Probabilistic structural modeling in linear dynamic analysis of complex mechanical systems, I - Theoretical elements

Christian Soize

PROBABILISTIC STRUCTURAL MODELING IN LINEAR DYNAMIC ANALYSIS OF COMPLEX MECHANICAL SYSTEMS I. THEORETICAL ELEMENTS

by

C. SOIZE

ABSTRACT

In this first of our paper we introduce theoretical elements of a probabilistic modeling of structural fuzzy in linear dynamic analysis of complex mechanical systems. The structural fuzzy is defined as the set of minor subsystems that are connected to the master structure but are not accessible by classical modeling.

For the low frequency (LF) dynamic analysis, the modeling of the structural fuzzy is commonly made with a system of masses. If the LF modeling of the structural fuzzy is applied in the medium frequency (MF) domain, there result some large differences between calculations and experiment. It is therefore necessary to take into account internal degrees of freedom of the structural fuzzy. We are proposing a global probabilistic modeling of the structural fuzzy to improve the calculated estimates of the MF vibrations into the master structure. In this paper we (1) develop a probabilistic modeling of the structural fuzzy and the fuzzy finite elements, (2) build a probabilistic behavior law of the structural fuzzy and (3) study the random vibrations in the master structure with a structural fuzzy.

The numerical developments and a few examples are presented in part II of this publication.

I. – INTRODUCTION

I,1. – CONCEPT OF MASTER STRUCTURE AND STRUCTURAL FUZZY OF A COMPLEX MECHANICAL SYSTEM

An industrial mechanical system such as an aeronautical construction, an aerospace construction, a marine engineering construction is generally a complex system. For predicting the static or dynamic behavior of the complete mechanical system or one of its parts by computation, we will use the term MASTER STRUCTURE to designate the mechanical system which is accessible to conventional modeling, i. e. the system whose mechanical properties, geometry, boundary conditions and excitations are known with sufficient accuracy and whose necessary modeling requires implementation and leads to a numerical approach at a cost which remains reasonable and is consistent with the results that can be expected from the model.

The complement to the master structure with respect to the complete mechanical system or the part thereof analyzed is designated the STRUCTURAL FUZZY. Considering the definition given above of the master structure, the structural fuzzy is the part which is not accessible to conventional modeling.

For instance, for the structures mentioned above, a distinction can be made between the primary structure which ensures the overall stiffness and forms the quasi-totality of the master structure and the structural fuzzy which consists of the many secondary mechanical systems "attached" to the primary structure and which contribute to the functionality of the construction.

In accordance with the terminology adopted, the concept of master structure extends to other elements in addition to those involved in the stiffness. For instance, where there is strong interaction between the primary structure and dense fluids, it may be necessary to globally model the primary structure and the fluids in order to analyse the dynamics. The system which can be modeled in this way is the master structure (Fig. 1).

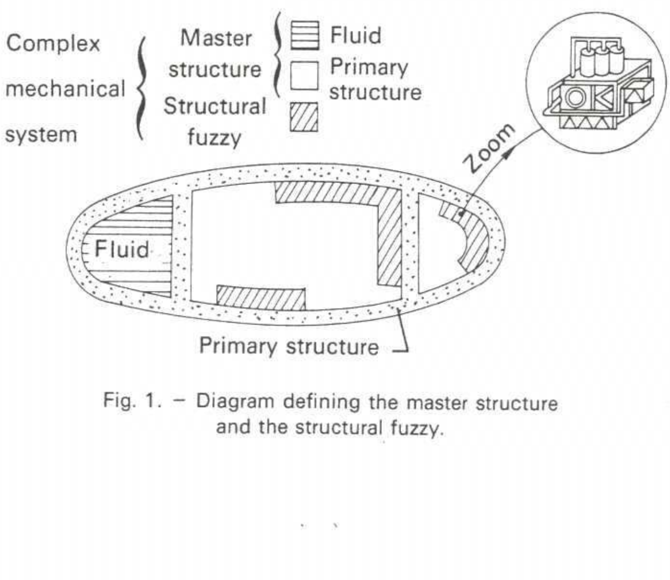

Fig. 1. – Diagram defining the master structure and the structural fuzzy.

I,2. – ROLE OF THE STRUCTURAL FUZZY IN THE DYNAMIC BEHAVIOR OF THE MASTER STRUCTURE

In this paper, we discuss modeling and prediction by computation of the dynamic behavior of WEAKLY DAMPED complex mechanical systems.

A. For analyzing the dynamic behavior in the low frequency (LF) domain, modeling generally consists of taking as master structure the primary structure, the fluids, the solid masses, etc. which are conventionally modeled and a system equivalent to the structural fuzzy, modeled globally by pure masses. In this case, the dynamics specific to the structural fuzzy are not modeled. Such modeling is legitimate. The LF response functions (applied force-displacement) exhibit resonance peaks which correspond to the response of the initial natural modes of vibration of the associated conservative system. Generally, the experimental results agree very well with with the results predicted by computation (Fig. 2).

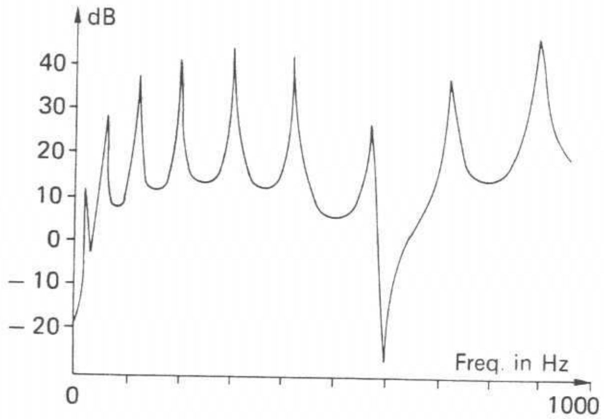

Fig. 2. – Crossed frequency response function in the master structure for the low frequency domain.

B. Let us now examine the case of dynamic behavior in the medium frequency (MF) domain.

B.1. Let us first consider a "pure" mechanical system, i. e. with no great complexity, which can therefore be fully modeled. There is no structural fuzzy in the above sense, there is only a master structure. This is the case, for instance, of a closed, stiffened cylindrical shell with finite length, with a small number of internal subsystems, the entire system being placed in a dense compressible fluid. Such a system can be modeled conventionally to predict the hydroelastoacoustic behavior by computation in the MF domain. The model must obviously be suited to the MF domain and requires introducing a large number of degrees of freedom (DOF). Special methods must be used for numerical processing, since the methods commonly used for analysis of the low frequency domain are not efficient enough. We have developed such an MF method and the comparisons of the direct and crossed MF frequency response functions obtained by computation with experimental results are satisfactory [17, 18, 20, 79, 82, 83]. The MF frequency response functions obtained do not have the same morphology as those obtained in the LF domain (Fig. 3).

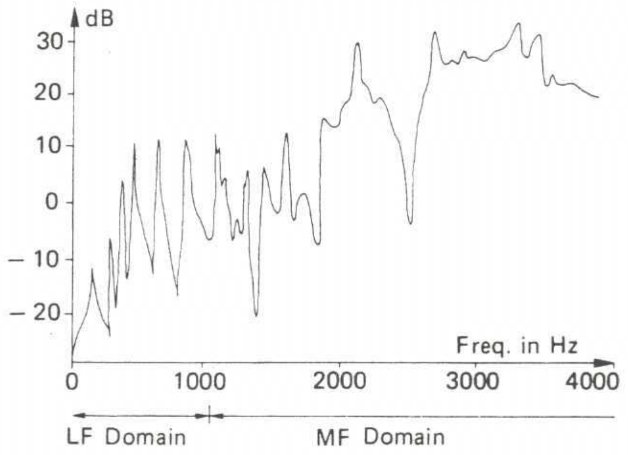

Fig. 3. – Crossed frequency response function in the master structure of a relatively pure system in the MF domain.

B.2. Now let us consider a mechanically complex system for which we wish to predict the dynamic MF behavior of the master structure in presence of the structural fuzzy. The mechanical system is always globally weakly damped. If the master structure is modeled for the MF domain like the "pure" mechanical system described above, and the structural fuzzy is modeled for the LF domain, i. e. by introducing equivalent pure masses, the frequency response functions of the master structure yielded by computation have the same morphology as the MF response functions of the "pure" mechanical systems mentioned above, which was to be expected. However, comparison of the predictions with experimental results shows that there are considerable differences. The frequency response functions of the master structure of experimentally obtained complex systems have a much smoother morphology than that yielded by predictions, as though dissipation were much higher (Fig. 4). However, the rates of dissipation which would correspond to such smoothing are much too high to allow this phenomenon to be explained by mechanical damping alone, which is always very weak in the master structure and in the structural fuzzy for the mechanical systems considered herein.

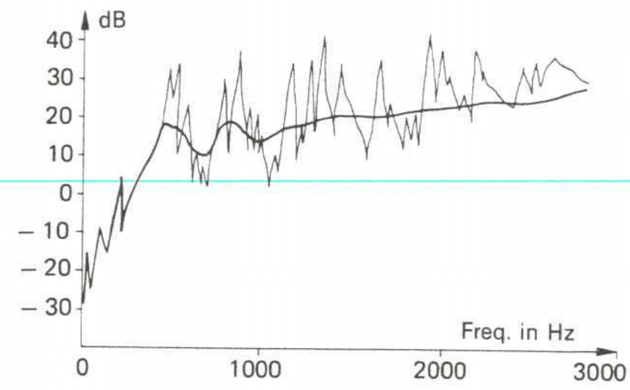

Fig. 4. – Frequency response function in the master structure of a complex system in the MF domain. ——— real structure; ——— computations without modeling of the structural fuzzy.

Actually, the apparent dissipation occurring on the master structure is due to the energy transfer (kinematic and potential) to the structural fuzzy which includes a large number of mechanical systems attached to the master structure entering in vibration in the MF domain and excited through the common boundary between the master structure and the structural fuzzy.

It is therefore the internal dynamics of the structural fuzzy which are involved and a pure mass model which does not allow modeling of the internal degrees of freedom of the fuzzy can in no case account for these phenomena. In addition, the weaker the damping specific to the mechanical subsystems comprising the structural fuzzy, the more energy will be stored by the fuzzy, which will result in a high apparent dissipation for the master structure. These observations mean that in order to predict the MF vibrations in the master structure of a complex mechanical system, we must model the dynamic effects that the structural fuzzy has on the master structure.

I,3. – METHOD FOR ACCOUNTING FOR THE EFFECTS OF THE STRUCTURAL FUZZY ON THE MASTER STRUCTURE. CONCEPT OF PROBABILISTIC IMPEDANCE OF THE FUZZY BOUNDARY

It should be noted that we are not attempting to find a model of structural fuzzy capable of predicting the internal vibrational state of the structural fuzzy, but only to model the effects of the structural fuzzy on the master structure through the common boundary. Moreover, it would be illusory to try to model the internal vibrational state of the structural fuzzy since by definition, the structural fuzzy is not accessible to conventional modeling. We therefore propose a probabilistic approach to obtain a structural fuzzy model globally accounting for the effects of the structural fuzzy on the master structure, so that the conventional model (adapted to the MF domain) of the master structure, plus the probabilistic model of the effects of the structural fuzzy qualitatively and quantitatively restore the average behavior of the master structure of the complex system.

The fuzzy consists of undefined mechanical subsystems formed of discrete mechanical systems with a finite number of degrees of freedom and continua with an infinite number of DOF. All the subsystems are very complex from the standpoint of possible detailed mechanical modeling. A probabilistic approach is therefore well suited in order to obviate the need for an "exact" description of the multiple subsystems. However, such an approach is not statistically valid unless the number of subsystems is large, which is assumed to be the case considering the definition of the structural fuzzy.

The state equation (formulated, for instance, with the displacement field) which governs the linear dynamics of the master structure is expressed conventionally as an impedance term in the Fourier space for the time variable. Therefore, the dynamic effects of the structural fuzzy on the master structure must be introduced by an impedance. Since the master structure sees the effects of the fuzzy only through the boundary it has in common with the structural fuzzy, the effects of the structural fuzzy on the master structure must be modeled by introducing a boundary impedance. As the model of the fuzzy is probabilistic, the effects of the fuzzy on the master structure will be modeled by setting a *probabilistic fuzzy boundary impedance*. This fuzzy impedance applies only to the trace of the master structure displacement field on the common boundary. Therefore, the number of DOF in the master structure model is not increased by taking into account the effects of the fuzzy.

I,4. – CONCEPT OF STRUCTURAL FUZZY PROBABILISTIC CONSTITUTIVE LAW AND FUZZY FINITE ELEMENTS

The probabilistic fuzzy boundary impedance is not intrinsic, since it depends on the local geometry of the common boundaries on which the fuzzy is "attached" and the degrees of freedom of the master structure displacement field on the same boundaries. We therefore introduce the concept of *probilistic fuzzy constitutive law* which can be assimilated to an impedance. The constitutive laws are used to construct the probabilistic fuzzy boundary impedances.

To model the structural fuzzy of a complex system, the simultaneous use of several fuzzy constitutive laws may be necessary, each of which describes a class of fuzzy. It is clear that a fuzzy class and the constitutive law describing it must be intrinsic. It would be of no interest to establish particular fuzzy constitutive laws for each mechanical system studied. The approach proposed requires identifying the classes comprising the structural fuzzy to be analyzed during modeling of a complex mechanical system and using for each class the constitutive law which must have been constructed intrinsically. In other words, a "library" of constitutive laws is used, as is the case for materials.

In addition, in complex mechanical systems, the structural fuzzy is generally not spatially homogeneous. For instance, nonhomogeneousness of the fuzzy can result simply from the absence of fuzzy in certain areas of the structure. Also, probabilistic modeling of the fuzzy in a region may require superimposing several classes of fuzzy. This is why it is necessary to develop *fuzzy finite elements* which are used to discretize the probabilistic fuzzy boundary impedances created with the fuzzy constitutive laws. Thus, the "library" of fuzzy constitutive laws is associated with a "library" of fuzzy finite elements. Then, conventional finite element methods can be used to superimpose the classes and take into account the spatial nonhomogeneousness, the isotropy, the orthotropy or the anisotropy of the fuzzy and also to introduce fuzzy finite macroelements created by the substructuring method, etc. These finite elements are obviously specific since they discretize random operators.

The fuzzy finite elements and the fuzzy macroelements are assembled and lead to the random matrix of the fuzzy boundary random impedance operator which reflects all the effects of the structural fuzzy on the master structure.

I,5. – LINEAR DYNAMICS OF THE MASTER STRUCTURE IN PRESENCE OF A STRUCTURAL FUZZY

As was explained in Sec. I,2, the aim of probabilistic modeling of the structural fuzzy is to obtain a more realistic model of the master structure in MF linear dynamics, i. e. to achieve better prediction of:

– the spatial propagation of the vibrational energy through the master structure;

— the direct and crossed frequency response functions for any parameters of the master structure;

— impedance matrices relative to specified parameters of the master structure, to establish boundary conditions for the dynamic analysis specific to an internal subsystem particularized in the fuzzy, taking into account the presence of the rest of the structural fuzzy.

As the master structure is modeled conventionally and the effects of the fuzzy are introduced by a random impedance operator (*see* Sec. I,4), the MF linear dynamics of the master structure with its structural fuzzy are governed by a random operator equation. However, we know that MF modeling of the master structure alone leads to models with a large number of DOF and that it is necessary to use appropriate numerical methods to construct solutions [78, 82, 83].

For the discretized model, introducing the structural fuzzy model leads to matrix equations with large dimension random matrices whose solution requires construction of a specific method.

I,6. — REMARKS CONCERNING CONSTRUCTION OF THE FUZZY CONSTITUTIVE LAWS

A basic problem is the construction or identification of the fuzzy constitutive laws according to the nature of the structural fuzzies, i. e., the classes of fuzzy.

II. — NOTATIONS AND DEFINITIONS

A number of notations are used frequently below. To avoid repetitions and make the developments clearer, we have grouped the main notations in this section.

II,1. — SPACES $E^N_{\mathbb{K}}$ AND $\mathbb{K}^N$

Let N be a positive finite integer and $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$. The conjugate of $z \in \mathbb{C}$ is conventionally noted $\bar{z}$. Below, $E^N_{\mathbb{R}}$ is used to designate a Euclidian vector space with dimension N, equipped with the scalar product noted $\langle U, V \rangle_N$ and the associated norm $\| U \|_N = \langle U, U \rangle_N^{1/2}$.

The symmetrical bilinear form $U, V \mapsto \langle U, V \rangle_N$ of $E^N_R \times E^N_{\mathbb{R}} \to \mathbb{R}$ is extended on $\mathbb{C}$ and we will note as $E^N_{\mathbb{C}}$ the complexification of $E^N_{\mathbb{R}}$. The vector space $E^N_{\mathbb{C}}$ is then equipped with the scalar product:

$$(U, V)_N = \langle U, \bar{V} \rangle_N, \qquad U \text{ and } V \in E^N_{\mathbb{C}} \tag{1}$$

and the associated norm:

$$\| U \|_N = (U, U)_N^{1/2}, \qquad U \in E^N_{\mathbb{C}}. \tag{2}$$

When $E^N_{\mathbb{R}}$ is referenced to an orthonormal basis (real) $\{ b_1, b_2, \ldots, b_N \}$, $E^N_{\mathbb{R}}$ will be identified with $\mathbb{R}^N$ and the complexification $E^N_{\mathbb{C}}$ with $\mathbb{C}^N$. In this case, we will therefore have $E^N_{\mathbb{K}} \sim \mathbb{K}^N$.

The vector space $\mathbb{K}^N$ is always referenced to the canonical basis of $\mathbb{R}^N$, and $\mathbb{C}^N$ will be considered as the complexification of $\mathbb{R}^N$. Therefore, for $U=(U_1, U_2, \ldots, U_N) \in \mathbb{K}^N$ and $V=(V_1, \ldots, V_N) \in \mathbb{K}^n$, we have:

$$\langle U, V \rangle_N = \sum_{j=1}^{N} U_j V_j \tag{3}$$

$$(U, V)_N = \sum_{j=1}^{N} U_j \bar{V}_j. \tag{4}$$

II,2. — LINEAR OPERATOR OF $E^N_{\mathbb{K}}$ AND $\mathbb{K}^N$

In this section, $\mathbb{V}_{\mathbb{K}}$ designates $E^N_{\mathbb{K}}$ or $\mathbb{K}^N$.

Let $L(\mathbb{V}_{\mathbb{K}})$ be the vector space of the linear mappings Q of $\mathbb{V}_{\mathbb{K}}$ and $[Q] \in \mathrm{Mat}_{\mathbb{K}}(N, N)$ be the matrix of elements $Q_{jk} \in \mathbb{K}$ of linear operator Q relative to an orthonormal basis of $E^N_{\mathbb{R}}$ if $\mathbb{V}_{\mathbb{K}} = E^N_{\mathbb{K}}$ or to the canonical basis of $\mathbb{R}^N$ if $\mathbb{V}_{\mathbb{K}} = \mathbb{K}^N$. We will note as ${}^tQ$ the transposed operator of Q defined by:

$$\langle {}^tQU, V \rangle_N = \langle U, QV \rangle_N, \qquad \forall U \text{ and } V \in \mathbb{V}_{\mathbb{K}}. \tag{5}$$

The matrix of ${}^tQ$ is $[{}^tQ]=[Q]^T$, where T designates the transposition of the matrix.

Operator Q is said to be symmetrical if $Q={}^tQ$ (for $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$).

The adjoint operator of Q is noted Q^*. In the case $\mathbb{K}=\mathbb{R}$, the adjoint Q^* of Q is the transposed operator ${}^tQ$.

If $\mathbb{K}=\mathbb{C}$, Q^*, which is defined by:

$$(Q^* U, V)_N = (U, QV)_N, \qquad \forall U \text{ and } V \in \mathbb{V}_{\mathbb{K}} \tag{6}$$

can be written $Q^* = {}^t\bar{Q}$.

Operator Q is said to be self-adjoint if $Q^*=Q$.

The vector space $L(\mathbb{V}_{\mathbb{K}})$ is equipped with the norm with operator:

$$\mathbf{I}\, Q\, \mathbf{I}_N = \operatorname*{Sup}_{\| U \|_N \leq 1} \| QU \|_N \tag{7}$$

and we will note as $\mathscr{L}(\mathbb{V}_{\mathbb{K}})$ the Banach space of continuous linear mappings Q of $\mathbb{V}_{\mathbb{K}}$ into $\mathbb{V}_{\mathbb{K}}$ which are such that $\| Q \|_N < +\infty$.

II,3. — SPACE $G(\mathbb{R}, \mathscr{L}(\mathbb{K}^N))$

It is recalled that a mapping $\omega \to Q(\omega))$ defined everywhere on $\mathbb{R}$ with values in a Banach space is continuous by parts if:

(1) In any point ω of $\mathbb{R}$, ω is either a point of continuity or a point of jump discontinuity, i.e., in any point, the limits from the right and left always exist.

(2) In any compact interval of $\mathbb{R}$, there is a finite number of points of discontinuity.

Such a mapping is locally bounded, i.e. it is bounded over any compact interval of $\mathbb{R}$.

We note as $G(\mathbb{R}, \mathscr{L}(\mathbb{K}^N))$ the space of mappings $Q: \omega \to Q(\omega)$ defined everywhere on $\mathbb{R}$ with values in $\mathscr{L}(\mathbb{K}^N)$ such that:

(*a*) $\forall \omega \in \mathbb{R}$, $Q(\omega) = {}^tQ(\omega)$,

(*b*) $\omega \to Q(\omega)$ is a continuous mapping by parts. It is therefore locally bounded and for any compact interval B of $\mathbb{R}$, for any $\omega \in B$, $|Q(\omega)|_N \leqq c < +\infty$,

(*c*) $\forall \omega \in \mathbb{R}$, $Q(\omega)$ is an invertible operator and the mapping $\omega \to Q(\omega)^{-1}$ of $\mathbb{R}$ into $\mathscr{L}(\mathbb{K}^N)$ is continuous by parts. In addition, since $Q(\omega)^{-1} = {}^tQ(\omega)^{-1}$, it can be seen that $\omega \to Q(\omega)^{-1}$ belongs to $G(\mathbb{R}, \mathscr{L}(\mathbb{K}^N))$.

II,4. – SPACE $H_B(\mathbb{R}, \mathbb{C}^N)$

Let B be a compact interval of $\mathbb{R}^+$:

$$B = [\omega_1, \omega_2], \qquad 0 < \omega_1 < \omega_2 < +\infty. \tag{8}$$

Let $L^2(\mathbb{R}, \mathbb{C}^N)$ be the Hilbert space of mappings $t \to U(t)$ defined dt-almost everywhere on $\mathbb{R}$, with values in $\mathbb{C}^N$, with integrable square, equipped with the scalar product:

$$((U, V))_N = \int_{\mathbb{R}} (U(t), V(t))_N \, dt \tag{9}$$

and the associated norm:

$$|||U|||_N = ((U, U))_N^{1/2}. \tag{10}$$

The Fourier transform $\hat{U} \in L^2(\mathbb{R}, \mathbb{C}^N)$ of U is such that for $d\omega$-almost any $\omega \in \mathbb{R}$

$$\hat{U}(\omega) = \int_{\mathbb{R}} e^{-i\omega t} U(t) \, dt. \tag{11}$$

We have Plancherel's relation

$$\begin{aligned} |||U|||_N &= \frac{1}{\sqrt{2\pi}} |||\hat{U}|||_N, \\ ((U, V))_N &= \frac{1}{2\pi} ((\hat{U}, \hat{V}))_N. \end{aligned} \tag{12}$$

The space $H_B(\mathbb{R}, \mathbb{C}^N)$ is defined as the vector subspace of $L^2(\mathbb{R}, \mathbb{C}^N)$ such that:

$$H_B(\mathbb{R}, \mathbb{C}^N) = \{ U \in L^2(\mathbb{R}, \mathbb{C}^N) \mid \operatorname{Supp} \hat{U} = B, \ \omega \to \hat{U}(\omega) \text{ ess. bounded} \} \tag{13}$$

where Supp $\hat{U}$ designates the support of $\hat{U}$, i.e. $\hat{U}(\omega) = 0$ for $\omega \notin B$ and "ess." means essentially. Therefore, $\omega \to \hat{U}(\omega)$ is bounded for $d\omega$-almost any ω in B. Considering (13), we have:

$$\forall U \in H_B(\mathbb{R}, \mathbb{C}^N), \quad \int_{\mathbb{R}} \| U(t) \|_N^2 \, dt = \frac{1}{2\pi} \int_B \| \hat{U}(\omega) \|_N^2 \, d\omega. \tag{14}$$

II,5. – SURFACE Σ OF $\mathbb{R}^3$

Below, any surface Σ of $\mathbb{R}^3$:

– will have as generic point the point noted m of $\mathbb{R}^3$;

– will be equipped with a positive surface measure borne by Σ and noted $ds(m)$ such that the area of Σ is $|\Sigma| = \int_\Sigma ds(m)$;

– will be equipped with a distance $m, m' \to d(m, m') : \Sigma \times \Sigma \to \mathbb{R}$, i.e. Σ will be considered a metric space.

II,6. – SPACE $C^0(\Sigma, E_{\mathbb{K}}^N)$

Let Σ be a surface of $\mathbb{R}^3$. We will note as $C^0(\Sigma, E_{\mathbb{K}}^N)$ all the continuous mappings $m \to u(m)$ defined on Σ with values in $E_{\mathbb{K}}^N$

II,7. – SPACE $H(\Sigma, E_{\mathbb{K}}^N)$

Let Σ be a surface of $\mathbb{R}^3$. We will note as $H(\Sigma, E_{\mathbb{K}}^N)$ the Hilbert space of functions $m \to u(m)$ defined ds-almost everywhere on Σ with values in $E_{\mathbb{K}}^N$ with integrable square for ds, equipped with the scalar product:

$$((u, v))_\Sigma = \int_\Sigma (u(m), v(m))_N \, ds(m) \tag{15}$$

and the associated norm:

$$|||u|||_\Sigma = ((u, u))_\Sigma^{1/2}. \tag{16}$$

II,8. – SPACES $(\mathscr{A}, \mathscr{C}, \mathscr{P})$ AND $L^2(\mathscr{A}, E_{\mathbb{K}}^N)$.

All the random values will be defined on the probabilistic space $(\mathscr{A}, \mathscr{C}, \mathscr{P})$ where $\mathscr{P}$ is the probability measure on $(\mathscr{A}, \mathscr{C})$. The elements of $\mathscr{A}$ will be noted a, and $\mathscr{E}$

will designate the mathematical expectation. We will note as $L^2(\mathscr{A}, E_{\mathbb{K}}^N)$ the Hilbert space of second order random variables with values in $E_{\mathbb{K}}^N$ equipped with the scalar product:

$$X, Y \to \mathscr{E}\{(X, Y)_N\} = \int_{\mathscr{A}} (X(a), Y(a))_N \, d\mathscr{P}(a)$$

and the associated norm:

$$X \to (\mathscr{E}\{\|X\|_N^2\})^{1/2} = \left(\int_{\mathscr{A}} \|X(a)\|_N^2 \, d\mathscr{P}(a)\right)^{1/2}.$$

II,9. — FREQUENCY BAND B_n

Below, B_n designates the compact interval of $\mathbb{R}^+ : \left[\Omega_n - \frac{\Delta\omega}{2}, \Omega_n + \frac{\Delta\omega}{2}\right]$ where $\Omega_n > 0$ is the center frequency of the band and $\Delta\omega > 0$ is the bandwidth such that $\Omega_2 - \frac{\Delta\omega}{2} > 0$.

III. — PROBABILISTIC MODELING OF THE STRUCTURAL FUZZY

III,1. — GENERAL GEOMETRIC AND MECHANICAL HYPOTHESES ABOUT THE FUZZY

The master structure occupies an open, connected, bounded domain D_M of $\mathbb{R}^3$ with boundary $\partial D_M = \Gamma \cup \Gamma_M$, and the structural fuzzy occupies an open, bounded domain D_F of $\mathbb{R}^3$ with boundary $\partial D_F = \Gamma \cup \Gamma_F$, such that $D_M \cap D_F = \varnothing$. In the general case, Γ_M is nonempty, but the particular case can exist where $\Gamma_M = \varnothing$. These hypotheses are consistent with the explanations given in Sec. I, i. e. that the effect of the structural fuzzy is transmitted to the master structure through a boundary Γ common to domain D_M occupied by the master structure and to domain D_F occupied by the structural fuzzy. The boundary Γ is generally not connected since the structural fuzzy can very well occupy certain disjoint areas of the domain occupied by the complex mechanical system. In addition, for reasons of modeling, fuzzy subsystems can be connected to the master structure on a finite set of spatially determined points (on ∂D_M). It should be mentioned that domains D_M and D_F as well as their boundaries are assumed to be determined, in the deterministic sense, boundary Γ in the case at hand. This is fully compatible with modeling of the structural fuzzy. The multiple mechanical subsystems comprising the fuzzy have random mechanical characteristics, geometries and spatial distributions in domain D_F. The connections of the structural fuzzy subsystems on the master structure can therefore have a random spatial distribution on boundary Γ. We thus assume only that the areas where the fuzzy is attached to the master structure, defined by Γ, are given.

III,1.1. — *Definition of S and S_I*

Let S and S_I be parts of Γ and m be a point of space $\mathbb{R}^3$ belonging to Γ.

(*a*) Part S of Γ will be an open, bounded, connected domain of a surface of $\mathbb{R}^3$. S is therefore a surface of $\mathbb{R}^3$ which we will assume to be C^0 and C^1 by parts. We will note as ∂S the boundary of S, C^0 and C^1 by parts and $\bar{S} = S \cup \partial S$.

(*b*) The part S_I of Γ will be the discrete set $S_I = \{m_1, m_2, \ldots, m_I\}$ consisting of I points m_i belonging to ∂D_M.

III,1.2. — *Mechanical hypotheses concerning the structural fuzzy*

(1) All the mechanical subsystems comprising the fuzzy are assumed to have a linear behavior around a state of static equilibrium, taken as reference state.

Since D_F is bounded, each mechanical subsystem of the fuzzy necessarily occupies a bounded domain of space.

(2) No energy is added to the structural fuzzy; in other words, there are no "internal" excitations in the fuzzy. We therefore consider a "passive" fuzzy. It should be noted that if there are sources of mechanical excitation in the fuzzy, they must be "extracted" from the structural fuzzy and applied in an equivalent manner to the master structure according to the laws of mechanics. They therefore become excitations of the master structure.

(3) Each mechanical subsystem of the fuzzy is dissipative, with a very weak specific dissipation (Sec. I). The associated conservative mechanical subsystem is a linear dynamic system whose spectrum of natural vibration frequencies is discrete. The spectrum is therefore countable.

To simplify the description, we will assume that the coupling state variable of the master structure with the structural fuzzy, as defined on Γ, is a displacement field $m \to u(m, t)$ with values in $\mathbb{R}^3$. The dual variable is then a force density field $m \to f(m, t)$ with values in $\mathbb{R}^3$. Actually, we could reason on any other pair of variables. With the formulation chosen, we have the following mechanical interpretation: the master structure is in vibration under the effects of known mechanical excitations which can be deterministic or random. In the neighborhood of Γ, the state of the master structure is defined by its displacement field $m \to u(m, t)$. The trace of $u(m, t)$ on Γ is also noted $m \to u(m, t)$. As boundary Γ is deformed as a function of time, all the mechanical systems of the structural fuzzy, which are connected on Γ, are excited by their boundary. Coupling obviously exists between the master structure and the structural fuzzy.

The proposed model is designed to obviate detailed mechanical definition of each fuzzy subsystem (as the fuzzy is by definition not accessible to modeling) and instead to globally describe the effects of the fuzzy on the master structure by a probabilistic model (Sec. I).

III,2. — PROBABILISTIC STRUCTURAL FUZZY BOUNDARY IMPEDANCE

III,2.1. — *Case of fuzzy with locally continuous boundary*

We consider a part of the fuzzy relative to the part S of Γ, where S is the surface of $\mathbb{R}^3$ with the properties defined in III,1.1 *a*. This part of the fuzzy is said to be locally continuous because (1) surface S is continuous and (2) S is only part of boundary Γ.

Let $\mathbf{Z}(m, m', \omega)$ be the probabilistic boundary impedance of S. It is assumed that this impedance verifies the following hypotheses, necessary to ensure consistency of the theory. For any ω in $\mathbb{R}$ and for ds-almost any m and m' in S:

(*a*) $\mathbf{Z}(m, m', \omega)$ is a second order random variable defined on $(\mathscr{A}, \mathscr{C}, \mathscr{P})$ with values in $\mathscr{L}(E_{\mathbb{C}}^3)$, i. e.:

$$\mathscr{E}(\| \mathbf{Z}(m, m', \omega) \|_3^2) = \int_{\mathscr{A}} \| \mathbf{Z}(m, m', \omega, a) \|_3^2 \, d\mathscr{P}(a) < +\infty. \quad (17)$$

(*b*) We have the symmetry property:

$$\mathbf{Z}(m, m', \omega) = {}^t\mathbf{Z}(m', m, \omega), \qquad \mathscr{P}\text{-almost surely}. \quad (18)$$

(*c*) For any v in $H(S, E_{\mathbb{C}}^3)$, the integral $\int_S Z(m, m', \omega) v(m') \, ds(m')$ is defined $\mathscr{P}$-almost surely.

The random operator $\mathbf{Z}(\omega)$ defined for any v and w in $H(S, E_{\mathbb{C}}^3)$ by:

$$((\mathbf{Z}(\omega)) v, w))_S = \int_S \int_S (\mathbf{Z}(m, m', \omega) v(m'), w(m))_3 \, ds(m) \, ds(m') \quad (19)$$

is $\mathscr{P}$-almost surely linear and continuous from $H(S, E_{\mathbb{C}}^3)$ into $H(S, E_{\mathbb{C}}^3)$.

(*d*) For any integer $J \geqq 1$ and for any continuous function $m \to A(m)$ of S in $\mathscr{L}(\mathbb{R}^J, E_{\mathbb{R}}^3)$ and for any v and w in $H(S, E_{\mathbb{C}}^3)$ with the form:

$$\begin{cases} v(m) = A(m) V, & V \in \mathbb{C}^J \\ w(m) = A(m) W, & W \in \mathbb{C}^J \end{cases} \quad (20)$$

[therefore v and w are in $C^0(\bar{S}, E_{\mathbb{C}}^3)$], the random operator $\mathbf{Z}_J(\omega)$ which is defined $\mathscr{P}$-almost surely by:

$$\mathbf{Z}_J(\omega) = \int_S \times \int_S {}^tA(m) \mathbf{Z}(m, m', \omega) A(m') \, ds(m) \, ds(m') \quad (21)$$

is such that for $\mathscr{P}$-almost any $a \in \mathscr{A}$, the function $\{\omega \to Z_j(\omega, a)\}$ is in $G(\mathbb{R}, \mathscr{L}(\mathbb{C}^j))$. It can easily be verified that if v and w have form (20), we have:

$$((\mathbf{Z}(\omega) v, w))_S = (\mathbf{Z}_J(\omega) V, W)_J, \qquad \mathscr{P}\text{-a. s.} \quad (22)$$

Equation (19) shows that in the Fourier space, the equation relating the displacement field $m \to \hat{u}(m, \omega)$ on S to the force density field $m \to \hat{f}(m, \omega)$ on S is written for ds-almost any $m \in S$:

$$\hat{f}(m, \omega) = \int_S \mathbf{Z}(m, m', \omega) \hat{u}(m', \omega) \, ds(m'). \quad (23)$$

This equation (23) completely describes the behavior of the structural fuzzy relative to S with respect to part S of boundary Γ.

III,2.2. — *Case of the fuzzy with locally discrete boundary*

Let us consider a part of the fuzzy relative to part S_I of Γ, where S_I is the discrete set defined in III,1.1 *b*. This part is said to be locally discrete because (1) set S_I is discrete, i. e. the connections of fuzzy S_I to the master structure are discrete and (2) S_I is only part of boundary Γ. It is recalled that points m_i of S_I are given (deterministic).

Let $J = 3I$ and let $\mathbf{Z}_J(\omega)$ be the probabilistic impedance of boundary S_I. It is assumed that it verifies the following hypotheses.

For any $\omega \in \mathbb{R}$, $\mathbf{Z}_J(\omega)$ is a second order random variable defined on $(\mathscr{A}, \mathscr{C}, \mathscr{P})$ with values in $\mathscr{L}(\mathbb{C}^J)$, i. e.:

$$\mathscr{E}(\| \mathbf{Z}_J(\omega) \|_J^2) = \int_{\mathscr{A}} \| \mathbf{Z}_J(\omega, a) \|_J^2 \, d\mathscr{P}(a) < +\infty \quad (24)$$

and for $\mathscr{P}$-almost any $a \in \mathscr{A}$, function $\omega \to \mathbf{Z}_J(\omega, a)$ is in $G(\mathbb{R}, \mathscr{L}(\mathbb{C}^J))$.

Let $\hat{U}_J(\omega)=(\hat{u}(m_1, \omega), \ldots, \hat{u}(m_I, \omega))$ with values in $\mathbb{C}^J$ and $\hat{F}_J(\omega)=(\hat{F}_1(\omega), \ldots, \hat{F}_I(\omega))$ with values in $\mathbb{C}^J$ where the $\hat{F}_i(\omega)$ have values in $\mathbb{C}^3$. Then, in the Fourier space, the equation relating the displacement vector $\hat{U}_J(\omega)$ relative to S_I to the force vector $\hat{F}_J(\omega)$ relative to S_I is written:

$$\hat{F}_J(\omega)=\mathbf{Z}_J(\omega)\,\hat{U}_J(\omega). \tag{25}$$

Equation (25) completely describes the behavior of the fuzzy relative to S_I.

III,3. – PROBABILISTIC CONSTITUTIVE LAW OF THE STRUCTURAL FUZZY

In this section, we give the elements used to define the constitutive laws of the structural fuzzy. The boundary impedances are not intrinsic since they involve the geometry, the DOF of the boundaries, etc. The object is therefore to define intrinsic values which will be used to construct the probabilistic boundary impedances. The values will be called: probabilistic constitutive law of the structural fuzzy. As we will see, this is not always possible. Where this is the case, we will introduce the concept of fuzzy subsystem.

III,3.1. – *Probabilistic constitutive law for the fuzzy with locally continuous boundary*

We again use the notations and hypotheses of Sec. III,2.1. Space $E^3_{\mathbb{R}}$ is referenced to an orthonormal basis $\{b_1, b_2, b_3\}$ which allows $E^3_{\mathbb{R}}$ to be identified with $\mathbb{C}^3$ by complexification. Let $v=(v_1, v_2, v_3)$ be an element of $H(S, \mathbb{C}^3)$. In any point m of S, we associate a local orthonormal basis and note as $[\Phi(m)]$ the orthogonal (3×3) real matrix of the linear operator for transition from basis $\{b_1, b_2, b_3\}$ to basis $\{e_1(m), e_2(m), e_3(m)\}$ such that:

$$[\Phi(m)]^{-1}=[\Phi(m)]^T. \tag{26}$$

Let $v_0(m)\in\mathbb{C}^3$ such that:

$$v(m)=[\Phi(m)]\,v_0(m), \qquad v_0(m)=[\Phi(m)]^T\,v(m). \tag{27}$$

We will assume that the family of local bases is such that for any v in $H(S, \mathbb{C}^3)$, we have $v_0\in H(S, \mathbb{C}^3)$. We therefore do not require mapping $m\to[\Phi(m)]$ to be continuous on S. Moreover, in most practical cases, it is not continuous.

Let $[\mathbf{Z}(m, m', \omega)]$ be the (3×3) matrix of $\mathbf{Z}(m, m', \omega)$ relative to basis $\{b_1, b_2, b_3\}$. Under these conditions, for any v and w in $H(S, \mathbb{C}^3)$, equation (19) is written:

$$((\mathbf{Z}(\omega)\,v, w))_S = \iint_{S\times S} [\overline{w(m)}]^T [\mathbf{Z}(m, m', \omega)]\,[v(m')]\,ds(m)\,ds(m')$$
$$= \iint_{S\times S} [\overline{w_0(m)}]^T \times [\mathbf{Z}_0(m, m', \omega)]\,[v_0(m')]\,ds(m)\,ds(m') \tag{28}$$

with the relation:

$$[\mathbf{Z}(m, m', \omega)]=[\Phi(m)]\,[\mathbf{Z}_0(m, m', \omega)]\,[\Phi(m')]^T. \tag{29}$$

(*a*) Fuzzy orthotropic on S for a band B_n

The structural fuzzy is said to be orthotropic on S for band B_n if there exists $m\to[\Phi(m)]$, i.e. a family of local bases on S such that $[Z_0(m, m', \omega)]$ is $\mathscr{P}$-almost surely diagonal for any ω in B_n and for ds-almost any m and m'in S. We then have:

$$[\mathbf{Z}_0(m, m', \omega)]_{jk}=\delta_{jk}\,\mathbf{z}_j(m, m', \omega). \tag{30}$$

The constitutive law for the fuzzy orthotropic on S for band B_n is therefore the given of the family of second order random variables with values in $\mathbb{C}^3$:

$$\{\mathbf{z}_j(m, m', \omega), j\in\{1, 2, 3\}\} \quad \text{for} \quad \omega\in B_n, \quad m \text{ and } m'\in S$$

The given represented by such a constitutive law is used to construct the probabilistic boundary impedance operator using (28), (29) and (30).

(*b*) Fuzzy isotropic on S for a band B_n

The structural fuzzy is said to be isotropic on S for a band B_n if it is orthotropic on S and for band B_n and if $\mathscr{P}$-almost surely:

$$\mathbf{z}_1(m, m', \omega)=\mathbf{z}_2(m, m', \omega)=\mathbf{z}_3(m, m', \omega). \tag{31}$$

Under these conditions, the given represented by the constitutive law for the fuzzy isotropic on S for B_n is a given of the family of second order random variables with values in $\mathbb{C}$: $\mathbf{z}(m, m', \omega)$ for $\omega\in B_n$, m and $m'\in S$ and it is true that:

$$[\mathbf{Z}_0(m, m', \omega)]_{jk}=\mathbf{z}(m, m', \omega)\,\delta_{jk}. \tag{32}$$

(*c*) Fuzzy anisotropic on S for a band B_n

The structural fuzzy is said to be anisotropic on S for band B_n if it is neither orthotropic nor isotropic. In this case, the constitutive law of the fuzzy anisotropic on S for B_n is the given of the family of second order random variables $[\mathbf{Z}(m, m', \omega)]$ with values in the (3×3) matrix $\mathbb{C}$ for $\omega \in B_n$, m and $m' \in S$.

(*d*) Fuzzy homogeneous on S for a band B_n

– A fuzzy anisotropic on S for B_n is said to be homogeneous if:

$$\left.\begin{array}{l}[\mathbf{Z}(m, m', \omega)) = [\mathbf{Z}(\omega)], \\ \forall m, m' \in S \times S, \quad \mathscr{P}\text{-a. s.}\end{array}\right\} \tag{33}$$

– A fuzzy orthotropic on S for B_n is said to be homogeneous if for $j \in 1, 2, 3\}$:

$$\left.\begin{array}{l}\mathbf{z}_j(m, m', \omega) = \mathbf{z}_j(\omega), \\ \forall m, m' \in S \times S, \quad \mathscr{P}\text{-a. s.}\end{array}\right\} \tag{34}$$

– A fuzzy isotropic on S for B_n is said to be homogeneous if:

$$\left.\begin{array}{l}\mathbf{z}(m, m', \omega) = \mathbf{z}(\omega), \\ \forall m, m' \in S \times S, \quad \mathscr{P}\text{-a. s.}\end{array}\right\} \tag{35}$$

Consequently, for the fuzzy, the homogeneousness on S corresponds to an absence of spatial memory on S. However, it can be noted that in the orthotropic or isotropic case, the homogeneousness on S does not necessarily mean that $[\mathbf{Z}(m, m', \omega)]$ is independent of m and m' (contrary to the anisotropic case), since, according to (29), (30) and (32), for the homogeneous orthotropic fuzzy, we have:

$$[\mathbf{Z}(m, m', \omega)] = [\Phi(m)][\mathbf{z}(\omega)][\Phi(m')]^T \tag{36}$$

where $[z(\omega)]_{jk} = \delta_{jk} z_j(\omega)$, and for the homogeneous isotropic fuzzy, we have:

$$[\mathbf{Z}(m, m', \omega)] = [\Phi(m)][\Phi(m')]^T \mathbf{z}(\omega) \tag{37}$$

(*e*) Fuzzy locally homogeneous on S for band B_n

A fuzzy is said to be locally homogeneous on S for band B_n if there is a finite partition of $S: S = U_1 S_1$ such that the fuzzy is homogeneous on each part S_1 of S. It should be noted that the probabilistic characteristics of the fuzzy will be different on each part S_1.

(*f*) Fuzzy nonhomogeneous on S for band B_n

A fuzzy is said to be nonhomogeneous on S for B_n if it is neither locally homogeneous nor homogeneous. This situation is generally that of fuzzy substructures. We will come back to this point.

III,3.2. – *Constitutive law of the fuzzy with locally discrete boundary*

We will use the notations and hypotheses of Sec. III,2.2. This situation is fully similar to the case of the nonhomogeneous anisotropic fuzzy. Consequently, the constitutive law of the fuzzy on S_1 for band B_n is the given of the family of second order random variables with values in $\mathscr{L}(\mathbb{C}^J)$: $\mathbf{Z}_j(\omega)$ for $\omega \in B_n$.

III,3.3. – *Remarks on construction of the fuzzy laws*

It is necessary to make a distinction between at least two cases:

(*a*) The first concerns the intrinsic construction of the constitutive law for a specified class of fuzzy (Sec. I). In Section IV, we will give the complete developments for construction of a first fuzzy law used to create a model for the orthotropic or isotropic locally homogeneous fuzzy. The general problem remains open and is now being investigated. For instance, the construction of a second law of nonhomogeneous fuzzy, taking into account the spatial memory, is under development.

(*b*) The second case concerns the behavior of fuzzy substructures. This situation is similar to the conventional substructuring methods. In our case, a fuzzy substructure consists of a master substructure and the structural subfuzzy which is "attached" to it. The general program developed (*see* Part II of this paper) gives an automatic solving method consisting of substructuring complex mechanical systems with a structural fuzzy to facilitate use of large models and optimize numerical processing costs, with the possibility of taking into account internal excitations in the fuzzy substructures. This substructuring approach remains possible due to the use of the general method adopted for solving random operator equations (Sec. V). However, it is interesting to consider a fuzzy substructure from another angle than that of the general solving method. In effect, the overall model of a fuzzy substructure can result from an experimental and numerical identification. For instance, let us consider a fuzzy substructure without internal excitation. If the common boundary between this substructure and the master mechanical system is of the discrete type, i. e. of type S_l, the constitutive law for this fuzzy substructure is then given by the results of Sec. III,2.2. However, if the common boundary is of the continuous type, i. e. type S, the constitutive law is generally given by the model of the nonhomogeneous and anisotropic law (Sec. II,3.1 *c* and *f*), since the spatial memory affects S (in the same way as it affects S_l). It can be noted that these two situations can be assimilated to the above scheme. This is the angle from which we introduce fuzzy finite macroelements in Sec. III,4.

III,4. — FUZZY FINITE ELEMENTS

The use of fuzzy finite elements (FFE) is made necessary for the reasons mentioned in Sec. I,4. As the master structure is modeled by the finite element (FE) method, the mesh of boundary Γ is common to the finite elements of the master structure and to the fuzzy finite elements. The degrees of interpolation of the finite elements on either side of Γ must therefore be compatible.

In all of section III,3, space $E^3_{\mathbb{R}}$ is referenced to the orthonormal basis $\{b_1, b_2, b_3\}$. We will note as:

$$v=(v_1, v_2, v_3) \quad \text{and} \quad w=(w_1, w_2, w_3)$$

two fields defined on S with values in $\mathbb{R}^3$ such that v and $w \in C^0(\bar{S}\ \mathbb{R}^3)$.

III,4.1. — *Localized finite elements for the locally continuous homogeneous fuzzy*

Let S' be a surface finite element borne by S with I nodes m_i, $i \in \{1, 2, \ldots, I\}$.

We set $J=3I$. Let:

$$[V_J]=\begin{bmatrix} v(m_1) \\ \vdots \\ v(m_I) \end{bmatrix}, \quad , \quad [W_J]=\begin{bmatrix} w(m_1) \\ \vdots \\ w(m_I) \end{bmatrix} \tag{38}$$

be the $(J\times 1)$ column matrices of the nodal variables of fields v and w at the nodes of finite element $\bar{S}'$. The conventional formulation of the isoparametric finite elements is generally used. Let $m \to [A'(m)]$ be the continuous function of $\bar{S}'$ in the real matrices with dimension $(3\times J)$ such that:

$$\forall m \in \bar{S}', \quad [v(m)]=[A'(m)][V_J], \quad [w(m)]=[A'(m)][W_J]. \tag{40}$$

Matrix $[A'(m)]$ is constructed conventionally using the weighting functions of finite element $\bar{S}'$. Applying (19)-(22) gives $\mathscr{P}$-almost surely:

$$((\mathbf{Z}(\omega)\, v, w))_{S'}=[W_J]^T[\mathbf{Z}_J(\omega)][V_J] \tag{41}$$

where $[\mathbf{Z}_J(\omega)]$ is the random matrix of the fuzzy finite element with values in the $(J\times J)$ square symmetrical complex matrices such that:

$$[\mathbf{Z}_J(\omega)]=\int_{S'}\int_{S'}[A'(m)]^T \times[\mathbf{Z}(m, m', \omega)][A'(m')]\, ds(m)\, ds(m') \tag{42}$$

Applying equations (33), (36) and (37) yields expressions (43), (44) and (45) for the anisotropic, the orthotropic and the isotropic fuzzy respectively:

$$[\mathbf{Z}_J(\omega)]=[\tilde{H}]^T[\mathbf{Z}(\omega)][\tilde{H}] \tag{43}$$

$$[\mathbf{Z}_J(\omega)]=[H]^T[\mathbf{z}(\omega)][H] \tag{44}$$

$$[\mathbf{Z}_J(\omega)]=\mathbf{z}(\omega)\,[H]^T[H] \tag{45}$$

where:

$$[\tilde{H}]=\int_{S'}[A'(m)]\, ds(m); \quad [H]=\int_{S'}[\Phi(m)]^T[A'(m)]\, ds(m). \tag{46}$$

It can be seen that for the locally continuous, homogeneous fuzzy, the finite elements remain localized. They are constructed using a constitutive law represented by $[\mathbf{Z}(\omega)]$, $[\mathbf{z}(\omega)]$, or $\mathbf{z}(\omega)$ depending on the case, and the local geometry represented by $[\tilde{H}]$ or $[H]$.

II,4.2. — *Fuzzy finite macroelements*

(*a*) Case of fuzzy with locally discrete boundary

We use the hypotheses of Sec. III,2.2. The I points $m_1, \ldots, m_I$ of S_I are assumed to coincide with the nodes of the finite element mesh of the master structure. Under these conditions, equation (25) directly defines the matrix of the fuzzy finite macroelement with a locally discrete boundary. This is the $(J\times J)$ symmetrical complex random matrix $[\mathbf{Z}_J(\omega)]$, $J=3I$, of the operator $\mathbf{Z}_J(\omega)$ defined in Sec. III,2.2.

(*b*) Case of fuzzy with locally continuous nonhomogeneous boundary

Let us consider a locally continuous, nonhomogeneous fuzzy on S as defined in Sec. III,3.1*f*. The surface $\bar{S}$ common to the master structure is meshed by finite elements. Let I be the total number of nodes of the mesh of $\bar{S}$ and $J=3I$ be the corresponding total number of DOF. Let $m\to[A(m)]$ be the continuous function of $\bar{S}$ in the $(3\times J)$ real matrices such that:

$$\left.\begin{aligned}\forall m \in \bar{S}, \quad [v(m)]&=[A(m)][V_J],\\ [w(m)]&=[A(m)][W_J]\end{aligned}\right\} \tag{47}$$

where $[V_J]$ and $[W_J]$ are the $(J\times 1)$ column matrices of the nodal values of fields v and w at all the nodes of the mesh of surface $\bar{S}$ and where $m\to[A(m)]$ is constructed using the weighting functions of all the finite elements used on $\bar{S}$.

Applying (19)-(22) yields $\mathscr{P}$-almost surely:

$$((\mathbf{Z}(\omega))\, v, w))_S=[W_J]^T[\mathbf{Z}_J(\omega)][V_J] \tag{48}$$

where $[\mathbf{Z}_J(\omega)]$ is the symmetrical, complex random matrix with dimension $(J\times J)$ of the finite macroelement such that:

$$[\mathbf{Z}_J(\omega)]=\int_S\int_S[A(m)]^T \times[\mathbf{Z}(m, m', \omega)][A(m')]\, ds(m)\, ds(m'). \tag{49}$$

It will be noted that the spatial memory of the fuzzy law does not allow the construction of localized finite elements in this case.

(*c*) Case of fuzzy substructures

It will first be noted that discretizing of a nonhomogeneous fuzzy with locally continuous boundary leads to a fuzzy finite macroelement of the same type as that of the fuzzy with a locally discrete boundary, which is conventional. Similarly, a fuzzy substructure (Sec. III,3.3*b*) whose "attachments" to the master structure are discrete gives a finite macroelement as defined by Sec. III,4.2*a*. If the boundary between the fuzzy substructure and the master structure is continuous, we have a finite macroelement as defined by Sec. III,4.2*b*.

IV. – CONSTRUCTION OF A PROBABILISTIC FUZZY CONSTITUTIVE LAW

IV,1. – STATEMENT OF THE PROBLEM

In this section, we construct a fuzzy law which can be used for an orthotropic or isotropic locally homogeneous fuzzy. Consistently with the results of Sec. III,3.1*d*, this law is defined by one quantity, $\mathbf{z}(\omega)$ in the isotropic case, or by three quantities, $\mathbf{z}_1(\omega)$, $\mathbf{z}_2(\omega)$, $\mathbf{z}_3(\omega)$ in the orthotropic case. Actually, we will construct a law with scalar values of type $\mathbf{z}(\omega)$. This law will depend on mechanical parameters. We will therefore be able to use it for the orthotropic case by assigning different values to the parameters for each direction 1, 2 and 3. Similarly, the use of localized fuzzy finite elements (Sec. III,4.1) constructed using (44) or (45) allows the parameters of the law to be varied from one FFE to another in order to take into account the spatial variations of the parameters (this is the use of the concept of locally homogeneous fuzzy introduced in III,3.1*e*). It is understood below that the fuzzy will be considered locally as regards the space variable. In addition, we are attempting to construct a probabilistic impedance reflecting the mechanical hypotheses introduced in Sec. III,1.2. The most elementary mechanical model verifying these hypotheses is that of a simple linear oscillator excited by its support. We will use this as underlying deterministic basis for constructing the probabilistic constitutive law. This obviously does not mean that the law obtained will only be suitable for simulating behavior of a fuzzy consisting of simple oscillators. On the contrary, this law will be capable of representing the dynamic behavior of the complex mechanical systems comprising the fuzzy. This is due to the fact that the parameters of the simple oscillator will be modeled by random variables and the probabilistic law obtained will generate a random family of oscillators. Below, we recall the formulas for the dynamics of a simple linear oscillator excited by its support.

IV,2. – REVIEW OF AN ELEMENTARY DYNAMIC MODEL

We consider a simple linear oscillator in a reference system whose mechanical characteristics are the point mass μ_0, the viscous damping constant c and the stiffness constant k. Let $V(t)$ be the displacement of mass μ_0 and $U(t)$ be that of the support. When $U(t)=0$, $\forall t$, the support is fixed and the natural frequency $\omega_p>0$ of the associated conservative oscillator is such that $k=\mu_0\,\omega_p^2$. The damping constant is then referenced to the critical damping ratio ξ such that $c=2\,\xi\,\mu_0\,\omega_p$ where $0<\xi<1$ by hypothesis. We now assume that the support is free and that the only excitation force, noted $F(t)$, is applied to the support. The corresponding displacement of the support is $U(t)$.

Using the notation $\dot{f}(t)=df(t)/dt$, the linear vibrations of this system around a position of static equilibrium are determined by the following equations:

$$\begin{aligned}\mu_0\,\ddot{V}+c(\dot{V}-\dot{U})+k(V-U)&=0\\ -c(\dot{V}-\dot{U})-k(V-U)&=F.\end{aligned}\tag{50}$$

If $F\in H_{B_n}(\mathbb{R},\mathbb{C})$, then U, $\dot{U}$, V, $\dot{V}$, and $\ddot{V}$ are in $H_{B_n}(\mathbb{R},\mathbb{C})$. The Fourier transform of (50) yields the desired equation:

$$\hat{F}(\omega)=z(\omega)\,\hat{U}(\omega)\tag{51}$$

where, for $\omega>0$:

$$\begin{aligned}z(\omega)&=-\omega^2R(\omega)+i\,\omega\,I(\omega)\\ R(\omega)&=\frac{\mu_0(\omega_p^2/\omega^2)((\omega_p^2/\omega^2)-1+4\,\xi^2)}{((\omega_p^2/\omega^2)-1)^2+4(\omega_p^2/\omega^2)\,\xi^2}\\ I(\omega)&=\frac{2\,\mu_0\,\omega\xi\,(\omega_p/\omega)}{((\omega_p^2/\omega^2)-1)^2+4(\omega_p^2/\omega^2)\,\xi^2}.\end{aligned}\tag{52}$$

It should be noted that $\mathbf{z}(\omega)$ can also be written

$$z(\omega)=-\omega^2M(\omega)+i\,\omega\,I(\omega)+K(\omega)$$

with, for any $\omega>0$, $M(\omega)\geqq 0$, $I(\omega)>0$, $K(\omega)\geqq 0$. Therefore $I(\omega)$ is effectively the dissipative term of the impedance and $M(\omega)$ and $K(\omega)$ are the mass and stiffness terms respectively, such that:

$$-\omega^2R(\omega)=-\omega^2M(\omega)+K(\omega)$$

where:

(*a*) For $\xi\in]0,\,1/2[$

$$M(\omega)=R(\omega),\qquad K(\omega)=0,$$

$$\text{if}\quad \frac{\omega_p}{\omega}\geqq\sqrt{1-4\,\xi^2}.$$

$$M(\omega)=0,\qquad K(\omega)=-\omega^2R(\omega),$$

$$\text{if}\quad 0<\frac{\omega_p}{\omega}<\sqrt{1-4\,\xi^2}$$

(*b*) For $\xi \in [1/2, 1[$

$$M(\omega) = R(\omega), \qquad K(\omega) = 0, \quad \forall \omega > 0$$

In addition, we have:

$$\lim_{(\omega_p/\omega) \to +\infty} R(\omega) = \mu_0, \qquad \lim_{(\omega_p/\omega) \to +\infty} I(\omega) = 0.$$

Consequently, for $\frac{\omega_p}{\omega} \gg 1$, we have

$$z(\omega) \sim -\omega^2 \mu_0. \tag{53}$$

IV,3. — DETERMINISTIC BASES UNDERLYING THE FUZZY LAW

Let D_{FS} be a connected part of domain D_F (Sec. III,1) with boundary $\partial D_{FS} = S \cup \Gamma_{FS}$, where S is the part of Γ defined in III,1.1 *a*. The framework for construction of the fuzzy law described in IV,1 is that of the homogeneous fuzzy on S. This means that all the reasoning will concern surface S, knowing that all the points S are "equivalent".

From a dimensional standpoint, $z(\omega)$ must have dimension $MT^{-2}L^{-2}$ since, in the equation $\hat{F}(\omega) = z(\omega)\hat{U}(\omega)$, $F(t)$ is a surface density of forces applied to $S(ML^{-1}T^{-2})$. Therefore, $\hat{F}(\omega)$ has $ML^{-1}T^{-1}$ as dimension and $U(t)$ is a displacement (dimension L); therefore, $\hat{U}(\omega)$ has LT as dimension.

Considering the mechanical hypotheses on the fuzzy (Sec. III,1.2), there is a fundamental vibration frequency, noted ω_{p1}, of the fuzzy D_{FS}, which is the smallest natural frequency of the set of systems contained in domain D_{FS}. Thus, for $\omega \ll \omega_{p1}$, the fuzzy D_{FS} behaves like a pure mass as seen by S.

(*a*) Cutoff frequency

The last remark leads us to introducing in construction of the model a cutoff frequency noted Ω_c in the fuzzy law such that for $\omega < \Omega_c$, the constitutive law is of type (53).

(*b*) Underlying deterministic model of the fuzzy constitutive law below the cutoff frequency

For $\omega \in]0, \Omega_c[$, we will take the following model for impedance $z(\omega)$ in any point m of surface S:

$$z(\omega) = -\omega^2 R(\omega) + i\omega I(\omega) \tag{54}$$

where:

$$\left\{ \begin{aligned} R(\omega) &= \mu(\omega) \\ I(\omega) &= 0 \end{aligned} \right. \tag{55}$$

and where $\omega \to \mu(\omega)$ is a function of $]0, \Omega_c[$ in $\mathbb{R}^+ = [0, +\infty[$, $\mu(\omega)$ being designated the equivalent mass of the fuzzy per unit surface at (angular) frequency ω. The dimension of $\mu(\omega)$ is ML^{-2}.

(*c*) Modal density of the fuzzy above the cutoff frequency

The natural frequencies of the fuzzy systems contained in D_{FS} are above Ω_c. It is understood that by natural frequencies are meant the frequencies of the associated conservative systems, as the natural damping of the fuzzy systems is assumed to be weak (Sec. III,1.2).

The modal density $n(\omega)$ of the fuzzy is introduced, the function $\omega \to n(\omega)$ being defined on $[\Omega_c, +\infty[$ with values in $]0, +\infty[$. By definition of $n(\omega)$, the number of natural frequencies of the fuzzy in the vicinity $\Delta\omega$ of ω is $n(\omega)\Delta\omega$.

The distance, noted $2\varepsilon(\omega)$, between two natural frequencies of the fuzzy in the vicinity of ω is therefore:

$$2\varepsilon(\omega) = \frac{\Delta\omega}{n(\omega)\Delta\omega} = \frac{1}{n(\omega)} \tag{56}$$

(*d*) Deterministic model underlying the constitutive law of the fuzzy above the cutoff frequency

For ω fixed above the cutoff frequency, we will take the following model, of type (52), for the impedance $z(\omega)$ in any point m of surface S:

$$\left. \begin{aligned} z(\omega) &= -\omega^2 R(\omega) + i\omega I(\omega), \\ \omega &\in [\Omega_c, +\infty[\end{aligned} \right\} \tag{57}$$

where :

$$R(\omega) = \frac{\mu(\omega)(\omega_p^2(\omega)/\omega^2)((\omega_p^2(\omega)/\omega^2) - 1 + 4\xi^2(\omega))}{((\omega_p^2(\omega)/\omega^2) - 1)^2 + 4(\omega_p^2(\omega)/\omega^2)\xi^2(\omega)} \tag{58}$$

$$I(\omega) = \frac{2\mu(\omega)\omega\xi(\omega)(\omega_p(\omega)/\omega)}{((\omega_p^2(\omega)/\omega^2) - 1)^2 + 4(\omega_p^2(\omega)/\omega^2)\xi^2(\omega)} \tag{59}$$

in which μ, ξ and ω_p are three functions defined on $[\Omega_c, +\infty[$, with values in $[0, +\infty[$, $]0, 1[$ and $]0, +\infty[$ respectively. The scalar $\mu(\omega)$ is the equivalent mass of the fuzzy per unit surface (dimension ML^{-2}). The rate of natural dissipation of the fuzzy at frequency ω is $\xi(\omega)$ and verifies $0 < \xi(\omega) < 1$. Obviously, for ω fixed, this frequency does not necessarily coincide with a natural frequency ω_p of the fuzzy. It is the probabilistic model that we construct below which will allow us to introduce the probability for $\omega_p(\omega) \in (\omega, \omega + d\omega)$.

IV,4. – PROBABILISTIC HYPOTHESES FOR CONSTRUCTION OF THE FUZZY CONSTITUTIVE LAW

The deterministic bases for the fuzzy constitutive law are:

(*a*) For the mechanical parameters:

– the cutoff frequency $\Omega_c \geqq 0$;

– the equivalent mass of the fuzzy per unit surface described by the function $\omega \rightarrow \mu(\omega)$ of $]0, +\infty[$ into $[0, +\infty[$;

– the rate of natural dissipation of the fuzzy described by the function $\omega \rightarrow \xi(\omega)$ of $[\Omega_c, +\infty[$ into $]0, 1[$;

– the modal density of the fuzzy described by the function $\omega \rightarrow n(\omega)$ of $[\Omega_c, +\infty[$ into $]0, +\infty[$.

The three functions μ, ξ and n are assumed to be continuous by parts.

(*b*) The algebraic expressions of $z(\omega)$ defined by (54) and (55) for ω into $]0, \Omega_c[$ and by (57) and (58) for ω into $[\Omega_c, +\infty[$.

The probabilistic construction consists of modeling scalars $\mu(\omega)$, $\xi(\omega)$, $n(\omega)$ and $\omega_p(\omega)$ for each ω fixed by random variables, giving random impedance $\mathbf{z}(\omega)$. The cutoff frequency is assumed to be deterministic in this model.

IV,4.1. – *Random variables expressing dispersion of the mechanical parameters*

We introduce three random variables, $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$, defined on $(\mathscr{A}, \mathscr{C}, \mathscr{P})$, mutually independent, with real values, each following a uniform probability law with support $[-\sqrt{3}, \sqrt{3}]$.

Let $x \rightarrow \mathbf{1}_{[-\sqrt{3}, \sqrt{3}]}(x)$ be the characteristic function of the interval $[-\sqrt{3}, \sqrt{3}] \subset \mathbb{R}$ and $p(x)$ be the uniform probability density:

$$p(x) = \frac{1}{2\sqrt{3}} \mathbf{1}_{[-\sqrt{3}, \sqrt{3}]}(x). \tag{60}$$

Then, for $j \in \{1, 2, 3\}$, the probability law for $\mathbf{X}_j$ is written:

$$P_{\mathbf{X}_j}(dx) = p(x)\, dx \tag{61}$$

and each $\mathbf{X}_j$ is a normalized random variable, i. e. centered and with unit variance:

$$\begin{gathered} \underline{\mathbf{X}}_j = \mathscr{E}(\mathbf{X}_j) = 0; \\ \sigma_{\mathbf{X}_j} = \{ \mathscr{E}((\mathbf{X}_j - \underline{\mathbf{X}}_j)^2) \}^{1/2} = 1. \end{gathered} \tag{62}$$

Since $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$ are independent, we can infer from (62) that:

$$\mathscr{E}(\mathbf{X}_j \mathbf{X}_k) = \delta_{jk}, \qquad j \text{ and } k \text{ in } \{1, 2, 3\} \tag{63}$$

where $\delta_{jj} = 1$ and $\delta_{jk} = 0$ for $j \neq k$.

To control dispersion of the mechanical parameters, we introduce the function with vector values:

$$\omega \rightarrow \lambda(\omega) = (\lambda_1(\omega), \lambda_2(\omega), \lambda_3(\omega)) \tag{64}$$

defined on $]0, +\infty[$ with values in $(]0, 1[)^3 \subset \mathbb{R}^3$, continuous by parts.

It will be noted that λ is a bounded mapping.

Let $\mathbf{Y}(\omega) = \{ \mathbf{Y}_1(\omega), \mathbf{Y}_2(\omega), \mathbf{Y}_3(\omega) \}$ be the random variable defined on $(\mathscr{A}, \mathscr{C}, \mathscr{P})$ with values in $\mathbb{R}^3$ such that for ω fixed and for $j \in \{1, 2, 3\}$:

$$\mathbf{Y}_j(\omega) = \mathbf{X}_j \frac{\lambda_j(\omega)}{\sqrt{3}}. \tag{65}$$

For $j \in \{1, 2, 3\}$, the probability law for $\mathbf{Y}_j(\omega)$ has a density $p_j(\omega, y_j)$ which is written:

$$p_j(\omega, y_j) = \frac{1}{2\lambda_j(\omega)} \mathbf{1}_{[-\lambda_j(\omega), \lambda_j(\omega)]}(y_j). \tag{66}$$

Therefore, $\mathbf{Y}_j(\omega)$ $\mathscr{P}$-almost surely takes on its values in the interval $[-\lambda_j(\omega), \lambda_j(\omega)] \subset [-1, 1]$. It is centered:

$$\underline{\mathbf{Y}}_j(\omega) = \mathscr{E}(\mathbf{Y}_j(\omega)) = 0 \tag{67}$$

and its standard deviation is written:

$$\sigma_j(\omega) = \frac{1}{\sqrt{3}} \lambda_j(\omega). \tag{68}$$

Considering the impedance of the $\mathbf{X}_j$ values, the probability law for $\mathbf{Y}(\omega)$ has a density $p_{\mathbf{Y}(\omega)}(\omega, y)$, where $y = (y_1, y_2, y_3)$ which is written:

$$p_{\mathbf{Y}(\omega)}(\omega, y) = \prod_{j=1}^{3} p_j(\omega, y_j). \tag{69}$$

IV,4.2. – *Modeling of the mechanical parameters*

(*a*) Modeling of the equivalent mass

For any ω in $]0, +\infty[$, $\mu(\omega)$ is modeled by the random variable:

$$\mu(\omega) = \underline{\mu}(\omega)(1 + \mathbf{Y}_1(\omega)) \tag{70}$$

where $\omega \rightarrow \underline{\mu}(\omega) = \mathscr{E}(\mu(\omega))$ is the average equivalent mass function of the fuzzy per unit surface defined on $]0, +\infty[$ with values in $[0, +\infty[$ continuous by parts. Therefore, $\mu(\omega)$ is a random variable which takes on its values in $[0, +\infty[$ with mean $\underline{\mu}(\omega)$, standard deviation $\sigma_1(\omega)$ given by (67) for $j = 1$ and $\omega \rightarrow \mu(\omega)$ is $\mathscr{P}$-almost surely continuous by parts.

(*b*) Modeling of the rate of dissipation

For any ω in $[\Omega_c, +\infty[$, $\xi(\omega)$ is modeled by the random variable:

$$\xi(\omega) = \underline{\xi}(\omega)(1 + \mathbf{Y}_2(\omega)) \tag{71}$$

where $\omega \rightarrow \underline{\mu}(\omega) = \mathscr{E}(\mu(\omega))$ is the average equivalent mass function of the fuzzy per unit surface defined on $]0, +\infty[$ with values in $[0, +\infty[$ continuous by parts. Therefore, $\mu(\omega)$ is a random variable which takes on its values in $[0, +\infty[$ with mean $\underline{\mu}(\omega)$, standard deviation $\sigma_1(\omega)$ given by (67) for $j=1$ and $\omega \rightarrow \mu(\omega)$ is $\mathscr{P}$-almost surely continuous by parts.

(*c*) Modeling of the modal density

For any ω in $[\Omega_c, +\infty[$, $n(\omega)$ is modeled by the random variable:

$$n(\omega) = \underline{n}(\omega)(1 + \mathbf{Y}_3(\omega)) \tag{72}$$

where $\omega \rightarrow \underline{n}(\omega) = \mathscr{E}(n(\omega))$ is the mean function of the modal density of the fuzzy, defined on $[\Omega_c, +\infty[$ with values in $]0, +\infty[$, continuous by parts. The random variable $n(\omega)$ $\mathscr{P}$-almost surely takes on its values in $]0, +\infty[$ with mean $\underline{n}(\omega)$, standard deviation $\sigma_3(\omega)$ given by (67) for $j=3$ and $\omega \rightarrow n(\omega)$ is $\mathscr{P}$-almost surely continuous by parts.

(*d*) Summary of the parameters of the fuzzy constitutive law

– For $\omega \geqq \Omega_c$, the constitutive law of the fuzzy depends on:

(1) three deterministic functions, i. e.: the equivalent mean mass $\underline{\mu}(\omega)$ per unit surface, the mean rate of dissipation $\underline{\xi}(\omega)$ and the mean modal density $\underline{n}(\omega)$;

(2) random fluctuations around these mean values whose dispersions themselves depend on the function $\omega \rightarrow \lambda(\omega) = (\lambda_1(\omega), \lambda_2(\omega), \lambda_3(\omega))$.

For ω fixed, if $\|\lambda(\omega)\|_3 \rightarrow 0$, then the mechanical parameters approach the deterministic mean values and if $\|\lambda(\omega)\|_3 \rightarrow \sqrt{3}$, the dispersion introduced on the mechanical parameters approaches the maximum value.

– For $\omega < \Omega_c$, the constitutive law depends only on the parameter of equivalent mass per unit surface.

– The cutoff frequency and the six deterministic functions $\underline{\mu}$, $\underline{\xi}$, $\underline{n}$, λ_1, λ_2, λ_3 are the parameters of the law and are therefore considered givens.

IV,4.3. – *Modeling of the natural frequencies of the fuzzy*

In this section, we describe the probabilistic construction of the fuzzy natural frequencies $\omega_p(\omega)$ pour $\omega \geqq \Omega_c$. For ω fixed, $[\Omega_c, +\infty[$, $\omega_p(\omega)$ is modeled by a random variable defined on $(\mathscr{A}, \mathscr{C}, \mathscr{P})$ with values in $]\Omega_c, +\infty[$.

This random variable depends indirectly on the parameters $\underline{n}(\omega)$ and $\lambda_3(\omega)$. It is for this reason that $\omega_p(\omega)$ does not appear in the list of parameters of the fuzzy constitutive law summarized in IV,4.3 *d*. Considering (56) and (72), the distance $2\varepsilon(\omega)$ between two natural frequencies $\omega_p(\omega)$ in the vicinity of ω is a random variable which is written:

$$2\varepsilon(\omega) = \frac{1}{\underline{n}(\omega)(1 + \mathbf{Y}_3(\omega))}. \tag{73}$$

By construction, knowing that $\mathbf{Y}_3(\omega) = y_3$, we will assume that the conditional probability law:

$$P_{\omega_p(\omega)}(d\tilde{\omega}, \omega \mid y_3) = p_{\omega_p(\omega)}(\tilde{\omega}, \omega \mid y_3)\, d\tilde{\omega}$$

of random variable $\omega_p(\omega)$ has a density which is written:

$$p_{\omega_p(\omega)}(\tilde{\omega}, \omega \mid y_3) = \frac{1}{2\varepsilon(\omega)} \mathbf{1}_{[\omega - \varepsilon(\omega),\, \omega + \varepsilon(\omega)]}(\tilde{\omega}) \tag{74}$$

i. e. considering (73):

$$p_{\omega_p(\omega)}(\tilde{\omega}, \omega \mid y_3) = \underline{n}(\omega)(1 + y_3) \times \mathbf{1}_{[\omega - b(\omega, y_3),\, \omega + b(\omega, y_3)]}(\tilde{\omega}) \tag{75}$$

with $b(\omega, y_3)$ given by:

$$b(\omega, y_3) = \frac{1}{2\underline{n}(\omega)(1 + y_3)}. \tag{76}$$

The graph of the conditional density is given in Figure 5.

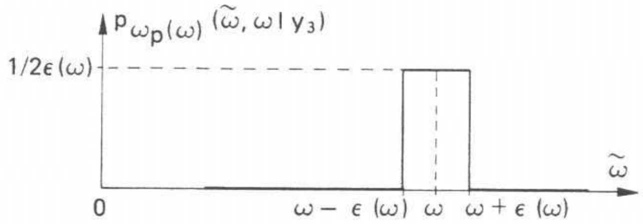

Fig. 5. – Graph of the conditional probability density

It is natural to introduce such a conditional probability law for $\omega_p(\omega)$ knowing that $\mathbf{Y}_3(\omega) = y_3$ because, since the distance between two natural frequencies of the fuzzy in the vicinity of ω is $\varepsilon(\omega)$, knowing that $\mathbf{Y}_3(\omega) = y_3$, the probability of presence of a natural frequency on an interval with measure $2\varepsilon(\omega) = (n(\omega)(1 + y_3))^{-1}$ is equal to 1. Furthermore, if the modal density $\underline{n}(\omega) \rightarrow +\infty$, the conditional probability must approach the Dirac measure at point ω, since the probability of presence of a natural frequency $\omega_p(\omega) = \omega$ then approaches 1. The model proposed has these properties.

A simple calculation gives:

$$\mathscr{E}\{\omega_p(\omega) \mid \mathbf{Y}_3(\omega) = y_3\} = \omega \tag{77}$$

$$\mathscr{E}\{(\omega_p(\omega) - \omega)^2 \mid \mathbf{Y}_3(\omega) = y_3\} = \frac{1}{12\,\underline{n}(\omega)^2 (1 + y_3)^2}. \tag{78}$$

The probability law $P_{\omega_p(\omega)}(d\tilde{\omega}, \omega)$ of $\omega_p(\omega)$ has a density given by:

$$p_{\omega_p(\omega)}(\tilde{\omega}, \omega) = \int_{\mathbb{R}} p_{\omega_p(\omega)}(\tilde{\omega}, \omega \mid y_3)\, p_3(\omega, y_3)\, dy_3 \quad (79)$$

which, considering (65) and (75), is written:

$$p_{\omega_p(\omega)}(\tilde{\omega}, \omega) = h(\tilde{\omega}, \omega)\, \mathbf{1}_{[\omega - a(\omega),\, \omega + a(\omega)]}(\tilde{\omega}) \quad (80)$$

where:

$$\left.\begin{aligned} & h(\tilde{\omega}, \omega) = \underline{n}(\omega) \quad \text{if} \quad \tilde{\omega} \in [\omega - b(\omega), \omega + b(\omega)]. \\ & h(\tilde{\omega}, \omega) = \frac{1}{16\lambda_3(\omega)\, \underline{n}(\omega)(\omega - \tilde{\omega})^2} - \underline{n}(\omega)\frac{(\lambda_3(\omega) - 1)^2}{4\lambda_3(\omega)} \\ & \text{if: } \tilde{\omega} \in [\omega - a(\omega), \omega - b(\omega)] \cup [\omega + b(\omega), \omega + a(\omega)] \end{aligned}\right\} \quad (81)$$

yielding:

$$a(\omega) = \frac{1}{2\underline{n}(\omega)(1 - \lambda_3(\omega))}, \quad b(\omega) = \frac{1}{2\underline{n}(\omega)(1 + \lambda_3(\omega))}. \quad (82)$$

The graph of $p_{\omega_p(\omega)}(\tilde{\omega}, \omega)$ is given in Figure 6.

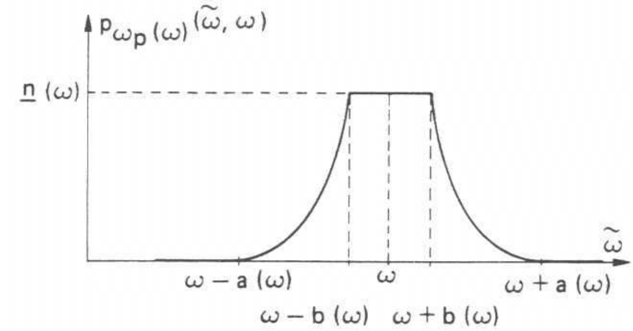

Fig. 6. – Graph of the probability density of natural frequencies.

Finally, the probability law for random variable $\{\mathbf{Y}(\omega), \omega_p(\omega)\}$ with values in $\mathbb{R}^4$ has a probability density which is written:

$$\begin{aligned} & p_{\mathbf{Y}(\omega),\, \omega_p(\omega)}(\omega, y, \tilde{\omega}) \\ & = p_1(\omega, y_1)\, p_2(\omega, y_2)\, p_3(\omega, y_3)\, p_{\omega_p(\omega)}(\tilde{\omega}, \omega \mid y_3). \end{aligned} \quad (83)$$

IV,5. – EXPLICIT FORM OF THE PROBABILISTIC FUZZY CONSTITUTIVE LAW

We obtained the following result on the basis of the above hypotheses.

For Ω_c, μ, ξ, $\underline{n}$ and λ given, the probabilistic constitutive law for the fuzzy is described by the family of second order random variables $\{\mathbf{z}(\omega, \lambda),\ \omega \in]0, +\infty[\}$, defined on $(\mathscr{A}, \mathscr{C}, \mathscr{P})$, with values in $\mathbb{C}$ such that for $\mathscr{P}$-almost any $a \in \mathscr{A}$, the function $\omega \to \mathbf{z}(\omega, \lambda, a)$ is continous by parts on $]0, +\infty[$ and $\mathbf{z}(\omega, \lambda)$ is written as follows for any $\omega \in]0, +\infty[$:

$$\mathbf{z}(\omega, \lambda) = \underline{\mathbf{z}}(\omega, \lambda) + \mathbf{z}_F^0(\omega, \lambda) \quad (84)$$

where:

(*a*) $\underline{\mathbf{z}}(\omega, \lambda) = \mathscr{E}(\mathbf{z}(\omega, \lambda)) \in \mathbb{C}$ is the mean of random variable $\mathbf{z}(\omega, \lambda)$ and is written:

$$\underline{\mathbf{z}}(\omega, \lambda) = -\omega^2 \underline{R}(\omega, \lambda) + i\omega \underline{I}(\omega, \lambda) \quad (85)$$

as functions $\omega \to \underline{R}(\omega, \lambda)$ and $\omega \to \underline{I}(\omega, \lambda)$, with values in $\mathbb{R}$ and $\mathbb{R}^+$ respectively, are continuous by parts on $]0, +\infty[$ and are given by equations (92), (93), (100), (101), and (104).

(*b*) Random variable $\mathbf{z}_F^0(\omega), \lambda) = \mathbf{z}(\omega, \lambda) - \underline{\mathbf{z}}(\omega, \lambda)$ is a centered, second order random variable.

Let $\mathscr{X}_{\mathbb{R}}$ be the vector subspace with dimension 3 of $L^2(\mathscr{A}, \mathbb{R})$, (Sec. II,8), generated by the orthonormal real random variables $\mathbf{X}_1$, $\mathbf{X}_2$ and $\mathbf{X}_3$ of $L^2(A, \mathbb{R})$, defined in Sec. IV,4. 1. Let $\mathscr{X}$ be the complexification of $\mathscr{X}_{\mathbb{R}}$. Then $\mathscr{X}$ is a vector subspace of $L^2(\mathscr{A}, \mathbb{C})$ whose elements are the centered, second order random variables ζ_F with complex values which are written:

$$\zeta_F = \sum_{j=1}^{3} \mathbf{X}_j c_j \quad (86)$$

where c_j are complex constants given by:

$$c_j = \mathscr{E}(\zeta_F \mathbf{X}_j), \qquad \mathbf{j} \in \{1, 2, 3\}. \quad (87)$$

Let $\mathbf{z}_F(\omega, \lambda)$ be the second order random variable with values in $\mathbb{C}$, centered, in $\mathscr{X}$, which is the best approximation of $\mathbf{z}_F^0(\omega, \lambda)$, i. e. for which the distance to $\mathbf{z}_F^0(\omega, \lambda)$ is the same as the distance from $\mathbf{z}_F^0(\omega, \lambda)$ to $\mathscr{X}$. Such a point $\mathbf{z}_F(\omega, \lambda)$ is said to be the projection of $\mathbf{z}_F^0(\omega, \lambda)$ on $\mathscr{X}$ or again the equivalent stochastic linearization of $\mathbf{z}_F^0(\omega, \lambda)$ by an element of $\mathscr{X}$.

The random variable $\mathbf{z}_F(\omega, \lambda)$ is then written:

$$\mathbf{z}_F(\omega, \lambda) = \sum_{j=1}^{3} \mathbf{X}_j(-\omega^2 R_j(\omega, \lambda) + i\omega I_j(\omega, \lambda)) \quad (88)$$

where, for any $j \in \{1, 2, 3\}$, the functions $\omega \to R_j(\omega, \lambda)$ and $\omega \to I_j(\omega, \lambda)$ have values in $\mathbb{R}$ and $\mathbb{R}^+$ respectively, are continuous by parts on $]0, +\infty[$ and are given explicitly by equations (94)-(106). Furthermore, by noting:

$$\|\lambda(\omega)\|_\infty = \operatorname{Sup}\{\lambda_1(\omega), \lambda_2(\omega), \lambda_3(\omega)\} \quad (89)$$

for any ω fixed and for any $j \in \{1, 2, 3\}$, we have:

$$\left.\begin{aligned} R_j(\omega, \lambda) &\to 0, \\ I_j(\omega, \lambda) &\to 0 \qquad \text{if} \quad \|\lambda(\omega)\|_\infty \to 0 \end{aligned}\right\} \quad (90)$$

(*c*) From the standpoint of construction of the probabilistic fuzzy law, we choose as law:

$$\mathbf{z}(\omega, \lambda) = \underline{\mathbf{z}}(\omega, \lambda) + \mathbf{z}_F(\omega, \lambda) \quad (91)$$

where $\underline{\mathbf{z}}(\omega, \lambda)$ is defined by (85) and $\mathbf{z}_F(\omega, \lambda)$ is defined by (88) (form (84) is not used directly).

(*d*) We have the following equations

$$\underline{R}(\omega, \lambda) = \begin{cases} \underline{\mu}(\omega) & \text{if} \quad \omega \in]0, \Omega_c[\\ \omega\, \underline{\mu}(\omega)\, \underline{n}(\omega)\, \underline{J}_3(\omega, \lambda) \\ \text{if} \quad \omega \in [\Omega_c, +\infty[\end{cases} \tag{92}$$

$$\underline{I}(\omega, \lambda) = \begin{cases} 0 & \text{if} \quad \omega \in]0, \Omega_c[\\ \frac{\pi}{2}\omega^2\, \underline{\mu}(\omega)\, \underline{n}(\omega)\, \underline{J}_0(\omega, \lambda) \\ \text{if} \quad \omega \in [\Omega_c, +\infty[\end{cases} \tag{93}$$

$$R_1(\omega, \lambda) = \begin{cases} \frac{1}{\sqrt{3}}\underline{\mu}(\omega)\,\lambda_1(\omega) & \text{if} \quad \omega \in]0, \Omega_c[\\ \omega\, \underline{\mu}(\omega)\, \underline{n}(\omega) \frac{1}{\sqrt{3}} \lambda_1(\omega)\, \underline{J}_3(\omega, \lambda) \\ \text{if} \quad \omega \in [\Omega_c, +\infty[\end{cases} \tag{94}$$

$$J_3(\omega, x, y) = \frac{1}{\omega\, \underline{n}(\omega)} - \frac{(1+y)}{4\sqrt{1-x^2}} \operatorname{Log}\left[\frac{[U_+(\omega, y) + W_+(\omega, x, y)][U_-(\omega, y) - W_-(\omega, x, y)]}{[U_+(\omega, y) - W_+(\omega, x, y)][U_-(\omega, y) + W_-(\omega, x, y)]}\right] \tag{104}$$

$$R_2(\omega, \lambda) = \begin{cases} 0 & \text{if} \quad \omega \in]0, \Omega_c[\\ \omega\, \underline{\mu}(\omega)\, \underline{n}(\omega) \frac{\sqrt{3}}{\lambda_2(\omega)} \underline{J}_5(\omega, \lambda) \\ \text{if} \quad \omega \in [\Omega_c, +\infty[\end{cases} \tag{95}$$

$$R_3(\omega, \lambda) = \begin{cases} 0 & \text{if} \quad \omega \in]0, \Omega_c[\\ \omega\, \underline{\mu}(\omega)\, \underline{n}(\omega) \frac{\sqrt{3}}{\lambda_3(\omega)} \underline{J}_4(\omega, \lambda) \\ \text{if} \quad \omega \in [\Omega_c, +\infty[\end{cases} \tag{96}$$

$$I_1(\omega, \lambda) = \begin{cases} 0 & \text{if} \quad \omega \in]0, \Omega_c[\\ \frac{\pi}{2}\omega^2\, \underline{\mu}(\omega)\, \underline{n}(\omega) \frac{1}{\sqrt{3}} \lambda_1(\omega)\, \underline{J}_0(\omega, \lambda) \\ \text{if} \quad \omega \in [\Omega_c, +\infty[\end{cases} \tag{97}$$

$$I_2(\omega, \lambda) = \begin{cases} 0 & \text{if} \quad \omega \in]0, \Omega_c[\\ \frac{\pi}{2}\omega^2\, \underline{\mu}(\omega)\, \underline{n}(\omega) \frac{\sqrt{3}}{\lambda_2(\omega)} \underline{J}_2(\omega, \lambda) \\ \text{if} \quad \omega \in [\Omega_c, +\infty[\end{cases} \tag{98}$$

$$I_3(\omega, \lambda) = \begin{cases} 0 & \text{if} \quad \omega \in]0, \Omega_c[\\ \frac{\pi}{2}\omega^2\, \underline{\mu}(\omega)\, \underline{n}(\omega) \frac{\sqrt{3}}{\lambda_3(\omega)} \underline{J}_1(\omega, \lambda) \\ \text{if} \quad \omega \in [\Omega_c, +\infty[\end{cases} \tag{99}$$

where for $k \in \{0, 1, 2, 3, 4, 5\}$:

$$\underline{J}_k(\omega, \lambda) = \frac{1}{4}\int_{-1}^{1} dy_2 \times \int_{-1}^{1} dy_3\, J_k(\omega, \underline{\xi}(\omega)[1 + \lambda_2(\omega) y_2], \lambda_3(\omega) y_3) \tag{100}$$

and:

$$J_0(\omega, x, y) = \frac{(1+y)}{\pi\sqrt{1-x^2}}[\operatorname{Arctan} Y_+(\omega, x, y) - \operatorname{Arctan} Y_-(\omega, x, y)] \tag{101}$$

$$Y_\pm(\omega, x, y) = \frac{\theta_\pm(\omega, y) + x^2}{x\sqrt{1-x^2}}$$

$$\theta_\pm(\omega, y) = 0{,}5\left[\left(1 \pm \frac{1}{2\,\omega\, \underline{n}(\omega)(1+y)}\right)^2 - 1\right]$$

$$J_1(\omega, x, y) = y\, J_0(\omega, x, y) \tag{102}$$

$$J_2(\omega, x, y) = \frac{1}{\underline{\xi}(\omega)}(x - \underline{\xi}(\omega))\, J_0(\omega, x, y) \tag{103}$$

$$U_\pm(\omega, y) = \left(1 \pm \frac{1}{2\,\omega\, \underline{n}(\omega)(1+y)}\right)^2 + 1$$

$$W_\pm(\omega, x, y) = 2\sqrt{1-x^2}\left(1 \pm \frac{1}{2\,\omega\, \underline{n}(\omega)(1+y)}\right)$$

$$J_4(\omega, x, y) = y\, J_3(\omega, x, y) \tag{105}$$

$$J_5(\omega, x, y) = \frac{1}{\underline{\xi}(\omega)}(x - \underline{\xi}(\omega))\, J_3(\omega, x, y). \tag{106}$$

(*e*) Elements of the proof

(A) *Proof of point IV,5 a*

Considering equation (54) or (57), we have:

$$\underline{\mathbf{z}}(\omega, \lambda) = \mathscr{E}(\mathbf{z}(\omega, \lambda)) = -\omega^2\, \underline{R}(\omega, \lambda) + i\,\omega\, \underline{I}(\omega, \lambda)$$

where:

$$\underline{R}(\omega, \lambda) = \mathscr{E}(R(\omega, \lambda)), \qquad \underline{I}(\omega, \lambda) = \mathscr{E}(I(\omega, \lambda))$$

which yields equation (85). $\underline{R}$ and $\underline{I}$ now remain to be calculated.

Case 1: $\omega \in]0, \Omega_c[$

Considering (55) and (70), we have:

$$\underline{R}(\omega, \lambda) = \mathscr{E}\,\mu(\omega) = \mathscr{E}(\underline{\mu}(\omega)(1 + \mathbf{Y}_1(\omega)) = \underline{\mu}(\omega)$$

since $\mathscr{E}\,\mathbf{Y}_1(\omega) = 0$. Furthermore, according to (55), for any ω and any λ, we have $I(\omega, \lambda) = 0$ and therefore $\underline{I}(\omega, \lambda) = 0$. For ω in$]0, \Omega_c[$. we can infer (92) and (93).

Case 2: $\omega \in [\Omega_c, +\infty[$

With the convention $y=(y_1, y_2, y_3) \in \mathbb{R}^3$, $dy = dy_1\, dy_2\, dy_3$:

$$r(\omega, y, \tilde{\omega}) = \frac{\underline{\mu}(\omega)(1+y_1)(\tilde{\omega}^2/\omega^2)[(\tilde{\omega}^2/\omega^2)-1+4\,\underline{\xi}(\omega)^2(1+y_2)^2]}{((\tilde{\omega}^2/\omega^2)-1)^2+4(\tilde{\omega}^2/\omega^2)\,\underline{\xi}(\omega)^2(1+y_2)^2} \tag{107}$$

$$i(\omega, y, \tilde{\omega}) = \frac{2\,\underline{\mu}(\omega)(1+y_1)\,\omega\,\underline{\xi}(\omega)(1+y_2)(\tilde{\omega}/\omega)}{((\tilde{\omega}^2/\omega^2)-1)^2+4(\tilde{\omega}^2/\omega^2)\,\underline{\xi}(\omega)^2(1+y_2)^2}. \tag{108}$$

Then, according to (58) and (83), we have:

$$\underline{R}(\omega, \lambda) = \int_{\mathbb{R}^4} r(\omega, y, \tilde{\omega})\, p_{\mathbf{Y}(\omega),\, \omega_p(\omega)}(\omega, y, \tilde{\omega})\, dy\, d\tilde{\omega}$$

$$\underline{I}(\omega, \lambda) = \int_{\mathbb{R}^4} i(\omega, y, \tilde{\omega})\, p_{\mathbf{Y}(\omega),\, \omega_p(\omega)}(\omega, y, \tilde{\omega})\, dy\, d\tilde{\omega}.$$

Using the following primitives for $x \in]0, 1[$:

$$\int \frac{u^2(u^2-1+4x^2)}{(u^2-1)^2+4u^2x^2}\,du = u - \frac{1}{4\sqrt{1-x^2}}\,\mathrm{Log}\left[\frac{u^2+2\sqrt{1-x^2}\,u+1}{u^2-2\sqrt{1-x^2}\,u+1}\right] \tag{109}$$

$$\int \frac{u\,du}{(u^2-1)^2+4u^2x^2} = \frac{1}{4x\sqrt{1-x^2}}\,\mathrm{Arctg}\left[\frac{u^2+2x^2-1}{2x\sqrt{1-x^2}}\right] \tag{110}$$

yields equations (92) and (93) for $\omega \in [\Omega_c, +\infty[$, J_0 and $\underline{J}_3$ being given by (100), (101) and (104). The last proof of point IV,5 *a* results from the hypotheses introduced.

(B) *Proof of point IV,5 b*

Case 1: $\omega \in]0, \Omega_c[$

Considering (54), (55), (65), (70) and (85), we have:

$$\mathbf{z}_F^0(\omega, \lambda) = -\omega^2\,\underline{\mu}(\omega)\,\frac{\lambda_1(\omega)}{\sqrt{3}}\,\mathbf{X}_1.$$

Identification with (88) yields $\mathbf{z}_F(\omega, \lambda) = \mathbf{z}_F^0(\omega, \lambda)$ where:

$$R_1(\omega, \lambda) = \underline{\mu}(\omega)\frac{\lambda_1(\omega)}{\sqrt{3}},$$

$$I_1(\omega, \lambda) = 0 \quad \text{and} \quad R_j(\omega, \lambda) = I_j(\omega, \lambda) = 0$$

for $j \in \{2, 3\}$, yielding (94) to (99) for $\omega \in]0, \Omega_c[$.

Case 2: $\omega \in [\Omega_c, +\infty[$

Using the theorem of orthogonal projection in the Hilbert spaces, we can infer that point $\mathbf{z}_F(\omega, \lambda)$ sought is such that:

$$\mathscr{E}\{[\mathbf{z}_F^0(\omega, \lambda) - \mathbf{z}_F(\omega, \lambda)]\,\bar{g}\} = 0, \qquad \forall g \in \mathscr{X}.$$

Since $\mathbf{z}_F(\omega, \lambda) \in \mathscr{X}$, it is expressed:

$$\mathbf{z}_F(\omega, \lambda) = \sum_{j=1}^{3} \mathbf{X}_j\, c_j(\omega, \lambda). \tag{111}$$

Furthermore, according to (84),

$$\mathbf{z}_F^0(\omega, \lambda) = \mathbf{z}(\omega, \lambda) - \underline{\mathbf{z}}(\omega, \lambda).$$

Taking $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$ respectively for g yields for $k \in \{1, 2, 3\}$:

$$\mathscr{E}\{(\mathbf{z}(\omega, \lambda) - \underline{\mathbf{z}}(\omega, \lambda) - \sum_{j=1}^{3} \mathbf{X}_j\, c_j(\omega, \lambda))\,\mathbf{X}_k\} = 0$$

which yields, according to (63) and since $\mathbf{X}_k$ is centered:

$$c_j(\omega, \lambda) = \mathscr{E}(\mathbf{z}(\omega, \lambda)\,\mathbf{X}_j), \quad j \in \{1, 2, 3\}. \tag{112}$$

Separating the real part from the imaginary part and identifying formulation (111) with (88), equations (112) give, with (65):

$$\left.\begin{aligned} R_j(\omega, \lambda) &= \frac{\sqrt{3}}{\lambda_j(\omega)}\,\mathscr{E}(R(\omega, \lambda)\,\mathbf{Y}_j), \; j \in \{1, 2, 3\} \\ I_j(\omega, \lambda) &= \frac{\sqrt{3}}{\lambda_j(\omega)}\,\mathscr{E}(I(\omega, \lambda)\,\mathbf{Y}_j), \quad j \in \{1, 2, 3\}. \end{aligned}\right\} \tag{113}$$

Equations (113) are expressed directly using (107), (108), (83):

$$R_j(\omega, \lambda) = \frac{\sqrt{3}}{\lambda_j(\omega)} \times \int_{\mathbb{R}^4} y_j\, r(\omega, y, \tilde{\omega}) \times p_{\mathbf{Y}(\omega),\, \omega_p(\omega)}(\omega, y, \tilde{\omega})\, dy\, d\tilde{\omega}$$

$$I_j(\omega, \lambda) = \frac{\sqrt{3}}{\lambda_j(\omega)} \times \int_{\mathbb{R}^4} y_j\, i(\omega, y, \tilde{\omega}) \times p_{\mathbf{Y}(\omega),\, \omega_p(\omega)}(\omega, y, \tilde{\omega})\, dy\, d\tilde{\omega}.$$

Conducting the calculations as for mean values yields equations (94) to (99) for $\omega \in [\Omega_c, +\infty[$. The proof of the last statement of point IV,5 *b* is simple to obtain.

V. – PROBABILISTIC ANALYSIS OF THE VIBRATIONS IN THE MASTER STRUCTURE IN PRESENCE OF STRUCTURAL FUZZY

V, 1. – DISCRETIZED IMPEDANCE OF THE MASTER STRUCTURE

As was indicated in paragraph I, 1, the master structure designates all the mechanical systems which can be modeled of the complex mechanical system. Furthermore, in Sec. II, 1, we assumed that domain D_M occupied by the master structure was bounded. This means that the state variables of the master structure are fields defined on the bounded domain D_M [1].

For the MF linear dynamic analysis, the master structure is modeled by the finite element method. The master structure state variables are therefore discretized on a finite element basis of domain D_M [2].

Let N be the number of DOFs of the finite element model of the master structure, $U(t)=(U_1(t), \ldots, U_N(t))$ be the element of $\mathbb{C}^N$ of the nodal unknowns and $F(t)=(F_1(t), \ldots, F_N(t))$ be the element of $\mathbb{C}^N$ of the applied equivalent nodal forces [3].

For any $\omega \in \mathbb{R}$, the three real operators of $\mathbb{R}^N$, $M(\omega)$, $C(\omega$ and $K(\omega)$, which are the mass, dissipation and stiffness operators respectively of the master structure, are symmetrical and positive definite. We will assume that:

$$M, C \text{ and } K \in G(\mathbb{R}, \mathscr{L}(\mathbb{R}^N)). \tag{114}$$

For any $\omega \in \mathbb{R}$, the discretized impedance of the master structure is written: [4]

$$Z_S(\omega) = -\omega^2 M(\omega) + i\omega C(\omega) + K(\omega). \tag{115}$$

Considering the above assumptions, it is easily verified that:

$$Z_S \in G(\mathbb{R}, \mathscr{L}(\mathbb{C}^N)). \tag{116}$$

V, 2. – DISCRETIZED IMPEDANCE OF THE STRUCTURAL FUZZY

The fuzzy is applied to the part Γ of boundary ∂D_M of D_M (Sec. III, 1). Using the results of Section III, the matrix $[Z(\omega)]$ of the probabilistic impedance operator $\mathbf{Z}(\omega)$ relative to the canonical basis of $\mathbb{R}^N$ is obtained by assembly of the fuzzy finite elements (Sec. III, 4.) [5], [6].

The following general hypotheses, compatible with the developments of Sections III and IV, are introduced on the fuzzy operator.

(*a*) Let L be a positive integer and $\omega \to \lambda(\omega) = (\lambda_1(\omega), \ldots, \lambda_L(\omega))$ be a mapping defined everywhere on $]0, +\infty[$ with values in $(]0, 1[)^L \subset \mathbb{R}^L$ (therefore λ is bounded), continuous by parts. We set:

$$\| \lambda(\omega) \|_\infty = \operatorname{Sup}(\lambda_1(\omega), \ldots, \lambda_L(\omega)) \tag{117}$$

(*b*) For any $\omega \in]0, +\infty[$, the probabilistic impedance operator, $\mathbf{Z}(\omega, \lambda(\omega))$, of the fuzzy depends on parameter $\lambda(\omega)$ [7], and is a second-order random variable defined on $(\mathscr{A}, \mathscr{C}, \mathscr{P})$ with values in $\mathscr{L}(\mathbb{C}^N)$ such that we have $\mathscr{P}$-almost surely:

$$\mathbf{Z}(\omega, \lambda) = {}^t\mathbf{Z}(\omega, \lambda), \qquad \forall \omega \in]0, +\infty[\tag{118}$$

$$\omega \to \mathbf{Z}(\omega, \lambda(\omega))$$

$$\text{is continuous by parts on }]0, +\infty[\tag{119}$$

$$\mathbf{Z}(\omega, \lambda) = \underline{\mathbf{Z}}(\omega, \lambda) + \mathbf{Z}_F(\omega, \lambda) \tag{120}$$

(*c*) The deterministic impedance

$$\underline{\mathbf{Z}}(\omega, \lambda) = \mathscr{E}\{\mathbf{Z}(\omega, \lambda)\} \in \mathscr{L}(\mathbb{C}^N)$$

is the mean of random variable $\mathbf{Z}(\omega, \lambda)$ and is written:

$$\underline{\mathbf{Z}}(\omega, \lambda) = -\omega^2 \underline{R}(\omega, \lambda) + i\omega \underline{I}(\omega, \lambda) \tag{121}$$

where: (1) for any ω in $]0, +\infty[$, $\underline{R}(\omega, \lambda(\omega))$ and $\underline{I}(\omega, \lambda(\omega))$ are two symmetrical linear operators of $\mathscr{L}(\mathbb{R}^N)$;

(2) $\underline{I}(\omega, \lambda(\omega))$ is a positive operator [8];

(3) Mappings $\omega \to \underline{R}(\omega, \lambda(\omega))$ and $\omega \to \underline{I}(\omega, \lambda(\omega))$ are continous by parts on $]0, +\infty[$.

---

(1) It should be noted that it is always possible to reduce the problem to this situation. For instance, if the master structure consists of a main structure which occupies a bounded domain located in a compressible fluid which occupies an unbounded domain, the equation for the coupled system vibrations can be written using only the state variables of the main structure, by introducing the hydrodynamic coupling operator formulated with the state variables of the primary structure. This operator is constructed, for instance, using an integral equation method [1, 81, 82, 83].

(2) In order to simplify the theoretical framework, the developments which follow will be formulated on discretized equations, so as not to have to manipulate operators with an infinite dimension. The functional aspects of the MF method have already been discussed [78, 79].

(3) If the excitation is random and stationary in time, then $F(t)$ is an intermediate deterministic quantity related to the second order moments or those of the stochastic excitation field. This quantity allows computation of the moments of the vector process which is the solution (*see* [80, 83]).

(4) For instance, the mass, dissipation and stiffness of the master structure are functions of ω if there are viscoelastic materials, if there is coupling with an inviscid compressible fluid occupying an unbounded domain of space, etc. (*see* [7, 83]).

(5) The fuzzy finite elements use only the DOFs of the master structure relative to boundary Γ.

(6) The assembly is made with respect to all the degrees of freedom of the master structure.

(7) To simplify the expression, we will in some cases use λ instead of $\lambda(\omega)$ below.

(8) $$\forall U \in \mathbb{R}^N, \quad \|U\|_N \neq 0,$$
$$\langle \underline{I}(\omega, \lambda(\omega))\, U, U \rangle_N \geqq 0.$$

(*d*) The random fluctuation $\mathbf{Z}_F(\omega, \lambda)$ of the impedance with values $\mathscr{L}(\mathbb{C}^N)$ is written:

$$\mathbf{Z}_F(\omega, \lambda(\omega)) = \sum_{l=1,\ldots,L} \mathbf{X}_l(-\omega^2 R_l(\omega, \lambda(\omega)) + i\omega I_l(\omega, \lambda(\omega))) \quad (122)$$

where:

(1) $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_L$ are random variables defined on $(\mathscr{A}, \mathscr{C}, \mathscr{P})$ with real values, mutually independent, each random variable $\mathbf{X}_l$ being centered, with unit variance and with a uniform probability law $P_{X_l}(dx) = p(x)\,dx$, where $p(x)$ is defined by (60) ([9]).

(2) For any $\omega \in]0, +\infty[$ and any $l \in \{1, \ldots, L\}$, $R_l(\omega, \lambda)$ and $I_l(\omega, \lambda)$ are linear operators of $\mathscr{L}(\mathbb{R}^N)$ and are symmetrical.

(3) $I_l(\omega, \lambda)$ is a positive operator.

(4) Mappings $\omega \to R_l(\omega, \lambda)$ and $\omega \to I_l(\omega, \lambda)$ are continuous by parts on $]0, +\infty[$.

(5) For any $\omega \in]0, +\infty[$ and any $l \in \{1, \ldots, L\}$, if $\|\lambda(\omega)\|_\infty \to 0$, then:

$$\begin{aligned} &\| R_l(\omega, \lambda(\omega)) \|_N \to 0; \\ &\| I_l(\omega, \lambda(\omega)) \|_N \to 0. \end{aligned} \quad (123)$$

V, 3. – REMARKS

(1) It can be verified that all the hypotheses introduced in V, 1 and V, 2 are compatible with those of Sections III and IV. The parameter λ introduced in point V, 2 *a* has the role of parameter λ defined by (64).

(2) If only one fuzzy law of type (91) is introduced in the modeling, then $L=3$. If M subsets of stochastically independent fuzzy finite elements, each of type (91), are introduced in the fuzzy model, then $L=3M$. In the general case, L can take on any value.

V, 4. – EQUATION FOR THE VIBRATIONS OF THE DISCRETIZED COMPLEX SYSTEM

In the Fourier space, the equation for the vibrations of the master structure with its structural fuzzy is written:

$$\left.\begin{aligned} (Z_S(\omega) + \mathbf{Z}(\omega, \lambda(\omega)))\, \hat{\mathbf{U}}(\omega) = \hat{F}(\omega), \\ \omega \in]0, +\infty[. \end{aligned}\right\} \quad (124)$$

Considering (120), this equation can be written:

$$(Z_{SF}(\omega, \lambda(\omega)) + \mathbf{Z}_F(\omega, \lambda(\omega)))\, \hat{\mathbf{U}}(\omega) = \hat{F}(\omega) \quad (125)$$

where:

$$Z_{SF}(\omega, \lambda(\omega)) = Z_S(\omega) + \underline{Z}(\omega, \lambda(\omega)) \quad (126)$$

where Z_{SF} is a deterministic impedance and Z_F is a centered random impedance.

Result 1: According to the above assumptions, $\omega \to Z_{SF}(\omega, \lambda(\omega))$ is a mapping belonging to $G(\mathbb{R}, \mathscr{L}(\mathbb{C}^N))$.

Proof: It can easily be seen that $Z_{SF}(\omega, \lambda) = {}^t Z_{SF}(\omega, \lambda)$ and that $\omega \to Z_{SF}(\omega, \lambda(\omega))$ is continuous by parts on $]0, +\infty[$. In addition, considering V, 2 *d*, it is always possible to construct the following breakdown of (121):

$$\underline{Z}(\omega, \lambda) = -\omega^2 \underline{M}_F(\omega, \lambda) + i\omega \underline{I}(\omega, \lambda) + \underline{K}_F(\omega, \lambda)$$

where, for any $\omega \in]0, +\infty[$, $\underline{M}_F(\omega, \lambda)$, $\underline{I}(\omega, \lambda)$ and $\underline{K}_F(\omega, \lambda)$ are three positive symmetric real operators (but not positive definite). Applying (115) and (126) yields:

$$Z_{SF}(\omega, \lambda) = -\omega^2 M_{SF}(\omega, \lambda) + i\omega C_{SF}(\omega, \lambda) + K_{SF}(\omega, \lambda)$$

where

$$M_{SF}(\omega, \lambda) = M(\omega) + \underline{M}_F(\omega, \lambda),$$
$$C_{SF}(\omega, \lambda) = C(\omega) + \underline{I}(\omega, \lambda)$$

and

$$K_{SF}(\omega, \lambda) = K(\omega) + \underline{K}_F(\omega, \lambda)$$

are three positive definite symmetrical operators of $\mathscr{L}(\mathbb{R}^N)$. From this it can be inferred that $\forall \omega \in]0, +\infty[$, $(Z_{SF}(\omega, \lambda(\omega)))^{-1}$ exists and belongs to $\mathscr{L}(\mathbb{C}^N)$ and that $\omega \to (Z_{SF}(\omega, \lambda(\omega)))^{-1}$ is continuous by parts on $]0, +\infty[$ and is therefore locally bounded.

V, 5. – EXISTENCE AND UNIQUENESS OF THE RANDOM SOLUTION

Let B_n be the compact interval of $\mathbb{R}^+$ defined in paragraph II, 9. Since Z_{SF} belongs to $G(\mathbb{R}, \mathscr{L}(\mathbb{C}^N))$ according to result 1, there exists a positive constant $c_{SF} > 0$, independent of ω such that:

$$\forall \omega \in B_n, \quad \| Z_{SF}(\omega, \lambda(\omega))^{-1} \|_N \leq \frac{1}{c_{SF}} \quad (127)$$

and, setting

$$\mathbf{T}(\omega, \lambda(\omega)) = -Z_{SF}(\omega, \lambda(\omega))^{-1} \mathbf{Z}_F(\omega, \lambda(\omega)) \quad (128)$$

we can write the following equation:

$$Z_{SF}(\omega, \lambda(\omega)) + \mathbf{Z}_F(\omega, \lambda(\omega)) = Z_{SF}(\omega, \lambda(\omega))(1 - \mathbf{T}(\omega, \lambda(\omega))). \quad (129)$$

([9]) The random variables $\mathbf{X}_l$ *are therefore uniform and orthonormal in* $L^2(\mathscr{A}, \mathbb{R})$.

However, for any ω fixed in B_n, the operator $(1-\mathbf{T}(\omega, \lambda(\omega)))$ will be $\mathscr{P}$-almost surely invertible if:

$$\|\mathbf{T}(\omega, \lambda(\omega))\|_N < 1, \quad \mathscr{P}\text{-a.s.} \tag{130}$$

i. e. considering (128), if:

$$\|\mathbf{Z}_F(\omega, \lambda(\omega))\|_N < \frac{1}{\|Z_{SF}(\omega, \lambda(\omega))^{-1}\|_N}, \quad \mathscr{P}\text{-a.s.} \tag{131}$$

However, according to assumption V, 2 *e*, there are two positive real constants $\lambda_F > 0$ and $c_F > 0$, independent of ω, such that for:

$$0 < \sup_{\omega \in B_n} \|\lambda(\omega)\|_\infty \leqq \lambda_F \tag{132}$$

we have:

$$\sup_{\omega \in B_n} \|\mathbf{Z}_F(\omega, \lambda(\omega))\|_N \leqq c_F < c_{SF} \; \mathscr{P}\text{-a.s.} \tag{133}$$

This is due to equation (122) for $\mathbf{Z}_F$, to equations (123) and to the fact that random variables $\mathbf{X}_l$ $\mathscr{P}$-almost surely have bounded values, since the support of the probability measure of $\mathbf{X}_l$ is $[-\sqrt{3}, \sqrt{3}]$. Under these conditions, inequalities (127), (131) and (133) show that the mapping:

$$\omega \to (Z_{SF}(\omega, \lambda(\omega)) + \mathbf{Z}_F(\omega, \lambda(\omega)))^{-1}$$

is $\mathscr{P}$-almost surely bounded on B_n. We therefore have the following result:

Result 2: There exists $\lambda_F > 0$ such that for $\mathrm{Sup}_{\omega \in B_n} \|\lambda(\omega)\|_\infty \leqq \lambda_F$ and for any element F in $H_{B_N}(\mathbb{R}, \mathbb{C}^N)$, the equation (125) has a unique solution for $d\omega$-almost any $\omega \in B_n \mathscr{P}$-a. s.

$$\hat{\mathbf{U}}(\omega, \lambda) = (Z_{SF}(\omega, \lambda) + \mathbf{Z}_F(\omega, \lambda))^{-1} \hat{F}(\omega), \quad \mathscr{P}\text{-a.s.} \tag{134}$$

Mapping $\omega \to \hat{\mathbf{U}}(\omega, \lambda(\omega))$ is $\mathscr{P}$-almost surely essentially bounded on B_n with values in $\mathbb{C}^N$.

V, 6. — EXPRESSION OF THE SOLUTION

Let λ_F be the positive real constant defined in result 2. Then, if $\sup_{\omega \in B_n} \|\lambda(\omega)\|_\infty \leqq \lambda_F$, for any ω in B_n, we have :

$$(1-\mathbf{T}(\omega, \lambda))^{-1} = \sum_{k=0}^{+\infty} \mathbf{T}(\omega, \lambda)^k, \quad \mathscr{P}\text{-a. s.} \tag{135}$$

the series of the right member of (135) being $\mathscr{P}$-almost surely convergent. Applying equations (128), (129) and (135) yields:

$$(Z_{SF}(\omega, \lambda) + \mathbf{Z}_F(\omega, \lambda))^{-1} = \sum_{k=0}^{+\infty} \mathbf{T}(\omega, \lambda)^k Z_{SF}(\omega, \lambda)^{-1}.$$

Solution (134) can therefore be written:

$$\hat{\mathbf{U}}(\omega, \lambda) = \sum_{k=0}^{+\infty} \mathbf{T}(\omega, \lambda)^k Z_{SF}(\omega, \lambda)^{-1} \hat{F}(\omega). \tag{136}$$

Result 3: If $\sup_{\omega \in B_n} \|\lambda(\omega)\|_\infty \leqq \lambda_F$, solution (134) is written for $d\omega$-almost any ω in B_n:

$$\hat{\mathbf{U}}(\omega, \lambda) = \hat{U}^{(0)}(\omega, \lambda) + \sum_{k=1}^{+\infty} \hat{\mathbf{U}}^{(k)}(\omega, \lambda), \quad \mathscr{P}\text{-a. s.} \tag{137}$$

with, for $k \geqq 1$:

$$\hat{\mathbf{U}}^{(k)}(\omega, \lambda) = \sum_{l_1=1}^{L} \times \ldots \times \sum_{l_k=1}^{L} \mathbf{X}_{l_1} \mathbf{X}_{l_2} \ldots \mathbf{X}_{l_k} \hat{U}^{(k)}_{l_1 l_2 \ldots l_k}(\omega, \lambda) \tag{138}$$

where $l_1 l_2 \ldots l_k$ is a multisubscript with length k and with the convention $\hat{U}^{(k)}_{l_1} \ldots l_k(\omega, \lambda) = \hat{U}^{(0)}(\omega, \lambda)$ for $k=0$.

(1) $\hat{U}^{(0)}(\omega, \lambda) \in \mathbb{C}^N$ is a solution of the following deterministic equation:

$$Z_{SF}(\omega, \lambda) \hat{U}^{(0)}(\omega, \lambda) = \hat{F}(\omega). \tag{139}$$

(2) Elements $\hat{U}^{(k)}_{l_1 l_2 \ldots l_k} \in \mathbb{C}^N$ for $k \geqq 1$ are the solutions of the following recurrent deterministic equations:

$$Z_{SF}(\omega, \lambda) \hat{U}^{(k)}_{l_1 l_2 \ldots l_k}(\omega, \lambda) = \hat{Q}_{l_1 l_2 \ldots l_k}(\omega, \lambda) \tag{140}$$

with $\hat{Q}_{l_1 \ldots l_k}(\omega, \lambda)$ in $\mathbb{C}^N$ such that:

$$\hat{Q}_{l_1 \ldots l_k}(\omega, \lambda) = -(-\omega^2 R_{l_k}(\omega, \lambda) + i\omega I_{l_k}(\omega, \lambda)) \hat{U}^{(k-1)}_{l_1 l_2 \ldots l_{k-1}}. \tag{141}$$

(3) Finally, we have:

$$\left.\begin{array}{l} \{ t \to Q_{l_1 \ldots l_k}(t) \} \in \mathrm{H}_{\mathrm{B}_n}(\mathbb{R}, \mathbb{C}^N) \\ \{ t \to U^{(k)}_{l_1 \ldots l_k}(t) \} \in H_{B_n}(\mathbb{R}, \mathbb{C}^N) \end{array}\right\} \tag{142}$$

and all the derivatives of $\mathrm{U}^{(k)}_{l_1 \ldots l_k}$ with respect to t in the sense of the generalized function are represented by function $H_{B_n}(\mathbb{R}, \mathbb{C}^N)$:

$$\frac{d^q}{dt^q} U^{(k)}_{l_1 \ldots l_k} \in H_{B_n}(\mathbb{R}, \mathbb{C}^N). \tag{143}$$

Proof: Identifying (136) and development (137) term for term yields:
$\hat{U}^{(0)}(\omega, \lambda) = Z_{SF}(\omega, \lambda)^{-1} \hat{F}(\omega)$ which gives (139), and

$\hat{\mathbf{U}}^{(k)}(\omega, \lambda) = \mathbf{T}(\omega, \lambda)^k \hat{U}^{(0)}(\omega, \lambda)$. This second equation is equivalent to the following recurrence:

$$\hat{\mathbf{U}}^{(k)}(\omega, \lambda) = \mathbf{T}(\omega, \lambda)\, \hat{\mathbf{U}}^{(k-1)}(\omega, \lambda), \qquad k \geqq 1$$
$$\hat{\mathbf{U}}^{(0)}(\omega, \lambda) = \hat{U}^{(0)}(\omega, \lambda).$$

Substituing equations (122) and (128) for $\mathbf{Z}_F$ and $\mathbf{T}$ gives, for $k \geqq 1$:

$$Z_{SF}(\omega, \lambda)\, \hat{\mathbf{U}}^{(k)}(\omega, \lambda) = -\sum_{l_k=1}^{L} \mathbf{X}_{l_k}(-\omega^2 R_{l_k}(\omega, \lambda) + i\omega I_{l_k}(\omega, \lambda)\, \hat{\mathbf{U}}^{(k-1)}(\omega, \lambda) \quad (144)$$

which gives (140) and (141) when the expressions of $\hat{\mathbf{U}}^{(k)}$ and $\hat{\mathbf{U}}^{(k-1)}$ given by (138) are substituted in (144).

According to result 1, and applying V, $2d(\varphi)(d)(4)$ and (139) to (141), it can be seen that for F in $H_{B_n}(\mathbb{R}, \mathbb{C}^N)$, functions $\omega \to \hat{\underline{Q}}_{l_1 \ldots l_k}(\omega, \lambda(\omega))$ and $\omega \to \hat{U}^{(k)}_{l_1 \ldots l_k}(\omega, \lambda(\omega))$ are essentially bounded and all have the same compact support B_n. They are therefore in $L^2(\mathbb{R}, \mathbb{C}^N)$, giving (142). Finally, since the support of $\hat{U}^{(k)}_{l_1 \ldots l_k}$ is the compact inverval B_n, $\forall q$ positive integer, the function:

$$\omega \to (i\omega)^q\, \hat{U}^{(k)}_{l_1 \ldots l_k}(\omega, \lambda(\omega))$$

has the same compact support B_n and is essentially bounded. It is therefore in $L^2(\mathbb{R}, \mathbb{C}^N)$, from which we can infer (143).

Result 4: Let c_{SF} be the positive real constant independent of ω defined by (127). Let $\lambda_F > 0$ such that for $\mathrm{Sup}_{\omega \in B_n} \|\lambda(\omega)\|_\infty \leqq \lambda_F$, the almost sure inequality (133) is verified. This inequality defines the real constant c_F independent of ω such that:

$$\alpha = \frac{c_F}{c_{SF}} < 1. \quad (145)$$

Let $\mathscr{K}$ be a positive integer $\geqq 1$, fixed and $\hat{\mathbf{U}}_{(\mathscr{K})}(\omega, \lambda)$ be the solution of (134) of order $\mathscr{K}$ such that:

$$\hat{\mathbf{U}}_{(\mathscr{K})}(\omega, \lambda) = \hat{U}^{(0)}(\omega, \lambda) + \sum_{k=1}^{\mathscr{K}} \hat{\mathbf{U}}^{(k)}(\omega, \lambda) \quad (146)$$

where $\hat{U}^{(0)}$ and $\hat{\mathbf{U}}^{(k)}$ are as defined in result 3. We then have the following estimation for $d\omega$-almost any ω in B_n:

$$\frac{\|\hat{\mathbf{U}}(\omega, \lambda) - \hat{\mathbf{U}}_{(\mathscr{K})}(\omega, \lambda)\|_N}{\|\hat{U}^{(0)}(\omega, \lambda)\|_N} \leqq \frac{\alpha^{\mathscr{K}+1}}{1-\alpha}, \quad \mathscr{P}\text{-a. s.} \quad (147)$$

Proof: Equations (125), (129), (135) and (139) show that solution (134), written with form (137), can also be written:

$$\hat{\mathbf{U}}(\omega, \lambda) = \hat{\mathbf{U}}_{(\mathscr{K})}(\omega, \lambda) + \sum_{k=\mathscr{K}+1}^{+\infty} \mathbf{T}(\omega, \lambda)^k \hat{U}^{(0)}(\omega, \lambda) \quad \mathscr{P}\text{-a. s.}$$

giving:

$$\|\hat{\mathbf{U}}(\omega, \lambda) - \hat{\mathbf{U}}_{(\mathscr{K})}(\omega, \lambda)\|_N \leqq \sum_{k=\mathscr{K}+1}^{+\infty} |\mathbf{T}(\omega, \lambda)|_N^k \times \|\hat{U}^{(0)}(\omega, \lambda)\|_N, \quad \mathscr{P}\text{-a. s.} \quad (148)$$

However, we have:

$$\sum_{k=\mathscr{K}+1}^{+\infty} |\mathbf{T}(\omega, \lambda)|_N^k = \frac{|\mathbf{T}(\omega, \lambda)|_N^{\mathscr{K}+1}}{1-|\mathbf{T}(\omega, \lambda)|_N}, \quad \mathscr{P}\text{-a. s.} \quad (149)$$

Finally, for any ω in B_n, we have

$$|\mathbf{T}(\omega, \lambda)|_N \leqq |Z_{SF}(\omega, \lambda)^{-1}|_N \times |\mathbf{Z}_F(\omega, \lambda)|_N \leqq \frac{c_F}{c_{SF}} = \alpha, \quad \mathscr{P}\text{-a. s.} \quad (150)$$

Equations (148) to (150) lead to result (147).

V, 7. – EXPLICIT CONSTRUCTION OF THE SOLUTION IN THE MF DOMAIN

In this section, we assume that λ is fixed such that result 3 is applicable. The solution exists, is unique and can be constructed using equations (137) to (141). Since N is generally large (several tens of thousands) and the solution is sought on a wide frequency band $[\omega_I, \omega_F] \subset \mathbb{R}^+$ (several thousand Hz), the MF method described in [78, 79, 80, 81, 83] is appropriate to numerically construct the solution. To be able to use the MF method, it is necessary to give the standard form to result 3.

This is the point which will be developed below.

To simplify the expression, we introduce the following notations:

$$\mathscr{M}(\omega) = M(\omega) + \underline{R}(\omega, \lambda(\omega)) \quad (151)$$
$$\mathscr{C}(\omega) = C(\omega) + \underline{I}(\omega, \lambda(\omega)). \quad (152)$$

Then, mapping Z_{SF} defined by (126) is written for any ω:

$$Z_{SF}(\omega) = -\omega^2 . \mathscr{M}(\omega) + i\omega\mathscr{C}(\omega) + K(\omega). \quad (153)$$

V, 7.1. – *Choice of frequency bands for the* MF *analysis*

In the MF method, the broadband is written $[\omega_I, \omega_F] = \cup_n B_n$ where B_n are narrow MF bands and the MF analysis is conducted for each band B_N [78, 83]. Such an analysis band, noted B_n, is the bounded closed interval of $\mathbb{R}^+$ defined in Sec. II, 9.

such that $\Delta\omega/\Omega_n \ll 1$. We note as:

$$\mathring{B}_n = \left]\Omega_n - \frac{\Delta\omega}{2},\ \Omega_n + \frac{\Delta\omega}{2}\right[,$$

the associated open interval and as $B_0 = \left[-\frac{\Delta\omega}{2}, \frac{\Delta\omega}{2}\right]$ the centered LF band associated with B_n.

Since all the mappings $\mathscr{M}$, $\mathscr{C}$, K, R_l and I_l, $l \in \{1, \ldots, L\}$ are continuous by parts on $]0, +\infty[$, the set of points of discontinuity in $[\omega_I, \omega_F]$ of all these mappings is finite. We can therefore always choose the partition of $[\omega_I, \omega_F]$ such that, for each MF band B_n, all the above mappings are continuous in any point ω in $\mathring{B}_n$. We now assume that the selected partition verifies this hypothesis.

V, 7.2. – *Frequency approximation of the operators on an analysis band*

Let A be any of mappings $\mathscr{M}$, $\mathscr{C}$, K, R_l or I_l. Since $\omega \to A(\omega)$ is the continuous, bounded mapping of $\mathring{B}_n$ into $\mathscr{L}(\mathbb{R}^n)$, we can define the mean operator relative to band B_n:

$$A_n = \frac{1}{\Delta\omega} \int_{\Omega_n - \Delta\omega/2}^{\Omega_n + \Delta\omega/2} A(\omega)\, d\omega. \qquad (154)$$

We will note as $\mathscr{M}_n$, $\mathscr{C}_n$, K_n, $R_{l,\,n}$ and $I_{l,\,n}$ the frequency approximations of the corresponding operators and define the following operators for any ω in B_n:

$$Z_{SF,\,n}(\omega) = -\omega^2 \mathscr{M}_n + i\,\omega\,\mathscr{C}_n + K_n \qquad (155)$$

$$\mathbf{Z}_{F,\,n}(\omega) = \sum_{l=1}^{L} \mathbf{X}_l(-\omega^2 R_{l,\,n} + i\,\omega\, I_{l,\,n}). \qquad (156)$$

V, 7.3. – *Definition of the approximated solution relative to an analysis band*

By construction, the approximated solution relative to the frequency approximation of V.7.2 and for F in $H_{B_n}(\mathbb{R}, \mathbb{C}^N)$ is such that:

$$\hat{\mathbf{U}}_n(\omega) = (Z_{SF,\,n}(\omega) + \mathbf{Z}_{F,\,n}(\omega))^{-1}\, \hat{F}(\omega), \quad \mathscr{P}\text{-a. s.} \qquad (157)$$

It can easily be verified that $\omega \to \hat{\mathbf{U}}_n(\omega)$ is $\mathscr{P}$-almost surely essentially bounded on B_n with values in $\mathbb{C}^N$ (*see* result 2).

V, 7.4. – *Convergence of the approximated solution on an analysis band*

Considering the above assumptions, it is easy to check that for any positive real ε, however small, there is a $\Delta\omega > 0$ such that:

$$|||\, \hat{\mathbf{U}} - \hat{\mathbf{U}}_n \,|||_N \leqq \varepsilon, \quad \mathscr{P}\text{-a. s.} \qquad (158)$$

where $\hat{\mathbf{U}}$ is the solution defined by (134) and developed in result 3.

Remarks: (1) The approximation method is obviously interesting from a numerical standpoint if $\mathscr{M}$, $\mathscr{C}$, K, R_l and I_l vary sufficiently slowly in ω for $\hat{\mathbf{U}}$ to be approximated sufficiently closely by $\hat{\mathbf{U}}_n$ with a bandwidth $\Delta\omega$ which is not too small. This method has already been used and is justified in [7, 81, 82, 83]. (2) Actually, criterion (158) also applies to the choice of the bandwidth of B_n.

V, 7.5. – *Construction of the approximated solution on an analysis band*

The construction is given directly by result 3 in which the frequency approximation V, 7.2 of the operators is used. For $d\omega$-almost any ω in B_n, we therefore obtain:

$$\hat{\mathbf{U}}_n(\omega) = \hat{U}_n^{(0)}(\omega) + \sum_{k=1}^{+\infty} \hat{\mathbf{U}}_n^{(k)}(\omega), \quad \mathscr{P}\text{-a. s.}$$

$$\hat{\mathbf{U}}_n^{(k)}(\omega) = \sum_{l_1=1}^{L} \times \ldots \qquad (159)$$

$$\times \sum_{l_k=1}^{L} \mathbf{X}_{l_1} \mathbf{X}_{l_2} \ldots \mathbf{X}_{l_k}\, \hat{U}^{(k)}_{l_1 l_2, \ldots l_k,\, n}(\omega)$$

where $U_n^{(0)}$ and $U^{(k)}_{l_1 \ldots l_k,\, n}$ belong to $H_{B_n}(\mathbb{R}, \mathbb{C}^N)$ and have Fourier transforms given by the following recurrence:

$$\left.\begin{aligned} &\mathscr{S}(\hat{U}_n^{(0)}, \hat{F}) = 0, \\ &\mathscr{S}(\hat{U}^{(k)}_{l_1 \ldots l_k,\, n}, \hat{Q}_{l_1 \ldots l_k,\, n}) = 0, \qquad k \geqq 1 \end{aligned}\right\} \qquad (160)$$

with $Q_{l_1 \ldots l_k,\, n}$ in $H_{B_N}(\mathbb{R}, \mathbb{C}^N)$ and given for $k \geqq 1$ by:

$$\hat{Q}_{l_1 \ldots l_k,\, n}(\omega) = -(-\omega^2 R_{l_k,\, n} + i\,\omega\, I_{l_k,\, n})\, \hat{U}^{(k-1)}_{l_1 l_2 \ldots l_{k-1}}(\omega) \qquad (161)$$

and where $\mathscr{S}$ is defined by:

$$\mathscr{S}(\hat{U}_n, \hat{Q}_n) = Z_{SF,\,n}(\omega)\, \hat{U}_n(\omega) - \hat{Q}_n(\omega). \qquad (162)$$

The recurrence calculation (160) therefore requires solving the following standard MF problem:

For Q_n in $H_{B_n}(\mathbb{R}, \mathbb{C}^N)$, calculate the solution U_n in $H_{B_n}(\mathbb{R}, \mathbb{C}^N)$ of the equation $\mathscr{S}(U_n, \hat{Q}_n) = 0$, i. e. of:

$$(-\omega^2 \mathscr{M}_n + i\,\omega\,\mathscr{C}_n + K_n)\, \hat{U}_n(\omega) = \hat{Q}_n(\omega). \qquad (163)$$

The solving method and the developments required for solving (160) are given in Part II of this paper.

VI. — CONCLUSION

We have given the theoretical developmental elements of an attempt to construct a probabilistic model of the structural fuzzy in linear dynamics of complex mechanical systems. In Part II of this paper, we give the additional developments concerning: numerical analysis of the problem, its implementation in a program, examples of processing on beams, plates and shells.

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