Functional reduction of stochastic fields for studying stationary random vibrations

Christian Soize, J.-M. David, A. Desanti

FUNCTIONAL REDUCTION OF STOCHASTIC FIELDS FOR STUDYING STATIONARY RANDOM VIBRATIONS

by

C. SOIZE (*), J. M. DAVID (**) and A. DESANTI (**)

ABSTRACT

We show how the cost of spectral analysis of linear stationary random vibrations of elastic media excited by nonhomogeous time stationary stochastic fields can be cut down when a spatial natural basis, such that the modal basis of the elastic media cannot be used.

The method proposed is based on a spatial functional reduction of the random field narrow frequency bands. This functional reduction basis is made up of the eigenfunctions of the covariance operator (associated with a narrow frequency band.

As long as the spatial correlation lengths of the field are not too small, the proposed method is very efficient. A numerical example is given and the results are compared with a known analytical case, validating the proposed method.

Keywords (NASA thesaurus): Structural vibration – Random vibration – Spectrum analysis – Elastic media.

(*) Engineer, Head of Research Group, ONERA.
(**) Research Engineer, ONERA.

I. – INTRODUCTION

For studying random linear vibrations in elastic media of any geometry, not isotropic or homogeneous, excited by a time-stationary stochastic field, the conventional approach has been to use modeling of the elastic medium by the finite element method aimed at computing the finite part of the modal basis of the associated conservative system involved in the response. The stationary random problem is then solved by spectrum analysis with a small number of general coordinates associated with the eigenmodes chosen, and the stochastic excitation field is projected on these eigenmodes. This leads to a natural reduction of the field on the modal basis. It is the conventional modal synthesis method, mainly used in the low frequency (LF) domain.

In certain cases, the modal synthesis method cannot be used or leads to excessive costs for modal extraction, for various reasons: study of certain coupled fluid-structures systems, large quasi-static contribution in the dynamic LF response, study of vibrations in the medium frequency (MF) domain, etc.

Herein, we therefore consider the situation in which the modal synthesis method cannot be used. In this case, the equations discretized on the finite element basis are solved by suitable direct methods and the stochastic excitation field is projected spatially on the finite element basis. For stationary, random vibrations, if it is not desired to use numerical simulation methods for the field, this leads to constructing for spectrum analysis the frequency response matrix function for the system whose input dimension is the number of degrees of freedom excited. This number is generally large, which results in a costly numerical effort.

This paper describes a method for functional reduction of the stochastic field assumed stationary in time but not necessarily homogeneous spatially. This reduction is designed to decrease the "random dimension" of the excitation, thereby similarly decreasing the input dimension of the frequency reponse function to be computed, allowing a very appreciable gain on the cost of spectrum analysis of the system.

III. – NOTATIONS

(*a*) For reasons which will be given below, the frequency domain $\mathbb{R}$ is considered to be the countable union of bands B_n such that:

$$B_n=[\Omega_n-\Delta\omega_n/2,\ \Omega_n+\Delta\omega_n/2]\subset\mathbb{R}^+, \qquad (1)$$

where $\Omega_n>0$ is the center frequency of band B_n and $\Delta\omega_n$ is the bandwidth. With band $B_n\subset\mathbb{R}^+$ is associated band $\underline{B}_n$, symmetrical with respect to the origin, such that:

$$\underline{B}_n=[-(\Omega_n+\Delta\omega_n/2),\ -(\Omega_n-\Delta\omega_n/2)]\subset\mathbb{R}^-. \qquad (2)$$

It can be seen that $\mathbb{R}$ is written:

$$\mathbb{R}=U_{n=1}^{\infty}(B_n\cup\underline{B}_n). \qquad (3)$$

(*b*) The Euclidian space $\mathbb{R}^3$ is equipped with the usual scalar product $\langle u, v\rangle=\sum_{j=1}^{3}u_jv_j$, u and v in $\mathbb{R}^3$ and the associated norm $\|u\|=\langle u, u\rangle^{1/2}$.

We note as $\mathrm{End}(\mathbb{R}^3)$ all the linear applications of $\mathbb{R}^3$, as $\|A\|_{\mathrm{End}(\mathbb{R}^3)}$ the norm of $A\in\mathrm{End}(\mathbb{R}^3)$, as $\mathrm{tr}\,A$ the trace of A and as ${}^TA$ the transpose of A. Space $\mathbb{C}^3$ is considered the complexification of $\mathbb{R}^3$. If $A\in\mathrm{End}(\mathbb{C}^3)$, $\bar{A}$ designates the conjugate of A and $A^*={}^T\bar{A}$ is the adjoint operator.

(*c*) Below, Σ designates a bounded surface of $\mathbb{R}^3$ with generic point M.

$d\sigma(M)$ designates the surface measure on Σ. Where $|\Sigma|$ designates the total area of surface Σ, we have:

$$\int_{M\in\Sigma}d\sigma(M)=|\Sigma|<+\infty. \qquad (4)$$

(*d*) Let $L^2(\Sigma, \mathbb{R}^3)$ be the Hilbert space of the functions defined $d\sigma$-almost everywhere on Σ with integrable square:

$$\int_{\Sigma}\|\varphi(M)\|^2\,d\sigma(M)<+\infty. \qquad (5)$$

This space is equipped with scalar product

$$\ll\varphi,\psi\gg=\int_{\Sigma}\langle\varphi(M),\psi(M)\rangle d\sigma(M) \qquad (6)$$

and the associated norm:

$$|||\varphi|||=\ll\varphi,\varphi\gg^{1/2}. \qquad (7)$$

(*e*) For any u and v in $\mathbb{R}^3$, the tensor $u\otimes v$ of $\mathbb{R}^3\otimes\mathbb{R}^3$ is identified with the linear operator of $\mathrm{End}(\mathbb{R}^3)$, also noted $u\otimes v$, by:

$$\forall w\in\mathbb{R}^3,\quad (u\otimes v)w=u\langle v, w\rangle. \qquad (8)$$

III. – DATA ON THE STOCHASTIC FIELD AND COVARIANCE OPERATOR

III,1. – GENERAL DATA

Let $\{p(M, t)=(p_1(M, t), p_2(M, t), p_3(M, t)), M\in\Sigma, t\in\mathbb{R}$ be a stochastic field defined on a probabilistic space $(\mathscr{U}, \mathscr{C}, P)$, indexed on $\Sigma\times\mathbb{R}$ with values in

$\mathbb{R}^3$, centered, of the second order, stationary in quadratic mean for variable t. The mathematic expectation is always E.

The mean function of this field is identical to zero since it is centered and its cross autocorrelation function, which is equal to its cross covariance function, is noted:

$$R_p(M, M', \tau) = E\{p(M, t+\tau) \otimes p(M', t)\}. \quad (8)$$

It is defined on $\Sigma \times \Sigma \times \mathbb{R}$, with values in $\mathrm{End}(\mathbb{R}^3)$. As the field is second order, we have:

$$E\{\|p(M, t)\|^2\} = \mathrm{tr}\, R_p(M, M, 0) < +\infty. \quad (9)$$

Since $p(M, t)$ has values in $\mathbb{R}^3$, we have the property:

$$R_p(M, M', -\tau) = {}^T R_p(M', M, \tau). \quad (10)$$

The following regularity assumptions are introduced on the field:

(*a*) The function $M \mapsto \mathrm{tr}\, R_p(M, M, O)$ is a bounded function of Σ in $\mathbb{R}^+$. Consequently, there is a real, positive, finite constant such that:

$$E\{\|p(M, t)\|^2\} \leqq K < +\infty, \qquad \forall M \in \Sigma. \quad (11)$$

(*b*) For any M and M' fixed in Σ, function $\tau \to R_p(M, M', \tau)$ is continuous of $\mathbb{R}$ in $\mathrm{End}(\mathbb{R}^3)$. It is second order continuity with respect to variable t.

(*c*) The cross spectral measure of field $p(M, t)$ noted $\mu_p(M, M', d\omega)$ accepts for any M and M' in Σ a density $S_p(M, M', \omega)$ with values in $\mathrm{End}(\mathbb{C}^3)$ with respect to the Lebesgue measure $d\omega$. Therefore, $\forall M$, M', τ in $\Sigma \times \Sigma \times \mathbb{R}$, we have:

$$R_p(M, M', \tau) = \int_{\mathbb{R}} e^{i\omega\tau} S_p(M, M', \omega)\, d\omega. \quad (12)$$

Considering equation (10), S_p has the following usual properties:

$$S_p(M, M', -\omega) = \overline{S_p(M, M', \omega)}, \quad (13)$$

$$S_p(M, M', \omega) = S_p(M', M, \omega)^* \quad (14)$$

and, for any function φ of Σ in $\mathbb{C}^3$, we have, for any ω in $\mathbb{R}$:

$$\left.\begin{aligned} &\int_\Sigma \int_\Sigma \langle S_p(M, M', \omega)\, \varphi(M'), \overline{\varphi(M)} \rangle \\ &d\sigma(M)\, d\sigma(M') \geqq 0, \end{aligned}\right\} \quad (15)$$

provided φ has regularity properties so that integral (15) has a meaning.

III,2. – COVARIANCE KERNEL PER BAND

The stationary linear vibrations of a dynamic system excited by a time-stationary stochastic field are governed by a linear convolution filter. The spectrum analysis can therefore be performed by bands. This is an advantage for two reasons. First, certain analysis methods use this technique, for instance the MF method [34 to 36]. In addition, the functional reduction is more accurate since it is adapted to each elementary band.

Below, we discuss the functional reduction of the field on each band $B_n \cup \underline{B}_n$. Thus, for a fixed band $B_n \cup \underline{B}_n$, function $M, M' \mapsto C_n(M, M')$ on $\Sigma \times \Sigma$ with values in $\mathrm{End}(\mathbb{R}^3)$ is defined, called covariance kernel on B_n and is such that:

$$C_n(M, M') = \int_{\omega \in B_n \cup \underline{B}_n} S_p(M, M', \omega)\, d\omega. \quad (16)$$

Accordingly, $\mathrm{tr}\, C_n(M, M)$ is the power of the field on band $B_n \cup \underline{B}_n$ in a point M.

Considering equations (1), (2) and (13), we have:

$$C_n(M, M') = 2 \int_{\omega \in B_n} \mathrm{Ré}\, S_p(M, M', \omega)\, d\omega \quad (17)$$

and considering (14):

$$C_n(M, M') = {}^T C_n(M', M). \quad (18)$$

II,3. – COVARIANCE OPERATOR ASSOCIATED WITH COVARIANCE KERNEL

For any φ and ψ in $L^2(\Sigma, \mathbb{R}^3)$, the covariance operator $\mathscr{C}_n$ is defined relative to $B_n \cup \underline{B}_n$ with function kernel C_n such that:

$$\ll \mathscr{C}_n \varphi, \psi \gg = \int_\Sigma \int_\Sigma \langle C_n(M, M')\, \varphi(M'), \psi(M) \rangle\, d\sigma(M)\, d\sigma(M'). \quad (19)$$

The following result is obtained with all the above assumptions:

$\mathscr{C}_n$ is a positive, symmetrical, compact, continuous, linear operator from $L^2(\Sigma, \mathbb{R}^3)$ into $L^2(\Sigma, \mathbb{R}^3)$. In addition, it is a Hilbert-Schmidt (H-S) operator.

The fact that $\mathscr{C}_n$ is compact means that its spectrum is countable. Since it is $H-S$, the series of the squares of its eigenvalues is convergent.

A few proofs are given below to demonstrate the role of the assumptions introduced.

(*a*) The function $M,\ M' \mapsto C_n(M,\ M')$ is bounded on $\Sigma \times \Sigma$:

$$\| C_n(M,\ M') \|^2_{\text{End}\,(\mathbb{R}^3)} \leqq \| R_p(M,\ M',\ 0) \|^2_{\text{End}\,(\mathbb{R}^3)}$$
$$\leqq [E\{\| p(M,\ t) \|^2\}][E\{\| p(M',\ t) \|^2\}] \leqq K^2 < +\infty,$$

according to (11).

(*b*) Considering (*a*) and (4),

$$C_n \in L^2(\Sigma \times \Sigma,\ \text{End}\,(\mathbb{R}^3))$$

since:

$$\int_\Sigma \int_\Sigma \| C_n(M,\ M') \|^2_{\text{End}\,(\mathbb{R}^3)}\, d\sigma(M)\, d\sigma(M') < +\infty. \quad (21)$$

(*c*) From (21), it results that $\mathscr{C}_n$ is continuous from $L^2(\Sigma,\ \mathbb{R}^3)$ into $L^2(\Sigma,\ \mathbb{R}^3)$.

(*d*) The symmetry of $\mathscr{C}_n$: $\ll \mathscr{C}_n \varphi,\ \psi \gg = \ll \varphi,\ \mathscr{C}_n \psi \gg$ results from (18) and the fact that it is positive results from (15).

(*e*) Finally, properties (*b*) and (*c*) allow it to be shown that $\mathscr{C}_n$ is an $H-S$ operator and is therefore compact [13, 23, 31, 33].

IV. – FUNCTIONAL REDUCTION OF THE FIELD BY BANDS

IV,1. – REPRESENTATION OF THE COVARIANCE OPERATOR

Since operator $\mathscr{C}_n$ is an $H-S$ operator, its spectrum is discrete and countable and there exists a normalization such that the eigenvectors $\left\{\psi_n^j\right\}$ of $\mathscr{C}_n$ form a Hilbert basis of $L^2(\Sigma,\ \mathbb{R}^3)$. If we note as λ_j^n the eigenvalue associated with ψ_n^j, we have:

$$\mathscr{C}_n \psi_n^j = \lambda_j^n \psi_n^j. \quad (20)$$

The eigenvalue equation (20) is an integral equation such that for $d\sigma$-almost every M in Σ:

$$\int_\Sigma C_n(M,\ M')\, \psi_n^j(M')\, d\sigma(M') = \lambda_j^n \psi_n^j(M). \quad (21)$$

Obviously, (21) will be solved numerically from the weak formulation. Since $\mathscr{C}_n$ is positive, the eigenvalues are real, positive values, the multiplicity of each is finite. Below, they are ordered by decreasing values:

$$\lambda_1^n \geqq \lambda_2^n \geqq \lambda_3^n \geqq \ldots \geqq 0, \qquad \sum_{j=1}^{\infty} (\lambda_j^n)^2 < +\infty. \quad (22)$$

The eigenvectors are such that:

$$\ll \psi_n^j,\ \psi_n^k \gg = \delta_{jk}, \quad (23)$$

$$\ll \mathscr{C}_n \psi_n^j,\ \psi_n^k \gg = \lambda_j^n \delta_{jk}. \quad (24)$$

Under these conditions, the representation of operator $\mathscr{C}_n$ is written:

$$\mathscr{C}_n = \sum_{j=1}^{\infty} \lambda_j^n \psi_n^j \otimes_{L^2} \psi_n^j, \quad (25)$$

with, for any u, v and φ in $L^2(\Sigma,\ \mathbb{R}^3)$:

$$(u \otimes_{L^2} v)\, \varphi = u \ll v,\ \varphi \gg. \quad (26)$$

IV, 2. – BREAKDOWN OF THE FIELD ON A BAND

Let $p_n(M,\ t)$ be the stochastic field defined on $(\mathscr{U},\ \mathscr{C},\ p)$, indexed on $\Sigma \times \mathbb{R}$, with values in $\mathbb{R}^3$, of the second order, centered, stationary in quadratic mean and continuous for variable t such that its cross spectral measure is written:
$\mu_{p_n}(M,\ M',\ d\omega = S_{p_n}(M,\ M',\ \omega)\, d\omega$, where:

$$S_{p_n}(M,\ M',\ \omega) = S_p(M,\ M',\ \omega)\, \mathbf{1}_{B_n \cup \underline{B}_n}(\omega), \quad (27)$$

where $\omega \mapsto \mathbf{1}_{\cdot}(\omega)$ is the indicator function of the I part of $\mathbb{R}$. Since p and p_n are centered and S_p coincides with S_{pn} on $B_n \cup \underline{B}_n$, any quantity resulting from linear filtering of p is a stationary process whose spectral power density coincides on $B_n \cup \underline{B}_n$ with the spectral density of the process resulting from the same filtering of field p_n. In other words, for computation of the spectral densities of the responses to linear filtering on $B_n \cup \underline{B}_n$, it is equivalent to use p_n or p, but this is wrong for any other probabilistic characteristic since fields p and p_n are different and it would even be wrong for the spectral densities if the filter were not linear. The total power of field $p_n(M,\ t)$ on Σ is written:

$$\mathscr{P}_n = E\{\| p_n \|^2\}$$
$$= \int_\Sigma \text{tr}\, C_n(M,\ M)\, d\sigma(M) < +\infty. \quad (28)$$

$p_n(M,\ t)$ can be projected on basis $\left\{\psi_n^j\right\} j$. This gives

$$p_n(M,\ t) = \sum_{j=1}^{\infty} X_{n,\ j}(t)\, \psi_n^j(M), \quad (29)$$

$$X_{n,\ j}(t) = \ll p_n,\ \psi_n^j \gg. \quad (30)$$

Processes $\{X_{n,\ j}(t)\}_j$ are indexed on $\mathbb{R}$, with values in $\mathbb{R}$, centered, second order, stationary in quadratic mean, correlated. The auto- and intercorrelation functions of processes $R_{jk}^n(\tau) = E\{X_{n,\ j}(t+\tau)\, X_{n,\ k}(t)\}$ are written:

$$R_{jk}^n(\tau) = \int_\Sigma \int_\Sigma \langle R_{p_n}(M,\ M',\ \tau)\, \psi_n^k(M'),\ \psi_n^j(M) \rangle$$
$$d\sigma(M)\, d\sigma(M'), \quad (31)$$

where:

$$R_{p_n}(M, M', \tau) = E\{p_n(M, t+\tau) \otimes p_n(M, t)\} = \int_{\omega \in B_n \cup \underline{B}_n} e^{i\omega\tau} S_p(M, M', \omega)\, d\omega. \quad (32)$$

The spectral and interspectral measures have densities with compact support $B_n \cup \underline{B}_n$ and are written for any j and k in $\mathbb{N}^*$:

$$S^n_{jk}(\omega) = \int_\Sigma \int_\Sigma \langle S_{p_n}(M, M', \omega)\, \psi^k_n(M'), \psi^j_n(M) \rangle \, d\sigma(M)\, d\sigma(M'). \quad (33)$$

Finally, it can easily be verified that the power of process $x_{n,\,j}(t)$ is written:

$$E\{X_{n,\,j}(t)^2\} = \lambda^n_j \quad (34)$$

and that the total power on Σ of field $p_n(M, t)$ defined by (28) is written:

$$\mathscr{P}_n = \sum_{j=1}^{\infty} \lambda^n_j. \quad (35)$$

Since $\mathscr{P}_n < +\infty$ and $\lambda^n_j \geqq 0$, equation (35) shows that $\mathscr{C}_n$ is also a nuclear operator.

IV, 3. – FUNCTIONAL REDUCTION OF THE FIELD ON A BAND AND APPROXIMATION

Let J_n be a positive, finite integer. Let us take $\tilde{p}_n(M, t)$ such that:

$$\tilde{p}_n(M, t) = \sum_{j=1}^{J_n} X_{n,\,j}(t)\, \psi^j_n(M). \quad (36)$$

Using (29), (30), (34) and (22) gives:

$$E\{\|\!|p_n - \tilde{p}_n|\!\|^2\} = \sum_{j=J_n+1}^{+\infty} \lambda^n_j = \mathscr{P}_n - \sum_{j=1}^{J_n} \lambda^n_j. \quad (37)$$

$$E\{\|\!|p_n - \tilde{p}_n|\!\|^2\} \leqq \varepsilon \mathscr{P}_n. \quad (38)$$

For a relative tolerance ε, the approximation criterion (38) is therefore written:

$$1 - \frac{\sum_{j=1}^{J_n} \lambda^n_j}{\mathscr{P}_n} \leqq \varepsilon, \quad (39)$$

where $\mathscr{P}_n$ is computed by (28), i. e.:

$$\mathscr{P}_n = 2 \int_\Sigma \int_{\omega \in B_n} \operatorname{tr} S_p(M, M', \omega)\, d\omega\, d\sigma(M). \quad (40)$$

V. – CONSTRUCTION OF FUNCTIONAL REDUCTION BY THE FINITE ELEMENT METHOD

V, 1. – INTERPOLATION BY THE FINITE ELEMENT METHOD

Surface Σ is meshed with isoparametric finite elements. Let N_0 be the number of nodes in the grid of Σ and $m = 3N_0$ be the number of degrees of freedom introduced. $\Phi = \{\Phi_1, \Phi_2, \ldots, \Phi_m\}$ are the nodal unknowns (three components per node).

Let $H(M)$ be the linear operator of $\mathbb{R}^m$ in $\mathbb{R}^3$ such that $M \mapsto H(M)$ is continuous from Σ into $L(\mathbb{R}^m, \mathbb{R}^3)$ and such that:

$$\varphi(M) = H(M)\Phi, \qquad M \in \Sigma. \quad (41)$$

Operator $H(M)$ is created conventionally using interpolation functions on the finite elements used.

V, 2. – DISCRETIZATION OF THE FIELD ON A BAND

For any $\varphi \in L^2(\Sigma, \mathbb{R}^3)$ with form (41), we have:

$$\ll p_n, \varphi \gg = \int_\Sigma \langle p_n(M, t), \varphi(M) \rangle\, d\sigma(M) = \int_\Sigma \langle p_n(M, t), H(M)\Phi \rangle\, d\sigma(M),$$

i. e.

$$\ll p_n, \varphi \gg = \langle F_n(t), \Phi \rangle_{\mathbb{R}^m}, \quad (42)$$

where $F_n(t)$ is a process with values in $\mathbb{R}^m$, indexed on $\mathbb{R}$, stationary in quadratic mean, centered, of the second order continuous, and which is written:

$$F_n(t) = \int_\Sigma {}^T H(M)\, p_m(M, t)\, d\sigma(M), \quad (43)$$

$F_n(t)$, which we will call discretized field, is actually the vector of equivalent nodal forces at the nodes of the finite element grid.

V, 3. – COVARIANCE OPERATOR OF THE DISCRETIZED FIELD

Combining (19) and (41) gives:

$$\ll \mathscr{C}_n \psi, \varphi \gg = \int_\Sigma \int_\Sigma \langle C_n(M, M')\, H(M')\Psi, H(M)\Phi \rangle\, d\sigma(M)\, d\sigma(M'),$$

i. e.:

$$\ll \mathscr{C}_n \psi, \varphi \gg = \langle C_{F_n} \Psi, \Phi \rangle_{\mathbb{R}^m}, \quad (44)$$

where $C_{F\,n}$ is the approximation of $\mathscr{C}_n$ which is a positive symmetrical continuous operator of End$(\mathbb{R}^m)$ and which is written:

$$C_{F_n}=\int_\Sigma\int_\Sigma {}^T H(M)\,C_n(M,\,M')\,H(M')\,d\sigma(M)\,d\sigma(M'). \tag{45}$$

It can be seen that $C_{F_n}=E\{F_n(t)\otimes F_n(t)\}$ is effectively the covariance operator of $F_n(t)$ defined by (43).

V, 4. – REPRESENTATION OF THE DISCRETIZED COVARIANCE OPERATOR

Considering (44), eigenvalue problem (20) becomes a symmetrical, positive definite operator of End$(\mathbb{R}^m)$ by setting, by the very construction of the interpolation:

$$\psi_n^j(M)=H(M)\,\Psi_n^j, \tag{46}$$

and for any $\varphi(M)=H(M)\,\Phi$:

$$\langle C_{F_n}\Psi_n^j,\,\Phi\rangle_{\mathbb{R}^m}=\lambda_n^j\langle A\,\Psi_n^j,\,\Phi\rangle_{\mathbb{R}^m},\qquad \forall\Phi\in\mathbb{R}^m, \tag{47}$$

where:

$$A=\int_\Sigma {}^T H(M)\,H(M)\,d\sigma(M), \tag{48}$$

Problem (47) is generalized to the eigenvalues since A is not the identity in the general case. It is necessary to solve:

$$C_{F_n}\Psi_n^j=\lambda_n^j A\,\Psi_n^j. \tag{49}$$

The condition of normalization on $\psi_n^j\in L^2(\Sigma,\,\mathbb{R}^3)$ results on $\Psi_n^j\in\mathbb{R}^m$ in:

$$\langle A\,\Psi_n^j,\,\Psi_n^k\rangle_{\mathbb{R}^m}=\delta_{jk}. \tag{50}$$

Operator $C_{F\,n}$ is therefore represented as follows:

$$C_{F_n}=\sum_{j=1}^{m}\lambda_n^j(A\,\Psi_n^j)\otimes_{\mathbb{R}^m}(A\,\Psi_n^j). \tag{51}$$

V, 5. – DECOMPOSITION OF THE DISCRETIZED FIELD

Process $F_n(t)$ with values in $\mathbb{R}^m$ is projected on basis $\{\Psi_n^j\}_j$ of the eigenvalues.

Considering (30) and (42), we have:

$$X_{n,\,j}(t)=\ll p_n,\,\psi_n^j\gg=\langle F_n(t),\,\Psi_n^j\rangle_{\mathbb{R}^m}, \tag{52}$$

$$\mathrm{F}_n(t)=\sum_{j=1}^{m}X_{n,\,j}(t)\,A\,\Psi_n^j. \tag{53}$$

Processes $\{X_{n,\,j}(t),\,j=1,\ldots,m\}$ are indexed on $\mathbb{R}$, with values in $\mathbb{R}$, centered, of the second order continuous, stationary in quadratic mean and correlated.

Using equations (33) and (43), it can be seen that the spectral and interspectral density functions $S_{jk}^n(\omega)$ of these processes have compact support $B_n\cup\underline{B}_n$ and, for any j and k in $\mathbb{N}^*$, are written:

$$S_{jk}^n(\omega)=\langle S_{F_n}(\omega)\,\Psi_n^k,\,\Psi_n^j\rangle_{\mathbb{R}^m}, \tag{54}$$

$$S_{F_n}(\omega) \tag{55}$$

$$=\int_\Sigma\int_\Sigma {}^T H(M)\,S_{p_n}(M,\,M',\,\omega)\,H(M')\,d\sigma(M')\,d\sigma(M)$$

where $\omega\mapsto \mathrm{S}_{\mathrm{F}_n}(\omega)$ is the spectral density function of vector process $F_n(t)$ and this function has compact support $B_n\cup\underline{B}_n$ with values in the positive Hermitian operators of End$(\mathbb{C}^m)$. Finally, it is verified that equation (54) remains unchanged, i. e. that:

$$E\{X_{n,\,j}(t)^2\}=\lambda_j^n \tag{56}$$

and that the total power on Σ of process $F_n(t)$ is written:

$$\mathscr{P}_{n,\,m}=E\{\|F_n(t)\|^2\}=\mathrm{tr}\,C_{F_n}=\sum_{j=1}^{n}\lambda_j^n. \tag{57}$$

V, 6. – FUNCTIONAL REDUCTION OF THE DISCRETIZED FIELD

In accordance with the results of paragraphs IV, 3, V, 2 and V, 5 and noting as J_n the positive integer such that $1\leqq J_n\leqq m$ and as $\tilde{F}_n(t)$ the approximation of $F_n(t)$ of order J_n, we have:

$$\tilde{F}_n(t)=\sum_{j=1}^{J_n}X_{n,\,j}(t)\,A\,\Psi_n^j, \tag{58}$$

$$\mathrm{E}\{\|F_n(t)-\tilde{F}_n(t)\|^2\}\leqq\varepsilon\,\mathscr{P}_{n,\,m}, \tag{59}$$

if J_n and ε verify:

$$1-\frac{\sum_{j=1}^{J_n}\lambda_j^n}{\mathscr{P}_{n,\,m}}\leqq\varepsilon. \tag{60}$$

V,7. – REMARKS ON THE DECREASE IN THE EIGENVALUES

In the general case, it is not possible to obtain information on the rate of decrease of the eigenvalues of $\mathscr{C}_n$ as a function of their rank. To do so, it would be necessary to obtain information on the structure of operator $\mathscr{C}_n$. Nevertheless, examination of the following particular case gives an understanding of the trends.

Let $r \mapsto \Delta_\varepsilon(r)$ be the function defined on $\mathbb{R}$ with values in $\mathbb{R}^+$, with compact support $[-\varepsilon, \varepsilon]$ such that $\Delta_\varepsilon(r) = (2\varepsilon)^{-1}$ if $r \in [-\varepsilon, \varepsilon]$. For $\varepsilon \mapsto 0$, $\Delta_\varepsilon(r)\,dr$ approaches the Dirac measure $\delta_0(r)$.

Let us assume that field $p_n(M, t)$ is homogeneous (stationary in time and space) on $B_n \cup \underline{B}_n$ and that its covariance kernel on the band is written:

$$C_n(M, M') = Q\,\Delta_\varepsilon(\|M - M'\|), \qquad (61)$$

with ε positive fixed, Q a symmetrical positive defined operator of End $(\mathbb{R}^3)$. For $\varepsilon \to 0$, field $p_n(M, t)$ approaches a field which remains colored in time but which spatially approaches a white noise. For $\varepsilon \to 0$, the spatial correlation lengths of the field approach zero. For $\varepsilon > 0$ specified, equation (45) gives:

$$C_{F_n} = \int_\Sigma \int_\Sigma \Delta_\varepsilon(\|M - M'\|)\, {}^T H(M)\, Q H(M') \times d\sigma(M)\, d\sigma(M'). \qquad (62)$$

Let us assume that ε is sufficiently small that (62) can be written:

$$C_{F_n} \simeq \int_\Sigma {}^T H(M)\, Q H(M)\, d\sigma(M). \qquad (63)$$

If $Q = a\,\mathrm{Id}_{\mathbb{R}^3}$ with $a > 0$, $C_{F_n} \simeq a\,A$ and (45) shows that $\lambda_n^1 = \lambda_n^2 = \ldots = \lambda_n^m = a$.

There is no decrease in the eigenvalues and we must take $J_n = m$. This is the limit case. In this case, the three components $p_{n,1}(M, t)$, $p_{n,2}(M, t)$ and $p_{n,3}(M, t)$ are not correlated with one another, are not spatially correlated (but are correlated in time) and the total power on Σ of each component $p_{n,j}(M, t)$ is written:

$$\mathscr{P}_n^j = E\left\{\int_\Sigma p_{n,j}(M, t)^2\, d\sigma(M)\right\} = (2\varepsilon)^{-1} a\,|\Sigma|. \qquad (64)$$

The powers of each component are therefore equal. It should be noted that for $\varepsilon \to 0$, $\mathscr{P}_n^j \to +\infty$. The limit field of kernel $C_n(M, M') = Q\,\delta_0(\|M - M'\|)$ must be treated with the theory of generalized processes [24]. Accordingly, the longer the spatial correlation lengths of the field on band $B_n \cup \underline{B}_n$, the faster the rate of decrease of the eigenvalues.

Remark. – Let (e^1, e^2, e^3) be an orthonormal basis in $\mathbb{R}^3$. In this basis, the coherence tensor of field $p(M, t)$ has as matrix $[\Gamma(M, M', \omega)]$ whose elements are written:

$$\Gamma_{jk}(M, M', \omega) = \frac{|[S^p(M, M', \omega)]_{jk}|}{\sqrt{[S_p(M, M, \omega)]_{jj}\,[S_p(M', M', \omega)]_{kk}}}. \qquad (65)$$

The spatial correlation length in the direction of the vector with basis e^j, at frequency ω and at point M can be defined by:

$$L_{jj}(\omega, M) = \int_0^{+\infty} \Gamma_{jj}(M + e^j r, M)\, dr. \qquad (66)$$

V,8. – METHODOLOGY FOR CONSTRUCTING THE REDUCTION ON A BAND

The data are:

(1) band $B_n \cup \underline{B}_n$ which is fixed;

(2) the cross spectral density function $S_{p_n}(M, M', \omega) = S_p(M, M', \omega)$ for $\omega \in B_n \cup \underline{B}_n$;

(3) the relative tolerance ε of the reduction;

(4) the grid of Σ and the finite elements used to construct $H(M)$.

We then compute:

(1) kernel $C_n(M, M')$ by equation (17);

(2) operator C_{F_n} by equation (45);

(3) power $\mathscr{P}_{n,m} = \mathrm{tr}\, C_{F_n}$ (equation 57);

(4) the first J_n eigenvalues and eigenvectors of problem (49), where J_n is such that criterion (60) is verified;

(5) functions $S_{jk}^n(\omega)$ for $\omega \in B_n$ and $j, k \in \{1, 2, \ldots, J_n\}$ by equations (54) and (55). Actually, only $J_n(J_n + 1)/2$ functions are computed since $S_{jk}^n(\omega) = \overline{S_{jk}^n(\omega)}$.

This yields the functional reduction of the discretized field given by equation (58).

VI. – USE OF THE FUNCTIONAL REDUCTION FOR THE STUDY OF RANDOM VIBRATIONS

VI,1. – THE DATA

We consider a solid, linear viscoelastic medium with instantaneous memory which occupies a bounded open domain Ω of $\mathbb{R}^3$ with a relatively regular boundary $\partial\Omega = \Sigma \cup \Gamma$.

Let $u(M, t)$ be the displacement field, $M \in \bar{\Omega}$, and $p(M, t)$ be the above random force field applied on Σ. The linear vibrations around a position of static equilibrium are analyzed. To do so, medium Ω is modeled by the finite element method. The trace of the grid of Ω on Σ must be compatible with the grid of Σ used to discretize $p(M, t)$. Let m be the number of degrees of freedom of the modeled system and $F(t)$ be the stationary vector process of applied forces due to discretizing of the field. We assign zero to the DOFs where there is no nodal force. Noting as $U(t)$ the nodal unknowns of the displacement field, it is

known that the stationary solution is obtained by linear convolution filtering of $F(t)$ by the impulse response $h \in L^1(\mathbb{R}, \text{End}(\mathbb{R}^m))$ whose frequency response function:

$$\omega \mapsto T(\omega) = \hat{h}(\omega) = \int_{\mathbb{R}} e^{-i\omega t} h(t)\, dt$$

is a continuous function bounded in ω, integrable and which is written:

$$T(\omega) = [-\omega^2 M + i\omega C + K]^{-1},$$

where M, C and K are the real, symmetrical, positive-definite operators of mass, damping and stiffness in $\text{End}(\mathbb{R}^n)$. This filter is stable and causal.

The stationary solution is therefore process $U(t)$ indexed on $\mathbb{R}$, with values in $\mathbb{R}^m$, stationary in quadratic mean, centered, continuous in second order, and is written:

$$U(t) = (h * F)(t) = \int_{-\infty}^{t} h(t-t') F(t')\, dt'.$$

Its spectral measure $M_U(d\omega)$ admits a density $\omega \mapsto S_U(\omega)$: $\mathbb{R} \to \text{End}(\mathbb{C}^n)$ such that:

$$S_U(\omega) = T(\omega) S_F(\omega) T(\omega)^*. \tag{67}$$

For $\omega \in B_n \cup \underline{B}_n$, we set $S_{Fn}(\omega) = S_F(\omega)$ and for $\omega \notin B_n \cup \underline{B}_n$, we set $S_{Fn} = 0$. Considering process $F_n(t)$, with the same properties as $F(t)$ but with spectral density $S_{Fn}(\omega)$, process $U_n(t) = (h^* F_n)(t)$ has the same properties as $U(t)$ with spectral density $S_{Un}(\omega) = S_U(\omega)$ for $\omega \in B_n \cup \underline{B}_n$ and $S_{Un}(\omega) = 0$ for $\omega \notin B_n \cup \underline{B}_n$.

Therefore, the construction of $S_U(\omega)$ for $\omega \in B_n \cup \underline{B}_n$ is obtained by using process $F_n(t)$ and $U_n(t)$.

VI,2. – SOLUTION BY THE FUNCTIONAL REDUCTION METHOD

Let $\tilde{F}_n(t)$ be the functional reduction (58) of the discretized field. In this case, the spectral density function $\omega \mapsto S_{\tilde{F}_n}(\omega)$ of $\mathbb{R}$ in $\text{End}(\mathbb{C}^m)$ of process $\tilde{F}_n(t)$ is written:

$$S_{\tilde{F}_n}(\omega) = \sum_{j=1}^{J_n} \sum_{k=1}^{J_n} S_{jk}^n(\omega) (A\Psi_n^j) \otimes (A\Psi_n^k), \tag{68}$$

where S_{jk}^n is given by (54) and (55). For the corresponding approximation $\tilde{U}_n(t)$ of $U_n(t)$, this yields:

$$S_{\tilde{U}_n}(\omega) = \sum_{j=1}^{J_n} \sum_{k=1}^{J_n} S_{jk}^n(\omega) (T(\omega) A\Psi_n^j) \otimes \overline{(T(\omega) A\Psi_n^k)}. \tag{69}$$

VI,3. – EFFECTIVE CONSTRUCTION OF THE SOLUTION

Let $Z_m(B_m)$ be the set of functions $t \mapsto f_n(t)$ defined dt-almost everywhere on $\mathbb{R}$ such that:

$$Z_m(B_n) = \{ f_n \in L^2(\mathbb{R}, \mathbb{C}^m), \hat{f}_n(\omega) = 0, \forall \omega \notin B_n \}. \tag{70}$$

Let $\omega \mapsto \hat{\mathbf{1}}_{B_n}(\omega)$ be the function defined on $\mathbb{R}$ with values in $\mathbb{R}$ such that:

$$\hat{\mathbf{1}}_{B_n}(\omega) = 1 \quad \text{si } \omega \in B_n; \qquad \hat{\mathbf{1}}_{B_n}(\omega) = 0 \quad \text{si } \omega \notin B_n. \tag{71}$$

For any j in $\{1, 2, \ldots, J_n\}$, we consider the function $t \mapsto f_n^j(t)$ such that:

$$f_n^j \in Z_m(B_n); \qquad \hat{f}_n^j(\omega) = \hat{\mathbf{1}}_{B_n}(\omega) A\Psi_n^j. \tag{72}$$

Let u_n^j be the response due to the deterministic excitation f_n^j:

$$u_n^j(t) = (h * f_n^j)(t). \tag{73}$$

Then $u_n^j \in Z_m(B_n)$ and:

$$\hat{u}_n^j(\omega) = T(\omega) \hat{f}_n^j(\omega), \tag{74}$$

$$\hat{u}_n^j \in L^2(\mathbb{R}, \mathbb{C}^m), \qquad \text{Supp } \hat{u}_n^j = B_n. \tag{75}$$

Under these conditions, equation (69) is written:

$$S_{\tilde{U}_n}(\omega) = \sum_{j=1}^{J_n} \sum_{k=1}^{J_n} S_{jk}^n(\omega) \times \hat{u}_n^j(\omega) \otimes \overline{\hat{u}_n^k(\omega)}, \qquad \forall \omega \in B_n, \tag{76}$$

$$S_{\tilde{U}_n}(\omega) = \overline{S_{\tilde{U}_n}(-\omega)}, \qquad \forall \omega \in \underline{B}_n. \tag{77}$$

These last two equations show that the approximation $S_{\tilde{U}}(\omega)$ of $S_U(\omega)$ can be computed for any $\omega \in B_n \cup \underline{B}_n$ by solving the deterministic problem (74).

It can easily be verified that if approximation $\tilde{F}_n(t)$ of $F_n(t)$ is within ε with the meaning of (59)-(60), we then have:

$$E\{ \| U_n(t) - \tilde{U}_n(t) \|^2 \} \leqq \varepsilon \| T \|_n \operatorname{tr} C_{F_n}, \tag{78}$$

$$\| T_n \| = \sup_{\omega \in B_n \cup \underline{B}_n} \sup_{k,\, l \in \{1, \ldots, m\}} \left| \sum_{j=1}^{m} T_{jk}(\omega) \overline{T_{jl}(\omega)} \right|. \tag{79}$$

In addition, it must be ascertained that the fineness of the grid on Σ and the degree of interpolation of the finite elements used are compatible with the spatial correlation lengths of the field. Such a criterion is easy to construct.

In the *MF* domain, we will use method *MF* [34, 35, 36] in the multiloading case version to construct $\hat{u}_n^j(\omega)$ by band B_n, i.e. the system response is computed simultaneously for all deterministic excitations $F_n^1(t)$, $F_n^2(t)$, ..., $F_n^J n(t)$.

VII. – PARAMETRIC STUDY OF CONVERGENCE

The smaller J_n with respect to m, the more numerically efficient the method. The choice of J_n is related to the rate of decrease of the eigenvalues of operator C_{Fn}. As we know that this decrease is related to the spatial correlation lengths, we give below the results of a parametric analysis limited to a simple case to obtain quantified information of the value of J_n.

Let $Oxyz$ be a cartesian reference system and $M=(x, y, z)$ a generic point. Surface Σ is the plane rectangular domain with sides a and b along ox and oy. Field $p(M, t)$ defined on Σ with values in $\mathbb{R}$ is homogeneous. Its cross spectral density is written:

$$S_p(M, M', \omega) = K(\omega)\exp\left(-\frac{|\Delta x|}{L_x(\omega)}-\frac{|\Delta y|}{L_y(\omega)}+i\frac{\Delta x}{l(\omega)}\right), \quad (80)$$

where $\Delta x=x-x'$, $\Delta y=y-y'$. Dimensionless quantities $\bar{L}_x/a$, $\bar{L}_y/b$, $\bar{l}/a$ et $s_p(M, M', \omega)$ are introduced such that:

$$s_p(M, M', \omega)=\frac{S_p(M, M', \omega)}{\sqrt{S(M, M, \omega)\,S_p(M', M', \omega)}} = \exp\left(-\frac{|\Delta x|}{L_x(\omega)}-\frac{|\Delta y|}{L_y(\omega)}+i\frac{\Delta x}{l(\omega)}\right). \quad (81)$$

$$\bar{L}_x=\inf_{\omega\in B_n} L_x(\omega), \qquad \bar{L}_y=\inf_{\omega\in B_n} L_y(\omega), \qquad \bar{l}=\inf_{\omega\in B_n} l(\omega). \quad (82)$$

Under these conditions, the frequency aspect, i.e. the position of B_n in $\mathbb{R}^+$, becomes relative and the convergence results given apply to all bands B_n. For each case treated, the grid of Σ is constructed with elements with four nodes, the fineness of the grid being compatible with the spatial correlation lengths. The corresponding grids vary from 256 to 576 finite elements and m therefore varies from 289 to 625. The computation cases correspond to the following values:

$$\bar{L}_x/a=\bar{L}_y/b\in[0., 5.], \qquad \bar{l}/a\in[0.05 \text{ and } 0.5]$$

The relative tolerance ε on the accuracy of the reduction was taken equal to 0.1.

The results obtained are summarized in Figure 1. As m varies from 289 to 625, taking as average value of m $\bar{m}=450$, the relative average rate in percentage of the functional reduction is written $\bar{\theta}=100\times(\bar{m}-J_n)/\bar{m}$. For instance, for $\bar{l}/a$ with order 0.5 and $\bar{L}_x/a=\bar{L}_y/b$ with order 2, reduction $\bar{\theta}$ is of order 99.1% ($J_n=4$).

VIII. – EXAMPLE AND VALIDATION

VIII,1. – DEVELOPMENT OF THE PROGRAMS

A functional reduction program was developed for the fields. The finite elements have three or four nodes. A fineness criterion for the grid of Σ related to the spatial correlation lengths of the field is included. The problem generalized to the eigenvalues and eigenvectors is solved by the iteration algorithm in subspaces [3 to 6] and allows extraction of the J_n dominant eigenvalues and eigenvectors. This method is convergent if the eigenvalues are multiples and is very efficient if $J_n \ll m$.

This program interfaces with program ADINA-ONERA in which we have included dynamic MF analyses for excitation by random fields according to the method described in paragraph VI,3.

VIII,2. – EXAMPLE

We consider a flat rectangular elastic plate, simply supported on its edges, with sides $a=3$ m, $b=1$ m, thickness $h=0.01$, homogeneous and isotropic, with density $\rho=78.5$ kg/m^2, with surface dissipation constant $c=2\rho\xi_n\Omega_n$, where $\xi=0.01$, Poisson's ratio $\nu=0.3$ and Young's modulus $E=0.21\times10^{12}$ N/m^2. The band analyzed is $B_n=[200, 300]$ Hz, $\Omega_n=2\pi\times250$. This plate is subjected to a random pressure field whose transverse spectrum is given by (80) with $\bar{l}/a=0.35$, $L_x/a=\bar{L}_y/b=5$.

(a) Theoretical solution

In this case, the stationary solution can be constructed implicitly. Noting as $w(x, y, t)$ the transverse displacement of the plate, we have:

$$S_w(x, y, x', y', \omega)=\sum_{n, m, n', m'}\varphi_n(x)\,\varphi_m(y)\,\varphi_{n'}(x')\,\varphi_{m'}(y')\,T_{nm}(\omega)\,\overline{T_{n'm'}(\omega)}\,S_{nmn'm'}(\omega). \quad (83)$$

$$S_{nmn'm'}(\omega)=\int_0^a dx\int_0^b dy\int_0^a dx'\int_0^b dy'\,S_p(x, y, x', y', \omega)\times\varphi_n(x)\,\varphi_m(y)\,\varphi_{n'}(x')\,\varphi_{m'}(y'). \quad (84)$$

where:

$$T_{nm}(\omega)=\left[\rho\frac{ab}{4}(-\omega^2+\omega_{nm}^2+2\,i\,\xi\,\Omega_n\,\omega)\right]^{-1}, \quad (85)$$

$$\omega_{nm}^2=\frac{D}{\rho}\left[\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\,\pi}{b}\right)^2\right]^2. \qquad D=\frac{E\,h^3}{12(1-\nu^2)}, \quad (86)$$

$$\varphi_n(x)=\sin(n\,\pi\,x/a), \qquad \varphi_m(y)=\sin(m\,\pi\,y/b). \quad (87)$$

In the band analyzed, [200, 300] Hz, the eigenfrequencies of the associated conservative plate as well as the modes (pair (n, m)) are: [equations (86) and (87)]:

$f=\omega_{nm}/2\pi$	n	m
201,2	8	1
226,0	1	3
234,3	2	3
234,3	7	2
248,0	3	3
248,0	9	1
267,3	4	3
275,6	8	2
292,1	5	3
300,4	10	1

For the theoretical computation, the truncation in n and m in (83) was made with values $1\leqq n\leqq 30$ and $1\leqq m\leqq 10$.

(b) *Numerical solution by reduction*

The structural grid is made with 3456 finite plate elements with three nodes for calculation of the *MF* response by ADINA-ONERA (approximately 5100 DOF). The grid used for functional reduction is compatible with the structural grid and has $36\times 12=432$ grid points. The functional reduction is made with tolerance $\varepsilon=0.5$, which, in this case, led to a value of $J_n=4$.

(c) *Results and "theory-computation" comparisons*

The spectral density functions $S_W(M, M, \omega)$ are computed at the points indicated on Figure 2. The results given by "theoretical" computation and by the functional reduction and ADINA-ONERA are given in Figures 3-12. These figures show very good agreement of the results which validates the theory developed as well as the actual programming.

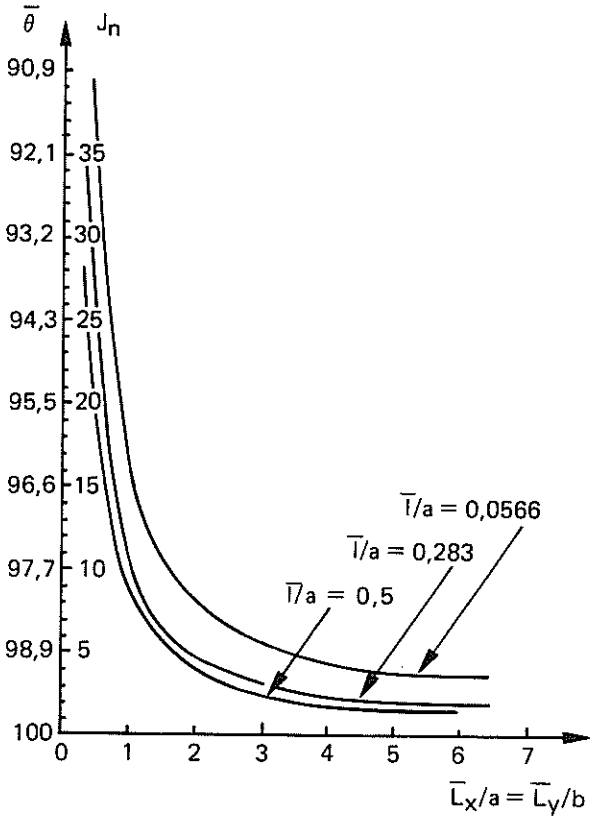

Fig. 1. – Curves representing the number J_n of eigenfunctions used for functional reduction as well as the average relative rate $\bar{\theta}$ for a relative tolerance of 0.1 according to different values of parameters $\frac{\bar{L}x}{a}=\frac{\bar{L}y}{b}$ and $\frac{\bar{l}}{a}$. J_n: number of eigenfunctions selected; θ: average relative rate of the functional reduction.

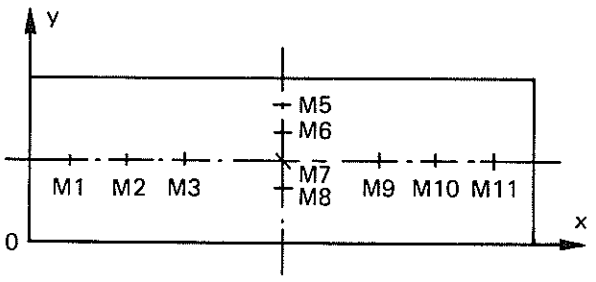

Fig. 2. – Geometric position of the observation points on the structure.

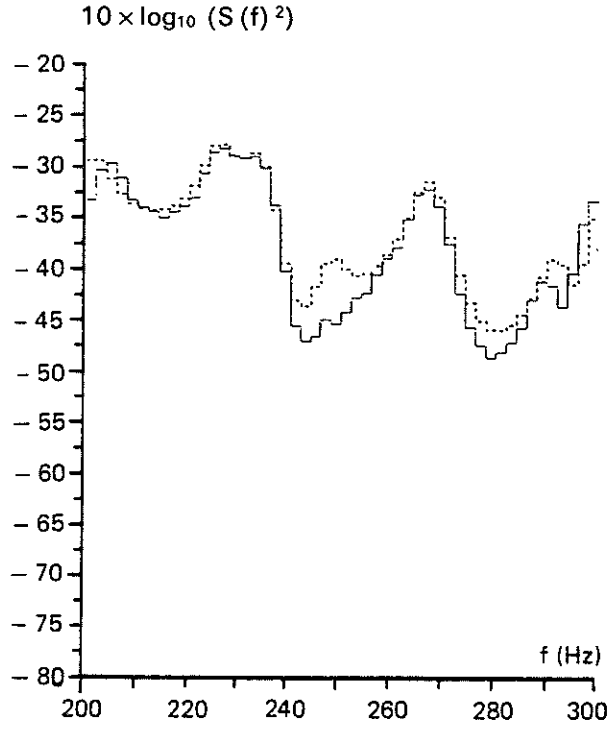

Fig. 3. – $S(f)$ = Spectral power density function of the acceleration normal to point *M1*. ——— computation by ADINA; - - - - - - - - analytic computation.

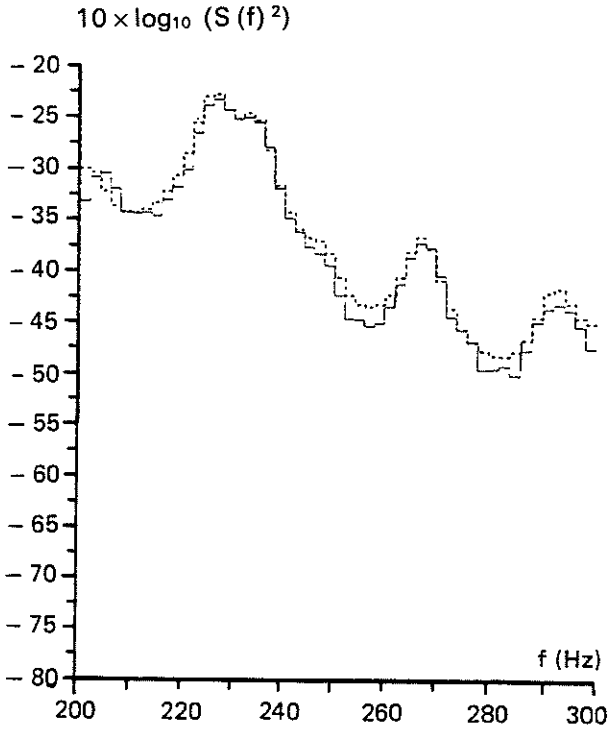

Fig. 5. – $S(f)$ = Spectral power density function of the acceleration normal to point *M3*. ——— computation by ADINA; - - - - - - - - analytic computation.

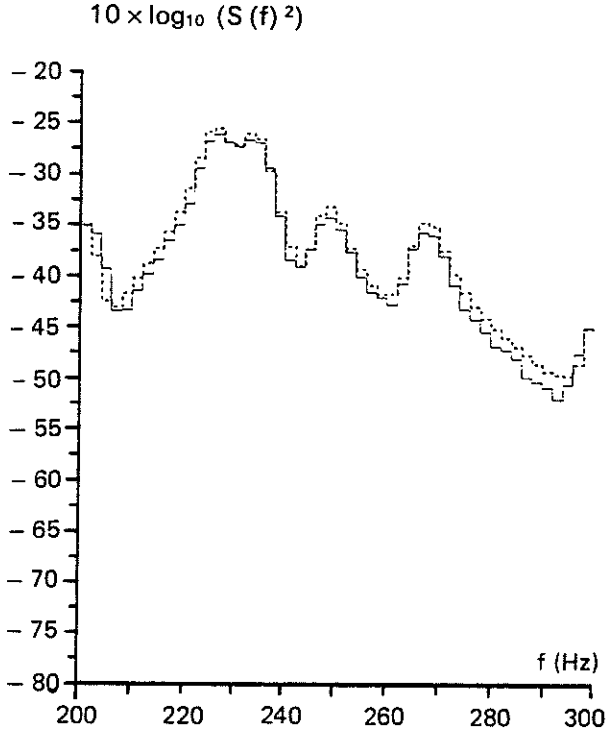

Fig. 4. – $S(f)$ = Spectral power density function of the acceleration normal to point *M2*. ——— computation by ADINA; - - - - - - - - analytic computation.

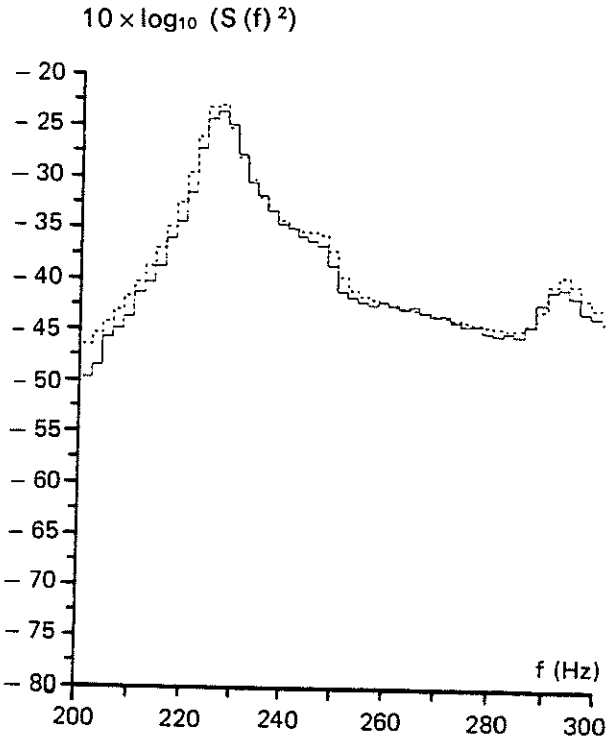

Fig. 6. – $S(f)$ = Spectral power density function of the acceleration normal to point *M5*. ——— computation by ADINA; - - - - - - - - analytic computation.

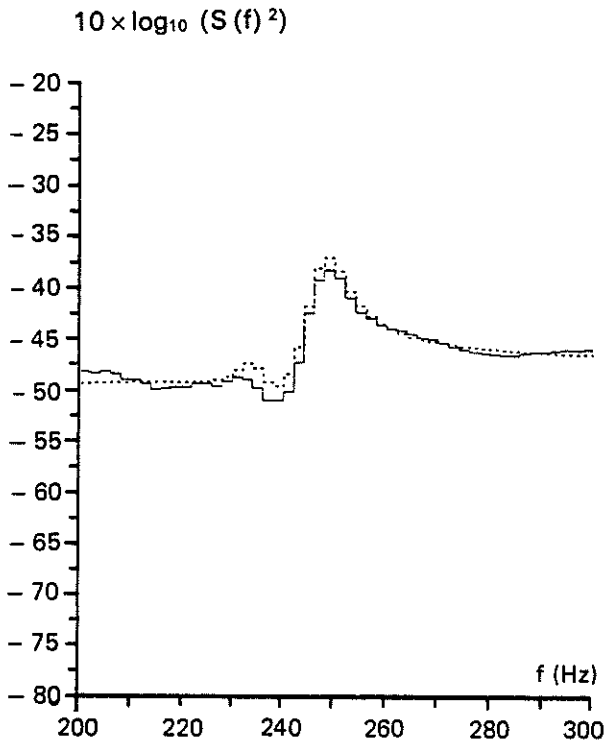

Fig. 7. – $S(f)$ = Spectral power density function of the acceleration normal to point *M6*. ——— computation by ADINA; - - - - - - - - analytic computation.

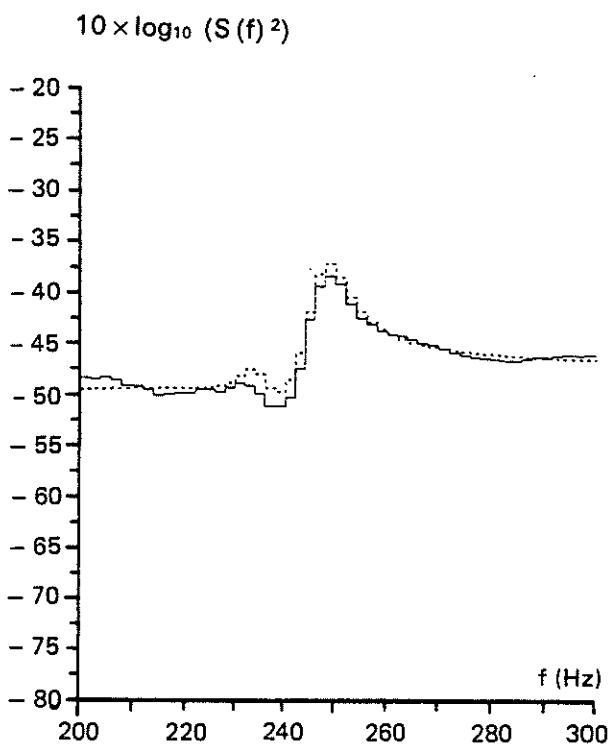

Fig. 9. – $S(f)$ = Spectral power density function of the acceleration normal to point *M8*. ——— computation by ADINA; - - - - - - - - analytic computation.

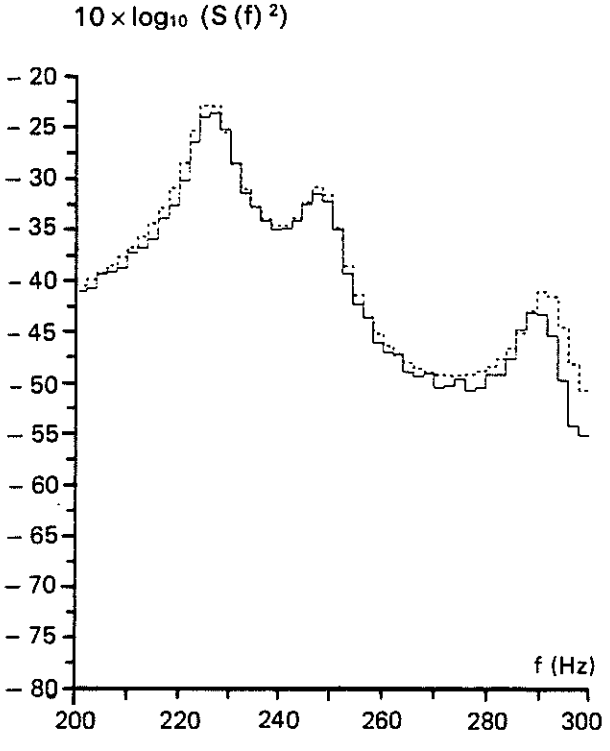

Fig. 8. – $S(f)$ = Spectral power density function of the acceleration normal to point *M7*. ——— computation by ADINA; - - - - - - - - analytic computation.

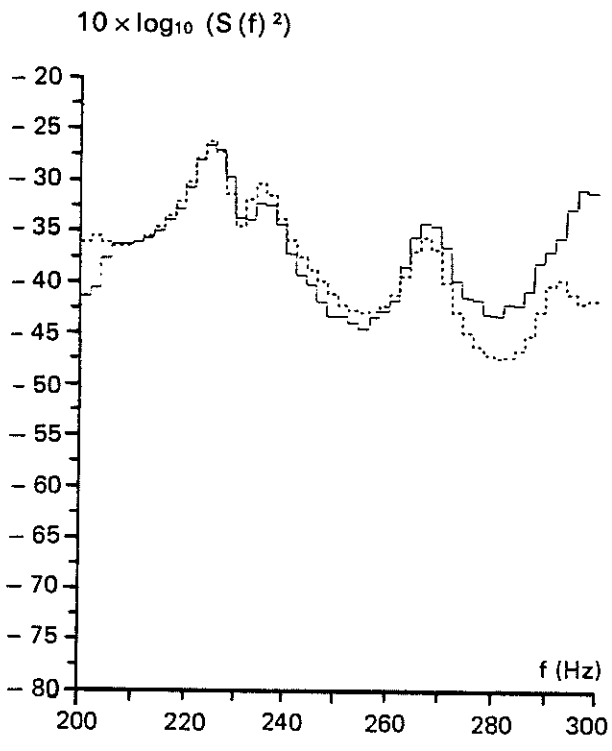

Fig. 10. – $S(f)$ = Spectral power density function of the acceleration normal to point *M9*. ——— computation by ADINA; - - - - - - - - analytic computation.

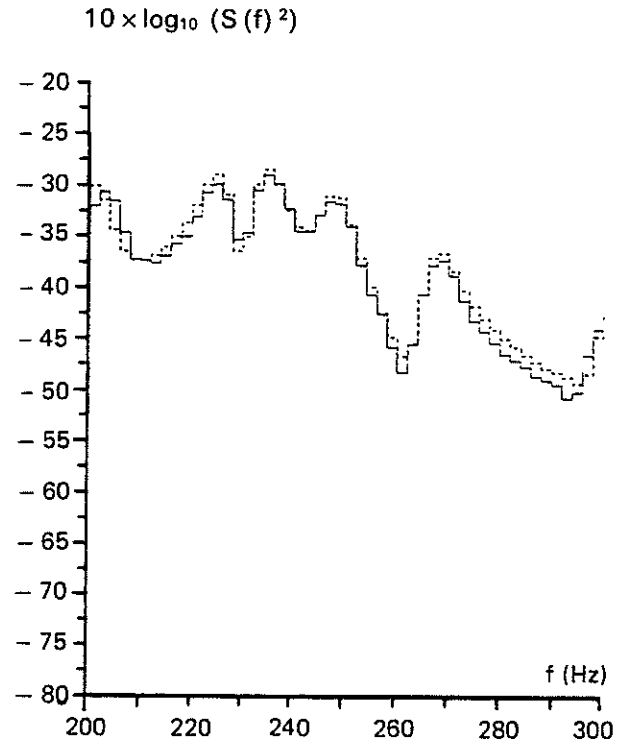

Fig. 11. – $S(f)$ = Spectral power density function of the acceleration normal to point *M 10*. ——— computation by ADINA; - - - - - - - - analytic computation.

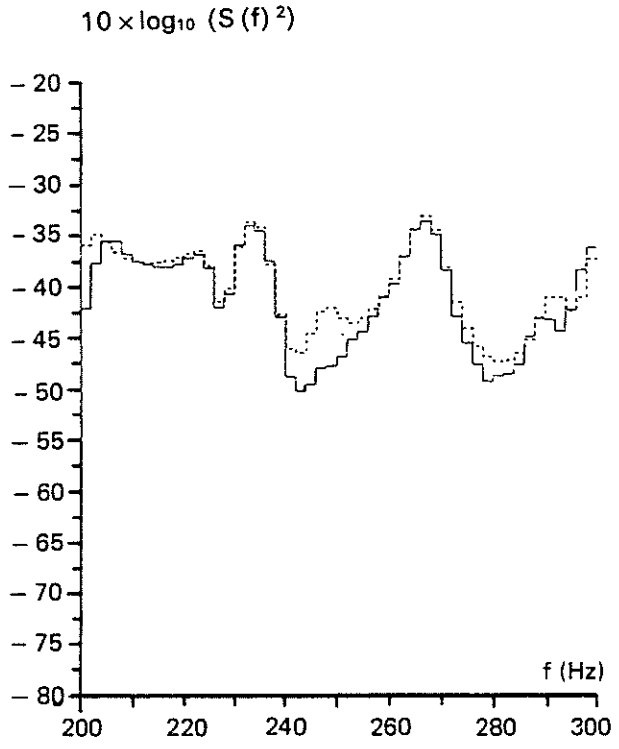

Fig. 12. – $S(f)$ = Spectral power density function of the acceleration normal to point *M 11*. ——— computation by ADINA; - - - - - - - - analytic computation.

IX. – CONCLUSION

The parametric analysis shows that if the spatial correlation lengths of the random field are not too small compared to the dimensions of the surface to which the field is applied, the functional reduction method is highly efficient. For instance, for the plate treated, there is very good agreement of the results when J_n is taken equal to 4. In this case, a conventional method would have led to taking 432 elementary loading cases. The numerical gain is therefore proportional to 432/4, i. e. about 100.

It should be noted that if the spatial correlation lengths become small, J_n increases fairly rapidly. Obviously, as long as J_n remains less than m, the method remains interesting since $m/J_n > 1$. However, considering the cost of extraction of the eigenvalues and eigenvectors, J_n must remain much smaller than m.

It should also be noted that the reduction is made by bands whose width is arbitrary. However, the narrower the bands, the more accurate the reduction since the reduction is then adapted in the frequency domain. In pratice, for a given excitation spectrum, very broadband, there are bands for which the reduction is highly efficient and others for which it is not. For the latter, other methods must be used, such as the conventional method.

Manuscript handed in on February 12, 1986

REFERENCES

[1] BADRIKIAN A. – *Séminaire sur les fonctions aléatoires linéaires et les mesures cylindriques.* Lecture notes in mathematics, n° 128, Springer-Verlag, Berlin, (1971).

[2] BATCHELOR G. K. – *The theory of homogeneous turbulence,* Cambridge at the university Press, (1967).

[3] BATHE K. J. – *Convergence of subspace iteration,* Formulations and numerical algorithms in finite element analysis, MIT Press, Massach., (1977).

[4] BATHE K. J. – *Finite element procedures in engineering analysis,* Prentice-Hall, Inc., Englewood Cliffs, N.J., (1982).

[5] BATHE K. J. – *Solution methods for large generalized eigenvalue problems in structural engineering, Report UC SESM 71-20, Civil Engineering Department, University of California, Berkely, (1971).*

[6] BATHE K. J. and RAMASWANY S. – *An accelerated subspace iteration method,* Journal Computer methods in applied mechanics and engineering, vol. 23, (1980), p. 313-331.

[7] BAKEWELL J. R. – *Turbulent Wall-pressure Fluctuations on a body of revolution,* JASA, vol. 43, n° 6, (1968).

[8] BENDAT J. S. and PIERSOL A. G. – *Random Data: Analysis and measurement procedures,* Wiley Interscience, New York, (1971).

[9] CIARLET P. – *Numerical analysis of the finite element method,* Presses de l'université de Montréal, (1976).

[10] CORCOS G. M. – *The structure of the turbulent pressure field in boundary-layer flows,* J. Fluid Mech. 18, (1964).

[11] CORCOS G. M. – *Resolution of Pressure in turbulence,* JASA, vol. 35, n° 2, (1963).

[12] CRAMER H. and LEADBETTER M. R. – *Stationary and related stochastic processes,* John Wiley and Sons, New York, (1967).

[13] DIEUDONNÉ J. – *Éléments d'analyse,* Gauthier-Vilars, Paris.

[14] DOOB J. L. – *Stochastic processes,* John Wiley and Sons, New York, (1967).

[15] DUNFORD N. and SCHWARTZ J. T. – *Linear operators* / Interscience, New York, (1967).

[16]FRANKS L. E. – *Signal theory,* Prentice-Hall, N.J., (1969).

[17] FRIEDMAN A. – *Stochastic differential equations and applications,* vol. 1 et 2, Academic Press, New York, (1975).

[18] GELFAND I. M. and VILENKIN N. Y. – *Les distributions,* Dunod, Paris, (1967).

[19] GUIKHMAN L. and SKOROKHOD A. V. – *The theory of stochastic processes,* Springer-Verlag, Berlin, (1979).

[20] HINZE J. O. – *Turbulence,* McGraw Hill, New York, (1959).

[21] IKEDA N. and WATANABE S. *Stochastic differential equations and diffusion processes,* North Holland, (1981).

[22] JORGENSEN D. W. and MAIDANIK G. – *Response of a system of point transducers to turbulent boundary-layer pressure field,* JASA, vol. 43, n° 6, (1968).

[23] KATO T. – *Perturbation theory for linear operators,* Springer-Verlag, New York, (1966).

[24] KREE P. and SOIZE C. – *Mécanique aléatoire,* Dunod, Paris, (1983).

[25] LAUCHLE G. C. – *Noise generated by axisymmetric turbulent boundary-layer flow,* JASA, vol. 61, n° 3, (1977).

[26] LIN Y. K. – *Probabilistic theory of structural dynamics,* McGraw Hill, New York, (1968).

[27] METIVIER M. – *Notions fondamentales de la théorie des probabilités,* Dunod, Paris, (1972).

[28] PAPOULIS A. – *Signal Analysis,* McGraw Hill, New York, (1977).

[29] RAVIART P. A. et FAURE P. – *Cours d'analyse numérique,* Paris, (1976).

[30] ROZANOV Y. A. – *Stationary random processes,* Holden Day, San Francisco, (1967).

[31] SCHWARTZ L. – *Analyse Hilbertienne,* Hermann, Paris, (1979).

[32] SKUDRZYK E. J. and HADDLE G. P. – *Noise production in a turbulent boundary-layer by smooth and rough surfaces,* JASA, vol. 32, n° 1, (1960).

33] SOIZE C. – *Éléments mathématiques de la théorie déterministe et aléatoire du signal,* ENSTA, Paris, (1985), (seconde édition).

[34] SOIZE C. – *Medium Frequency linear vibrations of anisotropic elastic structures,* La Recherche Aérospatiale, n° 1982-5, French and English edition.

[35] SOIZE C., DAVID J. M., DESANTI A. et HUTIN P. M. – *Étude dynamique moyenne fréquence de la poutre tubulaire, à sec et en immersion, dans plusieurs configurations et comparaisons expérimentales,* tomes 1 et 2, RT-ONERA n° 34/3454 RY 08 1 R, (1984).

[36] SOIZE C., HUTIN P. M., DESANTI A., DAVID J. M. and CHABAS F. – *Linear dynamic analysis of mechanical systems in the medium frequency range,* Journal of Computer and Structures. Pergamon Press, New York (to be published), (1986).

[37] SOONG T. T. – *Random differential equations in science and engineering,* Academic Press, New York, (1973).

[38] STRANWDERMAN W. A. – *Turbulence induced plate vibrations: an evaluation of finite and infinite plate models,* JASA, vol. 46, n° 5, (1969).

[39] VECCHIO E. A., and WILEY C. A. – *Noise radiated from a turbulent boundary-layer,* JASA, vol. 53, n° 2, (1973).

[40] WILLMARTH W. W. – *Pressure fluctuations beneath turbulent boundary-layers,* Annual Review of Fluid Mechanics, vol. 7, (1975).

[41] YOSIDA K. – *Functional Analysis,* Third edition, Springer-Verlag, Berlin, (1971).