Hybrid numerical method for solving the harmonic Maxwell equations: II - Construction of the numerical approximations

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HYBRID NUMERICAL METHOD FOR SOLVING THE HARMONIC MAXWELL EQUATIONS: II. CONSTRUCTION OF THE NUMERICAL APPROXIMATIONS

by

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ABSTRACT

Part II of this paper deals with the numerical aspects of the mathematical formulation, presented in Part I, for the solution of Maxwell's harmonic equations. Linear isoparametric finite elements are used for external integral and internal differential operators. Three-node and four-node finite elements are used for the surface and volume parts, respectively. Robust numerical approximations were constructed for the integral operators.

Keywords (NASA thesaurus): Electromagnetic wave – Scatter propagation – Numerical method – Finite element method – Integral equations.

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I. – INTRODUCTION

The present second part of our three-part paper is devoted to the numerical approximations of the various operators used in the formulation developed in the first Part, for solving Maxwell's harmonic equations. All of the notation remains the same. Details can be found in [1].

II. – DISCRETIZATION OPTIONS

For the reasons explained in section II, 2 of Part I, it is important to minimize the numerical costs of constructing elementary matrices of the various finite elements used in the formulation. So we chose finite elements of the lowest possible degree that are compatible with the minimum required regularity, namely isoparametric finite elements P_1, which are triangular for the surface elements and tetrahedrons for the volume elements, for the integral operators and for the differential surface and volume operators.

For the integral operators, we have constructed numerical approximations that are good for any geometrical configuration.

III. – ADAPTATION OF THE INTEGRAL FORMULATION TO A NUMERICAL PROCESS

The integral formulation operators for the external problem, defined by formulae (61) of Part I, are adapted by the finite element method.

III,1. – FINITE ELEMENTS ON THE EXTERNAL SURFACE

III,1.1. – *Grid Generation for the External Surface*

The surface $\partial\Omega_i = \Gamma_c \cup \Gamma_d$ is meshed with triangles. These must, of course, comply with the usual compatibility conditions for finite elements, which we will assume to hold from now on. We will also assume that the polyhedral surface defined by the grid is the true surface $\partial\Omega_i$ and will use the same notation for each, to simplify.

III,1.2. – *Triangular Finite Element P_1 and Approximation Space*

We cannot use P_0 elements because they result in an approximation space that does not have the regularity required for the integral operators of the problem at hand. P_1 elements will do, however, and the approximation space for all operators on $\partial\Omega_i$ will be the space of affine functions by triangle, and continuous on the surface $\partial\Omega_i$.

We will use T to denote an arbitrary finite element of the grid $\partial\Omega_i = \Gamma_c \cup \Gamma_d$. This is a triangle with apexes M_1, M_2 and M_3. The coordinates of the node M_j will be denoted (x_j, v_j, z_j) in the cartesian frame of reference introduced in section III,1 of Part I. The notation T will designate the element, but also the surface of $\mathbb{R}^3$ defined by the triangle (M_1, M_2, M_3). To simplify, we will use M_j to denote the vector OM_j of $\mathbb{R}^3$ of components (x_j, y_j, z_j). The area of the surface element T (area of surface T) is:

$$|T| = \int_T ds = \frac{1}{2} \| (M_2 - M_1) \times (M_3 - M_1) \| \quad (1)$$

in which $\| V \|$ designates the euclidian norm of $V \in \mathbb{R}^3$.

III,1.3. – *Element Size*

The fixed wave number k is associated with the wavelength $\Lambda = \frac{2\pi}{k}$. By hypothesis, the grid of $\partial\Omega_i$ is generated such that $\forall T$, $\sqrt{T} \ll \Lambda$. This hypothesis will be of constant use in developing the numerical process.

III,1.4. – *Normal and Orientation*

Let n_T be the unit normal to the planar surface T such that:

$$n_T = \frac{(M_2 - M_1) \times (M_3 - M_1)}{\| (M_2 - M_1) \times (M_3 - M_1) \|} = \frac{1}{2|T|} (M_2 - M_1) \times (M_3 - M_1) \quad (2)$$

We define the element $W_T = n_T |T|$ of $\mathbb{R}^3$ which, considering (1), is written:

$$W_T = n_T |T| = \frac{1}{2} (M_2 - M_1) \times (M_3 - M_1) \in \mathbb{R}^3 \quad (3)$$

The surface $\partial\Omega_i$ is oriented so that the unit normal is directed outward at $\partial\Omega_i$. We assume from here on that the triplet M_1, M_2, M_3 is ordered such that the normal n_T defined by (2) is outward at $\partial\Omega_i$. This ordered triplet M_1, M_2, M_3 defines the incidence of the element T and we deduce from this the sense of the curvilinear abscissa on the closest edge ∂T of T:

$$\partial T = M_1 M_2 + M_2 M_3 + M_3 M_1 \quad (4)$$

III,2. – P_1 INTERPOLATION FORMULAE

We will use (x, y, z) from now on to designate the cartesian coordinates of a given point M on T, where we will use the barycentric coordinates $\lambda_1, \lambda_2, \lambda_3$ for parametering the surface:

$$M = \lambda_1 M_1 + \lambda_2 M_2 + \lambda_3 M_3 \in T \quad (5\text{-}1)$$

$$\lambda_1 + \lambda_2 + \lambda_3 = 1, \quad 0 \leqq \lambda_i \leqq 1, \quad \forall i \in \{1, 2, 3\} \quad (5\text{-}2)$$

Let A_T be a function on T with values in $\mathbb{K}^3$, and A_T^1, A_T^2, A_T^3 the values of A_T in M_1, M_2, M_3:

$$A_T^i = A_T(M_i) \in \mathbb{K}^3, \qquad i \in \{1, 2, 3\} \quad (6)$$

We then have:

$$A_T = \lambda_1 A_T^1 + \lambda_2 A_T^2 + \lambda_3 A_T^3 \quad (7)$$

We will use the same conventions for expressing a function φ_T continuous on T with values in $\mathbb{K}^3$.

III,2.1. – *Integration Formula*

If $\alpha_1, \alpha_2, \alpha_3$ are three positive or zero integers, we have the formula:

$$\int_T \lambda_1^{\alpha_1} \lambda_2^{\alpha_2} \lambda_3^{\alpha_3} \, ds = 2\,|T|\, \frac{\alpha_1!\,\alpha_2!\,\alpha_3!}{(\alpha_1 + \alpha_2 + \alpha_3 + 2)!} \quad (8)$$

III,2.2. – *Barycentric Averages*

We will use $\underline{M}_T$ to denote the point T that is the barycenter of the triangle M_1, M_2, M_3:

$$\underline{M}_T = \frac{1}{|T|} \int_T M \, ds = \frac{1}{3}(M_1 + M_2 + M_3) \quad (9)$$

and we will use $\underline{A}_T \in \mathbb{K}^3$ to denote the barycentric average of A_T on T:

$$\underline{A}_T = \frac{1}{|T|} \int_T A_T \, ds = \frac{1}{3}(A_T^1 + A_T^2 + A_T^3) \quad (10)$$

We then get the following relations:

$$\int_T A_T \, ds = |T|\, \underline{A}_T \quad (11\text{-}1)$$

$$\int_T A_T \times n \, ds = -W_T^0 \underline{A}_T, \quad W_T^0 = (W_T \times) \quad (11\text{-}2)$$

$$\int_T A_T \, . \, n \, ds = \underline{A}_T \, . \, W_T \quad (11\text{-}3)$$

III.2.3. – *Calculation of the Integral over T of Differential Form $d(A_T\,dM)$*

Considering the orientation of the edge ∂T, we get from Stokes' theorem:

$$\int_T d(A_T \, . \, dM) = A_T^1 \, . \, V_T^1 + A_T^2 \, . \, V_T^2 + A_T^3 \, . \, V_T^3 \quad (12)$$

with $V_T^i \in \mathbb{R}^3$ for $i \in \{1, 2, 3\}$, and:

$$V_T^1 = \frac{1}{2}(M_2 - M_3) \quad (13\text{-}1)$$

$$V_T^2 = \frac{1}{2}(M_3 - M_1) \quad (13\text{-}2)$$

$$V_T^3 = \frac{1}{2}(M_1 - M_2) \quad (13\text{-}3)$$

III,3. – FORMULAE FOR THE KERNEL OF THE EXTERNAL PROBLEM

To state the expressions for the integral operators explicitly in numerical form, we will need information concerning the gradient of the kernel $N(r)$ defined by (34) in Part 1, the solid angle, the normal derivative of the kernel and lastly its average value.

III,3.1. – *Kernel Gradient*

Once all of the calculations are made, this is expressed:

$$\text{grad}' N = g(r) \frac{1}{r^3}(M' - M) \quad (14\text{-}1)$$

$$g(r) = -(1 + jkr)\, e^{-jkr} \quad (14\text{-}2)$$

III,3.2. – *Solid Angle*

Let M be a fixed point of $\mathbb{R}^3$. And let ω'_M be the solid-angle differential form, which is of class C^∞ and is closed in the complement of $M : d'\omega'_M = 0.$, expressed [17]:

$$\omega'_M = \frac{1}{r^3}((x' - x)\, dy' \wedge dz' + (y' - y)\, dz' \wedge dx' + (z' - z)\, dx' \wedge dy') \quad (15\text{-}1)$$

or, using the definitions of section III,4.2 in Part I:

$$\omega'_M = \frac{1}{r^3}(M' - M) \, . \, n' \, ds' \quad (15\text{-}2)$$

Moreover, we have:

$$\int_{\partial\Omega_i} \omega'_M = \begin{cases} 4\pi & \text{if } M \in \Omega_i \\ 0 & \text{if } M \in \Omega_e = \mathbb{R}^3 \; \overline{\Omega_i} \end{cases} \quad (15\text{-}3)$$

After the computations, we get the following formulae that can be used directly in the numerical process.

For $T \neq T'$:

$$\Omega_{TT'} = \int_{T'} \omega'_{M_T} = Sg\{(M_{T'} - M_T) . n_{T'}\} \times \left(2\pi - \sum_{i=1}^{3} \cos^{-1}(\alpha'_i . \alpha'_{i+1})\right) \quad (16\text{-}1)$$

$$\left.\begin{array}{ll} \alpha'_i = \dfrac{\rho'_i \times \rho'_{i+1}}{\|\rho'_i \times \rho'_{i+1}\|} & \text{if } \|\rho'_i \times \rho'_{i+1}\| \neq 0 \\ \text{et} \quad \alpha'_i = 0 & \text{if } \|\rho'_i \times \rho'_{i+1}\| = 0 \end{array}\right\} \quad (16\text{-}2)$$

$$\left.\begin{array}{l} \rho'_i = M'_i - M_T \in \mathbb{R}^3 \\ Sg(a) = \dfrac{a}{|a|} \quad \text{for } a \in \mathbb{R}^* \end{array}\right\} \quad (16\text{-}3)$$

For $T = T'$:

$$\Omega_{T'T'} = \int_{T'} \omega'_{M_T} = 0 \quad (16\text{-}4)$$

III,3.3. – *Integral of the Normal Derivative of the Kernel*

Using (14) and (15-2), we get

$$\frac{d'N}{dn'} ds' = g(r)\,\omega'_M \quad (17)$$

and deduce the following approximation from it using (16). For any fixed T and any fixed T' (with $T \neq T'$ or $T = T''$ and for any fixed $M = M_T \in T$:

$$\int_{T'} \frac{d'N}{dn'} ds' = \int_{T'} g(M_T, M')\,\omega'_{M_T} \simeq g(M_T, M_{T'})\,\Omega'_{TT'} \quad (18\text{-}1)$$

$$g(M_T, M_{T'}) = g(r(M_T, M_{T'})) \quad (18\text{-}2)$$

III,3.4. – *Average Value of the Kernel*

For any T and T' with $T \neq T'$ or $T = T'$, this average value is defined by:

$$\mathcal{N}_{TT'} = \frac{1}{|T\|T'|} \int_T \int_{T'} N\,ds\,ds' \quad (19)$$

The approximation of (19) that we have constructed not only preserves the symmetry but can also be counted on when elements exist face to face at a distance between barycenters that is smaller than the characteristic dimension of the two elements. The approximation is written:

$$\mathcal{N}_{TT'} \simeq \mathbb{N}_{TT'} + \frac{1}{2}(J_{TT'} + J_{T'T}) \quad (20\text{-}1)$$

$$\mathbb{N}_{TT} \simeq -jk \quad (20\text{-}2)$$

$$\mathbb{N}_{TT'} = \frac{e^{-jk\,r(M_T, M_{T'})} - 1}{r(M_T, M_{T'})} \quad (20\text{-}3)$$

$$J_{TT'} = \frac{1}{|T|} \int_T \frac{ds}{r}, \qquad r = \|M_{T'} - M\| \quad (20\text{-}4)$$

For $T \neq T'$, the integral of (20-4) is computed numerically by a Gauss method. For $T = T'$, we get an expression for (20-4) that is expressed using the log function (*see* [1]).

III,4. – $\tilde{\mathrm{P}}$ OPERATOR

This operator is defined by (61-1) of Part I as:

$$\langle \tilde{\mathrm{P}} A, \varphi \rangle = -\frac{jk}{2} \int_{\partial\Omega_i} \varphi . (n \times A)\,ds$$

So what we want is to compute, for an arbitrary finite element T of $\partial\Omega_i$:

$$\langle \tilde{\mathrm{P}}_{TT} A_T, \varphi_T \rangle = -\frac{jk}{2} \int_T \varphi_T . (n \times A_T)\,ds \quad (21)$$

Using (3) and (8) we get:

$$\langle \tilde{\mathrm{P}}_{TT} A_T, \varphi_T \rangle = \sum_{i=1}^{3} F^i_T(A_T) . \varphi^i_T \quad (22\text{-}1)$$

$$F^i_T(A_T) = -\frac{jk}{24} \sum_{l=1}^{3} (1 + \delta_{il})\, W^0_T A^l_T \in \mathbb{C}^3 \quad (22\text{-}2)$$

$$W^0_T = (W_T \times) \quad (22\text{-}3)$$

with $\delta_{il} = 0$ if $i \neq l$ and $\delta_{il} = 1$ if $i = l$.

III,5. – $\tilde{\mathrm{I}}$ OPERATOR

The operators $\tilde{\mathrm{I}}_{cc}$ and $\tilde{\mathrm{I}}_{dd}$ defined by (61-2) and (61-3) of Part I are the restrictions of the operator $\tilde{\mathrm{I}}$ to Γ_c and Γ_d, such that:

$$\langle \tilde{\mathrm{I}} A, \varphi \rangle = \int_{\partial\Omega_i} \varphi . A\,ds$$

For a given finite element T of $\partial\Omega_i$, the object is therefore to compute

$$\langle \tilde{\mathrm{I}}_{TT} A_T, \varphi_T \rangle = \int_T \varphi_T . A_T\,ds \quad (23)$$

Using (8), we get:

$$\langle \tilde{I}_{TT} A_T, \varphi_T \rangle = \sum_{i=1}^{3} F_T^i(A_T).\varphi_T^i \qquad (24\text{-}1)$$

$$F_T^i(A_T) = \frac{|T|}{12} \sum_{l=1}^{3} (1+\delta_{il}) A_T^l \in \mathbb{C}^3 \qquad (24\text{-}2)$$

III,6. – $\tilde{Q}$ OPERATOR

The $\tilde{Q}$ operator is defined by (61-4) in Part I:

$$\langle \tilde{Q} A, \varphi \rangle = \frac{jk}{4\pi} \int_{\partial\Omega_i} ds \int_{\partial\Omega_i} (\varphi \times n).[(n' \times A') \times \text{grad}' N]\, ds'$$

Let T be the "receiver" finite element and φ_T the continuous function defined on T with values in $\mathbb{R}^3$ such that:

$$\varphi_T = \lambda_1 \varphi_T^1 + \lambda_2 \varphi_T^2 + \lambda_3 \varphi_T^3, \quad \varphi_T^i = \varphi_T(M_i) \quad (25)$$

Let T' be the "transmitter" finite element and $A_{T'}$ the continuous function defined on T' with values in $\mathbb{C}^3$ such that

$$A_{T'} = \lambda_1 A_{T'}^1 + \lambda_2 A_{T'}^2 + \lambda_3 A_{T'}^3, \quad A_{T'}^i = A_{T'}(M_i) \quad (26)$$

What we compute is:

$$\langle \tilde{Q}_{TT'} A_{T'}, \varphi_T \rangle = \frac{jk}{4\pi} \int_T ds \int_{T'} (\varphi_T \times n_T).[(n_{T'} \times A'_{T'}) \times \text{grad}' N]\, ds' \quad (27)$$

III, 6.1. – *Expression of* $\tilde{Q}_{TT}$

Using the properties of the mixed product and the relation (14-1), and considering the fact that $(M'-M).n_T = 0$, for $\forall M$ and $M' \in T$, we get:

$$\langle \tilde{Q}_{TT} A_T, \varphi_T \rangle = 0 \qquad (28)$$

III,6.2. – *Expression of* $\tilde{Q}_{TT'}$

For $T \neq T'$, the $\tilde{Q}$ operator extracts the tangential part of φ and A. So we must preserve the expressions $\varphi_T \times n_T$ and $n_{T'} \times A'_{T'}$ when adapting to the numerical process. Moreover, we know that the $\tilde{Q}$ operator is algebraically symmetrical (*see* section VII, 4 in Part I). As this symmetry is hidden, it must be preserved when adapting to the numerical process. We will therefore have to write it explicitly. Expression (27) is not easy to adapt for numerical purposes because the expression to be integrated in the second member is not in differential form. Moreover, if T and T' are two elements that are face to face ($n_{T'} = -n_T$) and the distance $\|M_T - M_{T'}\|$ is small compared with $\sqrt{T}$ or $\sqrt{T'}$, then r^{-1} is very large and the computation would be very poorly conditioned. So we must transform the second member of (27) to bring out differential forms that will be easily adaptable to numerical computation, and in particular the solid angle differential form, to arrive at a numerical formula that is stable in all cases (*i.e.* the problem of face-to-face elements near each other as we have just mentioned). After a somewhat lengthy set of calculations that can be found in detail in [1], we get the new form we want for (27) for $T \neq T'$, which is written:

$$\langle \tilde{Q}_{TT'} A_{T'}, \varphi_T \rangle = \frac{jk}{8\pi} \int_T \int_{T'} \{ g(r)\,\omega'_M (U.n_{T'})\, ds - g(r)\,\omega_{M'}(U.n_T)\, ds' + (n_{T'} \times U).d'(N\, dM')\, ds - (n_T \times U).d(N\, dM)\, ds' \} \quad (29\text{-}1)$$

$$U = (\varphi_T \times n_T) \times (n_{T'} \times A'_{T'}) \qquad (29\text{-}2)$$

We can then construct the numerical approximation of (29) using the theorem of the average along the way, and we get [1]:

$$\langle \tilde{Q}_{TT'} A_{T'}, \varphi_T \rangle \simeq \sum_{i=1}^{3} F_{TT'}^i (A_{T'}).\varphi_T^i \qquad (30\text{-}1)$$

with $i \in \{1, 2, 3\}$ such that:

$$F_{TT'}^i(A_{T'}) = \frac{jk}{24\pi} \{ g(M_T, M_{T'}) [\Omega_{T'T} (n_T^0)^2 W_{T'}^0 - \Omega_{TT'} W_T^0 (n_{T'}^0)^2] \underline{A}_{T'} + W_T^0 [(n_{T'}^0 \mathbb{V}_{TT'}) \times (n_{T'}^0 \underline{A}_{T'})] - n_T^0 [(n_T^0 \mathbb{V}_{T'T}) \times (W_{T'}^0 \underline{A}_{T'})] \} \quad (30\text{-}2)$$

with $g(M_T, M_{T'})$ defined by (18-2), W_T^0 by (11-2), $\Omega_{TT'}$ by (16) and

$$n_T^0 = (n_T \times) \qquad (30\text{-}3)$$

$$(n_T^0)^2 = n_T \otimes n_T - I \qquad (30\text{-}4)$$

$$\mathbb{V}_{T'T} = \sum_{l=1}^{3} N(M_l, M_{T'})\, V_T^l \qquad (30\text{-}5)$$

in which V_T^l is defined by (13).

III,7. – $\tilde{B}$ OPERATOR

The $\tilde{B}$ operator is defined by (61-5) in Part I:

$$\langle \tilde{B} A, \varphi \rangle = -\frac{j}{4\pi} \int_{\partial\Omega_i} \int_{\partial\Omega_i} N\, d(\varphi.dM)\, d'(A'.dM').$$

Returning to the notation of (25) and (26), we compute:

$$\langle \tilde{B}_{TT'} A_{T'}, \varphi_T \rangle = -\frac{j}{4\pi} \int_T \int_{T'} N\, d(\varphi_T.dM)\, d'(A'_{T'}.dM') \quad (31)$$

It is observed that, in the framework of the P_1 elements, $d(\varphi_T . dM)$ is a "constant" on T, as $d'(A'_{T'} . dM')$ is on T'. So we apply the theorem of the average, which is in this case exact:

$$\langle \tilde{\mathrm{B}}_{TT'} A_{T'}, \varphi_T \rangle = \sum_{i=1}^{3} F^i_{TT'}(A_{T'}) . \varphi^i_T \quad (32\text{-}1)$$

with $F^i_{TT'}(A_{T'}) \in \mathbb{C}^3$ for $i \in \{1, 2, 3\}$ such that:

$$F^i_{TT'}(A_{T'}) = -\frac{j}{4\pi} \mathcal{N}_{TT'} V^i_T \left(\sum_{l=1}^{3} A^l_{T'} . V^l_{T'} \right) \quad (32\text{-}2)$$

in which $\mathcal{N}_{TT'}$ is computed by (20) and V^i_T by (13).

III,8. – $\tilde{\mathrm{S}}$ OPERATOR

This $\tilde{\mathrm{S}}$ operator is defined by (61-6) of Part I:

$$\langle \tilde{\mathrm{S}} A, \varphi \rangle = -\frac{jk^2}{4\pi} \int_{\partial\Omega_i} ds \int_{\partial\Omega_i} N \, ds' \, (\varphi \times n) . (A' \times n')$$

Using the notation of (25) and (26), we compute:

$$\langle \tilde{\mathrm{S}}_{TT'} A_{T'}, \varphi_T \rangle = -\frac{jk^2}{4\pi} \int_T ds \int_{T'} N \, ds' \, (\varphi_T \times n_T) . (A'_{T'} \times n_{T'}) \quad (33)$$

Using the theorem of the average and (11-2), we construct the following numerical approximation:

$$\langle \tilde{\mathrm{S}}_{TT'} A_{T'}, \varphi_T \rangle \simeq \sum_{i=1}^{3} F^i_{TT'}(A_{T'}) . \varphi^i_T \quad (34\text{-}1)$$

with $F^i_{TT'}(A_{T'}) \in \mathbb{C}^3$ for $i \in \{1, 2, 3\}$ such that:

$$F^i_{TT'}(A_{T'}) = \frac{jk^2}{12\pi} \mathcal{N}_{TT'} W^0_T W^0_{T'} \underline{A}_{T'} \quad (34\text{-}2)$$

in which the various terms of (34-2) are given by (10) (11-2) and (20).

III,9. – OPERATORS FOR COMPUTING THE SCATTERED ELECTROMAGNETIC FIELD

Here we must adapt the operators $\mathrm{R}_1(M)$, $\mathrm{R}_2(M)$ and $\mathrm{R}_3(M)$ defined by (78) in Part I. Let M be a fixed-point in Ω_e, T' the "transmitter" finite element, $A_{T'}$ (resp. $B_{T'}$) the continuous function defined on T' with values in $\mathbb{C}^3$ such that:

$$\left. \begin{aligned} A_{T'} &= \lambda_1 A^1_{T'} + \lambda_2 A^2_{T'} + \lambda_3 A^3_{T'}, \\ A^i_{T'} &= A_{T'}(M_i) \end{aligned} \right\} \quad (35\text{-}1)$$

$$\left. \begin{aligned} B_{T'} &= \lambda_1 B^1_{T'} + \lambda_2 B^2_{T'} + \lambda_3 B^3_{T'}, \\ B^i_{T'} &= B_{T'}(M_i) \end{aligned} \right\} \quad (35\text{-}2)$$

We then have:

$$\mathrm{R}_{1, T'}(M) A_{T'} = \int_{T'} A'_{T'} \frac{d' N}{dn'} ds'$$

$$\mathrm{R}_{2, T'}(M) A_{T'} = \int_{T'} A'_{T'} \times d'(N \, dM')$$

$$\mathrm{R}_{3, T'}(M) B_{T'} = jk \int_{T'} N B'_{T'} \, ds'$$

Using the theorem of the average, (17) and (18), we get the following numerical approximation:

$$\mathrm{R}_{1, T'}(M) A_{T'} \simeq g(M, M_{T'}) \Omega_{MT'} \underline{A}_{T'} \quad (36\text{-}1)$$

$$\mathrm{R}_{2, T'}(M) A_{T'} \simeq \underline{A}_{T'} \times \mathbb{V}_{MT'} \quad (36\text{-}2)$$

$$\mathrm{R}_{3, T'}(M) B_{T'} \simeq jk \, |T'| \, N(M, M_{T'}) \underline{B}_{T'} \quad (36\text{-}3)$$

with $g(M, M_{T'})$ given by (18-2), $\Omega_{MT'}$ by (16) by replacing the point M_T by the point M, and:

$$\mathbb{V}_{MT'} = \sum_{l=1}^{3} N(M'_l, M) V^l_{T'} \quad (36\text{-}4)$$

with $V^l_{T'}$ given by (13) and $\underline{A}_{T'}$ and $\underline{B}_{T'}$ by:

$$\underline{A}_{T'} = \frac{1}{|T'|} \int_{T'} A_{T'} \, ds' \quad (37\text{-}1)$$

$$\underline{B}_{T'} = \frac{1}{|T'|} \int_{T'} B_{T'} \, ds' \quad (37\text{-}2)$$

It will be noted that the relations (37) can be expressed using (10), but we will also need integral expressions because we will not directly know the value of certain fields at the grid nodes of $\partial\Omega_i$

III,10. – COMPUTATION OF AVERAGE VALUES OF THE FIELDS OVER THE EXTERNAL SURFACE

Relations (36) use average values $\underline{A}_{T'}$ and $\underline{B}_{T'}$ of $A_{T'}$ and $B_{T'}$ on T'. Relations (79) of Part I show that the fields in question are H_e and E_e on $\partial\Omega_i$, the expressions for which are given by relations (75) and (76) in Part I.

We spell out the computation of these average fields in the following sections.

III,10.1. — *Expression of the Average Fields on the External Surface for Computing the Scattered Magnetic Field*

Using (75) and (76) from Part I, and taking $A=H_e$ and $B=n\times E_e$, we get directly:

$$T'\in\Gamma_c: \quad \underline{A}_{T'}=(\underline{H_{e,c}})_{T'}$$

$$=(I-n_{T'}\otimes n_{T'})(\underline{H_{i,c}})_{T'}-n_{T'}\times(\underline{H_{r,0}})_{T'} \quad (38\text{-}1)$$

$$\underline{B}_{T'}=n_{T'}\times(\underline{E_{e,c}})_{T'}=0 \quad (38\text{-}2)$$

$$T'\in\Gamma_d: \quad \underline{A}_{T'}=(\underline{H_{e,d}})_{T'}$$

$$=(I+(n_{T'}\otimes n_{T'})(\mu^*_{T'}-I))(\underline{H_{bd}})_{T'} \quad (38\text{-}3)$$

$$\underline{B}_{T'}=n_{T'}\times(\underline{E_{e,d}})_{T'}=n_{T'}\times(\underline{E_{i,d}})_{T'}+(\underline{E_{r,0}})_{T'} \quad (38\text{-}4)$$

III,10.2. — *Expression of the Average Fields on the External Surface for Computing the Scattered Electric Field*

We therefore have $A=E_e$ and $B=n\times H_e$ and we use (75) and (76) from Part I to get:

$$T'\in\Gamma_c \quad \underline{A}_{T'}=(\underline{E_{e,c}})_{T'}$$

$$=\frac{1}{jk|T'|}n_{T'}\sum_{l=1}^{3}V^l_{T'}.\{(H_{i,c})^l_{T'}-n_{T'}\times(H_{r,0})^l_{T'}\} \quad (39\text{-}1)$$

$$\underline{B}_{T'}=n_{T'}\times(\underline{H_{e,c}})_{T'}=n_{T'}\times(\underline{H_{i,c}})_{T'}+(\underline{H_{r,0}})_{T'} \quad (39\text{-}2)$$

$$\underline{A}_{T'}=\frac{1}{jk|T'|}n_{T'}\sum_{l=1}^{3}V^l_{T'}.(H_{bd})^l_{T'}$$

$$-(I-n_{T'}\otimes n_{T'})(\underline{E_{i,d}})_{T'}-n_{T'}\times(\underline{E_{r,0}})_{T'} \quad (39\text{-}3)$$

$$\underline{B}_{T'}=n_{T'}\times(\underline{H_{e,d}})_{T'}=n_{T'}\times(\underline{H_{bd}})_{T'} \quad (39\text{-}4)$$

III,11. — m_c AND m_d OPERATORS

The m_c and m_d operators are defined by (62) in Part I. The operator m can be considered such that:

$$\langle m A, \delta\lambda\rangle=\int_{\partial\Omega_i}\delta\lambda\, A.n\, ds$$

and for a given finite element T of $\partial\Omega_i$ we then compute:

$$\langle m_{TT}A_T, (\delta\lambda)_T\rangle=\int_T(\delta\lambda)_T(A_T.n_T)\,ds. \quad (40)$$

We obtain the following expressions adapted for numerical computations:

$$\langle m_{TT}A_T, (\delta\lambda)_T\rangle=\sum_{i=1}^{3}F^i_T(A_{T'}).(\delta\lambda)^i_T \quad (41\text{-}1)$$

in which $F^i_T(A_{T'})\in\mathbb{C}$ for $i\in\{1, 2, 3\}$ such that:

$$F^i_T(A_T)=\sum_{l=1}^{3}\left(\frac{1+\delta_{il}}{12}\right)(A^l_{T'}.W_T) \quad (41\text{-}2)$$

with W_T given by (3).

III,12. — OPERATORS FOR COMPUTING THE FAR FIELD

Here we want to adapt the operators of equations (82) from Part I, to compute the reflected electromagnetic far field $\mathscr{E}_r$ and $\mathscr{H}_r$, asymptotically. The calculations are spelled out in [18]. Only $\mathscr{E}_r$ is computed since $\mathscr{H}_r$ is deduced immediately from it. According to (82-1) of Part I, we need only to put the following integral in numerical form:

$$\mathbb{I}_{T'}(u)=\int_{T'}\mathscr{N}_u A'\,ds' \quad (42)$$

in which $\mathscr{N}_u$ is defined by (80) in Part I and A' is written:

$$A'=u\times(n'\times E'_e)-[I-u\otimes u](H'_e\times n'). \quad (43)$$

The theorem of the average is not used here, but we compute the integral of (42) numerically using a seven-point Gauss method. The fields $n'\times E'_e$ and $H'_e\times n'$ are given by the relations (14-2), (75-2) and (16-2), (76-3) of Part I, and are computed at the Gaussian integration points using the interpolation P_1.

IV. — NUMERICAL FORM OF DIFFERENTIAL OPERATORS FOR THE INTERIOR PROBLEM

These operators, introduced in section IV of Part I, will be adapted to the numerical process by the finite element method. The problem is therefore to construct elementary matrices.

IV,1. — VOLUME AND SURFACE FINITE ELEMENTS FOR THE INTERNAL BOUNDARIES

IV,1.1. — *Meshing of Internal Interfaces*

The external interface $\partial\Omega_i$ has been meshed with triangles. The internal interfaces are of two types: Γ_{cd} and $\Gamma_{nn'}$ (*see* Part I). All of these internal interfaces are meshed with triangles exactly like the $\partial\Omega_i$ grid. We assume that the usual comptability is ensured between the grid of $\partial\Omega_i$ and that of the Γ_{cd} and

$\Gamma_{nn'}$ surfaces (matching of edges with the various intersections of these surfaces).

IV,1.2. – *Grid of Dielectric Volume Domains*

Each dielectric domain Ω_n has a boundary $\partial\Omega_n$ (*see* Part I) which is now meshed with triangles, since it is the union of the surfaces $\partial\Omega_n \cap (\Gamma_c \cup \Gamma_d)$, $\partial\Omega_n \cap \Gamma_{cd}$ and $\partial\Omega_n \cap \Gamma_{nn'}$, which are all meshed.

Each domain Ω_n is then filled with tetrahedra and the volume grid of Ω_n on $\partial\Omega_n$ coincides with the existing grid of $\partial\Omega_n$. This ensures compatibility, and the surface grid of $\partial\Omega_n$ consists of the triangular faces of the tetrahedra.

So there is only a series of nodes on the surfaces $\partial\Omega_n \cap (\Gamma_c \cup \Gamma_d)$ and $\partial\Omega_n \cap \Gamma_{cd}$, and there will be two series of geometrically matching nodes (the triangulation nodes of $\Gamma_{nn'}$ are doubled up) overlaying each other on the $\Gamma_{nn'} = \partial\Omega_n \cap \partial\Omega_n$ surfaces, in order to construct the $T_{\Gamma_{dd}}$ operator defined by (28-4) in Part I.

Notation Convention for all of Section IV:

We have to develop the caculation of the elementary matrices for the internal operators linked to each Ω_n domain. To simplify the notation, we drop the n superscript whenever there is no possible confusion, writing Ω, H, δH, μ^*, $\underline{\varepsilon}^*$, $\delta\lambda_\Omega$, $\delta\lambda_{cd}$, instead of Ω_n, H^n, δH^n, μ^{*n}, $\underline{\varepsilon}^{*n}$, $\delta\bar{\lambda}_{cd,n}$.

On the other hand, to deal with the $T_{\Gamma_{nn'}}$ operator for $\Gamma_{nn'}$, we have to keep the n and n' subscripts.

IV,1.3. – *P_1 Tetrahedral Finite Element and Approximation Space*

We take P_1 elements. The approximation space for the internal operators on Ω will be the space of affine functions by tetrahedron, and continuous in Ω (it is each Ω_n here, as the H field is discontinuous through the $\Gamma_{nn'}$ boundaries). This choice is compatible with the approximation space for the operators of the external problem on $\partial\Omega_i$, since the tetrahedral elements P_1 yield triangular P_1 elements for each of their faces.

We will use V to denote any finite element in the grid of Ω. This is a tetrahedron whose apexes will be referred to as M_1, M_2, M_3 and M_4. The notation V will designate the element, but also the part of $\mathbb{R}^3$ defined by the tetrahedron (M_1, M_2, M_3, M_4).

The volume element V will be denoted:

$$|V| = \int_V dx = \frac{1}{6}|det| \qquad (44\text{-}1)$$

$$det = [M_2 - M_1, M_3 - M_1, M_4 - M_1] \qquad (44\text{-}2)$$

in which [. . ., . . ., . . .] always designates the mixed product.

We introduce the following condensed notation:

$$\left.\begin{aligned} N^1 &= \frac{1}{det}(M_3 - M_4)\times(M_2 - M_4)\\ N^2 &= \frac{1}{det}(M_3 - M_1)\times(M_4 - M_1)\\ N^3 &= \frac{1}{det}(M_1 - M_2)\times(M_4 - M_2)\\ N^4 &= \frac{1}{det}(M_1 - M_3)\times(M_2 - M_3)\end{aligned}\right\} \qquad (45)$$

For $j \in \{1, 2, 3, 4\}$, $N^j \in \mathbb{R}^3$. The vector N^j is normal to the face opposite the node j.

IV,2. – P_1 INTERPOLATION FORMULAE

On V we use the barycentric coordinates $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ to parameter V:

$$M = \lambda_1 M_1 + \lambda_2 M_2 + \lambda_3 M_3 + \lambda_4 M_4 \in T \qquad (46\text{-}1)$$

$$\left.\begin{aligned} \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1,\\ 0 \leq \lambda_i \leq 1, \quad \forall i \in \{1, 2, 3, 4\}\end{aligned}\right\} \qquad (46\text{-}2)$$

Let φ be a continuous function on V with values in $\mathbb{K}^3$, and φ^1, φ^2, φ^3, φ^4 be the values of φ at M_1, M_2, M_3 and M_4:

$$\varphi^i = \varphi(M_i) \in \mathbb{K}^3, \qquad i \in \{1, 2, 3, 4\}. \qquad (47)$$

We then have:

$$\varphi = \lambda_1 \varphi^1 + \lambda_2 \varphi^2 + \lambda_3 \varphi^3 + \lambda_4 \varphi^4 \qquad (48)$$

As before, we have the following integration formula. If α_1, α_2, α_3 and α_4 are four positive or zero integers:

$$\int_V \lambda_1^{\alpha_1} \lambda_2^{\alpha_2} \lambda_3^{\alpha_3} \lambda_4^{\alpha_4}\, ds = 6|V| \frac{\alpha_1!\,\alpha_2!\,\alpha_3!\,\alpha_4!}{(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + 3)!}. \qquad (49)$$

IV,2.1. – *Computation of Gradient. Divergence and Rotational Operators*

Let φ be a function of the type (48) on V with values $\mathbb{K}$. The gradient of φ is written using (45):

$$\nabla\varphi = \sum_{i=1}^{4} \varphi^i . N^i. \qquad (50\text{-}1)$$

Let Ψ be a function on V with values $\mathbb{K}^3$ of the type (48). The divergence and rotational of Ψ is written using (45):

$$\operatorname{div}\Psi = \sum_{i=1}^{4} \Psi^i . N^i \qquad (50\text{-}2)$$

$$\operatorname{rot}\Psi = \sum_{i=1}^{4} N^i \times \Psi^i . \qquad (50\text{-}3)$$

IV,3. – NUMERICAL EXPRESSION OF THE $\mathbb{M}^n$

For a dielectric domain Ω_n, the $\mathbb{M}^n$ operator is defined by (21) of Part I as:

$$\langle\!\langle \mathbb{M}^n H^n, \delta H \rangle\!\rangle = jk^2 \int_{\Omega_n} (\underline{\mu}^* H) . \delta H \, dx$$
$$- j \int_{\Omega_n} ([\underline{\varepsilon}^*]^{-1} \operatorname{rot} H) . \operatorname{rot} \delta H \, dx$$

in which $\underline{\mu}^*$ and $\underline{\varepsilon}^*$ are the complex symmetrical (3 × 3) matrices defined by (4) and (12) in Part I, and are continuous on Ω_n. Let V be a finite element of Ω_n, and H_V and δH_V be the functions defined on V with values in $\mathbb{C}^3$, of the type (48). What we then have to state is:

$$\langle\!\langle \mathbb{M}_V^n H_V^n, \delta H_V \rangle\!\rangle = jk^2 \int_V (\underline{\mu}^* H_V) . \delta H_V \, dx$$
$$- j \int_V ([\underline{\varepsilon}^*]^{-1} \operatorname{rot} H_V) . \operatorname{rot} \delta H_V \, dx \qquad (51)$$

Using (49), (50-3), we deduce the expression we are looking for:

$$\langle\!\langle \mathbb{M}_V^n H_V^n, \delta H_V \rangle\!\rangle = \sum_{l=1}^{4} F_V^l (H_V) . \delta H_V^l \qquad (52\text{-}1)$$

With (49) for $l \in \{1, 2, 3, 4\}$ such that:

$$F_V^l (H_V = j \,|\, V \,|\, \sum_{i=1}^{4} \left\{ k^2 \left(\frac{1+\delta_{il}}{20} \right) [\underline{\mu}_V^*] \right.$$
$$\left. + [N^{l0}] [\underline{\varepsilon}_V^*]^{-1} [N^{i0}] \right\} H_V^i \qquad (52\text{-}2)$$

$$[N^{l0}] = (N^l \times) \qquad (52\text{-}3)$$

$$[\underline{\mu}_V^*] = [\underline{\mu}_V^* (\underline{M}_V)]; \quad [\underline{\varepsilon}_V^*]^{-1} = [\underline{\varepsilon}_V^* (\underline{M}_V)]^{-1} \qquad (52\text{-}4)$$

in which $\underline{M}_V$ designates the barycenter of the element V.

IV,4. – NUMERICAL EXPRESSION OF THE Div^n OPERATOR

For a dielectric domain Ω_n, the Div^n operator is defined by (28-2) of Part I as:

$$\langle\!\langle \operatorname{Div}^n H, \delta\lambda_\Omega \rangle\!\rangle = \int_{\Omega_n} \delta\lambda_\Omega \operatorname{div} (\underline{\mu}^* \mathrm{H}) \, dx.$$

Let V be a finite element of Ω_n, and let H_V and $\delta\lambda_V$ be the continuous functions defined on V with values in $\mathbb{C}^3$ and $\mathbb{C}$, of the type (48). What we calculate is:

$$\langle\!\langle \operatorname{Div}_V^n H_V, \delta\lambda_V \rangle\!\rangle = \int_V \delta\lambda_V \operatorname{div} (\underline{\mu}_V^* H_V) \, dx \qquad (53)$$

Using (50-2), we get:

$$\langle\!\langle \operatorname{Div}_V^n H_V^n, \delta\lambda_V \rangle\!\rangle = \sum_{l=1}^{4} F_V^l (H_V) . \delta\lambda_V^l \qquad (54\text{-}1)$$

with, for $l \in \{1, 2, 3, 4\}$ and $F_V^l (H_V) \in \mathbb{C}$, and independent of l, such that:

$$F_V^l (H_V) = \frac{1}{4} |\mathrm{V}| \sum_{i=1}^{4} (\underline{\mu}_V^* H_V^i) . N^i . \qquad (54\text{-}2)$$

IV,5. – NUMERICAL EXPRESSION FOR THE $T_{\Gamma_{nn'}}$, AND U_{cd}^n OPERATORS

IV,5.1. – U_{cd}^n *Operator*

This operator is relative to the surface Γ_{cd}, which is the union of the interfaces between the perfect internal conductor Ω_c and the internal dielectric Ω_n. It is defined by (28-5) in Part I as:

$$\langle \mathrm{U}_{cd}^n H^n, \delta\lambda_{cd} \rangle = \int_{\Gamma_{cd}} \delta\lambda_{cd,n} (\underline{\mu}^{*n} H^n) . n \, ds.$$

We will therefore use T to denote an arbitrary three-node P_1 finite of Γ_{cd}. This element is oriented in accordance with the convention adopted in Section III,1.4, so that the normal to T, n_T, defined by (2), is directed toward the middle of the conductor Ω_c. We then have to compute:

$$\langle \mathrm{U}_T H_T, \delta\lambda_T \rangle = \int_T \delta\lambda_T (\underline{\mu}^* \mathrm{H}_T) . n_T \, ds \qquad (55)$$

with H_T and $\delta\lambda_T$ being the continuous functions on T with values in $\mathbb{C}^3$ and $\mathbb{C}$, respectively, of type (7). Using (8), we get:

$$\langle \mathrm{U}_T H_T, \delta\lambda_T \rangle = \sum_{l=1}^{3} F_T^l (H_T) . \delta\lambda_T^l \qquad (56\text{-}1)$$

with, for $l \in \{1, 2, 3\}$ and $F_T^l (H_T) \in \mathbb{C}$, such that:

$$F_T^l (H_T) = \sum_{i=1}^{3} \left(\frac{1+\delta_{il}}{12} \right) (\underline{\mu}^* \mathrm{H}_T^i) . W_T \qquad (56\text{-}2)$$

with W_T defined by (3) and $\underline{\mu}_T^* = \underline{\mu}^* (\underline{\mathrm{M}}_T)$ where $\underline{M}_T$ is given by (9).

IV,5.2. — $T_{\Gamma_{nn'}}$ *Operator*

This operator is relative to the surface $\Gamma_{nn'}$, which is the interface between Ω_n and $\Omega_{n'}$. It is defined by (28-4) in Part I, *i.e.* for (n, n') fixed in $\mathcal{N}_\Gamma$ (*see* section IV-2 in Part I):

$$\langle T_{\Gamma_{nn'}} H, \delta\lambda_{\Gamma_{nn'}} \rangle = k \int_{\Gamma_{nn'}} (R^n_{\Gamma_{nn'}} H^n - R^{n'}_{\Gamma_{nn'}} H^{n'}) \, . \, \delta\lambda_{\Gamma_{nn'}} \, ds$$

$$R^n_{\Gamma_{nn'}} = (n \otimes n) \underline{\mu}^{*\,n} - (n \times)$$

$$R^{n'}_{\Gamma_{nn'}} = (n' \otimes n') \underline{\mu}^{*\,n'} + (n' \times).$$

The surface $\Gamma_{nn'}$ is oriented as in section III and is meshed by three-node P_1 finite elements. Let T be an arbitrary element of $\Gamma_{nn'}$ whose nodes are M_1, M_2, M_3. We choose this orientation so that the normal n_T defined by (2) is directed from Ω_n toward $\Omega_{n'}$. We introduce two P_1 finite elements denoted T_n and $T_{n'}$, such that T_n coincides with T and with one face of a tetrahedron of Ω_n, and such that $T_{n'}$ coincides with T and with one face of a tetrahedron of $\Omega_{n'}$. The normal of T_n (resp. $T_{n'}$) is $n_{T_n} = n_T$ (resp. $n_{T_{n'}} = -n_T$).

This is equivalent to doubling the grid of $\Gamma_{nn'}$. Geometrically, the T_n and $T_{n'}$ elements coincide, so what we compute for any finite element T of $\Gamma_{nn'}$ is:

$$\langle T_T H_T, \delta\lambda_T \rangle = k \int_T (R^n_T H_{T_n} - R^{n'}_T H_{T_{n'}}) \, . \, \delta\lambda_T \, ds \quad (57\text{-}1)$$

$$R^n_T = (n_T \otimes n_T) \underline{\mu}^{*\,n}_T - (n_T \times) \quad (57\text{-}2)$$

$$R^{n'}_T = (n_T \otimes n_T) \underline{\mu}^{*\,n'}_T - (n_T \times) \quad (57\text{-}3)$$

with $\delta\lambda_T$, H_{T_n} and $H_{T_{n'}}$ being three continuous functions on T with values in $\mathbb{C}^3$, of type (7). Using (8), we get:

$$\langle T_T H_T, \delta\lambda_T \rangle = \sum_{l=1}^{3} F^l_T(H_T) \, . \, \delta\lambda^l_T \quad (58\text{-}1)$$

with $F^l_T(H_T) \in \mathbb{C}^3$, for $l \in \{1, 2, 3\}$ such that:

$$F^l_T(H_T) = k \, |T| \sum_{i=1}^{3} \left(\frac{1+\delta_{il}}{12} \right) (R^n_T H^i_{T_n} - R^{n'}_T H^i_{T_{n'}}) \quad (58\text{-}2)$$

V. — COMPLEMENTARY REMARKS CONCERNING THE TREATMENT OF EDGES AND THEIR INTERSECTIONS

Let us recall that each dielectric domain Ω_n is meshed on its own, so the edge $\partial\Omega_n$ of Ω_n might consist of interfaces of the type $\Gamma_{cd,n}$, $\Gamma_{nn'}$ and $\Gamma_{d,n}$. We are concerned here with the treatment of edges consisting of the intersection of the outer surface with a surface separating two internal media ($\partial\Omega_i \cap \Gamma_{cd,n}$ or $\partial\Omega_i \cap \Gamma_{nn'}$).

V,1. — $\partial\Omega_i \cap \Gamma_{cd,n}$ TYPE EDGE

We proceed by continuity with the $\Gamma_{cd,n}$ interface. The nodes of this edge belong to the dielectric and the unknowns are H^n_{bd} and $E^{\vee}_{e,\tau}$. But we need to know $n \times H_{e,c}$ at the nodes of this edge for the integral operators. So we use the transmission condition (16-2) of Part I between Ω_e and Ω_n concerning the continuity of the magnetic tangential.

V,2. — $\partial\Omega_i \cap \Gamma_{nn'}$ TYPE EDGE

On an edge like this, the electric and magnetic fields are continuous. That is, we assume that the surface is locally C_1 on this edge so that it admits a tangent plane on the edge, and we use n_e to denote the unit normal to $\partial\Omega_i$ at a given node on the edge. The tangent part of H^n_{bd} and $H^{n'}_{bd}$ is continuous through $\Gamma_d : n_e \times H^n_{bd} = n_e \times H_e$ and $n_e \times H^{n'}_{bd} = n_e \times H_e$, whence $n_e \times (H^n_{bd} - H^{n'}_{bd}) = 0$, making $H^n_{bd} - H^{n'}_{bd}$ colinear at n_e.

Moreover, calling n the normal to $\Gamma_{nn'}$ at this node, we have $n \times H^n_{bd} = n \times H^{n'}_{bd}$, whence $H^n_{bd} - H^{n'}_{bd}$ is colinear at n.

Yet $n_e \neq n$ (the tangent planes at Γ_d and $\Gamma_{nn'}$ are assumed not to be identical), which implies $H^n_{bd} = H^n_{bd'}$.

An analogous line of reasoning applies to the electric field, so we have $E^n_{e,\tau} = E^{n'}_{e,\tau}$, in particular.

V,3. — INTERSECTION OF SEVERAL EDGES ON THE OUTER SURFACE

The procedure of V,1 and V,2 applies here too. The only difference stems from the fact that there may be more than two nodes matching at this point of the edge intersection.

VI. — CONCLUSION

We have given explicit expressions for the numerical form of the various terms in the formulation developed in Part I. These expressions minimize the numerical cost, considering the constraints we adopted (*see* section II of Part I) while maintaining the reliability of the approximations for the various possible geometric configurations.

The validation of these approximations is presented in Part III.

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