# Hybrid numerical method for solving the harmonic Maxwell equations: I - Mathematical formulation

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# HYBRID NUMERICAL METHOD FOR SOLVING THE HARMONIC MAXWELL EQUATIONS: I. MATHEMATICAL FORMULATION

by

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ABSTRACT

This article describes a mixed numerical method for solving harmonic Maxwell equations in the classical electrodynamic context. This formulation allows us to treat any body of general three-dimensional geometry. The body is considered to consist of perfectly conducting or dielectric materials described by locally heterogeneous and anisotropic constitutive equations, and possibly with dielectric losses. We present our developments in three parts, in three separate articles. Part I below is devoted to the mathematical formulation. Numerical approximations are treated in Part II and, finally, Part III concerns (1) an iterative method for solving the linear equations of the problem; (2) computer code development aspects; (3) applications which validate the whole development. The developed formulation is hybrid. The external problem is treated by integral equations. Local equations are preserved for the dielectric parts of the body. A global variational formulation of the coupled problem is developed, and enables us to apply the finite element method. Boundary finite elements are used for integral operators connected with the external problem. Localized finite elements are used for the internal problem. Irregular frequency difficulties due to the integral formulation are analyzed in detail and an efficient solving method is developed.

*Keywords (NASA thesaurus): Electromagnetic wave — Scatter propagation — Numerical method — Finite element method — Integral equations.*

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## I. — INTRODUCTION

The present article, the first in a series of three, presents a hybrid numerical method for solving Maxwell's harmonic equations in the context of classical electrodynamics involving conducting or dielectric objects of arbitrary geometry.

Due to the bulk of the explanations, we felt it necessary to present this work in the form of a three-article series. The first Part explains the mathematical formulation. The second deals with the construction of the numerical approximations, and Part III with the iterative solver algorithm, the computer code, and the numerical data validating the whole theory.

The formulation presented here draws upon our developments in the related field of elastoacoustics [31], in particular as concerns the treatment of the irregular frequencies arising from the integral-equation formulations of the external problem [2].

Lastly, for the sake of simplifying the explanation, we have omitted the functional framework [10, 25] that would be necessary if we were to concentrate on the mathematical analysis of the existence, uniqueness and regularity of the solutions.

## II. — FOUNDATIONS OF THE FORMULATION

### II, 1. — UNDERLYING HYPOTHESES

We are interested here only in the three-dimensional problem. The system we are studying consists of one or more bodies of arbitrary geometry impinged by a harmonic electromagnetic wave. The unbounded external medium is assumed to be homogeneous and isotropic, and to contain no currents or electrical charges. Each body is a bounded region of arbitrary surface geometry, consisting of an arbitrary set of perfectly conducting or dielectric domains.

Each dielectric domain is defined by a single material, and is assumed in principle to be anisotropic and inhomogeneous, so that the permittivity, permeability and conductivity tensors of this material do not reduce to a scalar and, moreover, are continuous functions of space. They are discontinuous from one domain to another, though. The magnetic losses can be modeled by introducing an imaginary part in the permeability tensor. Lastly, the equations are formulated in the harmonic domains, so that we can take into account the variations of these parameters as a function of the frequency.

What we want to find is the electromagnetic fields, the charges and the currents in the dielectrics; the surface charges and currents on the conductors; and the electromagnetic field in the external medium.

### II, 2. — CHOICE OF FORMULATION

There are a multitude of formulations possible for the problem defined above [6, 10, 13, 14, 16, 18, 22, 23, 34]. The one we present here is a hybrid, *i.e.* it is based on an integral equation (and thus nonlocal) formulation for the external problem, and also on a differential (local) equation for the internal problem.

This allows us to (1) consider the internal problems under very general conditions, *i.e.* anisotropic, inhomogeneous materials with magnetic losses, and so on; (2) have a body/external medium interface of arbitrary geometry with conductors and dielectrics; (3) avoid having to mesh the unbounded external domain.

We are aware that the size of this type of three-dimensional problem leads to very high numerical costs, if we decide to assemble the matrix of the discretized system, because of the inputs and outputs generated in solving it. An interative method is used to avoid this input-output load. To increase the efficiency of this iterative method, the formulation has to be constructed in such a way as to yield an operator having good properties. This is the objective we set for ourselves. Moreover, an iterative method without assembly makes it easy to implant the resulting code on massively parallel computers.

So the formulation we present here was constructed in such a way as to (1) minimize the number of unknown fields, or more precisely so we could eliminate the electric field from the dielectrics and consider only the magnetic fields; (2) get around the problem of the irregular frequencies induced by the integral-equation formulation of the external problem; (3) obtain a complex operator that would be symmetrical and accretive, for use in an efficient iterative solver algorithm.

It will be seen that, while there are a multitude of possible formulations, we found only one that would give us a symmetrical accretive operator, considering that we eliminated the electric field from the dielectrics and wanted some automatic way of getting around the problem of the irregular frequencies.

This new hybrid formulation we have developed is treated by a variational method. This way, we can apply a finite-element numerical process to the internal problem, with its differential operators, transmission boundary conditions on the dielectric interfaces, and so on, as well as to the external problem, with its integral operators on the surfaces. So we have localized finite elements for the internal problem and unlocalized ones for the external.

Once we have decided to solve the problem using an iterative method without matrix assembly, we should then expect to have to recalculate the elementary matrices at each iteration. But in fact, some of these can be kept in storage in partially assembled or elementary form, to reduce the numerical cost of the floating-point calculations. This means striking a compromise between memory space and computation time. Moreover, certain fields concerning the external problem are going to have to belong to the space tangent to the external surface. Here again, to minimize the numerical costs, we did not discretize these fields in the tangent space but rather opted for a cartesian representation. We will see in the present article that the integral operators involved extract the tangent part and project into the tangent space. So the only effect this decision has is to increase the dimension of the operator kernel. We will see in Part III that the iterative solving method we use automatically solves this problem too. So this approach has no determental repercussions of any kind on the solution.

However, we also have fields that we must force to belong to the tangent space, in which case we use Lagrange multipliers. We will do the same for the functional constraints, which are zero divergence of the magnetic induction and the transmission conditions of the dielectric interfaces, as well as the boundary conditions on the internal interfaces of the conductors with the dielectrics. Of course an iterative solver is needed that will be able to consider the presence of Lagrange multipliers in the problem. A method has been developed for this (*see* Part III).

In light of all of this, we can choose finite elements of the lowest possible degree to minimize the numerical costs, as shall be seen in Part II. Considering the minimum regularity required for the formulation constructed, we chose $P_1$ elements (triangular patches for surfaces, tetrahedra for volumes) for the integral operators and the surface and volume differential operators.

### II, 3. — COMMENTS ON THE FORMULATION

For the internal problem, we eliminate, as we have said, the electric field in each internal dielectric domain. However, the weak formulation applied in each of these domains yields the tangent electric field on the dielectric boundary of the domain. So to couple this domain with the external medium through the dielectric boundary, the integral equation formulation of the external problem should yield the tangent electric field and the magnetic field too. Due to the set of formulation constraints defined above, the variables in the two integral equations obtained are the external tangent electric and magnetic fields. As these tangent fields are continuous through the dielectric surfaces, they couple naturally with the internal tangent electric and magnetic fields. Moreover, as the integral equation operators extract the tangent part and project it into the tangent space, we write them directly with the internal tangent electric and magnetic fields.

It will be seen that these two integral equations will be used to formulate the coupled problem, but also to suppress the ambiguity due to the irregular frequencies.

## III. — MODELING AND EQUATIONS

### III, 1. — NOTATION AND UNITS

The space $\mathbb{R}^3$ is referenced to a cartesian axis system $(x_1, x_2, x_3)$. We use $M$ or $x$ to denote a point in the $(x_1, x_2, x_3)$ coordinate system. Throughout the following discussion, $\mathbb{K}$ designates $\mathbb{R}$ or $\mathbb{C}$; $j=\sqrt{-1}$; $\bar{A}$ is the conjugate of $A$; $A \,.\, B=\sum_{j=1}^{3} A_j B_j$; $\|A\|=(A \,.\, \bar{A})^{1/2}$; $A \times B$ is the vector product of $A \times B \in \mathbb{K}^3$; and $\mathrm{Mat}_{\mathbb{K}}(m, m)$ is the set of $m \times m$ matrices whose elements are in $\mathbb{K}$. We use the quantities in homogenized units, *i.e.* the electric and magnetic fields have the same dimensions, which requires us to state that:

$$E=\sqrt{\varepsilon_0}\,\hat{E}, \qquad H=\sqrt{\mu_0}\,\hat{H}$$

$$D=\frac{1}{\sqrt{\varepsilon_0}}\,\hat{D}, \qquad B=\frac{1}{\sqrt{\mu_0}}\,\hat{B}$$

$$J=\sqrt{\varepsilon_0}\,\hat{J}, \qquad q=\frac{1}{\sqrt{\varepsilon_0}}\,\hat{q},$$

in which the electric field $E$ has values in $\mathbb{K}^3$, as do the magnetic field $H$, the electric displacement (or induction) $D$, the magnetic induction $B$ and the electric current density $J$, while the electric charge density $q$ has values in $\mathbb{K}$. We will be using $\varepsilon_0$ and $\mu_0$ in their classical notation for the permittivity and permeability in a vacuum, such that $c^2 \varepsilon_0 \mu_0 = 1$, in which $c$ is the speed of light in a vacuum. Those quantities that are not topped by a circumflex marking are in homogenized units.

## III. 2. — HYPOTHESES UNDERLYING THE DEVELOPMENTS

Although the formulation below can be extended without modification to multibody configurations, we will simplify the discussion by considering the case of a single multidomain body occupying the open domain $\Omega_i$, a bounded set of $\mathbb{R}^3$, simply connected, and we have $\bar{\Omega}_i = \Omega_i \cup \partial\Omega_i$, in which $\delta\Omega_i$ is the boundary of $\Omega_i$.

We use $\Omega_e = \mathbb{R}^3 \setminus \bar{\Omega}_i$ to denote the open external domain, an unbounded domain of $\mathbb{R}^3$.

The domain $\Omega_i$ is the union of two domains of empty intersection, $\Omega_i = \Omega_d \cup \Omega_c \cup \Sigma$, $\Omega_d \cap \Omega_c = \varnothing$, and $\Sigma$ is the union of the various boundaries internal to $\Omega_i$, while $\Omega_c$ is the domain representing the union of perfect conductive materials and $\Omega_d$ the union of dielectric materials. $\Omega_c$ and $\Omega_d$ are not necessarily connected. It should be noted, though, that if a given dielectric part is completely surrounded by a perfect conductor, it does not enter into the modeling.

The domain $\Omega_d$ is expressed as the union of dielectric domains $\Omega_n$, $n \in \{1, \ldots, N\}$ open bounded domains of empty intersection $\Omega_d = \bigcup_{n=1}^{N} \Omega_n$ such that for each dielectric $\Omega_n$ the permittivity and permeability tensors are continuous functions on $\Omega_n$. Consequently, the domain $\Omega_d$ is a medium of materials having characteristics that are piecewise continuous functions.

In all of the ensuing developments, we will use the following hypotheses [10, 12, 15, 34]:

*a) External Medium* $\Omega_\varepsilon$

This is air, but will be assumed to be the same as a vacuum. There is no volume charge density, so we have in $\Omega_\varepsilon$:

$$\left.\begin{array}{c} D_e = E_e \\ B_e = H_e \\ J_e = 0, \quad q_e = 0 \end{array}\right\} \qquad (1)$$

*b) Perfect Conductive Medium* $\Omega_c$

The electric and magnetic fields being zero within a perfect conductive medium, there can only be surface currents and charges on the boundary $\Omega_c$. So in $\Omega_c$ we have:

$$\left.\begin{array}{c} D_c = E_c = B_c = H_c = J_c = 0 \\ q_c = 0 \end{array}\right\} \qquad (2)$$

*c) Dielectric Medium* $\Omega_n$

Each dielectric medium $\Omega_n$, $n \in \{1, \ldots, \mathrm{N}\}$ can have magnetic losses if so desired, is made of a material exhibiting continuous, antisotropic paraeters, and has some finite resistivity. The electric current density is given by Ohm's law. We therefore have:

$$\left.\begin{array}{c} D = \underline{\varepsilon}\, E \\ B = \underline{\mu}^*\, H \\ J = \underline{\sigma}\, E \end{array}\right\} \qquad (3)$$

with, at each fixed point $M$ in $\Omega_n : \underline{\varepsilon}(M) \in \mathrm{Mat}_{\mathbb{R}}(3, 3)$, the dielectric matrix, which is symmetrical, real, positive definite (permittivity tensor) $\underline{\mu}^*(M) \in \mathrm{Mat}_{\mathbb{C}}(3, 3)$ such that:

$$\underline{\mu}^*(M) = \underline{\mu}(M) - j\,\underline{\mu}_l(M) \qquad (4)$$

with $\underline{\mu}(M) \in \mathrm{Mat}_{\mathbb{R}}(3, 3)$ the magnetic permeability matrix, which is square, real, symmetrical and positive definite, and $\underline{\mu}_l(M) \in \mathrm{Mat}_{\mathbb{R}}(3, 3)$ a square, real, symmetrical, positive matrix (it is identically zero if $\Omega_n$ has no magnetic loss, and is positive definite if it does).

Lastly, $\underline{\sigma}(M) \in \mathrm{Mat}_{\mathbb{R}}(3, 3)$ is a symmetrical, real, positive matrix. If the material is isotropic, then $\underline{\sigma} = \sigma^{-1}$, in which $\sigma$ is the conductivity and $\sigma^{-1}$ the resistivity.

*Remarks*

(1) In the present harmonic formulation, all of the parameters may depend on the frequency without changing the results any.

(2) It is recalled that, according to our hypothesis, $M \mapsto \underline{\varepsilon}(M)$, $\underline{\mu}^*(M)$ and $\underline{\sigma}\, M)$ are continuous functions on $\Omega_n$.

## III. 3. — EQUATIONS FOR THE HARMONIC DOMAIN

We introduce the wave number $k$:

$$k = \frac{\omega}{c}, \qquad (5)$$

in which $\omega$ is the angular frequency of the incident wave. We assume throughout the following that $k$ is strictly positive.

*a) For the external medium* $\Omega_\varepsilon$, we write:

$$\left.\begin{array}{c} E_e = E_i + E_r \\ H_e = H_i + H_r \end{array}\right\} \qquad (6)$$

in which $(E_r, H_r)$ and $(E_1, H_1)$ are the reflected and incident electromagnetic field, respectively. The field $(E_r, H_r)$ verifies Maxwell's harmonic equations in $\Omega_\varepsilon$;

$$\left.\begin{array}{c} jk\, H_r + \mathrm{rot}\ E_r = 0 \\ -jk\, E_r + \mathrm{rot}\ H_r = 0 \\ \mathrm{div}\ H_r = 0 \\ \mathrm{div}\ E_r = 0 \end{array}\right\} \qquad (7)$$

to which Sommerfeld's outward radiation condition at infinity must be added to obtain a unique solution.

This condition for $r=\|x\| \to +\infty$ is written [10, 15, 34]:

$$\left.\begin{aligned} &(\mathrm{rot}\ E_r) \times \frac{x}{r} + jk\,E_r = o\left(\frac{1}{r}\right) \\ &\mathrm{rot}\ H_r + jk\,H_r \times \frac{x}{r} = o\left(\frac{1}{r}\right) \\ &E_r = O\left(\frac{1}{r}\right), \qquad H_r = O\left(\frac{1}{r}\right) \end{aligned}\right\} \tag{8}$$

using the classical notation in which $y \in \mathbb{R}$ is the function that verifies $|y^{-1}\,o(y)| \to 0$ for $y \to 0$, and $y \in \mathbb{R}$ is the function that verifies $|y^{-1}\,O(y)| \leqq C$, with $C$ being a constant.

By hypothesis, the incident electromagnetic wave $(E_i, H_i)$ verifies Maxwell's harmonic equations at every point in space:

$$\left.\begin{aligned} jk\,H_i + \mathrm{rot}\ E_i &= 0 \\ -jk\,E_i + \mathrm{rot}\ H_i &= 0 \\ \mathrm{div}\ H_i &= 0 \\ \mathrm{div}\ E_i &= 0 \end{aligned}\right\} \tag{9}$$

*b) For each dielectric medium* $\Omega_n$, Maxwell's equations are written (in consideration of (3)):

$$jk\,\underline{\mu}^*\,H + \mathrm{rot}\ E = 0 \tag{10.1}$$

$$-jk\,\underline{\varepsilon}^*\,E + \mathrm{rot}\ H = 0 \tag{10.2}$$

$$\mathrm{div}\ (\underline{\mu}^*\,H) = 0 \tag{10.3}$$

$$\mathrm{div}\ (\underline{\varepsilon}\,E) = q \tag{10.4}$$

$$jkq + \mathrm{div}(\underline{\sigma}\,E) = 0, \tag{11}$$

in which we have let:

$$\underline{\varepsilon}^*(M) = \underline{\varepsilon}(M) - \frac{j}{k}\,\underline{\sigma}(M) \in \mathrm{Mat}_{\mathbb{C}}(3, 3), \tag{12}$$

in which the matrix $\underline{\varepsilon}^*(M)$ is symmetrical, complex and invertible for any $M \in \Omega_n$, because $\underline{\varepsilon}(M)$ and $\underline{\sigma}(M)$ are two symmetrical, real, positive definite matrices. We assume from here on that the function $M \mapsto \underline{\varepsilon}^*(M)$ and $M \mapsto \underline{\mu}^*(M)$ are continuous on each domain $\Omega_n$.

## III, 4. — BOUNDARY AND INTERFACE CONDITIONS

### III, 4. 1. — *Boundary Notation*

Figure 1 gives an example of a body geometry with the boundary notation we will be using.

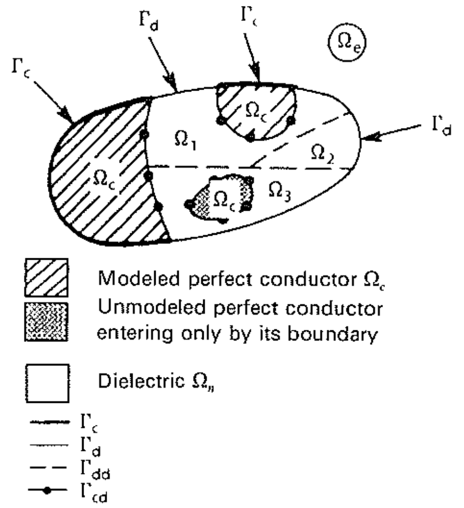


Fig. 1. – Example of geometry.

*a)* Boundary $\Omega_e$ between the body and the external medium. This is defined as $\partial\Omega_i = \Gamma_d \cup \Gamma_c$, in which $\Gamma_d$ is the interface of the external medium $\Omega_e$ with the union $\Omega_d$ of the internal dielectric media, and $\Gamma_c$ is the interface of the external medium $\Omega_e$ with the union $\Omega_c$ of the perfectly conductive media.

*b)* Internal boundary $\Gamma_{cd}$: the union of the interfaces of the perfect conducting media with the internal dielectrics.

*c)* Internal boundary $\Gamma_{dd} = \cup\,\Gamma_{nn'}$ between two dielectrics: the union of interfaces between the internal dielectrics, in which $\Gamma_{nn'}$ is the interface between $\Omega_n$ and $\Omega_{n'}$: $\Gamma_{nn'} = \partial\Omega_n \cap \partial\Omega_{n'}$.

*Note.* We assume there to be sufficient regularity over all of the boundaries for the Stokes theorem to apply.

### III, 4. 2. — *Notation Relating to the Surface Geometry*

From here on, $n$, $n'$, etc., will designate unit normals, *i.e.* $\|n\| = 1$, $\|n'\| = 1$, etc. The normal to $\delta\Omega_i$ will be external to $\Omega_i$. Let $\delta\Omega_n$ be the boundary of a dielectric domain $\Omega_n$. For $n \in \{1, \ldots, N\}$, we will use $n$ to indicate the normal at $\delta\Omega_n$, outward from $\Omega_n$. So a boundary such that $\Gamma_{nn'}$ will carry two opposite normals. We have $n' = -n$.

To simplify the writing, we will also use $M$ to denote the vector $OM$ in $\mathbb{R}^3$. We have:

$$M = \begin{bmatrix} x_1 \\ x_3 \\ x_2 \end{bmatrix}, \qquad dM = \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}.$$

Let $\Gamma$ be a surface in $\mathbb{R}^3$ of class $C^1$ oriented by the unit normal $n = (n_1, n_2, n_3)$ at each point $M$ of $\Gamma$. Let $ds$ be the surface measure on $\Gamma$. Then if $M \in \Gamma$ we have:

$$\left.\begin{aligned} dx_2 \wedge dx_3 &= n_1\,ds \\ dx_3 \wedge dx_1 &= n_2\,ds \\ dx_1 \wedge dx_2 &= n_3\,ds. \end{aligned}\right\}$$

Let $\Sigma$ be a surface of the type $\partial\Omega_i = \Gamma_c \cup \Gamma_d$ or $\Gamma_{nn'}$ or $\Gamma_{cd}$ (or any portion of these surfaces). For any field $M \mapsto A(M)$ defined on $\Sigma$ from values $\mathbb{K}^3$, we say that:

$$\left.\begin{aligned} A &= A_\nu n + A_\tau \\ A_\nu &= A \,.\, n \\ A_\tau \,.\, n &= 0 \end{aligned}\right\} \qquad (13)$$

with $A_\nu(M)$ having values in $\mathbb{K}$, the projection of $A(M)$ on $n(M)$, and $A_\tau(M)$ having values in $\mathbb{K}^3$, the projection of $A(M)$ on the plane tangent to $\Sigma$ at the point $M$. The decomposition (13) of $A$ is equivalent to considering the direct sum of the space tangent to $\Sigma$ with its orthogonal space. The normal $n$ being defined $ds$ almost everywhere on $\Sigma$, the local relation $A_\tau(M) \,.\, n(M) = 0$ for $M \in \Sigma$ is equivalent to

$$\int_\Sigma A_\tau(M) \,.\, n(M)\, ds(M) = 0.$$

*Remark:* For purposes of notation, we will sometimes have to use $A^\nu$ and $A^\nu{}_\tau$.

### III, 4.3. — *Transmission Conditions*

We recall the transmission conditions at the interface $\Sigma$ between two continuous, perfect and different media $\Omega$ and $\Omega'$ [10, 12, 15, 34]. We use $n$ to denote the unit normal at $\Sigma$ external to $\Omega$ and $n' = -n$ (Fig. 2).

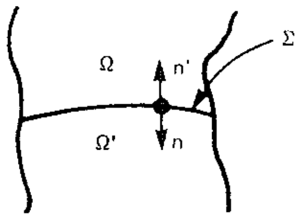


Fig. 2. – Diagram of principle.

The fields relative to $\Omega$ (resp. $\Omega'$) are denoted $D$, $E$, $B$ and $H$ (resp. $D'$, $E'$, $B'$, $H'$). The conditions at the interface $\Sigma$ are then:

$$\left.\begin{aligned} (D'-D)\,.\,n &= q_\Sigma \\ (H'-H)\times n &= -J_\Sigma \\ (B'-B)\,.\,n &= O \\ (E'-E)\times n &= 0 \end{aligned}\right\}$$

in which $q_\Sigma$ and $J_\Sigma$ are, respectively, the charge and current densities concentrated on the surface $\Sigma$.

*a) Conditions on the body-external medium interface* $\Omega_e$

This interface consists of $\Gamma_c$ and $\Gamma_d$ and the normal $n$ is external to $\Omega_1$. Using (2), we get on $\Gamma_c$:

$$H_e \,.\, n = 0 \qquad (14.1)$$
$$E_e \times n = 0 \qquad (14.2)$$
$$E_e \,.\, n = q_{\Gamma_c} \qquad (15.1)$$
$$H_e \times n = -J_{\Gamma_c}. \qquad (15.2)$$

The charge and current densities $q_{\Gamma_c}$ and $J_{\Gamma_c}$ concentrated on $\Gamma_c$ are not known. The conditions to be imposed are therefore (14). Once we have solved, we can calculate $q_{\Gamma_c}$ and $J_{\Gamma_c}$ using (15), knowing $E_e$ and $H_e$ on $\Gamma_c$.

We assume that there is no charge or current concentrated on the interface $\Gamma_d$, so $q_{\Gamma_d} = 0$ and $J_{\Gamma_d} = 0$. Using the relations (3), we get for the interface $\Gamma_d$:

$$E_e \times n = E \times n \qquad (16.1)$$
$$H_e \times n = H \times n \qquad (16.2)$$
$$E_e \,.\, n = (\underline{\varepsilon}\, E)\,.\, n \qquad (17.1)$$
$$H_e \,.\, n = (\underline{\mu}^* H)\,.\, n. \qquad (17.2)$$

The transmission conditions (16) will be used for the external-internal coupling, while (17) are used to compute the other quantities of the problem after the solution.

*b) Conditions on the* $\Gamma_{cd}$ *interface between a perfect internal conductor and an internal dielectric*

Using (2) and (3), we get:

$$(\underline{\mu}^* H)\,.\, n = 0 \qquad (18.1)$$
$$E \times n = 0 \qquad (18.2)$$
$$(\underline{\varepsilon}\, E)\,.\, n = -q_{\Gamma_{cd}} \qquad (19.1)$$
$$H \times n = J_{\Gamma_{cd}}. \qquad (19.2)$$

As before, the charge and current densities $q_{\Gamma_{cd}}$ and $J_{\Gamma_{cd}}$ concentrated on $\Gamma_{cd}$ are not known. The conditions to be set are therefore (18). Once solved, we can calculate $q_{\Gamma_{cd}}$ and $J_{\Gamma_{cd}}$ using (19), knowing $E$ and $H$ on $\Gamma_{cd}$.

*c) Conditions on the* $\Gamma_{nn'}$ *interfaces between two internal dielectrics*

These are interfaces between $\Omega_n$ and $\Omega_{n'}$.

The fields will be denoted $H^n$, $E^n$, etc. (resp. $H^{n'}$, $E^{n'}$, etc.) for those relating to $\Omega_n$ (resp. to $\Omega_{n'}$).

Like for the $\Gamma_d$ interface, we assume that there is no charge or current concentrated on the $\Gamma_{nn'}$ interface, so that $q_{\Gamma_{nn'}} = 0$ and $J_{\Gamma_{nn'}} = 0$. Using relations (3), we get for the $\Gamma_{nn'}$ interface:

$$E^n \times n + E^{n'} \times n' = 0 \qquad (20.1)$$
$$H^n \times n + H^{n'} \times n' = 0 \qquad (20.2)$$
$$(\underline{\mu}^{*n} H^n)\,.\, n + (\underline{\mu}^{*n'} H^{n'})\,.\, n' = 0 \qquad (20.3)$$
$$(\underline{\varepsilon}^n E^n)\,.\, n + (\underline{\varepsilon}^{n'} E^{n'})\,.\, n' = 0. \qquad (20.4)$$

The transmission conditions (20.1) to (20.3) have to be set. Relation (20.4) is used after solving, to compute the auxiliary quantities.

## IV. — WEAK FORMULATION FOR THE INTERNAL DIELECTRIC DOMAINS

Throughout the following, if $A$ designates an element of a certain functional space, then $\delta A$ will designate another element of this same functional space.

### IV, 1. — WEAK FORMULATION FOR A SINGLE INTERNAL DIELECTRIC DOMAIN

We consider here the dielectric domain $\Omega_n$ with $n$ fixed in $\{1, \ldots, N\}$.

We define the bilinear form $H^n \Psi^n \mapsto M^n(H^n, \Psi^n)$ on $C^1(\Omega_n, \mathbb{C}^3) \times C^1(\Omega_n, \mathbb{C}^3)$ by:

$$M^n(H^n, \Psi^n) = jk^2 \int_{\Omega_n} (\underline{\mu}^{*n} H^n) \cdot \Psi^n \, dx - j \int_{\Omega_n} ([\underline{\varepsilon}^{*n}]^{-1} \operatorname{rot} H^n) \cdot \operatorname{rot} \Psi^n \, dx. \quad (21)$$

Then the weak formulation for the first of Maxwell's equations (10.1) yields, for any differentiable $\delta H^n$ of $\Omega_n$ in $\mathbb{C}^3$:

$$M^n(H^n, \delta H^n) + k \int_{\partial\Omega_n} \delta H^n \cdot (n \times E^n) \, ds = 0. \quad (22)$$

We have:

$$\operatorname{Im}([\underline{\varepsilon}^*]^{-1}) = k[\underline{\sigma} + k^2 \underline{\varepsilon} \underline{\sigma}^{-1} \underline{\varepsilon}]^{-1} \stackrel{\text{def}}{=} \underline{\beta}.$$

The matrix $\underline{\beta}(M) \in \mathrm{Mat}_{\mathbb{R}}(3, 3)$ is symmetrical, definite and positive, so we can say that:

$$\operatorname{Re} M^n(H, \bar{H}) = a_n(H, H) + b_n(H, H) \quad (23.1)$$

$$a_n(H, H') = k^2 \int_{\Omega_n} (\underline{\mu}_I H) \cdot \bar{H}' \, dx \quad (23.2)$$

$$b_n(H, H') = \int_{\Omega_n} (\underline{\beta} \operatorname{rot} H) \cdot \operatorname{rot} \bar{H}' \, dx. \quad (23.3)$$

Since for any $M \in \Omega_n$, $\underline{\mu}_i(M)$ and $\underline{\beta}(M)$ are, respectively, positive and positive definite matrices, we conclude that:

$$\operatorname{Re} M^n(H, \bar{H}) \geqq 0. \quad (24)$$

If the dielectric $\Omega_n$ is lossy, then $\underline{\mu}_T(M)$ is positive definite for any $M \in \Omega_n$, and this implies $a_n(H, H) > 0$ and therefore $\operatorname{Re} M^n(H, \bar{H}) > 0$ for any nonzero $H$. So in this case it is not necessary to set the constraint (10.3) for the dielectric $\Omega_n$. But if the dielectric $\Omega_n$ is not lossy, then:

$$\underline{\mu}_I = 0 \quad \text{and} \quad \operatorname{Re} M^n(H, \bar{H}) = b_n(H, H)$$

and constraint (10.3) must be set [10], which we will do using a Lagrange multiplier for the reasons mentioned in section II, 2.

In this latter case, we use $\lambda_{\Omega_n}$ to denote the Lagrange multiplier, which is continuous of $\Omega_n$ in $\mathbb{C}$.

If $\Gamma_{cd,n} = \Gamma_{cd} \cap \partial\Omega_n$ is not empty, the boundary condition (18.1) is included by introducing a Lagrange multiplier, denoted $\lambda_{cd,n}$, which is continuous of $\Gamma_{cd,n}$ in $\mathbb{C}$, and the condition (18.2) is entered directly in (22).

If $\Gamma_d \cap \partial\Omega_n$ is not empty, on the other hand, it is the transmission conditions (16.1) and (16.2) that are entered in (22) instead.

### IV, 2. — TREATMENT OF TRANSMISSION CONDITIONS BETWEEN TWO INTERNAL DIELECTRIC DOMAINS

Let $N_\Gamma$ be the number of $\Gamma_{nn'}$ type interfaces. The paired subscript $(n, n')$ is assumed to be ordered and describes the subset $\mathcal{N}_\Gamma$ of

$$\{1, \ldots, N\} \times \{1, \ldots, N\},$$

the cardinal of which is $N_\Gamma$. It is the first subscript, $n$, that corresponds to the normal $n$ at $\Gamma_{nn'}$ and which is used to orient the interface $\Gamma_{nn'}$ (that is, there are two normals on $\Gamma_{nn'}$: $n$ and $n' = -n$). Thus

$$\Gamma_{dd} = \bigcup_{(n, n') \in \mathcal{N}_\Gamma} \Gamma_{nn'}.$$

For the transmission conditions on the interfaces $\Gamma_{nn'}$, we proceed as follows. The component $E_\tau^n$ of the tangential electric field on $\Gamma_{nn'}$ is unknown, so it must be conserved and we are then one equation short. So we use the transmission condition (20.2) for the tangential component of the magnetic fields as an equation that will be written weakly on the interface $\Gamma_{nn'}$ and we then have the right number of equations. Condition (20.3) relating to the normal magnetic induction at $\Gamma_{nn'}$ must been be taken as a constraint. We will use a Lagrange multiplier for this, denoting it $\lambda_{\Gamma_{nn'}}^{\nu}$, and which is a continuous function on $\Gamma_{nn'}$ with values in $\mathbb{C}$.

To condense the expression, we are then led to define the field $\lambda_{\Gamma_{n,n'}}$. This is continuous on $\Gamma_{nn'}$ and has values in $\mathbb{C}^3$ such that:

$$\lambda_{\Gamma_{nn'}} = \lambda_{\Gamma_{nn'}}^{\nu} \, n + E_\tau^n, \qquad \lambda_{\Gamma_{nn'}}^{\nu} = \lambda_{\Gamma_{nn'}} \cdot n,$$

and we must also introduce, for any $M \in \Gamma_{nn'}$, the two linear operators $R_{\Gamma_{nn'}}^n(M)$ and $R_{\Gamma_{nn'}}^{n'}(M)$ of $\mathbb{C}^3$ in $\mathbb{C}^3$, such that:

$$R^n_{\Gamma_{nn'}}(M)=[(n(M)\otimes n(M))\,\underline{\mu}^{*n}(M)-n(M)\times] \quad (25)$$

$$R^{n'}_{\Gamma_{nn'}}(M)=[(n(M)\otimes n(M))\,\underline{\mu}^{*n'}(M)-n(M)\times]$$
$$=[(n'(M)\otimes n'(M))\,\underline{\mu}^{*n'}(M)+n'(M)\times]. \quad (26)$$

It will be noted that $R^n_{\Gamma_{nn'}}(M)$ and $R^{n'}_{\Gamma_{nn'}}(M)$ are different, event though $n=-n'$, because in the general case, $\underline{\mu}^{*n}$ is different from $\underline{\mu}^{*n'}$.

## IV, 3. — WEAK FORMULATION FOR THE COMPLETE INTERNAL PROBLEM

Starting with the weak formulation (22), for each dielectric domain of equation (20.2) written weakly on each $\Gamma_{nn'}$ interface, we get the weak expression for a linear equation and we show that the associated linear operator is symmetrical, complex and has a real kernel.

The constraints have to be added to this equation. Their Lagrange multipliers are $\lambda_{\Omega_n}$, $\lambda^{v}_{\Gamma_{nn'}}$, and $\lambda_{\Gamma_{cd,n}}$. Since the part of the operator associated with (20.1) and (20.2) is real, we can replace its transport by its adjoint, and can then regroup it with the operators associated with the constraints by using the definition of $\lambda_{\Gamma_{nn'}}$ in (25) and (26). We then arrive at the following operational weak expression:

$$\ll \mathbb{M}\,H,\,\delta H \gg + \ll \mathrm{Div}^*\lambda_{\Omega_d},\,\delta H \gg$$
$$+2j<P_{dd}\,E_{e,\tau},\,\delta H>$$
$$+\langle T^*_{\Gamma_{dd}}\lambda_{\Gamma_{dd}},\,\delta H\rangle+\langle U^*_{cd}\lambda_{cd},\,\delta H\rangle=0 \quad (27.1)$$

$$\langle T_{\Gamma_{dd}}H,\,\delta\lambda_{\Gamma_{dd}}\rangle=0 \quad (27.2)$$

$$\ll \mathrm{Div}\,H,\,\delta\lambda_{\Omega_d} \gg =0 \quad (27.3)$$

$$\langle U_{cd}H,\,\delta\lambda_{cd}\rangle=0, \quad (27.4)$$

in which $\mathbb{M}$, Div. $P_{dd}$, $T_{\Gamma_{dd}}$ and $U_{cd}$ are the linear operators defined by the following bilinear forms:

$$\ll \mathbb{M}\,H,\,\delta H \gg = \sum_{n=1}^{N} M^n(H^n,\,\delta H^n), \quad (28.1)$$

$$\ll \mathrm{Div}\,H,\,\delta\lambda_{\Omega_d} \gg$$
$$=\sum_{n=1}^{N}\int_{\Omega_n}\delta\lambda_{\Omega_n}\,\mathrm{div}(\underline{\mu}^{*n}H^n)\,dx, \quad (28.2)$$

$$\langle P_{dd}\,E_{e,\tau},\,\delta H\rangle$$
$$=-\frac{jk}{2}\sum_{n=1}^{N}\int_{\Gamma_{d,n}}\delta H^n.(n\times E_{e,\tau})\,ds, \quad (28.3)$$

$$\langle T_{\Gamma_{dd}}H,\,\delta\lambda_{\Gamma_{dd}}\rangle$$
$$=\sum_{(n,n')\in A\Gamma}k\int_{\Gamma_{nn'}}(R^n_{\Gamma_{nn'}}H^n$$
$$-R^{n'}_{\Gamma_{nn'}}H^{n'}).\,\delta\lambda_{\Gamma_{nn'}}\,ds, \quad (28.4)$$

$$\langle U_{cd}H,\,\delta\lambda_{cd}\rangle$$
$$=\sum_{n=1}^{N}\int_{\Gamma_{cd,n}}\delta\lambda_{cd,n}(\underline{\mu}^{*n}H^n).n\,ds, \quad (28.5)$$

in which ${}^tA$ designates the algebraic transport of the linear operator $A$, and $A^*$ its algebraic adjoint ($A^*={}^t\bar{A}$).

$H$, $\lambda_{\Omega_d}$, $\lambda_{\Gamma_{dd}}$ and $\lambda_{cd}$ are the functions defined on $\Omega_n$, $\Omega_d$, $\Gamma_{dd}$ and $\Gamma_{cd}$, the restrictions of which are $H^n$, $\lambda_{\Omega_n}$, $\lambda_{\Gamma_{nn'}}$ and $\lambda_{cd,n}$ at $\Omega_n$, $\Omega_n$, $\Gamma_{nn'}$ and $\Gamma_{cd,n}$. We will note for example that if $H_n$ is continuous on $\Omega_n$, then $H$ is discontinuous on $\Omega_d$.

To couple the internal problem with the external one, we need to make the trace of $H$ appear explicitly on $\Gamma_d$. Calling this trace $H_{bd}$. We write the following block decompositions:

$$\mathbb{M}\,H=\begin{bmatrix}\mathbb{M}_{bb} & \mathbb{M}_{b,\,\mathrm{int}}\\ \mathbb{M}_{\mathrm{int},\,b} & \mathbb{M}_{\mathrm{int},\,\mathrm{int}}\end{bmatrix}\begin{bmatrix}H_{bd}\\ H_{\mathrm{int}}\end{bmatrix}, \quad (29)$$

$$H=\begin{bmatrix}H_{bd}\\ H_{\mathrm{int}}\end{bmatrix};\qquad \lambda_{\mathrm{int}}=\begin{bmatrix}\lambda_{\Omega_d}\\ \lambda_{\Gamma_{dd}}\\ \lambda_{cd}\end{bmatrix}. \quad (30)$$

With these notations, the symbolic operatorial form of the system of equations (27) is:

$$\begin{bmatrix}\mathbb{M}_{bb} & \mathbb{M}_{b,\,\mathrm{int}} & \mathcal{M}^*_{b,\,\mathrm{int}}\\ \mathbb{M}_{\mathrm{int},\,b} & \mathbb{M}_{\mathrm{int},\,\mathrm{int}} & \mathcal{M}^*_{\mathrm{int},\,\mathrm{int}}\\ \mathcal{M}_{b,\,\mathrm{int}} & \mathcal{M}_{\mathrm{int},\,\mathrm{int}} & 0\end{bmatrix}\begin{bmatrix}H_{bd}\\ H_{\mathrm{int}}\\ \lambda_{\mathrm{int}}\end{bmatrix}$$
$$+\begin{bmatrix}2j\,P_{dd}\,E_{e,\tau}\\ 0\\ 0\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}, \quad (31)$$

with:

$$\mathcal{M}^*_{b,\,\mathrm{int}}=[(\mathrm{Div})^*_b,\,T^*_{\Gamma_{dd}},\,0] \quad (32)$$

$$\mathcal{M}^*_{\mathrm{int},\,\mathrm{int}}=[(\mathrm{Div})^*_{\mathrm{int}},\,0,\,U^*_{cd}]. \quad (33)$$

The operator $\mathbb{M}$ is symmetrical and complex. We have ${}^t\mathbb{M}=\mathbb{M}$ and therefore $\mathbb{M}_{\mathrm{int},\,b}={}^t\mathbb{M}_{b,\,\mathrm{int}}$.

# V. — INTEGRAL EQUATION FORMULATION OF THE EXTERNAL PROBLEM

## V,1. — NOTATION

Let $M$ and $M'$ be two points in $\mathbb{R}^3$. Let $r=\|M'-M\|$ be the distance from $M$ to $M'$ and $r\mapsto N(r)$ such that:

$$N(r)=\frac{1}{r}\,e^{-jkr}. \tag{34}$$

Then $N(r)$ verifies the Helmoltz equation and the radiation to infinity condition in the sense of the distributions, at fixed $M$:

$$k^2 N+\Delta' N=-4\pi\delta_M, \qquad M'\in\mathbb{R}^3. \tag{35}$$

We recall the property:

$$\operatorname{grad} N=-\operatorname{grad}' N \tag{36}$$

In all of the following, the differential operators are primed when the derivatives are taken with respect to the point $M'$ (otherwise they are taken with respect to the point $M$). Similarly, a primed quantity is one that is a function of $M'$ (and otherwise it is a function of $M$). $N$ is an exception, as it depends on $M$ and $M'$.

## V,2. – INTEGRAL EQUATIONS USED TO COMPUTE THE EXTERNAL ELECTROMAGNETIC FIELD

The first step in the integral equation formulation is to obtain the integral relations that express the electromagnetic field radiated at any point $M$ in the open, unbounded set $\Omega_e$ as a function of the electromagnetic field on the surface $\partial\Omega_i$. These relations will also later be used to compute the electromagnetic field reflected into $\Omega_e$ (near and far field), once the coupled external-internal system is solved. These relations are established in [3] and are written:

$$4\pi H_r=\int_{\partial\Omega_i}\left(H'_e\,\frac{d'N}{dn'}\,ds'-H'_e\times d'(N\,dM')\right.$$
$$\left.-jk\,N(E'_e\times n')\,ds'\right),\qquad \forall M\in\Omega_e \tag{37.1}$$

$$4\pi E_r=\int_{\partial\Omega_i}\left(E'_e\,\frac{d'N}{dn'}\,ds'-E'_e\times d'(N\,dM')\right.$$
$$\left.+jk\,N(H'_e\times n')\,ds'\right),\qquad \forall M\in\Omega_e. \tag{37.2}$$

## V,3. – INTEGRAL EQUATIONS USED TO COMPUTE THE ELECTROMAGNETIC FIELD ON THE EXTERNAL SURFACE

The second step consists in constructing the limit of equations (37) as $M$ approaches the surface $\partial\Omega_i$. We then obtain the integral equations on $\partial\Omega_i$ that are established in [3], and which are written:

$$2\pi H_e-\oint_{\partial\Omega_i}((n'\times H'_e)\times\operatorname{grad}' N\,ds'$$
$$+(H'_e.n')\operatorname{grad}' N\,ds'$$
$$-jk\,N(E'_e\times n\pi)\,ds')=4\pi H_i, \tag{38.1}$$

$$2\pi E_e-\oint_{\partial\Omega_i}((n'\times E'_e)\times\operatorname{grad}' N\,ds'$$
$$+(jk)^{-1}((\operatorname{rot}' H'_e).n')\operatorname{grad}' N\,ds'$$
$$-jk\,N(n'\times H'_e)\,ds')=4\pi E_i. \tag{38.2}$$

The notation $\oint$ indicates that the main part of the integral is taken, *i.e.* $\oint_{\partial\Omega_i}=\int_{\partial\Omega_i\setminus M}$. It will be noted that, for the third term of the integrals in (38), the integral can be taken in its usual sense, since the term is regular.

# VI. – WEAK FORMULATION FOR THE EXTERNAL PROBLEM

The purpose of this section is to construct the integral operators of the external problem from the weak formulation of equations (38), and also to give the main algebraic properties. For the functional properties, the reader may refer to [6, 10, 13].

## VI,1. – NOTATION

Let $\mathscr{C}_1=C^1(\partial\Omega_i,\mathbb{R}^3)$ be the space of functions $C^1$ defined on $\partial\Omega_i$ with values in $\mathbb{R}^3$. The scalar product is defined on $\mathscr{C}^1$:

$$A,B\mapsto\langle A,B\rangle=\int_{\partial\Omega_i}A(M).B(M)\,ds(M).$$

Let $\mathscr{C}_1^c$ be the complexified form of $\mathscr{C}_1$. The bilinear form $A,B\mapsto\langle A,B\rangle$ *on* $\mathscr{C}_1\times\mathscr{C}_1$ is extended on $\mathscr{C}_1^c\times\mathscr{C}_1^c$. For a given field $A\in\mathscr{C}_1$, we use decomposition (13) and define $\mathscr{C}$ such that:

$$\mathscr{C}=\{A_\tau\in\mathscr{C}_1;\ \langle A_\tau,n\rangle=0\},$$

and we use $\mathscr{C}^c$ to denote the complexified form of $\mathscr{C}$. We will also be needing the identity operator $I$ on $\mathscr{C}^c$. This operator is such that, for any $A$ and $B$ in $\mathscr{C}^c$, we have $\langle IA, B \rangle = \langle A, B \rangle$.

### VI,2. – WEAK FORMULATION FOR THE TWO INTEGRAL EQUATIONS ON THE EXTERNAL SURFACE

Starting with equations (38), for any $\varphi_\tau \in \mathscr{C}^c$ we get their weak formulation as:

$$-\frac{k}{2}\langle n\times H_e, \varphi_\tau\rangle + \frac{k}{4\pi}\int_{\partial\Omega_i} ds \fint_{\partial\Omega_i} (\varphi_\tau \times n).[(n'\times H'_e)\times \text{grad}' N]\,ds' + \frac{j}{4\pi}\int_{\partial\Omega_i} ds \fint_{\partial\Omega_i} [(\text{rot}' E'_e).n'](\varphi_\tau\times n).\text{grad}' N\,ds' - \frac{jk^2}{4\pi}\int_{\partial\Omega_i} ds \int_{\partial\Omega_i} N(\varphi_\tau\times n).(E'_{e,\tau}\times n')\,ds' = -k\langle n\times H_i, \varphi_\tau\rangle, \quad (39.1)$$

$$-\frac{k}{2}\langle n\times E_{e,\tau}, \varphi_\tau\rangle + \frac{k}{4\pi}\int_{\partial\Omega_i} ds \fint_{\partial\Omega_i} (\varphi_\tau \times n).[(n'\times E'_e)\times \text{grad}' N]\,ds' - \frac{j}{4\pi}\int_{\partial\Omega_i} ds \fint_{\partial\Omega_i} [(\text{rot}' H'_e).n'](\varphi_\tau\times n).\text{grad}' N\,ds' + \frac{jk^2}{4\pi}\int_{\partial\Omega_i} ds \int_{\partial\Omega_i} N(\varphi_\tau\times n).(H'_e\times n')\,ds' = -k\langle n\times E_i, \varphi_\tau\rangle. \quad (39.2)$$

The two weak equations (39) are written symbolically in the following operatorial form. For any $\varphi_\tau \in \mathscr{C}^c$:

$$-j\langle (P+Q)H_e, \varphi_\tau\rangle - \langle (B-S)E_{e,\tau}, \varphi_\tau\rangle = -2j\langle PH_i, \varphi_\tau\rangle, \quad (40.1)$$

$$-j\langle (P+Q)E_{e,\tau}, \varphi_\tau\rangle + \langle (B-S)H_e, \varphi_\tau\rangle = -2j\langle PE_i, \varphi_\tau\rangle. \quad (40.2)$$

with, for any $A\in\mathscr{C}_1^c$ and for any $\varphi_\tau\in\mathscr{C}^c$:

$$\langle PA, \varphi_\tau\rangle = -\frac{jk}{2}\int_{\partial\Omega_i}\varphi_\tau.(n\times A)\,ds \quad (41)$$

$$\langle QA, \varphi_\tau\rangle = \frac{jk}{4\pi}\int_{\partial\Omega_i} ds \fint_{\partial\Omega_i}(\varphi_\tau\times n) \times[(n'\times A')\times\text{grad}' N]\,ds' \quad (42)$$

$$\langle BA, \varphi_\tau\rangle = -\frac{j}{4\pi}\int_{\partial\Omega_i}\int_{\partial\Omega_i} N\,d(\varphi_\tau.dM)\,d'(A'.dM') \quad (43)$$

$$\langle SA, \varphi_\tau\rangle = -\frac{jk^2}{4\pi}\int_{\partial\Omega_i} ds\int_{\partial\Omega_i} N\,ds'(\varphi_\tau\times n).(A'\times n'). \quad (44)$$

the following property will be observed. If $\text{Op}\in\{P, Q, B, S\}$, we have for any $A\in\mathscr{C}_1^c$ and $\varphi_\tau\in\mathscr{C}^c$:

$$\langle \text{Op}\,A, \varphi_\tau\rangle = \langle \text{Op}\,A_\tau, \varphi_\tau\rangle, \quad (45)$$

in which $A_\tau$ is in $\mathscr{C}^c$. We can therefore consider these operators on $\mathscr{C}^c$, and as such, $P$ is algebraically antisymmetrical:

$$^tP = -P, \quad (46)$$

and the operators $Q$, $B$ and $S$ are algebraically symmetrical:

$$^tQ = Q, \qquad ^tB = B, \qquad ^tS = S. \quad (47)$$

The reader may find further details on these developments in [3], in particular the way the operator $B$ is put in the form (43).

## VII. – WEAK FORMULATION FOR THE COUPLED PROBLEM

### VII,1. – UNKNOWN FIELDS IN THE COUPLED PROBLEM

For reasons that will be explained when we study the irregular frequencies in section VIII, we will be considering the following unknown fields for the assembled coupled problem:

$$E_{r,0} = n\times E_{r,d}\in\mathscr{C}^c \text{ on } \Gamma_d \text{ only } (E_{r,0}.n=0) \quad (48.1)$$

$$H_{r,0} = n\times H_{r,c}\in\mathscr{C}^c \text{ on } \Gamma_c \text{ only } (H_{r,0}.n=0) \quad (48.2)$$

$$E_{e,\tau}\in\mathscr{C}^c \text{ on } \Gamma_d \text{ only } (E_{e,\tau}.n=0) \quad (48.3)$$

$$H_{e,c}\in\mathscr{C}^c \text{ on } \Gamma_c \text{ only } (H_{e,c}.n=0) \quad (48.4)$$

$H_{bd}\in\mathscr{C}_1^c$ on $\Gamma_d$ only,

where $H_{bd}$ is the trace of $H$ on $\Gamma_d$ (48.5)

$H_{\text{int}}$ and $\lambda_{\text{int}}$ inside $\Omega_d$. (48.6)

## VII,2. – NEW BLOCKED EXPRESSION OF THE INTEGRAL EQUATIONS FOR COUPLING PURPOSES

The field $H$ can be expressed in blocks (30) and the field $H_e$ on $\Gamma_c \cup \Gamma_d$ is written:

$$H=\begin{bmatrix} H_{bd} \\ H_{\text{int}} \end{bmatrix}; \qquad H_e=\begin{bmatrix} H_{e,c} \\ H_{e,d} \end{bmatrix}, \tag{49}$$

with $H_{e,c}$ (resp. $H_{e,d}$) being the restriction $H_e$ to $\Gamma_c$ (resp. $\Gamma_d$).

To simplify the presentation, we will now rewrite (40.1) and (40.2) in operatorial form using the block breakdowns and coupling boundary conditions defined by (14) and (16). Starting with equation (40.1) and using the breakdown (6) of $H_e$, and the definition (48.2) of $H_{r,0} \in \mathscr{C}^c$, we get:

$$\begin{bmatrix} -P_{cc}+Q_{cc} & Q_{cd} \\ Q_{dc} & P_{dd}+Q_{dd} \end{bmatrix} \begin{bmatrix} H_{e,c} \\ H_{bd} \end{bmatrix} - \begin{bmatrix} j(B_{cd}-S_{cd})\,E_{e,\tau} \\ j(B_{dd}-S_{dd})\,E_{e,\tau} \end{bmatrix} + \begin{bmatrix} -jk\,I_{cc}\,H_{r,0} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2P_{dd}\,H_{i,d} \end{bmatrix} \tag{50}$$

in which $H_{i,c}$ and $H_{i,d}$ are, respectively, the restriction of $H_i$ to $\Gamma_c$ and $\Gamma_d$.

We rewrite equation (40.2) directly by stating explicitly the boundary conditions, which yields:

$$\begin{bmatrix} B_{cc}-S_{cc} & B_{cd}-S_{cd} \\ B_{dc}-S_{dc} & B_{dd}-S_{dd} \end{bmatrix} \begin{bmatrix} H_{e,c} \\ H_{bd} \end{bmatrix} + \begin{bmatrix} 0 \\ -j\,P_{dd}\,E_{e,\tau} \end{bmatrix} - \begin{bmatrix} j\,Q_{cd}\,E_{e,\tau} \\ j\,Q_{dd}\,E_{e,\tau} \end{bmatrix} = \begin{bmatrix} -2j\,P_{cc}\,E_{i,c} \\ -2j\,P_{dd}\,E_{i,d} \end{bmatrix}, \tag{51}$$

where we have used $E_{i,c}$ and $E_{i,d}$ to denote the restriction of $E_i$ to $\Gamma_c$ and $\Gamma_d$, respectively. Starting with the second block line of equation (51) and using (6) and definition (48.1) of $E_{r,0} \in \mathscr{C}^c$, we get:

$$j(B_{dc}-S_{dc})\,H_{e,c}+j(B_{dd}-S_{dd})\,H_{bd} + (-P_{dd}+Q_{dd})\,E_{e,\tau}-jk\,I_{dd}\,E_{r,0}=0, \tag{52}$$

*Note*: The blocked form of the $P$ operator in equation (50) uses notation that is misleading in that, as the function $\varphi_\tau$ is continuous on $\partial\Omega_i$, a coupling block should normally appear. Of course all of the terms will be considered when we adapt the formulation to the numerical process. However, for the time being, this sleight of hand makes for easier reading, and this is why we use it everywhere, even for the block form of the other integral operators in (50), (51) and (52).

## VII,3. – OVERALL ASSEMBLY OF THE COUPLED EXTERNAL-INTERNAL PROBLEM

We are going to construct a complex, symmetrical operator by introducing the new field $E^{\vee}_{e,\tau}$ such that:

$$E^{\vee}_{e,\tau}=-j\,E_{e,\tau}. \tag{53}$$

The equations of the problem are the three integral equations (50), (51) and (52) for the external domain, and equation (31) for the internal. The problems are assembled by putting the equations in the following order: (52), the first equation of (50) and then the second of (50), the first equation of (51) and then the second, with the first equation of (31) and finally the second and third equations of (31). Considering the fact that

$$Q_{cd}={}^tQ_{dc}, \qquad {}^tP_{dd}=-P_{dd}$$

and $B_{dc}-S_{dc}={}^t(B_{cd}-S_{cd})$, using (46) and (47) we get:

$$\begin{bmatrix} -jk\,I_{dd} & 0 & j(-P_{dd}+Q_{dd}) & j(B_{dc}-S_{dc}) & j(B_{dd}-S_{dd}) & 0 & 0 \\ 0 & -jk\,I_{cc} & (B_{cd}-S_{cd}) & -P_{cc}+Q_{cc} & Q_{cd} & 0 & 0 \\ 0 & 0 & B_{dd}-S_{dd} & Q_{dc} & P_{dd}+Q_{dd} & 0 & 0 \\ 0 & 0 & {}^tQ_{dc} & B_{cc}-S_{cc} & B_{cd}-S_{cd} & 0 & 0 \\ 0 & 0 & {}^tP_{dd}+Q_{dd} & {}^t(B_{cd}-S_{cd}) & B_{dd}-S_{dd}+\mathbb{M}_{bb} & \mathbb{M}_{b,\text{int}} & \mathscr{M}^*_{b,\text{int}} \\ 0 & 0 & 0 & 0 & {}^t\mathbb{M}_{b,\text{int}} & \mathbb{M}_{\text{int},\text{int}} & \mathscr{M}^*_{\text{int},\text{int}} \\ 0 & 0 & 0 & 0 & \mathscr{M}_{b,\text{int}} & \mathscr{M}_{\text{int},\text{int}} & 0 \end{bmatrix} \begin{bmatrix} E_{r,0} \\ H_{r,0} \\ E^{\vee}_{e,\tau} \\ H_{e,c} \\ H_{bd} \\ H_{\text{int}} \\ \lambda_{\text{int}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2P_{dd}\,H_{i,d} \\ -2j\,P_{cc}\,E_{i,c} \\ -2j\,P_{dd}\,E_{i,d} \\ 0 \\ 0 \end{bmatrix}. \tag{54}$$

It will be noted that the first two block lines of system (54) decouple. These will be used to calculate $H_{r,0}=n\times H_r$ on $\Gamma_c$ and $E_{r,0}=n\times E_r$ on $\Gamma_d$. It might seem that there is some redundancy in this system (54) but in fact this is not at all the case, as this formulation will be used to eliminate the irregular frequency problem due to the integral equations on $\Gamma_d \cup \Gamma_c$. This problem is discussed in section VIII.

The system (54) is therefore solved for the $(5\times 5)$ solution block in $E^{\vee}_{e,\tau}$, $H_{e,c}$, $H_{bd}$, $H_{\text{int}}$ and $\lambda_{\text{int}}$. This leads us to introduce the following notation to clarify the presentation:

$$m^*=\begin{bmatrix} 0 \\ 0 \\ \mathscr{M}^*_{b,\text{int}} \\ \mathscr{M}^*_{\text{int},\text{int}} \end{bmatrix}; \quad \mathbb{X}=\begin{bmatrix} E^{\vee}_{e,\tau} \\ H_{e,c} \\ H_{bd} \\ H_{\text{int}} \end{bmatrix};$$

$$\mathbb{S}=\begin{bmatrix} 2P_{dd}\,H_{i,d} \\ -2j\,P_{cc}\,E_{i,c} \\ -2j\,P_{dd}\,E_{i,d} \\ 0 \end{bmatrix}. \tag{55}$$

With this notation, the $(5\times 5)$ solver block of (54) is written:

$$\begin{bmatrix} \mathbb{A} & m^* \\ m & 0 \end{bmatrix}\begin{bmatrix} \mathbb{X} \\ \lambda_{\text{int}} \end{bmatrix}=\begin{bmatrix} \mathbb{S} \\ 0 \end{bmatrix}. \qquad (56)$$

It is shown that the complex operator $\mathbb{A}$ is algebraically symmetrical:

$$\mathbb{A}={}^t\mathbb{A} \qquad (57)$$

that its nonzero kernel $\mathscr{A}$ is a real vectorial subspace, and lastly that $\mathbb{A}$ is accretive, *i.e.* that:

$$\text{Ré}\langle \mathbb{A}\mathbb{X}, \bar{\mathbb{X}}\rangle \geqq 0, \qquad \forall \mathbb{X}. \qquad (58)$$

It is even shown that for any $\mathbb{X}\notin \ker \mathbb{A}$, we have

$$\text{Ré}\langle \mathbb{A}\mathbb{X}, \bar{\mathbb{X}}\rangle > 0. \qquad (59)$$

It will be noted that the global operator of (56) is neither symmetrical nor hermitian.

## VII,4. – INCLUSION OF CONSTRAINTS DUE TO THE WRITING OF THE COUPLED PROBLEM IN CARTESIAN REFERENCE

Formulation (54) assumes (48.1) to (48.4), *i.e.* that $H_{r,0}$ and $H_{e,c}$ are in the space tangent to $\Gamma_c$, and that $E_{r,0}$ and $E^{\vee}_{e,\tau}$ are in the space tangent to $\Gamma_d$. The integral equations (40.1) and (40.2) used to construct system (54) each correspond to two scalar equations on a variety. For the adaptation to the finite element method on this variety (*see* the Part II of this paper), to keep from complicating the finite element formulation (as we indicated in section II,2), we will use a description in cartesian coordinates $x_1 x_2 x_3$ of the $E_{r,0}$, and, $H_{r,0}$, $E^{\vee}_{e,\tau}$ and $H_{e,c}$ fields, which will give us three unknowns per mesh node for each of these fields. We should then require that $E_{r,0}$, $H_{r,0}$, $E^{\vee}_{e,\tau}$ and $H_{e,c}$ be in the space tangent to the variety, *i.e.*:

$$\left.\begin{array}{ll} E_{r,0}\cdot n=0, & E^{\vee}_{e,\tau}\cdot n=0 \ \text{ sur } \Gamma_d \\ H_{r,0}\cdot n=0, & H_{e,c}\cdot n=0 \ \text{ sur } \Gamma_c \end{array}\right\} \qquad (60)$$

But in fact, we will see below that it is possible to simplify this. For any $A\in\mathscr{C}^c_1$ and for any $\varphi\in\mathscr{C}^c_1$, we find the linear operators $\tilde{P}$, $\tilde{I}_{cc}$, $\tilde{I}_{dd}$, $\tilde{Q}$, $\tilde{B}$ and $\tilde{S}$, by the following bilinear forms on $\mathscr{C}^c_1\times\mathscr{C}^c_1$.

$$\langle \tilde{P}A, \varphi\rangle = -\frac{jk}{2}\int_{\partial\Omega_i}\varphi\,.(n\times A)\,ds = +\frac{jk}{2}\int_{\partial\Omega_i} A\,.(n\times\varphi)\,ds \qquad (61.1)$$

$$\langle \tilde{I}_{cc}A, \varphi\rangle = \int_{\Gamma_c}\varphi\,.A\,ds \qquad (61.2)$$

$$\langle \tilde{I}_{dd}A, \varphi\rangle = \int_{\Gamma_d}\varphi\,.A\,ds \qquad (61.3)$$

$$\langle \tilde{Q}A, \varphi\rangle = \frac{jk}{4\pi}\int_{\partial\Omega_i} ds\int_{\partial\Omega_i}(\varphi\times n)\,.\,[(n'\times A')\times \text{grad}' N]\,ds' \qquad (61.4)$$

$$\langle \tilde{B}A, \varphi\rangle = -\frac{j}{4\pi}\int_{\partial\Omega_i}\int_{\partial\Omega_i} N\,d(\varphi\,.\,dM)\,d'(A'\,.\,dM') \qquad (61.5)$$

$$\langle \tilde{S}A, \varphi\rangle = -\frac{jk^2}{4\pi}\int_{\partial\Omega_i} ds\int_{\partial\Omega_i} N\,ds'\,(\varphi\times n)\,.\,(A'\times n'). \qquad (61.6)$$

Algebraically, the $\tilde{P}$ operator is antisymmetrical, *i.e.* ${}^t\tilde{P}=-\tilde{P}$, while the operators $\tilde{I}_{cc}$, $\tilde{I}_{dd}$, $\tilde{Q}$, $\tilde{B}$ and $\tilde{S}$ are symmetrical but nonhermetian. Moreover, definitions (61.1) and (61.4) to (61.6) show that the operators $\tilde{P}$, $\tilde{Q}$, $\tilde{B}$ and $\tilde{S}$ extract the tangent part from the cartesian fields and project it onto the tangent plane.

Because of the presence of the irregular frequencies, the kernel of the operator of (56) as well as the kernel $\ker\mathbb{A}$ are nonzero for certain values of $k$ (which are unknowns before any calculations are made). So we will have to use a method for solving the discretized linear system (56) that can operate under these conditions. The algorithm presented in Part III of this paper satisfies this property and constructs the unconstrained fields of the solution sought in $\text{Im}\,\mathbb{A}$. For these $\{E^{\vee}_{e,\tau}, H_{e,c}\}$ fields, $\text{Im}\,\mathbb{A}$ reduces to $\text{Im}\,(B-S)+\text{Im}\,(P+Q)$ and is therefore in the tangent plane. There is then no need to introduce the two Lagrange multipliers associated with the constraints $E^{\vee}_{e,\tau}\cdot n=0$ on $\Gamma_d$ and $H_{e,c}\cdot n=0$ on $\Gamma_c$. The $\{E^{\vee}_{e,\tau}, H_{e,c}\}$ part is well constructed in the space tangent to the varieties $\Gamma_d$ and $\Gamma_c$ by the solver algorithm.

On the other hand, we do have to introduce the constraints $E_{r,0}\cdot n=0$ on $\Gamma_d$ and $H_{r,0}\cdot n=0$ on $\Gamma_c$, because the $\tilde{I}_{dd}$ and $\tilde{I}_{cc}$ operators do not project in the tangent plane. Let $\lambda_{E_{r,0}}$ and $\lambda_{H_{r,0}}$ be the Lagrange multipliers used to set these constraints. For any continuous $A_c$ (resp. $A_d$) of $\Gamma_c$ (resp. $\Gamma_d$) in $\mathbb{C}^3$, and for any continuous $\delta\lambda_c$ (resp. $\delta\lambda_d$) of $\Gamma_c$ (resp. $\Gamma_d$) in $\mathbb{C}$, we define the operators $m_c$ and $m_d$ by the following forms:

$$\langle m_c A_c, \delta\lambda_c \rangle = \int_{\Gamma_c} \delta\lambda_c A_c . n \, ds \qquad (62.1)$$

$$\langle m_d A_d, \delta\lambda_d \rangle = \int_{\Gamma_d} \delta\lambda_d A_d . n \, ds. \qquad (62.2)$$

Then, for the cartesian representation, the first two equations of (54), which are used to calculate $E_{r,\theta}$ and $H_{r,\theta}$, become:

$$\begin{bmatrix} -jk\,\tilde{I}_{dd} & {}^t m_d & 0 & 0 \\ m_d & 0 & 0 & 0 \\ 0 & 0 & -jk\,\tilde{I}_{cc} & {}^t m_c \\ 0 & 0 & m_c & 0 \end{bmatrix} \begin{bmatrix} E_{r,\theta} \\ \lambda_{E_{r,\theta}} \\ H_{r,\theta} \\ \lambda_{H_{r,\theta}} \end{bmatrix} = \begin{bmatrix} F_d \\ 0 \\ F_c \\ 0 \end{bmatrix}, \qquad (63)$$

with

$$F_d = -j(-P_{dd} + Q_{dd}) E^{\vee}_{e,\tau} - j(B_{dc} - S_{dc}) H_{e,c} - j(B_{dd} - S_{dd}) H_{bd} \qquad (64.1)$$

$$F_c = -(B_{cd} - S_{cd}) E^{\vee}_{e,\tau} - (-P_{cc} + Q_{cc}) H_{e,c} - Q_{cd} H_{bd}. \qquad (64.2)$$

It will be noted that each of the block diagonal operators of (63) is symmetrical complex. Lastly, with the cartesian representation, the system (56) to be solved is written:

$$\begin{bmatrix} \tilde{\mathbb{A}} & m^* \\ m & 0 \end{bmatrix} \begin{bmatrix} \mathbb{X} \\ \lambda_{\text{int}} \end{bmatrix} = \begin{bmatrix} \mathbb{S} \\ 0 \end{bmatrix}, \qquad (65.1)$$

with

$$\tilde{\mathbb{A}} = \begin{bmatrix} \tilde{B}_{dd} - \tilde{S}_{dd} & \tilde{Q}_{dc} & \tilde{P}_{dd} + \tilde{Q}_{dd} & 0 \\ {}^t\tilde{Q}_{dc} & \tilde{B}_{cc} - \tilde{S}_{cc} & \tilde{B}_{cd} - \tilde{S}_{cd} & 0 \\ {}^t\tilde{P}_{dd} + q t_{dd} & {}^t(\tilde{B}_{cd} - \tilde{S}_{cd}) & \tilde{B}_{dd} - \tilde{S}_{dd} + \mathbb{M}_{bb} & \mathbb{M}_{b,\text{int}} \\ 0 & 0 & {}^t\mathbb{M}_{b,\text{int}} & \mathbb{M}_{\text{int},\text{int}} \end{bmatrix} \qquad (65.2)$$

$$\mathbb{S} = \begin{bmatrix} 2\,\tilde{P}_{dd} H_{i,d} \\ -2j\,\tilde{P}_{cc} E_{i,c} \\ -2j\,\tilde{P}_{dd} E_{i,d} \\ 0 \end{bmatrix}. \qquad (65.3)$$

## VIII. – IRREGULAR FREQUENCIES AND METHOD OF SOLUTION

We are going to show that there exists a countable set of real values of $k$ for which the operator of (56) is singular. These values are the "irregular frequencies" introduced by the integral formulation retained for the external problem, whereas we know that the problem always as a unique solution for any real $k$ [10]. We will also be showing that the formulation constructed (54) can be used to construct this unique solution, even when $k$ is an "irregular frequency".

To study this problem, we will be considering the following two configurations:

– Configuration 1. External reflection on a perfect conductor: $\partial\Omega_i = \Gamma_c$, $\Gamma_d = \varnothing$.

– Configuration 2. External reflection on a dielectric: $\partial\Omega_i = \Gamma_d$, $\Gamma_c = \varnothing$.

### VIII,1. – REVIEW OF THE SPECTRAL PROBLEM OF THE VACUUM CAVITY

To analyze the irregular frequencies, we need to recall certain data concerning the internal cavity resonances of a vacuum domain $\Omega_i$ with "conducting" boundary [10, 15, 34]. We consider the Maxwell equations in $\Omega_i$:

$$\left.\begin{aligned} jk\,H^0 + \text{rot}\,E^0 = 0 \\ -jk\,E^0 + \text{rot}\,H^0 = 0 \end{aligned}\right\} \text{ in } \Omega_i, \qquad (66)$$

with the following boundary conditions on $\partial\Omega_i$.

*a) Electric type modes* (*E*): The boundary conditions on $\partial\Omega_i$ associated with (66) are:

$$H^0 . n = 0 \qquad (67.1)$$

$$E^0 \times n = 0 \qquad (67.2)$$

$$E^0 . n = q, \qquad (67.3)$$

with $q$ being the charge surface density on $\partial\Omega_i$ that we assume to be nonidentically zero. The spectrum of eigencalues $k$ of the discrete roblem (66)-(67), consisting of real positive values, is denoted $\mathscr{S}_E = \{k_1, k_2, \ldots\}$. The associated eigenvectors, which are real, are the "electrical" type electromagnetic modes $(H^0, E^0)$.

*b) Magnetic types modes* (*H*): The boundary conditions on $\partial\Omega_i$, associated with (66) are:

$$H^0 . n = 0 \qquad (68.1)$$

$$E^0 \times n = 0 \qquad (68.2)$$

$$E^0 . n = 0. \qquad (68.3)$$

In the same way, we will use $\mathscr{S}_H = \{k'_1, k'_2, \ldots\}$, to denote the discrete spectrum of the real positive eigenvalues $k$ of the problem (66)-(68), and the electromagnetic modes ($H^0$, $E^0$), which are also real, are set to be of the "magnetic" type.

*Remark*: For a sphere of radius $R$ (54), the values of $k \in \mathscr{S}_E$ are solutions of

$$\frac{d}{dk_1}(k_1 R j_n(k_1 R)) = 0, \qquad n \in \mathbb{N},$$

and the values of $k \in \mathscr{S}_H$ are solutions of

$$j_n(k_1' R) = 0, \qquad n \in \mathbb{N},$$

with $j_n$ being the first species spherical Bessel function, of order $n$.

## VIII,2. – INTEGRAL FORMULATION ASSOCIATED WITH THE VACUUM CAVITY PROBLEM

A calculation analogous to the one for the external problem leads to following equations, which are written symbolically with the operators defined in section V:

$$j(P-Q)H^0 - (B-S)E^0 = 0 \qquad (69.1)$$

$$j(P-Q)E^0 + (B-S)H^0 = 0. \qquad (69.2)$$

By setting the boundary conditions (67) or (68) we get:

$$j(P-Q)H^0 = 0 \qquad (70.1)$$

$$(B-S)H^0 = 0. \qquad (70.2)$$

– If $k \notin \mathscr{S}_E \cup \mathscr{S}_H$ then $j(P-Q)$ and $(B-S)$ are not singular, and the solution is $H^0 = 0$, $E^0 = 0$.

– If $k \in \mathscr{S}_E \cup \mathscr{S}_H$, then $j(P-Q)$ and $(B-S)$ are simultaneously singular ($\ker j(P-Q)$ and $\ker(B-S)$).

We will also show the result as in [3].

For any electromagnetic field, ($H_i$, $E_i$) verify the Maxwell equations in $\Omega_i$:

$$\left.\begin{aligned} jk H_i + \text{rot } E_i &= 0 \\ -jk E_i + \text{rot } H_i &= 0 \end{aligned}\right\} \text{ in } \Omega_i, \qquad (71)$$

If $E^0 \times n = 0$ on $\partial\Omega_i$ then we have:

$$\langle n \times E_i, H^0 \rangle = 0 \qquad (72.1)$$

$$\langle n \times H_i, H^0 \rangle = 0 \qquad (72.2)$$

## VIII,3. – CONFIGURATION 1. PROBLEM OF EXTERNAL REFLECTION ON A PERFECT CONDUCTOR

In this case, we have $\partial\Omega_i = \Gamma_c$, $\Gamma_d = \varnothing$ and the system (54) reduces to (dropping the $c$ subscripts):

$$\begin{bmatrix} -jk\,\mathrm{I} & -(P-Q) \\ 0 & B-S \end{bmatrix} \begin{bmatrix} H_{r,0} \\ \mathbb{H}_e \end{bmatrix} = \begin{bmatrix} 0 \\ -2jPE_i \end{bmatrix}, \qquad (73)$$

*First configuration*: $k \notin \mathscr{S}_E \cup \mathscr{S}_H$. Then $B-S$ is regular and there is a unique solution. For the component $\mathbb{H}_e$ we get $\mathbb{H}_e = H_e$.

*Second configuration*: $k \in \mathscr{S}_E \cup \mathscr{S}_H$. Then $B-S$ is singular. The $\mathbb{H}_e$ part of an element of the kernel of the operator of (73) necessarily belongs to the kernel of $B-S$ and verifies (70), which shows that the $H_{r,0}$ part of this element is zero ($I$ is bijective). The elements of the kernel of the operator in (73) are therefore of the form $\{0, H^0\}$, with $H^0$ verifying (70), and we have (72.1), *i.e.* $\langle PE_i, H^0 \rangle = 0$. The Fredholm alternative then applies. There exists a solution $\{H_{r,0}, \mathbb{H}_e\}$ to (73) in the form

$$\{H_{r,0}, \mathbb{H}_e\} = \{H_{r,0}, H_e\} + \{0, H^0\}.$$

The solution to (73) gives a unique solution for $H_{r,0}$, but not for $\mathbb{H}_e$.

## VIII,4. – CONFIGURATION 2. PROBLEM OF EXTERNAL REFLECTION ON A DIELECTRIC

In this case, we have $\partial\Omega_i = \Gamma_d$, $\Gamma_c = \varnothing$ and the system (54), when the $d$ subscripts are dropped and we use ${}^tP = -P$ and (53), reduces to:

$$\begin{bmatrix} -jkI & -(P-Q) & j(B-S) & 0 \\ 0 & -j(B-S) & P+Q & 0 \\ 0 & j(P-Q) & B-S+\mathbb{M}_{bb} & \mathbb{M}_{b,\text{int}} \\ 0 & 0 & {}^t\mathbb{M}_{b,\text{int}} & \mathbb{M}_{\text{int,int}} \\ 0 & 0 & \mathscr{M}_{b,\text{int}} & \mathscr{M}_{\text{int,nt}} \end{bmatrix} \begin{bmatrix} 0 & E_{r,0} \\ 0 & \mathbb{E}_{e,\tau} \\ \mathscr{M}^*_{b,\text{int}} & H_{bd} \\ \mathscr{M}^*_{\text{int,int}} & H_{\text{int}} \\ 0 & \lambda_{\text{int}} \end{bmatrix} = \begin{bmatrix} 0 \\ 2PH_i \\ -2jPE_i \\ 0 \\ 0 \end{bmatrix}. \qquad (74)$$

*First configuration*: $k \notin \mathscr{S}_E \cup \mathscr{S}_H$. Then the operator is regular and there is a unique solution such that, for the component $\mathbb{E}_{e,\tau}$, we have $\mathbb{E}_{e,\tau} = E_{e,\tau}$.

*Second configuration*: $k \in \mathscr{S}_E \cup \mathscr{S}_H$. There exists a nonidentically zero $H^0$ that verifies (70) and such that we have (72.2), *i.e.* $\langle PH_i, H^0 \rangle = 0$. It can be seen that the Fredholm alternative again applies since the kernel of the operator (74) consists of elements of the type (0, $H^0$, 0, 0), which are orthogonal to the second member of (74). There therefore exists a solution to equation (74) of the form ($E_{r,0}$, $E_{e,\tau} + H^0$, $H_{bd}$, $H_{\text{int}}$, $\lambda_{\text{int}}$) with $H^0$ being an arbitrary element in $\ker(B-S) = \ker j(P-Q)$. So the solution to (74) yields a unique solution for $E_{r,0}$, $H_{bd}$, $H_{\text{int}}$ and $\lambda_{\text{int}}$, but not for $\mathbb{E}_{e,\tau}$.

## VIII,5. – GENERAL CONFIGURATION. EXTENSION OF RESULTS

The general problem is governed by equation (54). We will assume that the results obtained in sections VIII,3 and VIII,4 for the two limiting cases remain true for the combined case $\partial\Omega_i = \Gamma_c \cup \Gamma_d$ with $\Gamma_c \neq \emptyset$ and $\Gamma_d \neq \emptyset$. Consequently, for $\forall k \in \mathbb{R}$, the operator of (54) is not necessarily regular, but the solution to the linear system (54) yields the unique solution for $E_{r,0}$, $H_{r,0}$, $H_{bd}$ and $H_{\text{int}}$ $\lambda_{\text{int}}$ and offers no way of finding $E^{\vee}_{e,\tau}$ or $H_{e,c}$.

# IX. – COMPUTATION OF CHARGES, SURFACE CURRENTS AND THE REFLECTED ELECTROMAGNETIC FIELD

## IX,1. – COMPUTATION OF THE ELECTROMAGNETIC FIELD ON THE EXTERNAL SURFACE

The $H_i$ and $E_i$ fields are known everywhere in the space, so their trace on $\partial\Omega_i$ is known. According to the preceding, all that we can compute for any $k \in \mathbb{R}^{+*}$ are:

$$E_{r,0} = n \times E_{r,d} \quad \text{on } \Gamma_d$$

$$H_{r,0} = n \times H_{r,c} \quad \text{on } \Gamma_c$$

$$H_{bd} = \text{trace of } H \quad \text{on } \Gamma_d.$$

We must therefore establish relations for computing $H_r$ and therefore $H_e = H_r + H_i$ on $\partial\Omega_i$, and also $E_e$ on $\partial\Omega_i$. For details, refer to [3].

*a) On the conductive part* $\Gamma_c$ we have:

$$H_{e,c} = (I - n \otimes n) H_{i,c} - n \times H_{r,0} \quad (75.1)$$

$$n \times H_{e,c} = n \times H_{i,c} + H_{r,0} \quad (75.2)$$

$$E_{e,c}\, ds = (jk)^{-1}\, n d((H_{i,c} - n \times H_{r,0}) \cdot dM) \quad (75.3)$$

*b) On the dielectric part* $\Gamma_d$ we have:

$$H_{e,d} = (I + (n \otimes n)(\mu^* - I)) H_{bd} \quad (76.1)$$

$$E_{e,d}\, ds = (jk)^{-1}\, n d(H_{bd} \cdot dM) + (I - n \otimes n) E_{i,d}\, ds - n \times E_{r,0}\, ds \quad (76.2)$$

$$n \times E_{e,d} = n \times E_{i,d} + E_{r,0}. \quad (76.3)$$

## IX,2. – CHARGE AND CURRENT DENSITIES CONCENTRATED ON THE CONDUCTIVE PART OF THE OUTER SURFACE

Let $J_{\Gamma_c}$ and $q_{\Gamma_c}$ be the current and charge densities, respectively, concentrated on $\Gamma_c$. We then have:

$$J_{\Gamma_c} = n \times H_{e,c} \text{ computed from (75.2)} \quad (77.1)$$

$$q_{\Gamma_c}\, ds = (E_{e,c} \cdot n)\, ds = (jk)^{-1}\, d(H_{e,c} \cdot dM). \quad (77.2)$$

## IX,3. – COMPUTATION OF THE REFLECTED ELECTROMAGNETIC FIELD

### IX,3.1. – *Scattered Electromagnetic Field at a Given Point in the External Domain*

Considering (37.1) and (37.2), we are led to introduce the following integral operators. For any fixed $M$ in $\Omega_e$, for any bounded field $A$ of $\partial\Omega_i$ in $\mathbb{C}^3$, and for any bounded field $B_\tau$ of $\partial\Omega_i$ in $\mathbb{C}^3$ and in the space tangent to $\partial\Omega_i$, we define the operators $R_1(M)$, $R_2(M)$ and $R_3(M)$ such that:

$$R_1(M) A = \int_{\partial\Omega_i} A' \frac{d' N}{dn'}\, ds' \quad (78.1)$$

$$R_2(M) A = \int_{\partial\Omega_i} A' \times d'(N\, dM') \quad (78.2)$$

$$R_3(M) B = jk \int_{\partial\Omega_i} N B'\, ds' \quad (78.3)$$

Then the equations (37.1) and 37.2), which are used to compute $H_r(M)$ and $E_r(M)$, respectively, and any fixed point $M$ in $\Omega_e$, are written:

$$4\pi H_r(M) = R_1(M) H_e - R_2(M) H_e + R_3(M)(n \times E_e) \quad (79.1)$$

$$4\pi E_r(M) = R_1(M) E_e - R_2(M) E_e - R_3(M)(n \times H_e) \quad (79.2)$$

in which:
$H_e$ is given by (75.1) on $\Gamma_c$, and (76.1) on $\Gamma_d$ for $R_1(M)$ and $R_2(M)$;
$n \times H_e$ is given by (75.2) on $\Gamma_c$, and (76.2) on $\Gamma_d$ for $R_3(M)$;
$E_e$ is given by (75.3) on $\Gamma_c$, and (76.2) on $\Gamma_d$ for $R_1(M)$ and $R_2(M)$;
and finally $n \times E_e$ by (14.2) on $\Gamma_c$, and (76.2) on $\Gamma_d$ for $R_3(M)$.

### IX,3.2. – *Scattered Electromagnetic Farfield*

Strictly speaking, formulae (79) can be used to compute the electromagnetic field, near and far. However, the numerical formulations that we will be establishing in Part II of this paper raise a few difficulties for the field at great distances, and we then prefer to use an asymptotic form, which recalls the farfield form, or relations (79). We have drawn up asymptotic expressions that are detailed in [33].

We consider a direction of observation defined by the unit vector $u \in \mathbb{R}^3$ and consider an observation point $M$ that tends toward infinity in the $u$ direction. We let $R = \|OM\|$ and $r = \|MM'\|$, in which $M'$ is some given point of $\partial\Omega_i$. The method consists in constructing the asymptotic kernel $\mathcal{N}_u$ defined by:

$$\mathcal{N}_u = e^{-jk(u \, . \, OM')} \tag{80}$$

and we show that:

$$\mathcal{N}_u = \lim_{R \to \infty} R e^{jkR} \frac{e^{-jkr}}{r}. \tag{81}$$

Starting with (79) and with the limit (81) we get the expressions for the asymptotic fields:

$$\mathcal{E}_r = \lim_{R \to \infty} E_r R e^{jkR} = -\frac{jk}{4\pi} \int_{\partial\Omega_i} \mathcal{N}_u \times \{ u \times (n' \times E'_e) - [I - u \otimes u](H'_e \times n') \} \, ds' \tag{82.1}$$

$$\mathcal{H}_r = \lim_{R \to \infty} H_r R e^{jkR} = -\frac{jk}{4\pi} \int_{\partial\Omega_i} \mathcal{N}_u \times \{ u \times (n' \times H'_e) + [I - u \otimes u](E'_e \times n') \} \, ds' \tag{82.2}$$

We show that the asymptotic farfield has a local plane wave structure:

$$u \times \mathcal{E}_r = \mathcal{H}_r \tag{83.1}$$

$$u \times \mathcal{H}_r = -\mathcal{E}_r \tag{83.2}$$

## X. – CONCLUSION

We have established a hybrid formulation mixing local and integral equations for the harmonic domain, which automatically gets around the problem of the irregular frequencies and also leads to a complex operator having the right properties of symmetry and accretivity, which can be used in an efficient iterative method for solving the problem.

The remainder of these explanations concerning numerical analysis, the iterative solver algorithm, the general 3-D code (HEM 3 D), and the validation with examples are presented in Parts II and III.

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