$\| P \|^2 = \sum_{j=1}^{5} | P^{(j)} |^2 = 1.$$

For figure 15, the maximum vibratory states for the defined configuration is obtained for $P^{(1)} = 0.391$, $P^{(2)} = 0.475$, $P^{(3)} = 0.493$, $P^{(4)} = 0.476$, $P^{(5)} = 0.388$.

For figure 16, it is obtained for $P^{(1)} = 0.626$, $P^{(2)} = 0.500$, $P^{(3)} = 0.123$, $P^{(4)} = -0.288$, $P^{(5)} = -0.509$.

We note that, for each calculation considered, the total excitation energy is the same for vibrations (1) and (2). Furthermore, the spatial localization is roughly the same. Nevertheless, the vibration intensities obtained are very different in the figure 15.

We thus see the great interest of the maximum vibratory state calculations, because the vibration (1) in the figure 15, for example, directly yields the absolute maximum of the vibration that can exist in band B_n at the specified point on the plate when the space component of the excitation is not fixed but describes a subspace, the total energy of the excitation of course being fixed. This aspect is extremely useful in predictions.

IX – CONCLUSIONS

The difficulties encountered in studying the dynamics of linear anisotropic elastic media in the medium frequency range with modal techniques led us to develop a new numerical method. Although the method proposed is supported by numerical processes, it led us to introduce new mechanical concepts to study the vibrations of the structures in the higher frequency domain (energy characteristics per band, extremum vibratory states, etc.), for both deterministic and random excitations, It should be noted that similar concepts have been in use for a long time in other fields of physics (theoretical physics), and in the probabilistic approach to the dynamics of structures.

The numerical analysis that we have carried out shows that this method can be used without difficulty in the main existing structural computation codes. We are currently attracted to this method so as to be able to process practical cases entailing modelization with a large number of degrees of freedom.

Finally, we believe that this approach can be used to address problems of fluid-structure interaction in the same frequency ranges, and in particular elasto-acoustical problems, as long as the problems remain linear.

Manuscript submitted July 19th, 1982.

APPENDIX

MODAL DENSITY

Below we give the proof of relations (61) to (64) relative to the modal density in the band B_n.

We again use here the set of notations and hypotheses given in the article.

Let $\{\varphi_j\}, j \in \mathbb{N}$ be the modal basis of the associated conservative elastic medium, and $\{\Omega_j\}$, $j \in \mathbb{N}$ be the spectrum of associated angular eigen frequencies such that :

$$\Omega_1 \leqq \Omega_2 \leqq \Omega_3 \leqq \dots$$

The set $\{\varphi_j\}$, $j \in \mathbb{N}$ constitutes a real orthonormal basis in H. We have :

$$(M \varphi_j, \varphi_k)_H = \delta_{jk} m_j, \quad \text{(A.1)}$$

$$(K \varphi_j, \varphi_k)_H = \delta_{jk} m_j \Omega_j^2, \quad \text{(A.2)}$$

where δ_{jk} is the Kronecker symbol.

Considering (59) and (A.1), we have :

$$\forall j \in \mathbb{N}, \quad \xi_j = \xi \frac{n \Delta\omega}{\Omega_j}, \quad \text{(A.3)}$$

$$(C \varphi_j, \varphi_k)_H = \delta_{jk} 2 \xi_j \Omega_j m_j, \quad \text{(A.4)}$$

as $\sup_j \xi_j = \xi \frac{n \Delta\omega}{\Omega_1} = \xi_1$, when $\xi_1 \to 0, \forall j, \xi_j \to 0$.

Yet when $\xi \to 0$, $\xi_1 \to 0$. Thus :

$$\forall j \in \mathbb{N}, \quad \lim_{\xi \to 0_+} \xi_j = 0. \quad \text{(A.5)}$$

Under these conditions, the frequency response operator T_ω defined by (5) is expressed

$$T_\omega = \sum_{j=1}^{+\infty} h_j(\omega) \varphi_j \otimes \varphi_j, \quad \text{(A.6)}$$

where

$$h_j(\omega)=\frac{1}{m_j(\Omega_j^2-\omega^2+2\,i\,\omega\xi_j\,\Omega_j)}. \qquad (A.7)$$

We thus have :

$$T_\omega^* M T_\omega=(\sum_j \bar{h}_j(\omega)\,\varphi_j\otimes\varphi_j)(M)(\sum_{j'} h_{j'}(\omega)\,\varphi_{j'}\otimes\varphi_{j'})$$

$$=\sum_{j,j'} \bar{h}_j(\omega)\,h_{j'}(\omega)\,(M\,\varphi_{j'},\,\varphi_j)_H\,\varphi_j\otimes\varphi_{j'}.$$

Considering (A.1), we conclude that :

$$T_\omega^* M T_\omega=\sum_{j=1}^{\infty} |h_j(\omega)|^2\,m_j\,\varphi_j\otimes\varphi_j. \qquad (A.8)$$

Substituting (A.8) in (58), considering (60), (17) and (A.3), we obtain, by letting :

$$I(\Omega,\,\xi)=\frac{1}{4\pi}\int_{\omega\in B_n}\frac{\omega^2\,d\omega}{(\Omega^2-\omega^2)^2+4\,\omega^2\,\xi^2\,(n\,\Delta\omega)^2}, \qquad (A.9)$$

$$\mathscr{E}^n_{c,\,\xi}=\sum_{j=1}^{\infty} m_j^{-1}\,I(\Omega_j,\,\xi)\,\varphi_j\otimes\varphi_j. \qquad (A.10)$$

By composing on the left the two sides of (A.10) with the mass operator M, and since :

$$\mathrm{tr}\,\{M\,(\varphi_j\otimes\varphi_j)\}=(M\,\varphi_j,\,\varphi_j)_H=m_j,$$

we deduce the relationship :

$$\mathrm{tr}\,(M\,\mathscr{E}^n_{c,\,\xi})=\sum_{j=1}^{\infty} I(\Omega_j,\,\xi). \qquad (A.11)$$

Let $\mathscr{N}$ be the positive measure on $\mathbb{R}$ such that $\mathscr{N}=\sum_{j=1}^{\infty}\delta_{\Omega_j}$, where δ_{Ω_j} is the Dirac measure at point Ω_j of $\mathbb{R}^+$. We then have :

$$\int_{\mathbb{R}} I(\Omega,\,\xi)\,\mathscr{N}(d\Omega)=\sum_{j=1}^{+\infty} I(\Omega_j,\,\xi). \qquad (A.12)$$

For any interval J bounded in $\mathbb{R}$,

$$\mathscr{N}(J)=\int_{\Omega\in J}\mathscr{N}(d\Omega)<+\infty$$

is equal to the number of eigenmodes there are in the interval J. From relationships (A.11) and (A.12) we can state :

$$\mathrm{tr}\,(M\,\mathscr{E}^n_{c,\,\xi})=\int_{\mathbb{R}} I(\Omega,\,\xi)\,\mathscr{N}(d\Omega)<+\infty, \qquad (A.13)$$

because as $M\,\mathscr{E}^n_{c,\,\xi}$ is a nuclear operator in H, $\mathrm{tr}\,(M\,\mathscr{E}^n_{c,\,\xi})<+\infty$ the integral of (A.13) can be written as :

$$\int_{\mathbb{R}} I(\Omega,\,\xi)\,\mathscr{N}(d\Omega)=K_1+K_2, \qquad (A.14)$$

where :

$$K_1=\int_{\Omega\in B_n} I(\Omega,\,\xi)\,\mathscr{N}(d\Omega)<+\infty \qquad (A.15)$$

$$K_2=\int_{\Omega\in\mathbb{R}\setminus B_n} I(\Omega,\,\xi)\,\mathscr{N}(d\Omega)$$

$$=\sum_{j;\,\Omega_j\notin B_n} I(\Omega_j,\,\xi)<+\infty. \qquad (A.16)$$

To simplify the demonstration, we suppose that the band $B_n=[(n-1/2)\,\Delta\omega,\,(n+1/2)\,\Delta\omega]$ is such that

$$\left.\begin{array}{c}\forall j\in\mathbb{N},\\ \left(n-\frac{1}{2}\right)\Delta\omega\neq\Omega_j,\qquad \left(n+\frac{1}{2}\right)\Delta\omega\neq\Omega_j.\end{array}\right\} \qquad (A.17)$$

Note that if the hypothesis (A.17) is not verified, there still exists a band B_n' in the neighborhood of B_n, for (A.17) to be satisfied.

We will thus limit ourselves to the hypothesis (A.17).

For $\Omega_j\notin B_n$, $(\Omega_j^2-\omega^2)^2$ is strictly positive for any $\omega\in B_n$. Therefore $I(\Omega_j,\,\xi)\leqq I(\Omega_j,\,0)$ and, considering hypothesis (A.17), we have $I(\Omega_j,\,0)<+\infty$, whence :

$$\lim_{\xi\to 0_+}\xi\,I(\Omega_j,\,\xi)\leqq\lim_{\xi\to 0_+}\xi\,I(\Omega_j,\,0)=0.$$

Consequently :

$$\forall\,\Omega_j\notin B_n,\quad \lim_{\xi\to 0_+}\xi\,I(\Omega_j,\,\xi)=0. \qquad (A.18)$$

Considering (A.16), we conclude :

$$\lim_{\xi\to 0_+}\xi\,K_2=0. \qquad (A.19)$$

By construction, the modal density on band B_n is the positive constant N_n such that :

$$\mathscr{N}(d\Omega)=N_n\,d\Omega. \qquad (A.20)$$

From relationships (A.15) (A.9) and A.20) we conclude :

$$K_1=\frac{N_n}{4\pi}\int_{\Omega\in B_n}\int_{\omega\in B_n} \times\frac{\omega^2\,d\omega\,d\Omega}{(\Omega^2-\omega^2)^2+4\,\omega^2\,\xi^2\,(n\,\Delta\omega)^2}. \qquad (A.21)$$

When the changes of variables of integration $\omega=(n+\varepsilon)\,\Delta\omega$ and $\Omega=(n+\varepsilon')\,\Delta\omega$ are made, (A.21) is expressed

$$K_1=\frac{N_n}{16\pi}\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} + \frac{d\varepsilon\,d\varepsilon'}{\eta^2\,\xi^2+(\varepsilon'-\varepsilon)^2\,((1/2)+(1/2)\,(n+\varepsilon')/(n+\varepsilon))^2} \qquad (A.22)$$

As for $(\varepsilon;\,\varepsilon')\in\left(-\frac{1}{2},\frac{1}{2}\right)\times\left(-\frac{1}{2},\frac{1}{2}\right)$, we have for $n\geqq 1$:

$$\max\frac{n+\varepsilon'}{n+\varepsilon}=\frac{n+1/2}{n-1/2}>0;$$

$$\min\frac{n+\varepsilon'}{n+\varepsilon}=\frac{n-1/2}{n+1/2}>0,$$

we obtain the following inequalities from (A.22) :

$$\frac{N_n}{16\pi b_n^2}\int_{-1/2}^{1/2}\int_{-1/2}^{1/2}\frac{d\varepsilon\, d\varepsilon'}{b_n^{-2} n^2 \xi^2+(\varepsilon-\varepsilon')^2}<K_1$$

$$<\frac{N_n}{16\pi a_n^2}\int_{-1/2}^{1/2}\int_{-1/2}^{1/2}\frac{d\varepsilon\, d\varepsilon'}{a_n^{-2} n^2 \xi^2+(\varepsilon-\varepsilon')^2}. \quad \text{(A.23)}$$

Let :

$$\gamma=\int_{-1/2}^{1/2}\int_{-1/2}^{1/2}\beta(x-x')\,dx\,dx',$$

where $\beta(y)=(\lambda^2+y^2)^{-1}$.

We note that $\beta(-y)=\beta(y)$. The change of variable $x-x'=y$ gives

$$\gamma=\int_{-1/2}^{1/2}F(x')\,dx',$$

where

$$F(x')=\int_{-1/2-x'}^{1/2-x'}\beta(y)\,dy.$$

We integrate the integral defining γ by parts to obtain

$$\int_{-1/2}^{1/2}\int_{-1/2}^{1/2}\beta(x-x')\,dx\,dx'$$

$$=\frac{2}{\lambda}\tan^{-1}\frac{1}{\lambda}-\operatorname{Log}\left(1+\frac{1}{\lambda^2}\right). \quad \text{(A.24)}$$

By applying (A.24) to calculate (A.23) and by multiplying each member of the inequalities (A.23) by ξ, we have :

$$L_1<\xi K_1<L_2, \quad \text{(A.25)}$$

where :

$$L_1=\frac{N_n}{16\pi b_n^2}\left(2\frac{b_n}{n}\tan^{-1}\left(\frac{b_n}{n\xi}\right)-\xi\operatorname{Log}\left(1+\frac{b_n^2}{n^2\xi^2}\right)\right), \quad \text{(A.26)}$$

$$L_2=\frac{N_n}{16\pi a_n^2}\left(2\frac{a_n}{n}\tan^{-1}\left(\frac{a_n}{n\xi}\right)-\xi\operatorname{Log}\left(1+\frac{a_n^2}{n^2\xi^2}\right)\right), \quad \text{(A.27)}$$

where a_n and b_n are defined by (62). We conclude that :

$$\lim_{\xi\to 0_+} L_1=\frac{N_n}{16\,nb_n}; \quad \text{(A.28)}$$

$$\lim_{\xi\to 0_+} L_2=\frac{N_n}{16\,na_n}.$$

Considering (A.13), (A.14), (A.19) and (A.25), we have:

$$\frac{N_n}{16\,nb_n}<\lim_{\xi\to 0_+}\xi\operatorname{tr}(M\,\mathscr{E}^n_{c,\,\xi})<\frac{N}{16\,na_n},$$

whence :

$$a_n\alpha_n<N_n<b_n\alpha_n, \quad \text{(A.29)}$$

with α_n given by (63).

For $n\ll 1$, we have $a_n\simeq b_n\simeq 1$. Then (A.26) and (A.27) yield :

$$L_1\simeq L_2\simeq\frac{N_n}{16\pi}\left(\frac{2}{n}\tan^{-1}\left(\frac{1}{n\xi}\right)-\xi\operatorname{Log}\left(1+\frac{1}{n^2\xi^2}\right)\right).$$

Thus when ξ is positive and small ($\xi\ll 1$), we have

$$\xi\operatorname{tr}(M\,\mathscr{E}^n_{c,\,\xi})\simeq\frac{N_n}{8\pi n}\left[\tan^{-1}\left(\frac{1}{n\xi}\right)+n\xi\operatorname{Log}\left(\frac{n\xi}{\sqrt{1+n^2\xi^2}}\right)\right],$$

whence the relationship (64).

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