Modeling mechanical subsystems by boundary impedance in the finite element method

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MODELING MECHANICAL SUBSYSTEMS BY BOUNDARY IMPEDANCE IN THE FINITE ELEMENT METHOD

by

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ABSTRACT

We study the linear vibrations in the medium frequency range of a mechanical system consisting of two coupled subsystems, the first of them being classically modelled by the finite element method and the second one being described by a given boundary impedance found from experimental data or numerical calculation.

Because the system may have a very high number of degrees of freedom, we want to stay with a multiple scale algorithm, which considerably brings down the numerical costs, compared with a direct method in which the equations are solved frequency by frequency in the frequency domain.

To do this, we present an approximation method which consists of establishing a system of second-order linear differential equations that governs vibrations of the coupled system in the temporal domain, and which allows us, by introducing hidden variables, to use the fast algorithm mentioned above.

Three numerical applications are given.

Keywords (NASA thesaurus): Vibration–Finite element analysis.

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I. — INTRODUCTION

The dynamic analysis of mechanical systems with a linear behavior by the finite element method leads to solving second order linear differential equations using suitable numerical schemes. With this type of method, the substructuring concept is widely used.

In the special case of a substructure described by its boundary impedance, obtained experimentally or from a numerical model, the problem can be solved directly for the assembled system operating in the frequency domain, frequency by frequency. For this purpose, it is sufficient to assemble the boundary impedance of the substructure with the impedance of the initial mechanical system, constructed conventionally. However, this approach can lead to excessive numerical costs if the number of degrees of freedom of the complete system is high (several tens of thousands) and if the analysis is to be made on a wide band (several thousand hertz). An attempt is therefore made to preserve a second-order differential equation structure after assembly of the subsystem to allow the use of less costly numerical schemes.

In the general case, a boundary impedance cannot be interpreted as the impedance of a mechanical system governed by a second-order differential equation because of the condensation of the internal degrees of freedom on the boundary. In other words, the transfer function associated with this impedance is not the transfer function of a causal second-order linear convolution filter, i. e. associated with a second-order differential equation with constant coefficients. In most cases, the associated filter cannot even be approximated by a second-order filter. It is therefore necessary to introduce hidden variables which are directly related to the internal degrees of freedom eliminated from the subsystem considered and about which all information has been lost.

In this paper, we describe an approach which can be used to solve this problem. We first state the problem in detail, recalling why it is desirable to preserve an underlying second-order differential equation structure with constant coefficients for the assembled system (Section III) and we introduce hypotheses, for reasons explained later, on the stability of the assembled system (Section IV). We then describe the method proposed; it is based on appropriate smoothing of the impedance of the subsystem which has been coupled and the introduction of additional variables, called hidden variables.

Finally, we give three numerical examples which validate the method developed.

Remark. — Considering the objectives, we will reason on discretized mechanical systems with a finite number of observed degrees of freedom. Thus, all the linear operators have finite dimensions and are identified with their matrix relative to suitable bases.

II. — NOTATIONS AND REMINDERS

To facilitate reading, we summarize below the main notations we will use.

Let $U=(U_1, \ldots, U_m)$ be a vector of $\mathbb{C}^m$. We will identify U with the column matrix $(m, 1)$ of its components U_j. We note as $\mathrm{Mat}_K(n, m)$ the set of matrices with dimension (n, m) on body K, where K can be $\mathbb{R}$ or $\mathbb{C}$.

Let $Q\in\mathrm{Mat}_{\mathbb{C}}(n, m)$. We will note as Q^T the transposed matrix of Q and $Q=\bar{Q}^T$.

If $n=m$ and $Q^T=Q$, the complex matrix Q is said to be symmetrical.

$\mathbb{C}^m$ is equipped with the usual scalar product and associated norm:

$$(U, V)_m=\sum_{j=1}^{m} U_j \bar{V}_j \quad (1)$$

$$\| U\|_m=(U, U)_m^{1/2}. \quad (2)$$

We introduce the following notations and reviews.

(*a*) Let $H_m=L^2(\mathbb{R}, \mathbb{C}^m)$ be the Hilbert space of mappings $t\mapsto U(t)$ defined almost everywhere on $\mathbb{R}$ with values in $\mathbb{C}^m$, with integrable square. H_m is equipped with the norm:

$$||| U |||_m=\left(\int_{\mathbb{R}} \| U(t)\|_m^2\, dt\right)^{1/2} \quad (3)$$

and the associated scalar product

$$((U, V))_m=\int_{\mathbb{R}} (U(t), V(t))_m\, dt \quad (4)$$

(*b*) For any U in H_m, we will note as $\omega\mapsto\hat{U}(\omega)$ the Fourier transform (FT) of $U(\hat{U}\in H_m)$ such that for $j\in\{1, \ldots, m\}$ and for almost every ω in $\mathbb{R}$:

$$\hat{U}_j(\omega)=\int_{\mathbb{R}} e^{-i\omega t}\, U_j(t)\, dt. \quad (5)$$

For U and V in H_m, we have:

$$((U, V))_m=\frac{1}{2\pi}((\hat{U}, \hat{V}))_m. \quad (6)$$

(*c*) Let B be any closed bounded interval of $\mathbb{R}$. We define subspace $H_m(B)$ of H_m such that:

$$H_m(B)=\{ U\in H_m ; \operatorname{supp}\hat{U}=B\}. \quad (7)$$

We note that the restriction of the support concerns $\hat{U}$ and not U. Under these conditions, we have:

$$|||U|||_m^2 = \int_{\mathbb{R}} \|U(t)\|_m^2 \, dt = \frac{1}{2\pi} \int_B \|\hat{U}(\omega)\|_m^2 \, d\omega. \quad (8)$$

(*d*) Let h be a function of $L^1_{loc}(\mathbb{R}, \mathbb{C})$ with $\mathbb{R}^+$ as support. We note as $\tilde{h}(p)$, $p \in \mathbb{C}$ the Laplace transform (unilateral) of h such that:

$$\tilde{h}(p) = \int_0^{+\infty} e^{-pt} h(t) \, dt. \quad (9)$$

This Laplace transform is defined in the open domain D_h of $\mathbb{C}$ (which can be empty), such that:

$$D_h = \{p \in \mathbb{C};\ \text{Ré}\, p > a\} \quad (10)$$

where $a \in \bar{\mathbb{R}}$. If a is finite, $D_h \neq \emptyset$ and $\tilde{h}$ is holomorphic in D_h.

If $a < 0$, D_h contains the imaginary axis and we have:

$$\hat{h}(\omega) = \tilde{h}(0 + i\omega) \quad (11)$$

where $\hat{h}$ is the FT of h. For instance, this is verified if $h \in L^1(\mathbb{R}, \mathbb{C})$ with $\text{supp}\, h \subset \mathbb{R}^+$. In this case, $\hat{h}$ is then the FT of h in L^1 and $\hat{h}$ is obviously a continuous function.

If $h \in L^2(\mathbb{R}, \mathbb{C})$ with $\text{supp}\, h \subset \mathbb{R}^+$, the Laplace transform (LT) belongs to the Hardy class and $\tilde{h}$ is holomorphic in $D_h = \{p \in \mathbb{C},\ \text{Ré}\, p > 0\}$.

In this case we have:

$$\hat{h}(\omega) = \lim_{a \to 0} \tilde{h}(a + i\omega) \quad \text{in } L^2(\mathbb{R}, \mathbb{C}). \quad (12)$$

Considering a function $t \mapsto h(t)$ of $\mathbb{R}$ in $\text{Mat}_{\mathbb{C}}(n, m)$ with $\text{supp}\, h \subset \mathbb{R}^+$, we define as above

$$\{h_{ij}(p), D_{h_{ij}}\}, (i, j) \in \{1, \ldots, n\} \times \{1, \ldots, m\},$$

and we have $\tilde{h}(p) = \{\tilde{h}_{ij}(p)\}_{ij}$, $D_h = \bigcap_{i,j} D_{h_{ij}}$.

(*e*) A linear convolution filter h_*, real, with an impulse response $t \mapsto h(t)$ from $\mathbb{R}$ into $\text{Mat}_{\mathbb{R}}(m, m)$ is causal if $\text{supp}\, h = \mathbb{R}^+$. Its transfer function is given by $\{H(p) = \tilde{h}(p), D_h\}$ with D_h of type (10). It is said to be stable (exponentially) if D_h contains the imaginary axis. In this case, $\omega \mapsto \hat{h}(\omega) = H(0 + i\omega)$ is a continuous function from $\mathbb{R}$ into $\text{Mat}_{\mathbb{C}}(m, m)$ and is called the frequency response function of the filter.

Below, we will consider causal and stable filters. The poles of the transfer function $H(p)$ will therefore all have a strictly negative real part and the frequency response function $\omega \mapsto \hat{h}(\omega) = H(0 + i\omega)$ will be in $C^0(\mathbb{R}, \text{Mat}_{\mathbb{C}}(m, m))$.

The filter will be said to be regular at the (angular) frequency ω if $\hat{h}(\omega)$ is reversible in this point. In this case, $Z(\omega) = \hat{h}^{-1}(\omega) \in \text{Mat}_{\mathbb{C}}(m, m)$ is called the associated impedance of the filter. We will then consider cases for which $\hat{h}(\omega)$ and therefore $Z(\omega)$ are complex symmetrical matrices.

III. — REVIEWS OF A MODEL EQUATION OF MF VIBRATIONS AND SOLVING METHODS

III, 1. — MODEL EQUATION OF MF VIBRATIONS

Let us consider an elastic structure S occupying an open domain Ω_e bounded by $\mathbb{R}^3$, with boundary $\partial\Omega_e$ located in a vacuum. The structure is heterogeneous, anisotropic, dissipative, with a linear constitutive law. It is modeled by finite elments and discretized for this purpose into a finite number m of degrees of freedom.

For ω fixed in $\mathbb{R}$, we will note as $\hat{U}(\omega)$ the vector of $\mathbb{C}^m$ representing the discretized displacement field of S, as $\hat{\mathscr{F}}(\omega)$ the vector of $\mathbb{C}^m$ representing the system of external forces applied to S and as $\mathscr{M}(\omega)$, $\mathscr{C}(\omega)$ and $\mathscr{K}(\omega)$ respectively the positive-defined real, symmetrical mass, damping and stiffness matrices (m, m) of S. It is noted that these matrices depend a priori on ω, which is the case for instance if S contains linear viscoelastic materials [5, 32] or if its model involves structural fuzzy finite elements [6, 31].

The MF vibrations of the system are then governed in the Fourier domain by the following linear equation of $\mathbb{C}^m$:

$$\mathscr{Z}(\omega)\, \hat{U}(\omega) = \hat{\mathscr{F}}(\omega), \qquad \omega \in \mathbb{R} \quad (13)$$

where

$$\mathscr{Z}(\omega) = -\omega^2 \mathscr{M}(\omega) + i\omega \mathscr{C}(\omega) + \mathscr{K}(\omega) \quad (14)$$

is the complex, symmetrical mechanical impedance matrix (m, m) of the structure.

We will assume below that the mappings $\mathscr{M}$, $\mathscr{C}$, $\mathscr{K}$ are continuous on $\mathbb{R}$ and therefore that mapping $\omega \mapsto \mathscr{Z}(\omega)$ is continuous on $\mathbb{R}$. Considering the assumptions, $\mathscr{Z}(\omega)$ is reversible for any ω in $\mathbb{R}$ and the mapping that defines the frequency response function of the system is written:

$$\omega \mapsto \hat{h}(\omega) = \mathscr{Z}(\omega)^{-1}. \quad (15)$$

For any $\omega \in \mathbb{R}$, $\hat{h}(\omega) \in \text{Mat}_{\mathbb{C}}(m, m)$ and is symmetrical. As function $\hat{h}$ is continuous on $\mathbb{R}$, it is bounded on B.

If we also assume that $\hat{\mathscr{F}} \in H_m(B)$, the solution of (13) is then written $\hat{U}(\omega) = \hat{h}(\omega)\, \hat{\mathscr{F}}(\omega)$ for almost every $\omega \in \mathbb{R}$ and is such that $U \in H_m(B)$.

Remark 1. — If the structure is not in a vacuum but is placed in a compressible, inviscid, unbounded fluid, it is known that the effects of the fluid on S can be introduced by the matrices of the operators of added mass and dissipation by radiation at infinity of the fluid, applied to the trace of the displacement field of S on the surface in contact with the fluid. If such is the case, we will assume herein that these matrices are included in $\mathcal{M}(\omega)$ and $\mathcal{C}(\omega)$ respectively, with (13) then representing the vibration equation of the elastic structure coupled with the external fluid [32].

Similarly, any internal compressible inviscid fluids which occupy bounded domains are included implicitly with the structure [11, 32].

Remark 2. — When the model of S includes structural fuzzy finite elements, $\mathcal{Z}(\omega)$ is modeled, for ω fixed, by a random variable defined on a suitably probabilistic space, with values in the complex symmetrical matrices (m, m); equation (13) is in this case a random operator equation.

Similarly, considering the case in which the excitation $t \mapsto \mathcal{F}(t)$ is a stochastic vector process stationary in time, we are led to seek the stationary solution.

However, we have shown in [31] and [32] that for the two above cases, the solution of the random problem amounts to solving several deterministic problems of type (13). Therefore, without losing the general character and to simplify presentation, we can restrict ourselves to the case in which all the values introduced are deterministic.

III, 2. — SOLVING OF THE MODEL EQUATION IN THE MF DOMAIN

Let $B=[\omega_1, \omega_2]$ be any closed bounded interval of $\mathbb{R}^+$ with width $\Delta\omega$, and let $\mathcal{F} \in H_m(B)$. When $\Delta\omega$ is large, which is the case for instance for wideband dynamic structure identification problems, we must a priori solve (13) for a high number of values of ω. However, the cost of constructing $\omega \mapsto \hat{U}(\omega)$ by directly solving (13) for a large number of ω becomes prohibitive when m is large. To decrease this cost, we can use a special solving method [29] outlined below in order to show why it is desirable to preserve an underlying second-order differential equation structure with constant coefficients.

Let $B = \bigcup_{n=1}^{N} B_n$ be a finite partition of B where for $n \in \{1, \ldots, N\}$, $B_n = \left[\Omega_n - \frac{\delta\omega}{2}; \Omega_n + \frac{\delta\omega}{2}\right]$ are compact intervals of $\mathbb{R}^+$ called MF narrow bands with center frequency $\Omega_n > 0$, width $\delta\omega$, verifying $\frac{\delta\omega}{\Omega_n} \ll 1$ and such that $\bigcap_{n=1}^{N} B_n$ is a set of $\mathbb{R}^+$ with measure zero.

Let $\omega \mapsto \mathbf{1}_{B_n}(\omega)$ be the indicator function of interval B_n ($\mathbf{1}_{B_n}(\omega)=1$ if $\omega \in B_n$, $\mathbf{1}_{B_n}(\omega)=0$ if $\omega \notin B_n$).

Then, for $\mathcal{F} \in H_m(B)$, the solution of (13) is written:

$$Q_n = \frac{1}{\delta\omega} \int_{\Omega_n - \delta\omega/2}^{\Omega_n + \delta\omega/2} Q(\omega)\, d\omega. \tag{18}$$

where $\omega \mapsto \underline{\hat{U}}_n(\omega)$ are solutions of the N independent problems:

$$[-\omega^2 \mathcal{M}(\omega) + i\omega \mathcal{C}(\omega) + \mathcal{K}(\omega)]\, \underline{\hat{U}}_n(\omega) = \hat{\mathcal{F}}_n(\omega) \tag{17}$$

where $\hat{\mathcal{F}}_n(\omega) = \mathbf{1}_{B_n}(\omega) \hat{\mathcal{F}}(\omega)$ for any n in $\{1, \ldots, N\}$. We note that $\mathcal{F}_n \in H_m(B_n)$ and therefore $\underline{U}_n \in H_m(B_n)$.

Let $Q(\omega) \in \{\mathcal{M}(\omega), \mathcal{C}(\omega), \mathcal{K}(\omega)\}$; for n fixed in $\{1, \ldots, N\}$, we set:

$$\tilde{h}(p) = \int_0^{+\infty} e^{-pt} h(t)\, dt. \tag{9}$$

Then, considering the assumptions, it is shown that we can always choose N (or, which amounts to the same thing, $\delta\omega$) such that the solution U_n, belonging to $H_m(B_n)$ of

$$\mathcal{Z}_n(\omega)\, \hat{U}_n(\omega) = \hat{\mathcal{F}}_n(\omega) \tag{19}$$

where:

$$\mathcal{Z}_n(\omega) = -\omega^2 \mathcal{M}_n + i\omega \mathcal{C}_n + \mathcal{K}_n \tag{20}$$

approaches the solution $\underline{U}_n$ of (17) within ε fixed positive, as small as desired, i. e. such that the following is true:

$$||| U_n - \underline{U}_n |||_m \leq \varepsilon. \tag{21}$$

We will note that considering the hypotheses on $\mathcal{M}$, $\mathcal{C}$ and $\mathcal{K}$, matrices $\mathcal{M}_n$, $\mathcal{C}_n$ and $\mathcal{K}_n$ are real, symmetrical, positive-defined.

From a numerical standpoint, it is obvious that the approximation is interesting insofar as $\delta\omega$ is not too small, i. e. when the number N of subbands of the partition of B is not too large. This condition is satisfied if mappings $\omega \mapsto \mathcal{M}(\omega)$, $\mathcal{C}(\omega)$, $\mathcal{K}(\omega)$ have "slow variations" on $\mathbb{R}$, which is the case for the various situations described in [32].

In the time domain, the equation associated with (19) is a second-order linear differential equation on $\mathbb{C}^m$ with constant coefficients, which is written:

$$\mathcal{M}_n \ddot{U}_n(t) + \mathcal{C}_n \dot{U}_n(t) + \mathcal{K}_n U_n(t) = \mathcal{F}_n(t) \tag{22}$$

where $\dot{U}_n(t) = \frac{dU_n(t)}{dt}$ and $\ddot{U}_n(t) = \frac{d^2 U_n(t)}{dt^2}$.

By a special multiple scale technique [29], this is reduced to a linear differential equation on $\mathbb{C}^m$ relative to the low frequency part of the solution; for given initial conditions, this equation is then solved by direct integration over time using an implicit, unconditionally stable numerical scheme.

This multiple scale method which uses the structure of differential equation (22) allows $\hat{U}_n(\omega)$ to be computed for any $\omega \in B_n$ for a numerical cost equivalent to that for solving (13) for a single value of ω. The cost ratio is therefore equal to the number of frequency points selected in B_n to construct $\hat{U}_n(\omega)$, generally high (around a hundred). It is for this reason that we wish to preserve this type of MF algorithm, even when coupling a system described by its boundary impedance on a system modeled as finite elements.

The aim of this paper is to develop a method to achieve this objective.

IV. – STATEMENT OF THE PROBLEM AND GENERAL HYPOTHESES

IV, 1. – THE GIVENS

(*a*) As we will use the method mentioned in Section III, 2, we will assume that the excitation belongs to $H_m(B_n)$ with n fixed and we will eliminate once and for all all subscripts n when there is no ambiguity.

(*b*) The initial mechanical system is modeled by m degrees of freedom and is described by the impedance $\mathscr{Z}(\omega) \in \mathrm{Mat}_{\mathbb{C}}(m, m)$, symmetrical, defined by (20). For an excitation $\mathscr{F} \in H_m(B_n)$, the vibrations of this system are governed by equation (19).

As function $\mathscr{Z}$ is defined by (20) and as $\mathscr{M}_n$, $\mathscr{C}_n$ and $\mathscr{K}_n$ are three positive-defined symmetrical real constant matrices, we infer that $\omega \mapsto \mathscr{Z}(\omega)$ is continuous from $\mathbb{R}$ into $\mathrm{Mat}_{\mathbb{C}}(m, m)$, that $\forall \omega \in \mathbb{R}$, $\hat{h}(\omega) = \mathscr{Z}(\omega)^{-1}$ exists, that function $\omega \mapsto \hat{h}(\omega)$ is continuous and integrable from $\mathbb{R}$ into $\mathrm{Mat}_{\mathbb{C}}(m, m)$ and that the impulse response h of $\mathbb{R}$ into $\mathrm{Mat}_{\mathbb{R}}(m, m)$ has $\mathbb{R}^+$ as support and is integrable on $\mathbb{R}$. The linear convolution filter $h*$ is therefore stable and causal and all the poles of its transfer function $H(p) = (p^2 \mathscr{M}_n + p\mathscr{C}_n + \mathscr{K}_n)^{-1}$, $p \in \mathbb{C}$ have a strictly negative real part.

(*c*) Among the m degrees of freedom, we will distinguish q degrees of freedom, called the boundary degrees, on which will be coupled the subsystem described by its boundary impedance $Z_F(\omega)$. We set $r = m - q$. We then naturally introduce the following breakdown of (19) into blocks:

$$\left(\begin{array}{c|c} \mathscr{Z}_{11}(\omega) & \mathscr{Z}_{12}(\omega) \\ \hline \mathscr{Z}_{21}(\omega) & \mathscr{Z}_{22}(\omega) \end{array}\right) \begin{pmatrix} \hat{U}_1(\omega) \\ \hat{U}_2(\omega) \end{pmatrix} = \begin{pmatrix} \hat{\mathscr{F}}_1(\omega) \\ \hat{\mathscr{F}}_2(\omega) \end{pmatrix} \tag{23}$$

where $\hat{U}_1(\omega)$ and $\hat{\mathscr{F}}_1(\omega)$ in $\mathbb{C}^r$, $\hat{U}_2(\omega)$ and $\hat{\mathscr{F}}_2(\omega)$ in $\mathbb{C}^q$, $\mathscr{Z}_{11}(\omega) \in \mathrm{Mat}_{\mathbb{C}}(r, r)$, $\mathscr{Z}_{22}(\omega) \in \mathrm{Mat}_{\mathbb{C}}(q, q)$, $\mathscr{Z}_{12}(\omega) \in \mathrm{Mat}_{\mathbb{C}}(r, q)$, $\mathscr{Z}_{11}(\omega)$ and $\mathscr{Z}_{22}(\omega)$ symmetrical and $\mathscr{Z}_{21}(\omega) = \mathscr{Z}_{12}^T(\omega)$..

(*d*) The boundary impedance $Z_F(\omega)$ of the subsystem is relative to the q degrees of freedom. We introduce the following hypotheses on Z_F: the function $\omega \mapsto Z_F(\omega)$ is continuous from B_n into $\mathrm{Mat}_{\mathbb{C}}(q, q)$ and $\forall \omega \in B_n$. $Z_F(\omega) = Z_F^T(\omega)$.

The function Z_F on B_n is therefore assumed given. However, we generally only have a finite sequence of matrices $\{ Z_F(\omega_l), l \in \{ 1, \ldots, L \} \}$ with $\omega_l \in B_n$, which is the basic given. This will be the case when $Z_F(\omega)$ is supplied by experiment or constructed by a numerical model [6]. It should be noted, and this is the practical situation, that $Z_F(\omega)$ is not assumed known on $\mathbb{R}$, but only on a compact interval of $\mathbb{R}$. No conclusion can therefore be drawn on the structure of the associated filter from the only given on $\omega \mapsto Z_F(\omega)$, $\omega \in B_n$.

IV, 2. – EQUATION FOR THE COUPLED SYSTEM

The vibration equation for the coupled system in the frequency domain for $\mathscr{F} \in H_m(B_n)$ is written:

$$\mathscr{Z}_T(\omega)\, \hat{U}(\omega) = \hat{\mathscr{F}}(\omega) \tag{24}$$

where

$$\mathscr{Z}_T(\omega) = \begin{pmatrix} \mathscr{Z}_{11}(\omega) & \mathscr{Z}_{12}(\omega) \\ \mathscr{Z}_{21}(\omega) & \mathscr{Z}_{22}(\omega) + Z_F(\omega) \end{pmatrix}. \tag{25}$$

Considering the explanations given above, the impedance $\mathscr{Z}_T(\omega)$ is not associated with a differential equation of type (22) owing to the presence of $Z_F(\omega)$. Obviously, we could write $Z_F(\omega)$ in a form similar to (14) then use approximation (18). In the general case, such an approach would however lead to choosing very small bandwidths $\delta\omega$ since function $\omega \mapsto Z_F(\omega)$ can exhibit large fluctuations due to the internal dynamics of the subsystem considered. As we wish to preserve sufficiently large bandwidths $\delta\omega$ for the method to remain efficient, we must forego this type of solution.

IV, 3. – GENERAL STABILITY HYPOTHESES

(*a*) Considering the hypotheses introduced on $\mathscr{Z}$ and Z_F, no conclusion can be drawn as to the existence of a solution $U \in H_m(B_n)$ such that $\hat{U}$ verifies (24). However, we are concerned here only with stable physical systems. This leads us to introduce the following hypothesis for the coupled system: for any ω in B_n, $\hat{h}_T(\omega) = \mathscr{Z}_T^{-1}(\omega)$ exists. Considering the hypotheses of Section I, this means that $\omega \mapsto \hat{h}_T(\omega)$ is continuous on the compact interval B_n. Under these conditions, there exists a unique solution $U \in H_m(B_n)$ such that $\hat{U}$ verifies (24). It should however be noted that at this level, there is no knowledge on the frequency response function $\hat{h}_T$, but only on its restriction to interval B_n.

(*b*) In the method which will be described, we will construct an approximation $\underset{\sim}{U} \in H_m(B_n)$ of $U \in H_m(B_n)$, this approximation $\underset{\sim}{U}$ being a solution of the following problem associated with (24):

$$\underset{\sim}{\mathscr{Z}}_T(\omega)\,\hat{\underset{\sim}{U}}(\omega) = \hat{\mathscr{F}}(\omega) \qquad (26)$$

with

$$\underset{\sim}{\mathscr{Z}}_T(\omega) = \begin{pmatrix} \mathscr{Z}_{11}(\omega) & \mathscr{Z}_{12}(\omega) \\ \mathscr{Z}_{21}(\omega) & \mathscr{Z}_{22}(\omega) + \underset{\sim}{Z}_F(\omega) \end{pmatrix} \qquad (27)$$

where $\omega \mapsto \underset{\sim}{Z}_F(\omega)$ is a continuous function of $\mathbb{R}$ in $\mathrm{Mat}_{\mathbb{C}}(q, q)$ such that $\forall \omega \in \mathbb{R}$, $\underset{\sim}{Z}(\omega) = \underset{\sim}{Z}_F^T(\omega)$, $\underset{\sim}{Z}_F(\omega)$ being an approximation of $Z_F(\omega)$ on band B_n (and not on $\mathbb{R}$ since $Z_F(\omega)$ is known only on B_n). Furthermore, $\underset{\sim}{Z}_F$ also has certain algebraic properties which will be described below. However, it is necessary to introduce a hypothesis which is similar to (but different from) that introduced above in (*a*), i. e. that $\forall \omega \in \mathbb{R}$ [and not $\forall \omega \in B_n$ as in (*a*)], $\hat{\underset{\sim}{h}}_T(\omega) = \underset{\sim}{Z}_T^{-1}(\omega)$ exists and function $\omega \mapsto \hat{\underset{\sim}{h}}_T(\omega)$, which is continuous on $\mathbb{R}$, approaches $O_{(m,\,m)} + i\,O_{(m,\,m)}$ when $|\omega| \to +\infty$. As, by the construction of $\underset{\sim}{Z}_F$, the frequency response filter $\omega \mapsto \hat{\underset{\sim}{h}}_T(\omega)$ is causal (because the equations will be differential), the hypothesis introduced implies the stability of this filter.

V. — CONSTRUCTION OF AN APPROXIMATION OF THE BOUNDARY IMPEDANCE

V, 1. — ALGEBRAIC STRUCTURE OF THE APPROXIMATION

As we indicated in the introduction, the basic principle of the method proposed consists of initially smoothing the boundary impedance of the mechanical subsystem, i. e. of constructing a mapping $\omega \mapsto \underset{\sim}{Z}_F(\omega)$ from $\mathbb{R}$ into $\mathrm{Mat}_{\mathbb{C}}(q, q)$, continuous on $\mathbb{R}$ and such that the matrix series $\{\underset{\sim}{Z}_F(\omega_l)\}_l$, (where for any $l \in \{1, \ldots, L\}$, $\omega_l \in B_n$) approaches, in the sense of a certain metric, series $\{Z_F(\omega_l)\}_l$ which is the given of the problem.

Obviously, the choice of the smoothing function, i. e. the choice of the algebraic function of mapping $\omega \mapsto \underset{\sim}{Z}_F(\omega)$ must be made carefully. First of all, this choice must lead to an appropriate underlying differential equation structure for the vibration equation of the coupled system and it must also allow the fluctuations of mapping $\omega \mapsto Z_F(\omega)$ on band B_n to be restored with sufficient accuracy.

In this section, we describe the basic algebraic structure of mapping $\omega \mapsto \underset{\sim}{Z}_F(\omega)$ which will be systematically used herein to construct an approximation of any boundary impedance. In the case at hand, we have chosen the algebraic structure of the boundary impedance of a mechanical system whose vibrations in the time domain are governed by a second-order differential equation with constant coefficients. It is clear that this structure is generally not that of the real boundary impedance of a mechanical system, but simply corresponds to a computation support for the approximation. We will however see that this structure is interesting because it satisfies the two criteria mentioned above; in addition, it is normal to use a model mechanical system with a physical reality.

Therefore, let us take a mechanical system whose dynamics are governed by a second-order differential equation with constant coefficients (for instance, a medium with a linear elastic constitutive law, weakly dissipative, placed in a vacuum). This system is discretized in q'' degrees of freedom. We set $q'' = q + q'$, where q designates the number of degrees of freedom which discretize the part of the boundary relative to which we wish to determine the impedance Z_F of the subsystem.

By breaking the total impedance matrix of the system down into blocks in the normal manner and reducing this matrix on the q degrees of freedom on the boundary by conventional substructuring methods, for any ω of $\mathbb{R}$ we obtain an expression of $Z_F(\omega)$ with the form:

$$Z_F(\omega) = Z_{22}(\omega) - Z_{12}^T(\omega)\,Z_{11}^{-1}(\omega)\,Z_{12}(\omega) \qquad (28)$$

where, considering the hypotheses

$$Z_{jk}(\omega) = -\omega^2 M_{jk} + i\,\omega\, C_{jk} + K_{jk}, \qquad (29)$$
$$j \text{ and } k \in \{1, 2\},$$

where M_{11}, C_{11}, $K_{11} \in \mathrm{Mat}_{\mathbb{C}}(q', q')$; M_{12}, C_{12}, $K_{12} \in \mathrm{Mat}_{\mathbb{C}}(q', q)$ and M_{22}, C_{22}, $K_{22} \in \mathrm{Mat}_{\mathbb{C}}(q, q)$, M_{11}, C_{11}, K_{11}, M_{22}, C_{22} and K_{22} are in addition positive-defined matrices.

Let $\{\psi_k\}_k$, $k \in \{1, \ldots, q'\}$ be the eigenvectors which are solutions of the generalized eigenvalues problem:

$$K_{11}\,\psi = \lambda\, M_{11}\,\psi$$

$\{\lambda_k\}_k$, $k \in \{1, \ldots, q'\}$ are the associated eigenvalues. From a mechanical standpoint, the $\{\psi_k\}_k$ correspond to the natural modes of vibration of the associated conservative discretized system and the q degrees of freedom on the boundary are blocked.

Assuming, as is usually the case, that damping matrix C_{11} is diagonalized by eigenvectors $\{\psi_k\}_k$ and introducing the generalized masses, dampings and stiffnesses associated with eigenmode ψ_k, k being fixed, such that:

$$\begin{aligned} m_k &= \psi_k^T M_{11} \psi_k \\ C_k &= \psi_k^T C_{11} \psi_k \\ k_k &= \psi_k^T K_{11} \psi_k \end{aligned} \tag{31}$$

the admittance relative to eigenmode ψ_k is then represented by the function $\omega \mapsto H_k(\omega)$ continuous from $\mathbb{R}$ into $\mathbb{C}$, defined for every ω in $\mathbb{R}$ by:

$$H_k(\omega) = \frac{1}{-\omega^2 m_k + i\omega C_k + k_k}. \tag{32}$$

Under these conditions, $Z_{11}^{-1}(\omega)$ is written, for every ω in $\mathbb{R}$:

$$Z_{11}^{-1}(\omega) = \sum_{k=1}^{q'} H_k(\omega)\, \psi_k \psi_k^T. \tag{33}$$

It can be noted that mapping $\omega \mapsto H_k(\omega)$ represents the frequency response function of a stable second-order linear filter.

Setting, for any $k \in \{1, \ldots, q'\}$ and any ω in $\mathbb{R}$, $A_k(\omega) = Z_{12}^T(\omega)\, \psi_k$, and using (33), (28) is then written:

$$Z_F(\omega) = Z_{22}(\omega) - \sum_{k=1}^{q'} H_k(\omega)\, A_k(\omega)\, A_k^T(\omega). \tag{34}$$

By reducing (34) to the same denominator, we then observe from equations (29) that the boundary impedance of the subsystem considered has an algebraic rational fraction structure of the type:

$$Z_F(\omega) = \frac{N(\omega)}{d(\omega)} \tag{35}$$

where $N(\omega)$ is a polynomial whose coefficients are complex matrices, symmetrical by construction, and $d(\omega)$ is a polynomial with scalar, complex coefficients. Both polynomials have positive degrees such that degree (N) = degree $(d)+2$. In the case at hand, we have degree $(N) = 2q'+2$ and degree $(d) = 2q'$.

It is this algebraic structure that we will use below.

It can be remarked that the degree of d is systematically even. Furthermore, its term of the lowest degree is a positive real number or zero. Below, we will assume it to be strictly positive.

V, 2. — SMOOTHING PROCEDURES

In order to simplify the expression, we will eliminate subscripts "F" which were used until now to characterize the boundary impedances.

Let $\{Z(\omega_l)\}_l$ be the matrix series forming the given of problem (4.1 d), where, for any $l \in \{1, \ldots, L\}$, $\omega_l \in B_n$. For ω_l fixed, $Z(\omega_l)$ is a complex, symmetrical matrix a priori undefined, i.e. it does not necessarily have the algebraic structure defined by (35).

From the above, we wish to construct a fractional approximation $\underset{\sim}{Z}$ of the boundary impedance represented by mapping $\omega \mapsto \dfrac{N(\omega)}{d(\omega)}$, from $\mathbb{R}$ into $\mathrm{Mat}_{\mathbb{C}}(q, q)$, where $N(\omega)$ and $d(\omega)$ are two complex polynomials, of the matrix and scalar type respectively such that degree (N) = degree $(d)+2$.

To construct this approximation, we naturally introduce distance $\delta(Z, \underset{\sim}{Z})$ of mappings Z and $\underset{\sim}{Z}$ relative to B_n and defined by:

$$\delta(Z, \underset{\sim}{Z}) = \left(\sum_{l=1}^{L} \| Z(\omega_l) - \underset{\sim}{Z}(\omega_l) \|_q^2 \right)^{1/2} \tag{36}$$

where $\| \,.\, \|_q$ represents the matrix norm on $\mathrm{Mat}_{\mathbb{C}}(q, q)$ such that for any Q in $\mathrm{Mat}_{\mathbb{C}}(q, q)$, we have:

$$\| Q \|_q^2 = \sum_{k=1}^{q} \sum_{k'=1}^{q} |Q_{kk'}|^2. \tag{37}$$

Under these conditions, to construct $\underset{\sim}{Z}$, we are led to determine polynomials N and d of variable ω which minimize the quantity:

$$\delta^2(Z, \underset{\sim}{Z}) = \sum_{k=1}^{q} \sum_{k'=1}^{q} \times \left(\sum_{l=1}^{L} \left| Z_{kk'}(\omega_l) - \frac{N_{kk'}(\omega_l)}{d(\omega_l)} \right|^2 \right). \tag{38}$$

As the quantity in parentheses in (38) is positive, it is therefore sufficient to minimize each term of the double sum, i.e. finally to construct a rational approximation $\underset{\sim}{Z}_{kk'}$ of $Z_{kk'}$ separately for each term of the initial matrix.

However, we will not use the smoothing approach defined by (38) directly. In effect, this equation shows that polynomial $d(\omega)$ is common to all the terms $\underset{\sim}{Z}_{kk'}$ of the approximation for k and $k' \in \{1, \ldots, q\}$. And it is this polynomial which is used to represent the fluctuations of the boundary impedance on band B_n.

As the fluctuations of each term of the initial impedance matrix are a priori independent and as we are working on subsystems which can have a high number q of degrees of freedom on the boundary (several hundred), it is clear that such a methodology would lead us to choose polynomials $d(\omega)$ of a very high degree, which would raise serious difficulties from a numerical standpoint.

We have therefore chosen to replace the smoothing technique based on the algebraic structure (35), which could be called global smoothing, by local smoothing consisting of constructing an approximated boundary impedance matrix, each term of which is a rational fraction of variable ω with its own denominator. We are thus led to determine q^2 pairs of polynomials $(N_{kk'}(\omega), D_{kk'}(\omega))$, k and $k' \in \{1, \ldots, q\}$ which minimize the quantity:

$$\delta^2(Z, \underset{\sim}{Z}) = \sum_{k=1}^{q} \sum_{k'=1}^{q} \times \left(\sum_{l=1}^{L} \left| Z_{kk'}(\omega_l) - \frac{N_{kk'}(\omega_l)}{D_{kk'}(\omega_l)} \right|^2 \right) \quad (39)$$

where, for any k and k' in $\{1, \ldots, q\}$, $N_{kk'}$ and $D_{kk'}$ are polynomials in ω with scalar, complex coefficients such that degree $(N_{kk'}) = \text{degree}(D_{kk'}) + 2$.

Of course, for reasons of symmetry, we will construct only $q(q+1)/2$ pairs $(N_{kk'}(\omega), D_{kk'}(\omega))$ and, for any k and k' in $\{1, \ldots, q\}$ and any ω in $\mathbb{R}$, we will have $N_{kk'}(\omega) = N_{k'k}(\omega)$ and $D_{kk'}(\omega) = D_{k'k}(\omega)$.

Under these conditions, it is clear that the numerical problems raised above do not appear since now, each polynomial $D_{kk'}(\omega)$ represents the fluctuations on B_n of a single term of the initial boundary impedance matrix and can therefore be chosen with a moderate degree (a few units). Furthermore, we do not first have to compute the series of determinants $\{\det Z(\omega_l)\}_l$. Obviously, we thereby lose the initial properties of the boundary impedance operator, but this is unimportant since we have not used any property of this type except the symmetry, which is recovered by construction.

V, 3. – CONSTRUCTION OF THE APPROXIMATION

In this section, we will initially reason on any given term of the boundary impedance matrix, i.e. for k and k' fixed in $\{1, \ldots, q\}$.

Let $\{Z_{kk'}(\omega_l)\}_l$ be the complex series representing the given of this term and $\omega \mapsto \dfrac{N_{kk'}(\omega)}{D_{kk'}(\omega)}$ be a rational approximation of $Z_{kk'}$ on B_n.

By hypothesis, $N_{kk'}(\omega)$ and $D_{kk'}(\omega)$ are polynomials on $\mathbb{C}$ such that degree $(N_{kk'}) = \text{degree}$ $(D_{kk'}) + 2$. Obviously, the choice of degrees, arbitrary a priori, will condition the accuracy of the smoothing and must be carefully made so that the fluctuations of mapping $\omega \mapsto Z_{kk'}(\omega)$ on B_n are correctly restored. It can be noted that, according to Section V, 1, and also for reasons which will appear below, an even degree is systematically chosen for $D_{kk'}$. It will be noted $2\tilde{n}_{kk'}$. To construct the approximation $\underset{\sim}{Z}_{kk'}$ of $Z_{kk'}$, we introduce polynomials with real coefficients $\underset{\sim}{N}_{kk'}(p)$ and $\underset{\sim}{D}_{kk'}(p)$ such that for any ω in $\mathbb{R}$, we have:

$$N_{kk'}(\omega) = \underset{\sim}{N}_{kk'}(i\omega), \qquad D_{kk'}(\omega) = \underset{\sim}{D}_{kk'}(i\omega). \quad (40)$$

Obviously, we have degree $(\underset{\sim}{N}_{kk'}) = \text{degree}(N_{kk'})$ and degree $(\underset{\sim}{D}_{kk'}) = \text{degree}(D_{kk'})$.

Furthermore, we will normalize the rational approximation of $Z_{kk'}$ such that the constant term of $\underset{\sim}{D}_{kk'}$ (and therefore of $D_{kk'}$), which is a strictly positive real number according to Section V, 1 is equal to 1.

The smoothing thus made is a smoothing by norm which consists, for $\varepsilon_{kk'}$ real, positive and fixed, chosen arbitarily and which defines the desired proximity on B_n of mappings $Z_{kk'}$ and $\underset{\sim}{Z}_{kk'}$, of determining the real polynomials $\underset{\sim}{N}_{kk'}$ and $\underset{\sim}{D}_{kk'}$ such that:

$$\sum_{l=1}^{L} \left| Z_{kk'}(\omega_l) - \frac{\underset{\sim}{N}_{kk'}(i\omega_l)}{\underset{\sim}{D}_{kk'}(i\omega_l)} \right|^2 \leqq \varepsilon^2 \sum_{l=1}^{L} |Z_{kk'}(\omega_l)|^2. \quad (41)$$

For this purpose we use an iterative algorithm developed for smoothing the admittance measures of linear system [10]. This algorithm consists of minimizing the following quantity by a conventional least squares method on each iteration:

$$\sum_{l=1}^{L} \left| \mathscr{P}_{kk'}^{(j)}(i\omega_l) \left[Z_{kk'}(\omega_l) - \frac{\underset{\sim}{N}_{kk'}^{(j)}(i\omega_l)}{\underset{\sim}{D}_{kk'}^{(j)}(i\omega_l)} \right] \right|^2 \quad (42)$$

where, for the jth iteration, $\underset{\sim}{N}_{kk'}^{(j)}(i\omega)$ and $\underset{\sim}{D}_{kk'}^{(j)}(i\omega)$ are the approximated solutions of $\underset{\sim}{N}_{kk'}(i\omega)$ and $\underset{\sim}{D}_{kk'}(i\omega)$, and $\mathscr{P}_{kk'}^{(j)}(i\omega_l)$ are the weights assigned to each frequency point ω_1 of B_n. Taking $\mathscr{P}_{kk'}^{(1)}(i\omega_l) = 1$, for any $l \in \{1, \ldots, L\}$ we can show [10] that the optimum weights are obtained on the jth iteration by taking $\mathscr{P}_{kk'}^{(j)}(i\omega_l) = \underset{\sim}{D}_{kk'}^{(j)}(i\omega_l)^{-1}$

For reasons which will appear below, we have imposed the condition that, for the smoothing algorithm, all the roots (complex in the general case) of polynomial $\underset{\sim}{D}_{kk'}(p)$ have a strictly negative real part.

Furhermore, from the standpoint of convergence of the algorithm, the iteration stop test is given by (41).

Let $\omega \mapsto \dfrac{\underset{\sim}{N}_{kk'}(i\omega)}{\underset{\sim}{D}_{kk'}(i\omega)}$ be the mapping resulting from the above smoothing. Taking the integer part of the division of $\underline{N}_{kk'}$ by $\underline{D}_{kk'}$, we obtain an expression of the following type for any ω in $\mathbb{R}$:

$$\frac{\underline{N}_{kk'}(i\omega)}{\underline{D}_{kk'}(i\omega)} = \underline{P}_{kk'}(i\omega) + \frac{\underline{N}_{kk'}(i\omega)}{\underline{D}_{kk'}(i\omega)} \qquad (43)$$

where $\underline{N}_{kk'}(p)$ is a real polynomial such that degreee $(\underline{N}_{kk'}) \leqq \text{degree}(\underline{D}_{kk'})-1$ and $\underline{P}_{kk'}(p)$ is a real polynomial of the second degree which we will note:

$$\underline{P}_{kk'}(i\omega) = M_{kk'}(i\omega)^2 + C_{kk'}(i\omega) + K_{kk'} \qquad (44)$$

where $M_{kk'}$, $C_{kk'}$ and $K_{kk'} \in \mathbb{R}$.

$\underline{D}_{kk'}(p)$ is a real polynomial with an even degree $2\tilde{n}_{kk'}$. Therefore, in the general case, it has $2\tilde{n}_{kk'}$ mutually conjugate complex roots. By breaking down the residual fraction of (43) into simple elements and grouping the simple elements associated with conjugate poles in pairs, we readily obtain, using equation (44), the expression of the approximation:

$$\underset{\sim}{Z}_{kk'}(\omega) = -\omega^2 M_{kk'} + i\omega C_{kk'} + K_{kk'} + \sum_{s=1}^{\tilde{n}_{kk'}} \frac{i\omega\alpha_{kk'}^{(s)} + \beta_{kk'}^{(s)}}{-\omega^2 + i\omega\gamma_{kk'}^{(s)} + \delta_{kk'}^{(s)}} \qquad (45)$$

where, for any s in $\{1, \ldots, \tilde{n}_{kk'}\}$, $\alpha_{kk'}^{(s)}$, $\beta_{kk'}^{(s)}$, $\gamma_{kk'}^{(s)}$ and $\delta_{kk'}^{(s)}$ are finite real constants.

In addition, we have $\gamma_{kk'}^{(s)} > 0$ and $\delta_{kk'}^{(s)} > 0$ by construction.

Let us now consider the set of approximations defined by (45) when k and k' describe $\{1, \ldots, q\}$. To condense the expression, we set $\tilde{n} = \sup_{k,k'} \tilde{n}_{kk'}$ and, for k and k' fixed in $\{1, \ldots, q\}$, by convention we will take $\alpha_{kk'}^{(s)} = 0$ and $\beta_{kk'}^{(s)} = 0$ if $\tilde{n}_{kk'} < s \leqq \tilde{n}$. Then, in matrix form, approximation $\omega \mapsto \underset{\sim}{Z}(\omega)$ of the initial boundary impedance $\omega \mapsto Z(\omega)$ is written, for any ω in $\mathbb{R}$:

$$\underset{\sim}{Z}(\omega) = -\omega^2 M + i\omega C + K + \sum_{s=1}^{\tilde{n}} R^{(s)}(\omega) \qquad (46)$$

where M, C and K are real, constant, symmetrical matrices (q, q) with generic terms $M_{kk'}$, $C_{kk'}$ and $K_{kk'}$ respectively, given by (44) and where, for any s in $\{1, \ldots, \tilde{n}\}$ and any ω in $\mathbb{R}$, $R^{(s)}(\omega)$ is a complex, symmetrical matrix (q, q) with generic terms given by:

$$R_{kk'}^{(s)}(\omega) = \frac{i\omega\alpha_{kk'}^{(s)} + \beta_{kk'}^{(s)}}{-\omega^2 + i\omega\gamma_{kk'}^{(s)} + \delta_{kk'}^{(s)}}. \qquad (47)$$

Under these conditions, we can see that for any k and k' in $\{1, \ldots, q\}$ and any s in $\{1, \ldots, \tilde{n}\}$, $\omega \mapsto R_{kk'}^{(s)}(\omega)$ is continuous on $\mathbb{R}$ and bounded.

The advantage of the smoothing technique is now obvious. The polynomial part of the right member of (46) is similar to (20) and therefore directly fits in the framework of the hypotheses of III, 2. As for the fractional part, equation (47) shows that for any k and k' in $\{1, \ldots, q\}$ and any s in $\{1, \ldots, \tilde{n}\}$, $R_{kk'}^{(s)}(\omega)$ is the frequency response function of a stable and causal second-order linear (monovariable) convolution filter, i. e. associated with a second-order differential equation with constant coefficients. By introducing hidden variables, we can therefore use the appropriate conventional numerical schemes to solve the vibration equation of the coupled system.

Remark

It can be noted that, considering the smoothing method used, nothing can be said about the positiveness of matrices M, C, K. Along the lines of the general stability hypotheses introduced in Section IV, 3 for the coupled system, we can possibly replace these matrices by their closest positive approximation (in the sense of the matrices). Thus, for instance, in the case where $\tilde{n} = 0$, i. e. if an approximation of the boundary impedance is sought directly with form (20), we will be sure of satisfying the hypotheses of Section IV, 3.

We will use the method described below to compute these new approximations.

Let A be a real, symmetrical matrix (q, q). We note as $\{\lambda_k\}_k$, $k \in \{1, \ldots, q\}$ the real series of its eigenvalues, ordered by decreasing values $(\lambda_q \geqq \lambda_{q-1} \geqq \ldots \geqq \lambda_1)$ and as $\{\varphi_k\}_k$, $k \in \{1, \ldots, q\}$ the series of associated eigenvectors. It is then known that A is expressed uniquely with the form:

$$A = \sum_{k=1}^{q} \frac{\lambda_k}{\|\varphi_k\|_q^2} \varphi_k \varphi_k^T. \qquad (48)$$

Let λ_{k_0} be the smallest eigenvalue of A, positive or zero, and let A^+ be the real, symmetrical matrix (q, q) defined by:

$$A^+ = \sum_{k=k_0}^{q} \frac{\lambda_k}{\|\varphi_k\|_q^2} \varphi_k \varphi_k^T. \qquad (49)$$

Then, A^+ is a positive matrix by construction and it is easily verified that it is the closest to A for the metric with norm 2.

From an algorithmic standpoint, the eigenvalues of A are computed by a Jacobi method.

VI. – SOLVING THE VIBRATION EQUATION FOR THE COUPLED SYSTEM

VI, 1. – BASIC VIBRATION EQUATION FOR THE COUPLED SYSTEM

The vibration equation (approximated) of the coupled system is given by (26) and (27). To simplify the expression, we will note as $\begin{pmatrix} \hat{U}(\omega) \\ \hat{V}(\omega) \end{pmatrix}$ the breakdown into blocks of the discretized displacement field of the structure, where, for ω fixed, $\hat{U}(\omega)\in\mathbb{C}^r$ and $\hat{V}(\omega)\in\mathbb{C}^q$, r and q being defined in Section IV, 1 c. It is noted that, considering the hypotheses, $U\in H_r(B_n)$ and $V\in H_q(B_n)$.

Thus, for $\mathscr{F}\in H_m(B_n)$, we are led to solving the following matrix equation:

$$\left(\begin{array}{c|c}\mathscr{Z}_{11}(\omega) & \mathscr{Z}_{12}(\omega)\\ \hline \mathscr{Z}_{21}(\omega) & \mathscr{Z}_{22}(\omega)+\underline{Z}(\omega)\end{array}\right)\begin{pmatrix}\hat{U}(\omega)\\ \hat{V}(\omega)\end{pmatrix}=\begin{pmatrix}\hat{\mathscr{F}}_1(\omega)\\ \hat{\mathscr{F}}_2(\omega)\end{pmatrix} \quad (50)$$

in which the dimensions and properties of the blocks of the matrix have already been specified.

By hypothesis [equation (20)], the impedance $\omega\mapsto\mathscr{Z}(\omega)$ is written, for any ω in B_n:

$$\mathscr{Z}(\omega)=-\omega^2\mathscr{M}+i\omega\mathscr{C}+\mathscr{K} \quad (51)$$

where matrices $\mathscr{M}$, $\mathscr{C}$ and $\mathscr{K}$, constant on B_n, have a block structure induced by that of $\mathscr{Z}(\omega)$.

Grouping the polynomial part of approximation $\underline{Z}$ given by (46) with equation (51) and substituting these two equations in (50) yields the basic explicit form for the vibration equation of the coupled system:

$$[-\omega^2\mathbb{M}+i\omega\mathbb{D}+\mathbb{K}]\begin{pmatrix}\hat{U}(\omega)\\ \hat{V}(\omega)\end{pmatrix}+\sum_{s=1}^{\tilde{n}}\left(\begin{array}{c|c}O_{(r,r)} & O_{(r,q)}\\ \hline O_{(q,r)} & R^{(s)}(\omega)\end{array}\right)\begin{pmatrix}\hat{U}(\omega)\\ \hat{V}(\omega)\end{pmatrix}=\hat{\mathscr{F}}(\omega) \quad (52)$$

with

$$\mathbb{M}=\left(\begin{array}{c|c}\mathscr{M}_{11} & \mathscr{M}_{12}\\ \hline \mathscr{M}_{21} & \mathscr{M}_{22}+M\end{array}\right),\quad \mathbb{D}=\left(\begin{array}{c|c}\mathscr{C}_{11} & \mathscr{C}_{12}\\ \hline \mathscr{C}_{21} & \mathscr{C}_{22}+C\end{array}\right),\quad \mathbb{K}=\left(\begin{array}{c|c}\mathscr{K}_{11} & \mathscr{K}_{12}\\ \hline \mathscr{K}_{21} & \mathscr{K}_{22}+K\end{array}\right) \quad (53)$$

where M, C, K and $R^{(s)}(\omega)$ for $s\in\{1,\ldots,\tilde{n}\}$ and $\omega\in B_n$ are the symmetrical matrices (q, q) defined by (46).

The term in brackets in (52) is conventional, i. e. has form (20). It therefore does not raise any particular problems. It is recalled that, considering the hypotheses and the remark of Section V,6, the three matrices comprising it, which are given by (53), are positive-defined symmetrical real matrices (m, m).

However, the terms of the sum do not have a structure similar to (20) and therefore require special processing. This is what we will now examine, introducing hidden variables for the problem considered.

VI, 2. – ADDITIONAL VARIABLES

We will reason here for s fixed in $\{1,\ldots,\tilde{n}\}$.

Let $\omega\mapsto\hat{\mathscr{J}}^{(s)}(\omega)$ be the mapping from $\mathbb{R}$ into $\mathbb{C}^q$ such that, for any ω in B_n, we have:

$$\hat{\mathscr{J}}^{(s)}(\omega)=R^{(s)}(\omega)\,\hat{V}(\omega). \quad (54)$$

For k fixed in $\{1,\ldots,q\}$ and ω fixed in B_n, the kth component of $\hat{\mathscr{J}}^{(s)}(\omega)$ is therefore written:

$$\hat{\mathscr{J}}_k^{(s)}(\omega)=\sum_{k'=1}^{q}\hat{\mathscr{G}}_{kk'}^{(s)}(\omega) \quad (55)$$

with

$$\hat{\mathscr{G}}_{kk'}^{(s)}(\omega)=R_{kk'}^{(s)}(\omega)\,\hat{V}_{k'}(\omega) \quad (56)$$

where $R_{kk'}^{(s)}(\omega)$ is given by (47).

We note that since $V\in H_q(B_n)$ and since $\omega\mapsto R^{(s)}(\omega)$ is bounded on B_n, $\mathscr{G}_{kk'}^{(s)}\in H_1(B_n)$. When k and k' describe $\{1,\ldots,q\}$, and for s fixed in $\{1,\ldots,\tilde{n}\}$, relation (56) defines q^2 mappings $\omega\mapsto\hat{\mathscr{G}}_{kk'}^{(s)}(\omega)$ from $\mathbb{R}$ into $\mathbb{C}$. These q^2 mappings are called the hidden, or additional, variables associated with mapping $\omega\mapsto R^{(s)}(\omega)$ from $\mathbb{R}$ into $\mathrm{Mat}_{\mathbb{C}}(q, q)$.

Considering (47) and since, for k and k' fixed in $\{1,\ldots,q\}$, $\gamma_{kk'}^{(s)}$ and $\delta_{kk'}^{(s)}$ are strictly positive, relation (56) shows that $\mathscr{G}_{kk'}^{(s)}$ results from second-order linear convolution filtering, the filter involved being stable and causal. In the time domain, the equation associated with this filter is a second-order differential equation with constant coefficients. To solve it, we can therefore use the integration scheme mentioned in Section III, 2 for solving (22).

VI, 3. – VIBRATION EQUATION OF THE COUPLED SYSTEM IN THE FREQUENCY DOMAIN

Considering (47), (52), (53), (54) and (55), the vibration equation of the coupled system in the frequency domain is written:

$$(-\omega^2\mathbb{M}+i\omega\mathbb{D}+\mathbb{K})\begin{pmatrix}\hat{U}(\omega)\\ \hat{V}(\omega)\end{pmatrix}+\sum_{s=1}^{\tilde{n}}\begin{pmatrix}0\\ \hat{\mathscr{J}}^{(s)}(\omega)\end{pmatrix}=\hat{\mathscr{F}}(\omega) \quad (57\,a)$$

$$\hat{\mathscr{J}}_k^{(s)}(\omega)=\sum_{k'=1}^{q}\hat{\mathscr{G}}_{kk'}^{(s)}(\omega), \quad (57\,b)$$

$$k\in\{1,\ldots,q\},\qquad s\in\{1,\ldots,\tilde{n}\}$$

$$(-\omega^2+i\omega\gamma_{kk'}^{(s)}+\delta_{kk'}^{(s)})\,\hat{\mathscr{G}}_{kk'}^{(s)}(\omega)=(i\omega\alpha_{kk'}^{(s)}+\beta_{kk'}^{(s)})\,\hat{V}_{k'}(\omega), \quad (57\,c)$$

$$k,k'\in\{1,\ldots,q\},\qquad s\in\{1,\ldots,\tilde{n}\}$$

VI, 4. — VIBRATION EQUATION OF THE COUPLED SYSTEM IN THE TIME DOMAIN

In the time domain, the equation associated with (57) is written:

$$\mathbb{M}\begin{pmatrix}\ddot{U}(t)\\ \ddot{V}(t)\end{pmatrix}+\mathbb{D}\begin{pmatrix}\dot{U}(t)\\ \dot{V}(t)\end{pmatrix}+\mathbb{K}\begin{pmatrix}U(t)\\ V(t)\end{pmatrix}+\sum_{s=1}^{\tilde{n}}\begin{pmatrix}0\\ \mathscr{J}^{(s)}(t)\end{pmatrix}=\mathscr{F}(t) \quad (58a)$$

$$\mathscr{J}_k^{(s)}(t)=\sum_{k'=1}^{q}\mathscr{G}_{kk'}^{(s)}(t), \quad (58b)$$

$$k\in\{1,\ldots,q\}, \qquad s\in\{1,\ldots,\tilde{n}\}$$

$$\ddot{\mathscr{G}}_{kk'}^{(s)}(t)+\gamma_{kk'}^{(s)}\dot{\mathscr{G}}_{kk'}^{(s)}(t)+\delta_{kk'}^{(s)}\mathscr{G}_{kk'}^{(s)}(t)=\alpha_{kk'}^{(s)}\dot{V}_{k'}(t)+\beta_{kk'}^{(s)}V_{k'}(t), \quad (58c)$$

$$k,k'\in\{1,\ldots,q\}, \qquad s\in\{1,\ldots,\tilde{n}\}$$

VI, 5. — SOLVING THE VIBRATION EQUATION OF THE COUPLED SYSTEM

As was indicated in Section III, 2, we will use a special multiple scale technique so that the process amounts to solving a system of differential equations similar to (58) but relative to the low frequency domain.

For this purpose, we introduce the mappings defined by the following relations for any t in $\mathbb{R}$:

$$\mathbb{U}(t)=U(t)\,e^{-i\Omega_n t} \quad (59)$$

$$\mathbb{V}(t)=V(t)\,e^{-i\Omega_n t} \quad (60)$$

$$\mathbb{F}(t)=\mathscr{F}(t)\,e^{-i\Omega_n t} \quad (61)$$

$$\mathbb{J}^{(s)}(t)=\mathscr{J}^{(s)}(t)\,e^{-i\Omega_n t} \quad (62)$$

for any s in $\{1,\ldots,\tilde{n}\}$.

Ω_n as defined in Section III,2.

For s fixed, relation (62) is equivalent to defining q^2 mappings $\mathbb{G}_{kk'}^{(s)}$, for k and $k'\in\{1,\ldots,q\}$ such that:

$$\mathbb{G}_{kk'}^{(s)}(t)=\mathscr{G}_{kk'}^{(s)}(t)\,e^{-i\Omega_n t}. \quad (63)$$

Substituting relations (59)-(62) in (58) gives the new system of differential equations

$$(a)\quad \mathbb{M}\begin{pmatrix}\ddot{\mathbb{U}}(t)\\ \ddot{\mathbb{V}}(t)\end{pmatrix}+(2i\Omega_n\mathbb{M}+\mathbb{D})\begin{pmatrix}\dot{\mathbb{U}}(t)\\ \dot{\mathbb{V}}(t)\end{pmatrix}+(-\Omega_n^2\mathbb{M}+i\Omega_n\mathbb{D}+\mathbb{K})\begin{pmatrix}\mathbb{U}(t)\\ \mathbb{V}(t)\end{pmatrix}+\sum_{s=1}^{\tilde{n}}\begin{pmatrix}0\\ \mathbb{J}^{(s)}(t)\end{pmatrix}=\mathbb{F}(t) \quad (64)$$

$$(b)\quad \mathbb{J}_k^{(s)}(t)=\sum_{k'=1}^{q}\mathbb{G}_{kk'}^{(s)}(t),$$

$$k\in\{1,\ldots,q\}, \qquad s\in\{1,\ldots,\tilde{n}\}$$

$$(c)\quad \ddot{\mathbb{G}}_{kk'}^{(s)}(t)+c_{kk'}^{(s)}\dot{\mathbb{G}}_{kk'}^{(s)}(t)+d_{kk'}^{(s)}\mathbb{G}_{kk'}^{(s)}(t)=a_{kk'}^{(s)}\dot{\mathbb{V}}_{k'}(t)+b_{kk'}^{(s)}\mathbb{V}_{k'}(t),$$

$$k,k'\in\{1,\ldots,q\}, \qquad s\in\{1,\ldots,\tilde{n}\}$$

where, for any k and k' in $\{1,\ldots,q\}$ and any s in $\{1,\ldots,\tilde{n}\}$, we set:

$$\begin{aligned} a_{kk'}^{(s)}&=\alpha_{kk'}^{(s)}\\ b_{kk'}^{(s)}&=\beta_{kk'}^{(s)}+i\Omega_n\alpha_{kk'}^{(s)}\\ c_{kk'}^{(s)}&=2i\Omega_n+\gamma_{kk'}^{(s)}\\ d_{kk'}^{(s)}&=i\Omega_n\gamma_{kk'}^{(s)}-\Omega_n^2+\delta_{kk'}^{(s)}. \end{aligned} \quad (65)$$

Since $U\in H_r(B_n)$, $V\in H_q(B_n)$, $\mathscr{F}\in H_m(B_n)$ and $\mathscr{G}_{kk'}^{(s)}\in H_1(B_n)$, for any s in $\{1,\ldots,\tilde{n}\}$, it is easily verified that $\mathbb{U}\in H_r(B_0)$, $\mathbb{V}\in H_q(B_0)$, $\mathbb{F}\in H_m(B_0)$ and $\mathbb{G}_{kk'}^{(s)}\in H_1(B_0)$ for any s in $\{1,\ldots,\tilde{n}\}$, where B_0 is the "low frequency" bounded closed interval of $\mathbb{R}$ associated with B_n which is written $B_0=\left[-\frac{\delta\omega}{2};\frac{\delta\omega}{2}\right]$.

Accordingly, for given initial conditions, we can now solve equation (64) by direct numerical integration over time, step by step. For this purpose, we use Newmark's algorithm with parameters $\alpha=\frac{1}{4}$ and $\beta=\frac{1}{2}$. It is a centered, implicit, unconditionally stable numerical scheme whose equations are given by:

$$\ddot{U}(t_{m+1})=\frac{4}{\Delta t^2}(U(t_{m+1})-U(t_m))-\frac{4}{\Delta t}\dot{U}(t_m)-\ddot{U}(t_m) \quad (66)$$

$$\dot{U}(t_{m+1})=\dot{U}(t_m)+\frac{\Delta t}{2}\ddot{U}(t_m)+\frac{\Delta t}{2}\ddot{U}(t_{m+1})$$

where $\Delta t=t_{m+1}-t_m$ is the integration time step.

Using (66) to solve (64 c) at time t_m, we obtain the following equations for hidden variables $\mathbb{G}_{kk'}^{(s)}$, k and $k'\in\{1,\ldots,q\}$ and $s\in\{1,\ldots,\tilde{n}\}$:

$$e_{kk'}^{(s)}\mathbb{G}_{kk'}^{(s)}(t_m)=\mathbb{H}_{kk'}^{(s)}(t_{m-1})+\left(a_{kk'}^{(s)}+\frac{2}{\Delta t}b_{kk'}^{(s)}\right)\mathbb{V}_{k'}(t_m)-\frac{2}{\Delta t}b_{kk'}^{(s)}\mathbb{V}_{k'}(t_{m-1})-b_{kk'}^{(s)}\dot{\mathbb{V}}_{k'}(t_{m-1})$$

$$\mathbb{H}^{(s)}_{kk'}(t_{m-1})=\left(\frac{4}{\Delta t^2}+\frac{2}{\Delta t}c^{(s)}_{kk'}\right)\mathbb{G}^{(s)}_{kk'}(t_{m-1}) \quad (67)$$
$$+\left(\frac{4}{\Delta t}+c^{(s)}_{kk'}\right)\dot{\mathbb{G}}^{(s)}_{kk'}(t_{m-1})$$
$$+\ddot{\mathbb{G}}^{(s)}_{kk'}(t_{m-1})$$

where

$$e^{(s)}_{kk'}=\frac{4}{\Delta t^2}+\frac{2}{\Delta t}c^{(s)}_{kk'}+d^{(s)}_{kk'}. \quad (68)$$

Applying the scheme in parallel to equation ($64\,a$) and using (67) gives the final recurrent solution equations which are written:

$$\underline{A}\,\mathbb{U}(t_m)=\mathbb{I}(t_m)+\underline{B}\,\mathbb{U}(t_{m-1})$$
$$+\underline{C}\,\dot{\mathbb{U}}(t_{m-1})+\underline{D}\,\ddot{\mathbb{U}}(t_{m-1}) \quad (69)$$

where, for any k in $\{1,\ldots,q\}$

$$\mathbb{I}_k(t_m)=\mathbb{F}_k(t_m)-\sum_{s=1}^{\tilde{n}}\sum_{k'=1}^{q}\mathbb{H}^{(s)}_{kk'}(t_{m-1}) \quad (70)$$

and where $\underline{A}$, $\underline{B}$, $\underline{C}$ and $\underline{D}$ are complex, symmetrical matrices (m, m) which accept the following breakdowns by blocks:

$$\underline{A}=\left(-\Omega_n^2+\frac{4i\Omega_n}{\Delta t}+\frac{4}{\Delta t^2}\right)\left(\begin{array}{c|c}\mathcal{M}_{11} & \mathcal{M}_{12}\\ \hline \mathcal{M}_{21} & \mathcal{M}_{22}+M\end{array}\right)$$
$$+\left(i\Omega_n+\frac{4i\Omega_n}{\Delta t}\right)\left(\begin{array}{c|c}\mathcal{C}_{11} & \mathcal{C}_{12}\\ \hline \mathcal{C}_{21} & \mathcal{C}_{22}+C\end{array}\right)$$
$$+\left(\begin{array}{c|c}\mathcal{K}_{11} & \mathcal{K}_{12}\\ \hline \mathcal{K}_{21} & \mathcal{K}_{22}+K\end{array}\right)$$
$$+\sum_{s=1}^{\tilde{n}}\left[\left(\begin{array}{c|c}0 & 0\\ \hline 0 & \mathcal{T}^{(s)}\end{array}\right)+\frac{2}{\Delta t}\left(\begin{array}{c|c}0 & 0\\ \hline 0 & \mathcal{S}^{(s)}\end{array}\right)\right) \quad (71)$$

$$\underline{B}=\left(\frac{4}{\Delta t^2}+\frac{4i\Omega_n}{\Delta t}\right)\left(\begin{array}{c|c}\mathcal{M}_{11} & \mathcal{M}_{12}\\ \hline \mathcal{M}_{21} & \mathcal{M}_{22}+M\end{array}\right)$$
$$+\frac{2}{\Delta t}\left(\begin{array}{c|c}\mathcal{C}_{11} & \mathcal{C}_{12}\\ \hline \mathcal{C}_{21} & \mathcal{C}_{22}+C\end{array}\right)$$
$$+\sum_{s=1}^{\tilde{n}}\frac{2}{\Delta t}\left(\begin{array}{c|c}0 & 0\\ \hline 0 & \mathcal{S}^{(s)}\end{array}\right) \quad (72)$$

$$\underline{C}=\left(\frac{4}{\Delta t}+2i\Omega_n\right)\left(\begin{array}{c|c}\mathcal{M}_{11} & \mathcal{M}_{12}\\ \hline \mathcal{M}_{21} & \mathcal{M}_{22}+M\end{array}\right)$$
$$+\left(\begin{array}{cc}\mathcal{C}_{11} & \mathcal{C}_{12}\\ \mathcal{C}_{21} & \mathcal{C}_{22}+C\end{array}\right)+\sum_{s=1}^{\tilde{n}}\left(\begin{array}{c|c}0 & 0\\ \hline 0 & \mathcal{S}^{(s)}\end{array}\right) \quad (73)$$

$$\underline{D}=\left(\begin{array}{c|c}\mathcal{M}_{11} & \mathcal{M}_{12}\\ \hline \mathcal{M}_{21} & \mathcal{M}_{22}+M\end{array}\right) \quad (74)$$

where, for any s in $\{1,\ldots,\tilde{n}\}$, $\mathcal{T}^{(s)}$ and $\mathcal{S}^{(s)}$ are complex, symmetrical matrices (q, q) with generic terms $\mathcal{T}^{(s)}_{kk'}$ and $\mathcal{S}^{(s)}_{kk'}$ given respectively by:

$$\mathcal{T}^{(s)}_{kk'}=\frac{a^{(s)}_{kk'}}{e^{(s)}_{kk'}} \quad (75)$$

$$\mathcal{S}^{(s)}_{kk'}=\frac{b^{(s)}_{kk'}}{e^{(s)}_{kk'}} \quad (76)$$

where $a^{(s)}_{kk'}$, $b^{(s)}_{kk'}$ and $e^{(s)}_{kk'}$ are given by (65) and (68).

Equations (69)-(76) are the new equations of the scheme to be used in the MF algorithm to solve our problem. As the remainder of the algorithm is not modified, we refer the reader to [29] for the additional detailed developments of the solution (choice of the integration time step, the initial time, the final time, etc.). We will not come back either on computation of the solution $\begin{pmatrix}U\\V\end{pmatrix}$ or $\begin{pmatrix}\hat{U}\\\hat{V}\end{pmatrix}$ from the low frequency solution $\begin{pmatrix}\mathbb{U}\\\mathbb{V}\end{pmatrix}$ which is also described in [29].

VII. — NUMERICAL APPLICATIONS

VII, 1. — ELEMENTARY EXAMPLE

In this section, we consider two simple linear oscillators respectively subscripted 1 and 2. Their mechanical characteristics are point masses m_1 and m_2, the viscous dissipation constants are c_1, c_2 and the stiffness constants are k_1 and k_2. We will note as ω_1 and ω_2 ($\omega_1>0$, $\omega_2>0$) the natural vibration frequencies of the associated conservative systems with fixed support; in this case, we have $k_1=m_1\omega_1^2$, $k_2=m_2\omega_2^2$, and by introducing the critical damping rates ξ_1 and ξ_2 (with $0<\xi_1<1$ and $0<\xi_2<1$ by hypothesis), we also have $c_1=2m_1\xi_1\omega_1$ and $c_2=2m_2\xi_2\omega_2$.

We will investigate the coupled mechanical system consisting of two oscillators in series in a reference frame, with the support of oscillator 1 taken as support of the coupled system and assumed fixed.

With the mechanical excitation force F applied to mass m_1, the displacement $\omega\mapsto\hat{X}(\omega)$ of this mass, taken as reference solution, is expressed for any ω and $\mathbb{R}$:

$$\hat{X}(\omega)=\hat{F}(\omega)\frac{-\omega^2 m_2+i\omega c_2+k_2}{[(-\omega^2 m_1+i\omega c_1+k_1)\times(-\omega^2 m_2+i\omega c_2+k_2)-\omega^2 m_2(i\omega c_2+k_2)]}. \quad (77)$$

We now assume that oscillator 2 is represented, with respect to oscillator 1, by its boundary impedance relative to the connection point of the two oscillators and we compute the displacement $\omega\mapsto\hat{\underline{X}}(\omega)$ of mass m_1 by the method proposed.

For this simple example, we can obtain the explicit algebraic expression of this boundary impedance. In the case at hand, we have, for any ω in $\mathbb{R}$:

$$Z(\omega) = -m_2\,\omega_2^2 \frac{((\omega_2^2/\omega^2) - 1 + 4\,\xi_2^2)}{((\omega_2^2/\omega^2) - 1)^2 + 4\,\xi_2^2\,(\omega_2^2/\omega^2)}$$

$$+ i\,\omega \frac{2\,m_2\,\xi_2\,\omega_2}{((\omega_2^2/\omega^2) - 1)^2 + 4\,\xi_2^2\,(\omega_2^2/\omega^2)}. \quad (78)$$

Here we can note that the boundary impedance initially exhibits a rational fraction algebraic structure.

The computations were made for $\omega \in [2\pi \times 2{,}000, 2\pi \times 2{,}100]$ with the following mechanical constants: $m_1 = 100$, $m_2 = 0.1$, $\xi_1 = 0.003$, $\xi_2 = 0.002$, $\omega_1 = 2\pi \times 2{,}034$. Concerning the substructure, we considered three cases: $\omega_2 = 2\pi \times 1{,}700$ (static), $\omega_2 = 2\pi \times 2{,}068$ (dynamic) and $\omega_2 = 2\pi \times 2{,}150$ (vibrational isolation).

For smoothing the impedance, we used 101 frequency points in the band.

The results are given in Figures 1 to 3 which illustrate mappings $\omega \mapsto 10 \log_{10} \left| \frac{\hat{X}(\omega)}{\hat{F}(\omega)} \right|^2$ (solid line) and $\omega \mapsto 10 \log_{10} \left| \frac{\underset{\sim}{X}(\omega)}{\hat{F}(\omega)} \right|^2$ (dashed line) for the three above cases. These results are satisfactory.

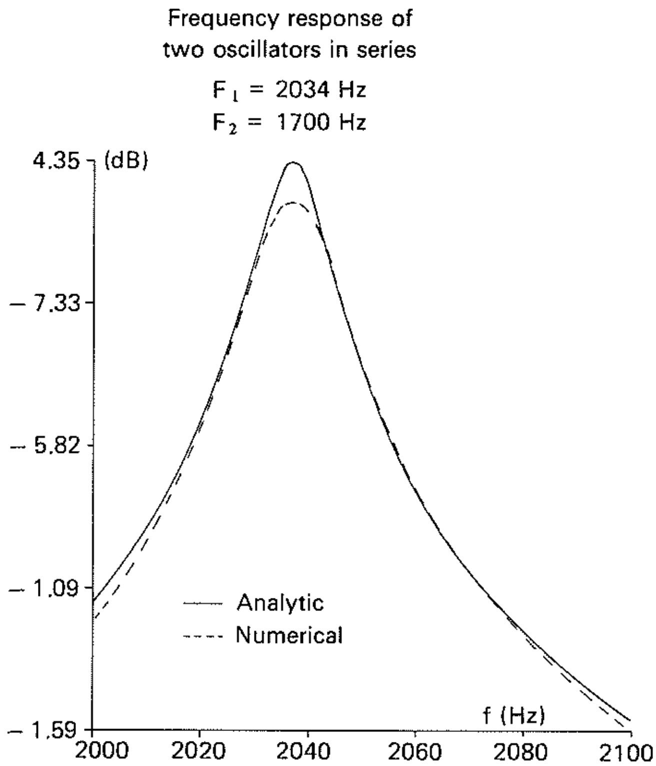

Fig. 1. – Elementary system.

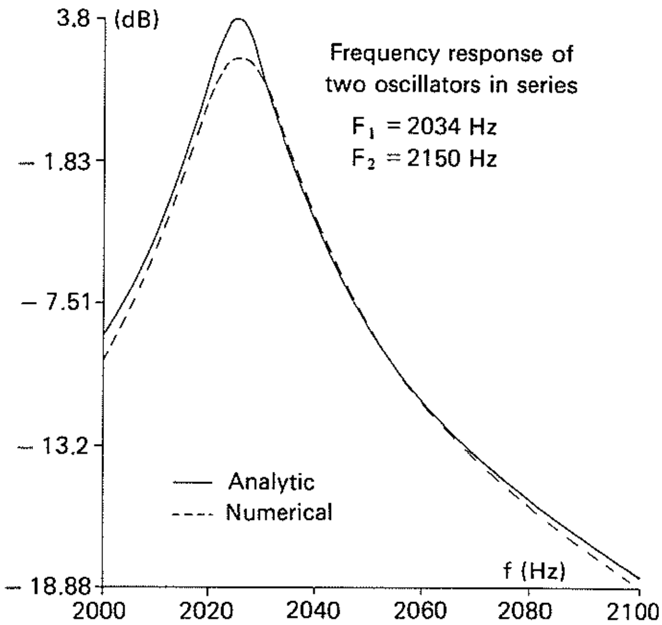

Fig. 2. – Elementary system.

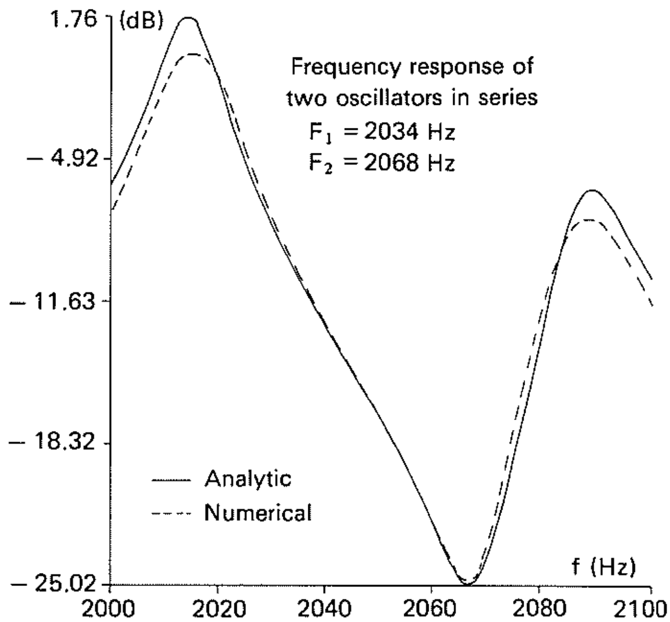

Fig. 3. – Elementary system.

VII, 2. – ONE-DIMENSIONAL SYSTEM

The second mechanical system investigated is a straight beam located in plane Oxy, with axis Ox, origin $x=0$, length $2L$, clamped in $x=0$ and free in $x=2L$.

The beam is homogeneous, isotropic, with length $2L=2$, density $\rho=31{,}400$, Young's modulus $E=2.1 \times 10^{11}$, Poisson ratio $\nu=0.3$ and structural damping $\xi=0.003$.

It is modeled by 150 straight beam finite elements with two nodes, each node with 2 degrees of freedom (a translation on Oy and a rotation around Oz). To complexify this beam, we added finite lineal elements with two structural fuzzy nodes, homogeneous, orthotropic, the mean lineal mass of the fuzzy being $\underline{m}=5.2\times 10^{-3}$ and its mean modal density being $\underline{n}=2.75$.

The analysis band chosen is $\omega\in[2\pi\times 100, 2\pi\times 1\,000]$. It is divided into ten subbands for the MF method. For this analysis band, we chose a point mechanical excitation $\omega\mapsto\hat{F}(\omega)$, with unit amplitude, applied to point $x=1$ in direction Oy.

First the response of the complexified mechanical system was computed. This response was the reference solution.

Then the beam was truncated in the middle and the truncated part $x\geqq 1$ was modeled by a boundary impedance applied to node $x=1$. We therefore replaced the initial mechanical system by a master system (the part of the beam $0\leqq x\leqq 1$) plus a substructure represented by its boundary impedance (diagram 1). It should be noted that in this operation, the structural fuzzy was left present in the master system and the substructure.

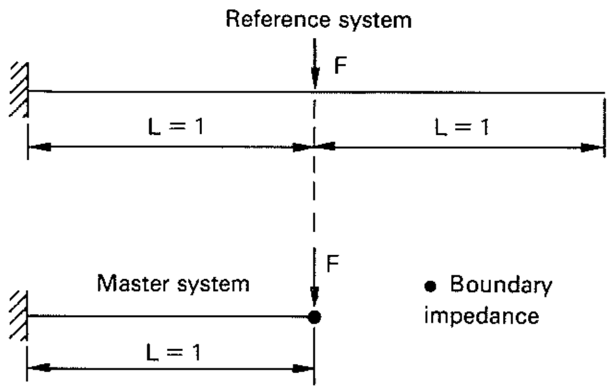

Diagram 1. – One-dimensional system.

The boundary impedance of the substructure is computed numerically by the finite element method [6]. For ω fixed, $Z(\omega)$ is a (2,2) matrix. Figure 4 shows the (1,1) term on the analysis band.

For computations with the method proposed, the boundary impedance was smoothed on each MF narrow band (with width $2\pi\times 100$) using twenty frequency points in each band. For each term of the matrix, the degree of the denominator was taken equal to 2 and the sequence mentioned in the remark of Section V, 6 was used.

The results are given in Figures 5 and 6 which represent the transverse acceleration energy in decibels computed on the elementary bands with width $2\pi\times 5$. The response of the initial structure (complete) without fuzzy is also shown for reference.

Globally, it can be said that the results are very good. It can however be noted that the situation was relatively unfavorable insofar as, for the analysis band chosen, the response of the complete pure structure is more of the modal type.

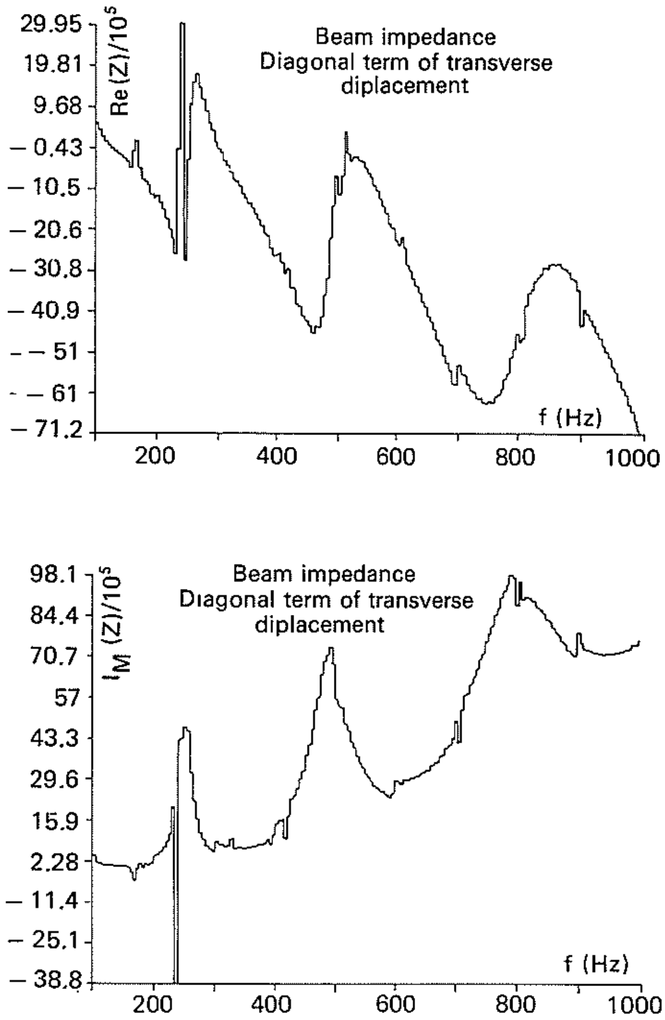

Fig. 4. – Example of boundary impedance.

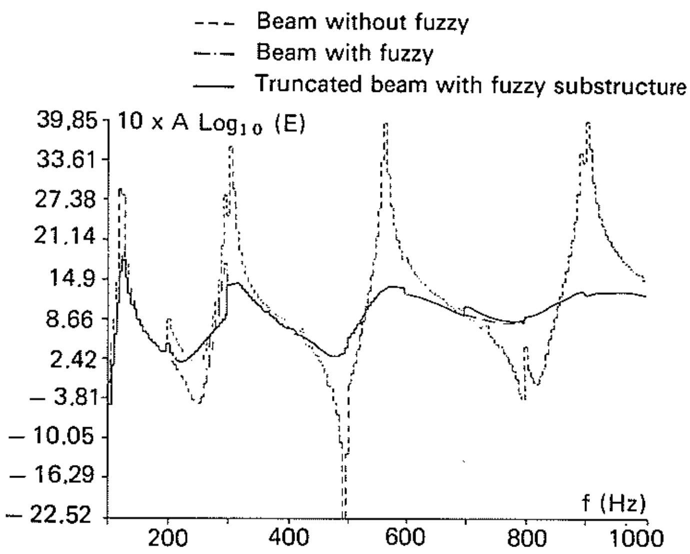

Fig. 5. – One-dimensional system. Excitation point.

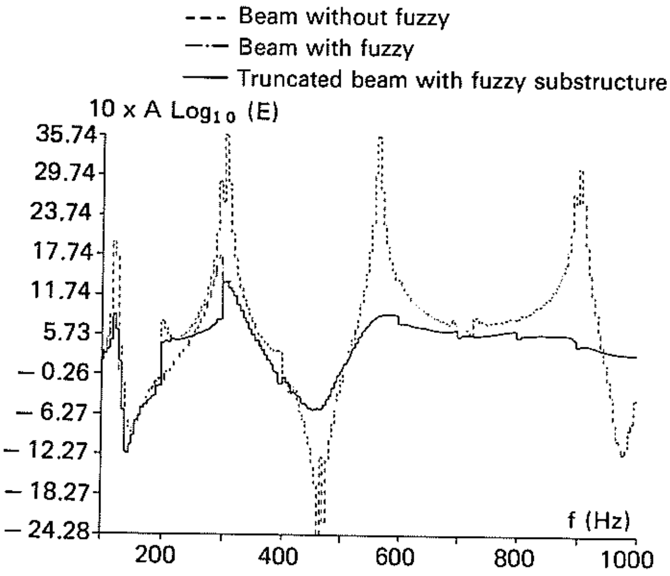

Fig. 6. – One-dimensional system.
Point $x=\frac{L}{2}$.

In effect, it is known that a certain "organization" of the vibrations occurs which it is difficult to restore when the system is truncated.

VII, 3. – THREE-DIMENSIONAL SYSTEM

The third numerical application concerns a smooth cylindrical shell, with axis Oz in a cylindrical reference system (r, θ, z), clamped at end $z=0$ and free at the other end. It is open at the ends.

The shell is homogeneous, isotropic, with total length $2L=2.52$, mean radius $R=0.619$, thickness $e=0.045$, Youngs's modulus $E=4.5\times10^{10}$, density $\rho=1{,}920$, Poisson ratio $\nu=0.3$ and structural damping $\xi=0.003$ constant on the analysis band.

Considering the plans of geometric symmetry, only a quarter of the shell $\left(0\leqq\theta\leqq\frac{\pi}{2}\right)$ was modeled by 96 finite thin shell elements with eight nodes. The cylinder was also complexified with a homogeneous, orthotropic structural fuzzy (96 finite surface fuzzy elements with eight nodes), the mean surface mass of the fuzzy being $\underline{m}=0.326$ and its mean modal density being $\underline{n}=0.03$.

The analysis band is $\omega\in[2\pi\times200,\ 2\pi\times1{,}200]$ and is divided into ten narrow bands for the MF method. The excitation is a point force with unit amplitude applied in the radial direction to point $(r=R,\ \theta=0,\ z=L)$.

The principle of the computations is strictly identical to that of VII, 2. The initial system was truncated at abscissa $z=L$, by a plane perpendicular to Oz and part $0\leqq z\leqq L$ was taken as substructure of the initial system (diagram 2). It can be noted that here, the substructure contains the boundary condition of the clamped end.

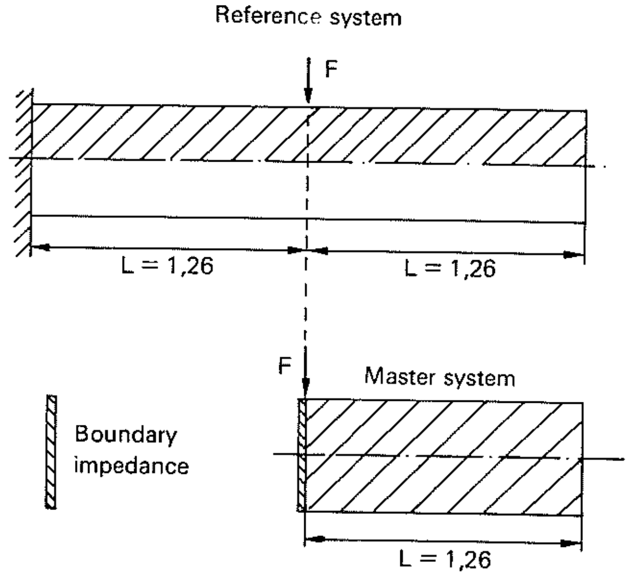

Diagram 2. – Three-dimensional system.

For a given ω in the analysis band, the boundary impedance was a complex (61,61) matrix (37 degrees of freedom in translation and 24 in rotation) which was numerically computed by the finite element method as in Section VII, 2. The smoothing was then carried out separately on each MF narrow band with 20 frequency points in each band. For each term of the impedance matrix, the degree of the denominator was taken equal to 8.

The results are given in Figures 7 to 9. They show the energy in decibels of the radial acceleration

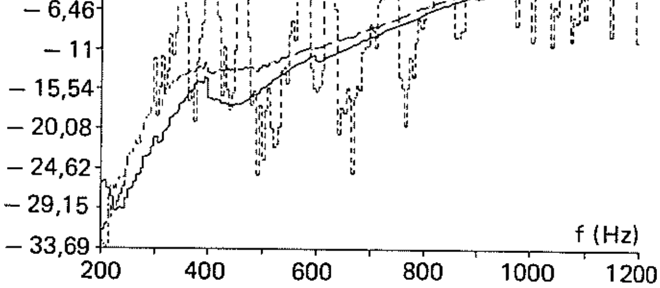

Fig. 7. – 3 D system.
Excitation point.

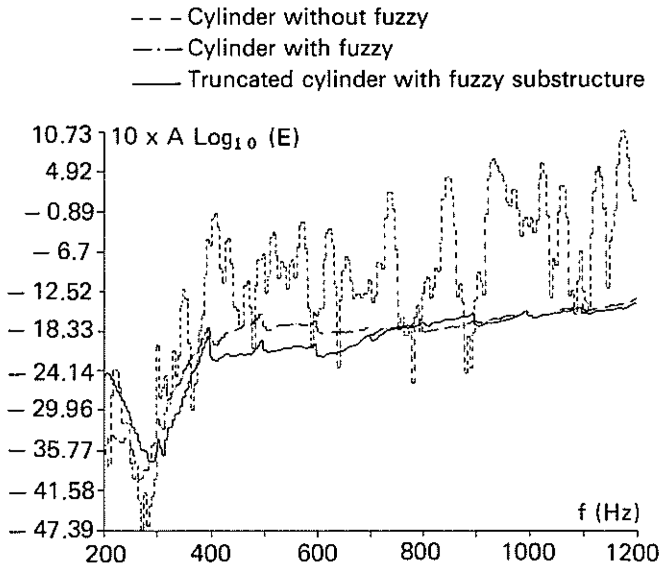

Fig. 8. − 3 D system.

Point $\left(r=R,\ \theta=0,\ z=\frac{6}{5}L\right)$.

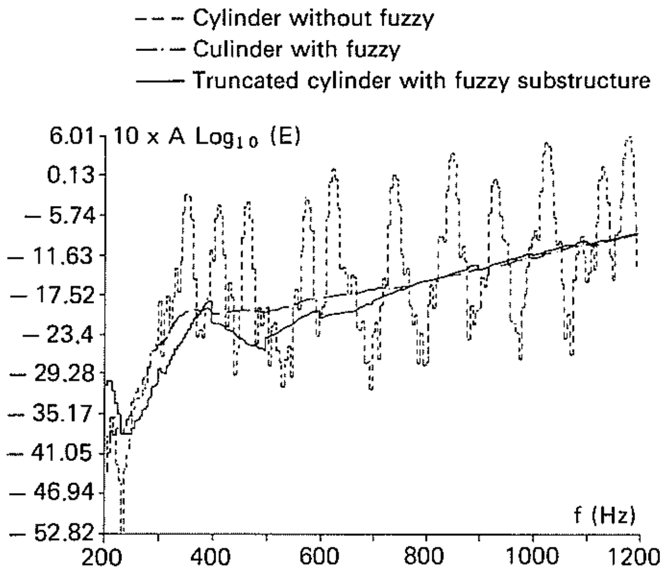

Fig. 9. − 3 D system.

Point $\left(r=R,\ \theta=\frac{\pi}{12},\ z=L\right)$.

computed on bands of width $2\pi\times 5$. The observation points are the excitation point, the point on generatrix $\left(r=R,\ \theta=0,\ z=\frac{6}{5}L\right)$ and the point of the truncation directrix $\left(r=R,\ \theta=\frac{\pi}{12}, z=L\right)$.

Here again, the results can be considered satisfactory taking into account the large mechanical disturbance contributed to the master system by the substructure. It can be noted that the prediction is even excellent in the typical MF domain which is of particular interest to us.

VIII. − CONCLUSION

This paper describes a method used to introduce mechanical substructures characterized by a boundary impedance in finite element computations. We described the technique used to systematically reduce the problem to second-order differential equations with constant coefficients. This technique allows the direct integration methods already developed in the time domain to be used to solve the problem.

For the dynamic analysis of complex mechanical systems, this method should make it possible to contemplate systematic parametric studies at lower cost, for instance in view of an optimum definition of the main parts of the system. Another interesting aspect is the possibility of using an experimental characterization of certain parts of a mechanical system to predict the vibrational behavior of such a system by computation.

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