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► **To cite this version:**

Christian Soize, O. Le Fur. Modal identification of weakly nonlinear multidimensional dynamical systems using a stochastic linearization method with random coefficients. *Mechanical Systems and Signal Processing*, Elsevier, 1997, 11 (1), pp.37-49. 10.1006/mssp.1996.0085 . hal-00770020

**HAL Id: hal-00770020**

**<https://hal-upec-upem.archives-ouvertes.fr/hal-00770020>**

Submitted on 4 Jan 2013

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# MODAL IDENTIFICATION OF WEAKLY NONLINEAR MULTIDIMENSIONAL DYNAMICAL SYSTEMS USING A STOCHASTIC LINEARIZATION METHOD WITH RANDOM COEFFICIENTS

by

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## ABSTRACT

It is known that an efficient approach for modal identification of a weakly nonlinear multidimensional second-order dynamical system consists in using a model based on equivalent stochastic linearization with constant coefficients. Such a model leads to a good identification of the total power of the stationary response but can give an incorrect identification of the matrix-valued spectral density functions. The objective of this paper is to present an identification procedure which is based on the use of a stochastic linearization method with random coefficients. The model is then defined as a multidimensional linear second-order dynamical system with random coefficients. An optimization procedure is developed to identify the parameters of the probability law of the random coefficients. The identification procedure is described step by step. Finally, an example is presented and shows the interest of the method proposed.

## 1. INTRODUCTION

This paper deals with modal identification of weakly nonlinear multidimensional second-order dynamical systems. In the case of linear dynamical behavior, the modal identification methods are well developed due to the use of the linear vibration theory of structures. For general nonlinear dynamical systems, some different strategies exist to identify such systems by nonlinear models (for instance, nonlinear ARMAX or Volterra series). The present paper concerns the weakly nonlinear case for which, the unique objective is the identification of a given nonlinear multidimensional dynamical system by an infinite family of linear multidimensional dynamical systems, this family being spanned by a second-order dynamical system with random operators. The eigenfrequencies and associated eigenmodes are then deduced from the linearized representation which is identified (consequently, the eigenfrequencies are random variables). Using broad-band stationary random excitation and constant coefficients of the model yields the classical Stochastic Linearization Method with Constant Coefficients (SLMCC). The SLMCC was introduced by Caughey in 1963 [1] within the context of prediction methods. Many developments have been proposed in this area since this date and an excellent synopsis was made by Roberts and Spanos in 1990 [2]. In the field of identification procedures, it should be noted that difficulties arise due to the presence of the mass matrix which is unknown and which must be identified. Fillâtre [3] developed a method for identifying such an equivalent linear model in which mass, damping and stiffness matrices are constant and unknown. His approach is based on an extension of Kozin's works [4,5] and can be considered as a method based on the SLMCC. Generally speaking, it is known that the SLMCC yields a very good approximation of the second-order statistical moments of the stationary response of second-order dynamical systems. Consequently, an identification method based on such a procedure yields an equivalent linear dynamical system which

can reconstitute the second-order moments. Unfortunately, in some cases, although the second-order moments are correctly estimated, the matrix-valued spectral density function of the response may be erroneous. This difficulty was first shown by Miles for a one-dimensional nonlinear dynamical system [6]. For such nonlinear dynamical systems, methods were proposed to calculate the power spectral density function of the stationary response without using Monte Carlo numerical simulation (Miles [6], Bouc [7,8] and Soize [9,10]). The matrix-valued spectral density function can generally not be calculated explicitly for multidimensional nonlinear dynamical systems, except for particular cases related to linear dynamical systems with random parametric excitations (see for instance Soize [11]). Recently, Bellizzi and Bouc [12] proposed an interesting method for multidimensional systems in the context of prediction methods.

The Stochastic Linearization Method with Random Coefficients (SLMRC) [9,10] is adapted to identification procedures and allows the identification to be improved with respect to the classical SLMCC. This fact was recently proved by Soize [13] for one-degree-of-freedom nonlinear second-order dynamical systems. This method, based on a linear dynamical model with random coefficients, has just been extended by Le Fur [14] for the identification of weakly nonlinear multidimensional second-order dynamical systems.

The purpose of this paper is to present a new approach for identifying weakly nonlinear multidimensional second-order dynamical systems based on the identification of a linear model with random coefficients. In order to improve the clarity of the paper, we have limited the presentation of the theory to the case of stiffness nonlinearities. This theory can be extended to the case of damping and stiffness nonlinearities as it is shown in Soize [10]. Nevertheless, the validation for the multidimensional case has been performed for stiffness nonlinearities. Below we present

- a multidimensional stochastic linear model with random coefficients which allows a set of parametric probabilities to be defined.
- a method for identifying the model parameters using a minimization criterion for the difference between the matrix-valued spectral density functions of the model responses and measured responses (so called reference responses or experimental responses). The model parameter identification problem leads us to solve an optimization problem.
- an appropriate numerical method to solve the optimization problem introduced above (which is not standard).
- an example which validates the method proposed.

## 2. CONSTRUCTION OF A STOCHASTIC DIFFERENTIAL EQUATION WITH RANDOM COEFFICIENTS

We consider a weakly nonlinear dynamical system of dimension  $n \geq 1$  subjected to an external random excitation. This dynamical system is written as the following stochastic differential equation

$$[M]\ddot{\boldsymbol{\chi}}(t) + [C]\dot{\boldsymbol{\chi}}(t) + [K]\boldsymbol{\chi}(t) + \varepsilon \mathbf{f}(\boldsymbol{\chi}(t)) = \mathbf{F}(t) \quad , \quad (1)$$

in which matrices  $[M]$ ,  $[C]$  and  $[K]$  are positive definite; function  $\mathbf{f}$  from  $\mathbb{R}^n$  into  $\mathbb{R}^n$  is odd, continuous and nonlinear; excitation force  $\mathbf{F}$  is a Gaussian, second-order, centered, stationary, mean-square continuous stochastic process indexed by  $\mathbb{R}$  with values in  $\mathbb{R}^n$ . The matrix-valued spectral density function of process  $\mathbf{F}$  is written as

$$[S_{\mathbf{F}}(\omega)] = s(\omega)[B] \quad , \quad (2)$$

where  $[B]$  is a positive matrix and  $s$  is a positive-valued function defined on  $\mathbb{R}$  having the required properties such that process  $\mathbf{F}$  is physically realizable and approaches an ideal normalized narrow-band noise.

Within the context of an identification problem, it is assumed that stochastic differential equation (1) has a unique stationary, second-order, centered stochastic solution  $\mathbf{X}$  having a matrix-valued spectral density function  $[S_{\mathbf{X}}]$ . Furthermore, function  $[S_{\mathbf{X}}]$  is assumed to be square integrable on  $\mathbb{R}$ .

Applying the identification procedure developed by Fillâtre [3] (based on a stochastic linearization method with constant coefficients) yields the following linear stochastic differential equation on  $\mathbb{R}^n$

$$[\underline{M}_c] \ddot{\mathbf{X}}(t) + [\underline{C}_c] \dot{\mathbf{X}}(t) + [\underline{K}_c] \mathbf{X}(t) = \mathbf{F}(t) \quad , \quad (3)$$

in which  $\mathbf{F}$  is the stochastic process used in Eq. (1) and matrices  $[\underline{M}_c]$ ,  $[\underline{C}_c]$  and  $[\underline{K}_c]$  result from the identification procedure and are positive definite.

We introduce the eigenmodes  $\underline{\varphi} \in \mathbb{R}^n$  and the associated eigenfrequencies  $\underline{\omega}$  of the conservative problem associated with Eq. (3), which are the solutions of the generalized eigenvalue problem  $[\underline{K}_c] \underline{\varphi} = \underline{\omega}^2 [\underline{M}_c] \underline{\varphi}$ . Let  $[\underline{\Phi}]$  be the  $(n \times n)$  real matrix of the eigenmodes such that  $[\underline{\Phi}]_{jk} = \{\varphi_k\}_j$ . We introduce modal coordinates  $\mathbf{Q}$  such that  $\mathbf{X} = [\underline{\Phi}] \mathbf{Q}$ . Substituting this change of coordinates in Eq. (3) yields

$$[\underline{M}_g] \ddot{\mathbf{Q}}(t) + [\underline{C}_g] \dot{\mathbf{Q}}(t) + [\underline{K}_g] \mathbf{Q}(t) = [\underline{\Phi}]^T \mathbf{F}(t) \quad , \quad (4)$$

where  $[\underline{M}_g] = [\underline{\Phi}]^T [\underline{M}_c] [\underline{\Phi}]$ ,  $[\underline{C}_g] = [\underline{\Phi}]^T [\underline{C}_c] [\underline{\Phi}]$  and  $[\underline{K}_g] = [\underline{M}_g] [\underline{\Omega}^2] = [\underline{\Phi}]^T [\underline{K}_c] [\underline{\Phi}]$  are  $(n \times n)$  real positive-definite matrices. Matrices  $[\underline{M}_g]$ ,  $[\underline{K}_g]$  and  $[\underline{\Omega}^2]$  are diagonal and  $[\underline{C}_g]$  is a dense matrix in the general case.

We associate with Eq. (4) the following stochastic differential equation with random coefficients

$$[\underline{M}_g] \ddot{\mathbf{Y}}(t) + [\underline{C}_g] \dot{\mathbf{Y}}(t) + [\underline{K}_g] ([I] + [\Lambda]) \mathbf{Y}(t) = [\underline{\Phi}]^T \mathbf{F}(t) \quad , \quad (5)$$

where  $[\Lambda]$  is a random variable with values in the  $(n \times n)$  real diagonal matrices. We introduce the vector  $\Lambda = (\Lambda_1, \dots, \Lambda_n)$  of its diagonal entries  $\Lambda_i = [\Lambda]_{ii}$ . It is assumed that  $\{\Lambda_1, \dots, \Lambda_n\}$  are independent real-valued random variables. The probability law  $P_{\Lambda_i}(d\lambda)$  of the real-valued random variable  $\Lambda_i$  is defined by a probability density function  $p_{\Lambda_i}(\lambda)$  on  $\mathbb{R}$  with respect to  $d\lambda$ :

$$P_{\Lambda_i}(d\lambda) = p_{\Lambda_i}(\lambda) d\lambda \quad , \quad (6)$$

in which for all  $\lambda \in \mathbb{R}$ ,

$$p_{\Lambda_i}(\lambda) = \alpha_{\Lambda_i} (1 + \lambda) W_{\Lambda_i}(\lambda) \quad . \quad (7)$$

Real function  $\lambda \mapsto W_{\Lambda_i}(\lambda)$  defined on  $\mathbb{R}$  is such that

$$W_{\Lambda_i}(\lambda) = \mathbb{1}_{[\lambda_i^{(1)}, +\infty[}(\lambda) (\lambda - \lambda_i^{(1)}) e^{-\beta_{\Lambda_i} (\lambda - \lambda_i^{(1)})^2} \quad . \quad (8)$$

Equations (6)-(8) define a parametric family of probabilities where the unknown parameters  $\alpha_{\Lambda_i}$ ,  $\beta_{\Lambda_i}$  and  $\lambda_i^{(1)}$  verify the conditions  $\alpha_{\Lambda_i} > 0$ ,  $\beta_{\Lambda_i} > 0$ ,  $1 + \lambda_i^{(1)} > 0$ . Since  $P_{\Lambda_i}(d\lambda)$  is a probability,  $P_{\Lambda_i}(\mathbb{R}) = 1$  and consequently, the three parameters  $\alpha_{\Lambda_i}$ ,  $\beta_{\Lambda_i}$  and  $\lambda_i^{(1)}$  are dependent. Calculating  $\alpha_{\Lambda_i}$  as a function of  $\beta_{\Lambda_i}$  and  $\lambda_i^{(1)}$  yields

$$\alpha_{\Lambda_i} = \frac{2\beta_{\Lambda_i}}{1 + \lambda_i^{(1)} + \frac{1}{2} \sqrt{\frac{\pi}{\beta_{\Lambda_i}}}} \quad . \quad (9)$$

Because of the independence of random variables  $\{\Lambda_1, \dots, \Lambda_n\}$ , the probability law of the  $\mathbb{R}^n$ -valued random variable  $\Lambda$  is written as

$$P_{\Lambda} = \otimes_{i=1}^n P_{\Lambda_i} \quad . \quad (10)$$

The right-hand side of Eq. (7) corresponds to a second-order expansion in orthogonal polynomials with respect to the weight  $W_{\Lambda_i}$  (see [9,10,14]). This means that for all  $i$  in  $\{1, \dots, n\}$ , probability law  $P_{\Lambda_i}$  has a probability density function with respect to a Gaussian measure. It should be noted that the support of  $P_{\Lambda_i}$  is interval  $[\lambda_i^{(1)}, +\infty[$ .

The stationary solution of Eq. (5) is constructed using the properties of conditional probabilities. We therefore introduce a family of linear stochastic differential equations corresponding to Eq. (5) with  $[\Lambda] = [\lambda]$  in which  $[\lambda]$  is the parameter of the family. Let  $\mathcal{S}_\Lambda$  be the set of  $(n \times n)$  real diagonal matrices such that

$$\mathcal{S}_\Lambda = \left\{ [\lambda] \mid [\lambda]_{ii} \geq \lambda_i^{(1)}, i \in \{1, \dots, n\} \right\} . \quad (11)$$

As above, matrix  $[\lambda] \in \mathcal{S}_\Lambda$  is identified to the  $\mathbb{R}^n$ -valued vector  $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$  such that  $\lambda_i = [\lambda]_{ii}$ . We introduce the family of second-order linear differential equations for  $[\lambda]$  in  $\mathcal{S}_\Lambda$ , such that

$$[\underline{M}_g] \ddot{\mathbf{Y}}_\Lambda(t; \boldsymbol{\lambda}) + [\underline{C}_g] \dot{\mathbf{Y}}_\Lambda(t; \boldsymbol{\lambda}) + [\underline{K}_g] ([I] + [\lambda]) \mathbf{Y}_\Lambda(t; \boldsymbol{\lambda}) = [\Phi]^T \mathbf{F}(t) . \quad (12)$$

Since for all  $i$  in  $\{1, \dots, n\}$ ,  $1 + [\lambda]_{ii} \geq 1 + \lambda_i^{(1)} > 0$  and stochastic process  $\mathbf{F}$  is independent of random vector  $\Lambda$ , the following results can be demonstrated [14].

(1)- Equation (12) has a unique Gaussian, second-order, centered, stationary solution denoted  $\mathbf{Y}_\Lambda(t; \boldsymbol{\lambda})$ . This process can be constructed by linear filtering of process  $[\Phi]^T \mathbf{F}$ , whose frequency response function  $\omega \mapsto [H_\Lambda(\omega; \boldsymbol{\lambda})]$  can be written as

$$[H_\Lambda(\omega; \boldsymbol{\lambda})] = [-\omega^2 [\underline{M}_g] + i\omega [\underline{C}_g] + [\underline{K}_g]([I] + [\lambda])]^{-1} . \quad (13)$$

Its matrix-valued spectral density function  $[S_{\mathbf{Y}_\Lambda}(\omega; \boldsymbol{\lambda})]$  can then be written as

$$[S_{\mathbf{Y}_\Lambda}(\omega; \boldsymbol{\lambda})] = [H_\Lambda(\omega; \boldsymbol{\lambda})] [\Phi]^T [S_{\mathbf{F}}(\omega)] [\Phi] [H_\Lambda(\omega; \boldsymbol{\lambda})]^* . \quad (14)$$

(2)- Equation (5) has a unique second-order, centered, stationary solution  $\mathbf{Y}$  defined by

$$\mathbf{Y}(t) = \mathbf{Y}_\Lambda(t; \Lambda) , \quad (15)$$

and stochastic process  $\mathbf{Y}$  has a square integrable matrix-valued spectral density function given by the relation

$$[S_{\mathbf{Y}}(\omega)] = \int_{\mathbb{R}^n} [S_{\mathbf{Y}_\Lambda}(\omega; \boldsymbol{\lambda})] P_\Lambda(d\boldsymbol{\lambda}) , \quad (16)$$

in which  $[S_{\mathbf{Y}_\Lambda}(\omega; \boldsymbol{\lambda})]$  is given by Eq. (14) and  $P_\Lambda$  by Eqs. (6)-(10).

### 3. IDENTIFICATION PROCEDURE

The identification procedure consists in calculating the parameters of probability law  $P_\Lambda$  in order to minimize the "distance" between the matrix-valued spectral density function of the model responses and measured (experimental) responses.

#### 3.1 Definition of the Measured Matrix-Valued Spectral Density Function Expressed in Modal Coordinates

Let  $\mathcal{Q}$  be the  $\mathbb{R}^n$ -valued stationary stochastic process such that  $\mathcal{Q} = [\Phi]^{-1} \mathcal{X}$  in which  $\mathcal{X}$  is the measured stationary stochastic process (experimental responses) and  $[\Phi]$  are the estimated eigenmodes introduced in Section 2 . We then deduce that for all real  $\omega$ , the matrix-valued spectral density function  $[S_{\mathcal{Q}}(\omega)]$  of process  $\mathcal{Q}$  can be written as

$$[S_{\mathcal{Q}}(\omega)] = [\Phi]^{-1} [S_{\mathcal{X}}(\omega)] [\Phi]^{-T} . \quad (17)$$

### 3.2 Rewriting the Model Matrix-Valued Spectral Density Function Expressed in Modal Coordinates

Let  $\boldsymbol{\lambda}^{(1)}$  and  $\mathbf{A}$  be the vectors in  $\mathbb{R}^n$  such that  $\boldsymbol{\lambda}^{(1)} = (\lambda_1^{(1)}, \dots, \lambda_n^{(1)})$  and  $\mathbf{A} = (A_1, \dots, A_n)$  with

$$A_i = 1/\sqrt{\beta_{\Lambda_i}} \quad , \quad i \in \{1, \dots, n\} \quad . \quad (18)$$

Let  $\boldsymbol{\xi} = (\xi_1, \dots, \xi_n)$  be the vector in  $\mathbb{R}^{2n}$  such that  $\boldsymbol{\xi} = (\mathbf{A}, \boldsymbol{\lambda}^{(1)})$  and  $\xi_i = (A_i, \lambda_i^{(1)})$ . Let  $\mathcal{D}$  be the domain of  $\boldsymbol{\xi}$  which is such that

$$\mathcal{D} = \left\{ (\mathbf{A}, \boldsymbol{\lambda}^{(1)}) \in \mathbb{R}^{2n} \quad | \quad A_i > 0, 1 + \lambda_i^{(1)} > 0, \forall i \in \{1, \dots, n\} \right\} \quad . \quad (19)$$

In order to indicate the dependence of  $P_\Lambda$  and  $[S_Y]$  in  $\boldsymbol{\xi}$ , we rewrite these quantities as  $P_\Lambda^\xi$  and  $[S_Y^\xi]$  respectively.

### 3.3 Definition of the Distance

Since the measured and the model matrix-valued spectral density functions are square integrable, the following cost function can be used to express the distance:

$$H(\boldsymbol{\xi}) = \sum_{i=1}^n \|[S_Q]_{ii} - [S_Y^\xi]_{ii}\|^2 = \sum_{i=1}^n \int_{\mathbb{R}} ([S_Q(\omega)]_{ii} - [S_Y^\xi(\omega)]_{ii})^2 d\omega \quad . \quad (20)$$

### 3.4 Definition of an Optimization Problem

The identification procedure is defined as the following optimization problem: find  $\boldsymbol{\xi}_0$  in  $\mathcal{D}$  such that

$$H(\boldsymbol{\xi}_0) = \min_{\boldsymbol{\xi} \in \mathcal{D}} H(\boldsymbol{\xi}) \quad . \quad (21)$$

It should be noted (see Eqs. (13), (14) and (16)) that diagonal terms  $[S_Y^\xi]_{ii}$  depend on all the components of  $\boldsymbol{\xi}$  due to the fact that matrix  $[\underline{C}_g]$  used in the calculation of  $[H_\Lambda(\omega; \boldsymbol{\lambda})]$  is not diagonal.

In order to replace problem (21) by  $n$  independent optimization problems in  $\mathbb{R}^2$ , elements  $[S_Y^\xi(\omega)]_{ii}$  are approximated by  $[S_Y^{\xi_i}(\omega)]_{ii}$  obtained by neglecting the extra-diagonal part in matrix  $[\underline{C}_g]$ . It should be noted that this approximation (introduced only to simplify the optimization problem) is not used in the final calculation of matrix  $[S_Y^\xi(\omega)]$  (see Section 4). Then, from Eqs. (2), (6)-(9), (13), (14) and (16), we deduce that for all  $i$  in  $\{1, \dots, n\}$

$$[S_Y^{\xi_i}(\omega)]_{ii} = \frac{2e_{ii}s(\omega)}{1 + \lambda_i^{(1)} + A_i\sqrt{\pi}/2} \int_0^{+\infty} \frac{(1 + A_ix + \lambda_i^{(1)})xe^{-x^2} dx}{((1 + A_ix + \lambda_i^{(1)})[\underline{K}_g]_{ii} - \omega^2[\underline{M}_g]_{ii})^2 + \omega^2[\underline{C}_g]_{ii}^2} \quad , \quad (22)$$

in which  $e_{ii} = \{[\Phi]^T [B] [\Phi]\}_{ii}$ . For each  $i$  in  $\{1, \dots, n\}$ , we define the functional on  $]0, +\infty[\times]-1, +\infty[\subset \mathbb{R}^2$  such that

$$J_i(\boldsymbol{\xi}_i) = \int_{\mathbb{R}} \left\{ [S_Q(\omega)]_{ii} - [S_Y^{\xi_i}(\omega)]_{ii} \right\}^2 d\omega \quad . \quad (23)$$

Consequently, the optimization problem on a subset of  $\mathbb{R}^{2n}$  defined by Eq. (21), is replaced by the  $n$  following optimization problems on a subset of  $\mathbb{R}^2$

$$J_i(\boldsymbol{\xi}_{i,0}) = \min_{\boldsymbol{\xi}_i \in ]0, +\infty[\times]-1, +\infty[} J_i(\boldsymbol{\xi}_i) \quad , \quad i \in \{1, \dots, n\} \quad . \quad (24)$$

### 3.5 Solving the Constraint Optimization Problems

It should be noted that each constraint optimization problem defined by Eq. (24) is not standard because  $J_i$  is not a convex function. Consequently the following method has been used:

- *Step1*: Determine a bounded subdomain

$$\mathcal{C}_i = [A_{2,i}, A_{1,i}] \times [\lambda_{1,i}^{(1)}, \lambda_{2,i}^{(1)}] \quad (25)$$

included in unbounded domain  $]0, +\infty[ \times ]-1, +\infty[$  such that  $\mathcal{C}_i$  contains the solution, in order to limit the space of the research for a solution.

- *Step 2*: Use a global optimization algorithm on  $\mathcal{C}_i$  based on an adaptive random search [15] which allows a first approximation  $\xi_{i,a}$  of the solution of (24) to be constructed.

- *Step 3*: Finally, use a local optimization method on  $\mathcal{C}_i$  based on the Gauss-Newton algorithm and initialized with  $\xi_{i,a}$ , giving solution  $\xi_{i,0}$  of problem (24).

### 3.6 Construction of Bounded Subdomain $\mathcal{C}_i$

To locate the center of subdomain  $\mathcal{C}_i$ , we directly construct an approximation of  $\xi_{i,0}$  in writing the equality of the second-order moments

$$E\{Y_i(t)^2\} = E\{Q_i(t)^2\} \quad (26)$$

Then, integrating Eq. (22) on  $\mathbb{R}$  with respect to  $\omega$  yields

$$E\{Y_i(t)^2\} \simeq \frac{e_{ii}}{2[\underline{C}_g]_{ii}[\underline{K}_g]_{ii}(1 + \lambda_i^{(1)} + A_i\sqrt{\pi}/2)} \quad (27)$$

Equality (26) is equivalent to writing (see Fig. 1) that  $\xi_i = (A_i, \lambda_i^{(1)})$  belongs to the line  $\Delta_i$  defined by the equation

$$A_i = \frac{2}{\sqrt{\pi}} \left( \frac{e_{ii}}{2[\underline{C}_g]_{ii}[\underline{K}_g]_{ii}E\{Q_i(t)^2\}} - \lambda_i^{(1)} - 1 \right) \quad (28)$$

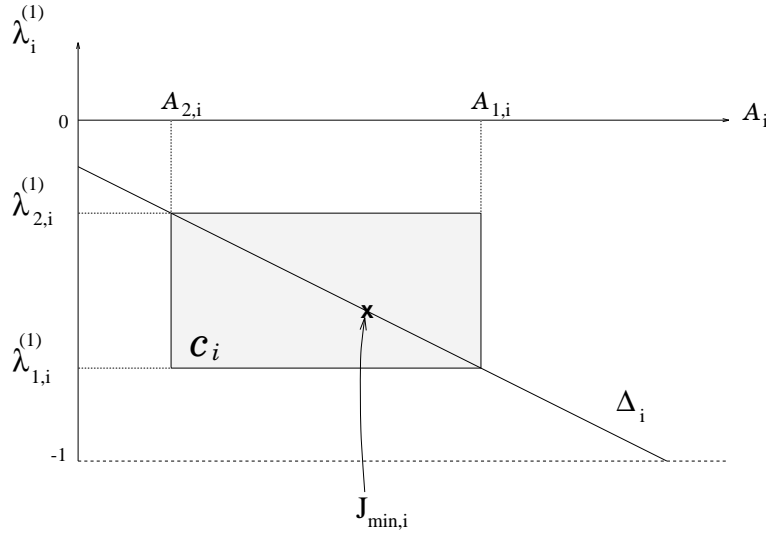


Fig. 1. Scheme defining the bounded subdomain  $\mathcal{C}_i$

Let  $J_{\min,i}$  be the minimum of  $J_i$  on  $\Delta_i$ . We then define the bounded domain  $\mathcal{C}_i$  such that  $J_i(\xi_i) < \varepsilon J_{\min,i}$  for all  $\xi_i$  varying on the segment  $[\xi_{1,i}, \xi_{2,i}]$  where  $\xi_{1,i} = (A_{1,i}, \lambda_{1,i}^{(1)}) \in \Delta_i$  and  $\xi_{2,i} = (A_{2,i}, \lambda_{2,i}^{(1)}) \in \Delta_i$ , and where  $\varepsilon$  is a given constant greater than 1.

#### 4. CALCULATION OF THE MODEL MATRIX-VALUED SPECTRAL DENSITY FUNCTION

In this section, we present the calculation of matrix-valued spectral density function  $[S_{\mathbf{Y}}(\omega)]$  for  $\boldsymbol{\xi} = \boldsymbol{\xi}_0$  where  $\boldsymbol{\xi}_0$  results from the identification procedure (see Section 3). Knowing  $[S_{\mathbf{Y}}(\omega)]$  which is the identified model of measured matrix  $[S_{\mathbf{Q}}(\omega)]$  expressed in terms of modal coordinates, we deduce the identified model  $[S_{\mathbf{Z}}(\omega)]$  of the measured matrix  $[S_{\mathbf{X}}(\omega)]$  relative to the physical coordinates. We have the relation

$$[S_{\mathbf{Z}}(\omega)] = [\Phi] [S_{\mathbf{Y}}(\omega)] [\Phi]^T \quad .$$

Since matrix  $[\underline{C}_g]$  is dense, the direct calculation of the model matrix-valued spectral density function defined by Eq. (16) requires calculating a  $n$ -uple integral on  $\mathbb{R}^n$  for each  $\omega$ . This calculation can only be carried out by numerical integration. Consequently, it cannot be made for large values of  $n$  (for instance when  $n$  is 10 or 20 (or more)). We therefore construct an approximation which leads us to calculate only  $n$  simple integrals on  $\mathbb{R}$ . This approximation is constructed as follows.

(1)- Damping matrix  $[\underline{C}_g]$  is written as  $[\underline{C}_g] = [\underline{C}_g^0] + [\Delta\underline{C}_g]$  in which  $[\underline{C}_g^0]$  is its diagonal part and  $[\Delta\underline{C}_g]$  its extra-diagonal part.

(2)- For all real  $\omega$  and  $[\lambda]$  in  $\mathcal{S}_\Lambda$ , we introduce the  $(n \times n)$  complex diagonal matrix

$$[H_\Lambda^0(\omega; \boldsymbol{\lambda})] = [-\omega^2[\underline{M}_g] + i\omega[\underline{C}_g^0] + [\underline{K}_g]([I] + [\lambda])]^{-1} \quad . \quad (29)$$

It should be noted that  $[H_\Lambda^0(\omega; \boldsymbol{\lambda})]_{ii}$  depends only on  $\lambda_i$ . Matrix  $[H_\Lambda(\omega; \boldsymbol{\lambda})]$  can then be rewritten as

$$[H_\Lambda(\omega; \boldsymbol{\lambda})] = \left( [I] + i\omega [H_\Lambda^0(\omega; \boldsymbol{\lambda})] [\Delta\underline{C}_g] \right)^{-1} [H_\Lambda^0(\omega; \boldsymbol{\lambda})] \quad . \quad (30)$$

(3)- For all real  $\omega$  and  $[\lambda]$  in  $\mathcal{S}_\Lambda$ , we can write

$$\left( [I] + i\omega [H_\Lambda^0(\omega; \boldsymbol{\lambda})] [\Delta\underline{C}_g] \right)^{-1} = [I] - i\omega [H_\Lambda^0(\omega; \boldsymbol{\lambda})] [\Delta\underline{C}_g] + O \left\{ \|\omega [H_\Lambda^0(\omega; \boldsymbol{\lambda})] [\Delta\underline{C}_g]\|^2 \right\} \quad . \quad (31)$$

Assuming that for all real  $\omega$  in the frequency band of analysis, the term in  $O$  in the right-hand side of Eq. (31) is negligible and substituting Eq. (31) into Eq. (30), we obtain the approximation

$$[H_\Lambda(\omega; \boldsymbol{\lambda})] \simeq [H_\Lambda^0(\omega; \boldsymbol{\lambda})] - i\omega [H_\Lambda^0(\omega; \boldsymbol{\lambda})] [\Delta\underline{C}_g] [H_\Lambda^0(\omega; \boldsymbol{\lambda})] \quad . \quad (32)$$

It should be noted that matrix  $[H_\Lambda(\omega; \boldsymbol{\lambda})]$  is a  $(n \times n)$  complex symmetric dense matrix.

(4)- Substituting Eq. (32) into Eq. (14) and using Eqs. (16) and (6)-(10) yields an explicit expression for matrix  $[S_{\mathbf{Y}}(\omega)]$ . For all  $i$  in  $\{1, \dots, n\}$ ,

$$[S_{\mathbf{Y}}(\omega)]_{ii} \simeq 2S_i^{(1)} s(\omega) \left\{ e_{ii} - 2\omega^2 \sum_{k \neq i} [\underline{C}_g]_{kk} [\underline{C}_g]_{ik} e_{ik} S_k^{(1)} \right\} \quad , \quad (33)$$

and for  $i$  and  $j$  in  $\{1, \dots, n\}$  with  $i \neq j$ ,

$$\begin{aligned} [S_{\mathbf{Y}}(\omega)]_{ij} \simeq & 4S_i^{(2)} \overline{S_j^{(2)}} s(\omega) \left\{ e_{ij} - i\omega \sum_{k \neq i, j} [\underline{C}_g]_{ik} e_{kj} S_k^{(2)} + i\omega \sum_{k \neq i, j} [\underline{C}_g]_{kj} e_{ik} \overline{S_k^{(2)}} \right\} \\ & + i\omega s(\omega) [\underline{C}_g]_{ij} \left\{ e_{ii} S_i^{(1)} \overline{S_j^{(2)}} - e_{jj} S_i^{(2)} S_j^{(1)} \right\} \quad , \quad (34) \end{aligned}$$

with

$$\begin{aligned} S_i^{(1)} &= \frac{1}{1 + \lambda_{i,0}^{(1)} + A_{i,0} \sqrt{\pi}/2} \int_0^{+\infty} \frac{(1 + A_{i,0}x + \lambda_{i,0}^{(1)}) x e^{-x^2} dx}{\left( (1 + A_{i,0}x + \lambda_{i,0}^{(1)}) [\underline{K}_g]_{ii} - \omega^2 [\underline{M}_g]_{ii} \right)^2 + \omega^2 [\underline{C}_g]_{ii}^2} \quad , \\ S_i^{(2)} &= \frac{1}{1 + \lambda_{i,0}^{(1)} + A_{i,0} \sqrt{\pi}/2} \int_0^{+\infty} \frac{(1 + A_{i,0}x + \lambda_{i,0}^{(1)}) x e^{-x^2} dx}{-\omega^2 [\underline{M}_g]_{ii} + i\omega [\underline{C}_g]_{ii} + (1 + A_{i,0}x + \lambda_{i,0}^{(1)}) [\underline{K}_g]_{ii}} \quad . \end{aligned}$$



## 5. EXAMPLE

### 5.1 Measured Quantities

For this example, we constructed an "experimental data base" using a Monte Carlo numerical simulation in the time domain of the second-order nonlinear dynamical system defined by Eq. (1) with  $n = 5$ , where the nonlinear mapping  $\mathbf{f}$  is defined by  $\mathbf{f}(\mathcal{X}(t)) = ([K]_{11}\mathcal{X}_1(t)^3, \dots, [K]_{55}\mathcal{X}_5(t)^3)$  with  $\varepsilon = 1875$ , where the frequency band of narrow-band process  $\mathbf{F}$  is [14 Hz, 28 Hz] and where  $[K]_{jj}$  are the diagonal terms of matrix  $[K]$  appearing in Eq. (1). Matrices  $[M]$ ,  $[C]$  and  $[K]$  were generated by the formulas  $[M] = [S]^{-T} [\mathcal{M}] [S]^{-1}$ ,  $[C] = [S]^{-T} [C] [S]^{-1}$  and  $[K] = [S]^{-T} [K] [S]^{-1}$ , where

$$[S] = \begin{bmatrix} 0.208513 & 0.333334 & 0.301512 & 0.447214 & 0.447214 \\ 0.208513 & -0.333334 & 0.301512 & -0.447214 & 0.447214 \\ 0.625543 & 0.577350 & 0.522233 & 0.447214 & 0.0 \\ 0.625543 & 0.577350 & -0.522233 & -0.447214 & 0.0 \\ 0.361158 & -0.333334 & -0.522233 & 0.447214 & 0.774597 \end{bmatrix},$$

and where  $[\mathcal{M}]$  is the identity matrix (generalized masses equal to 1),  $[K] = [\mathcal{M}] [\Omega]^2$  in which  $[\Omega]$  is the diagonal matrix whose diagonal is  $2\pi \times [18.0, 20.0, 20.4, 22.0, 23.0]$  and finally,  $[C]$  is a diagonal matrix whose diagonal is  $[4.5, 5.0, 5.13, 5.5, 5.8]$ . Digital signal processing on the time-simulated sample paths of the stationary response was applied to estimate the "measured" matrix-valued spectral density function  $[S_{\mathcal{X}}(\omega)]$  for  $\omega$  in the frequency band of analysis.

It should be noted that some eigenfrequencies  $\Omega_j$  of the underlying linear dynamical system associated with the nonlinear dynamical system are close (20.0 Hz and 20.4 Hz). In presence of nonlinearities, this kind of situation is generally recognized as a difficult problem within the context of structural dynamic identification. This is why we chose such an example, in order to demonstrate the interest of the identification method presented above. The elements  $[S_{\mathcal{X}}(\omega)]_{ii}$  for  $i$  in  $\{1, 2, 3, 4, 5\}$  and some elements of the matrix-valued coherence function  $[\gamma_{\mathcal{X}}(\omega)]_{ij} = |[S_{\mathcal{X}}(\omega)]_{ij}| ([S_{\mathcal{X}}(\omega)]_{ii} [S_{\mathcal{X}}(\omega)]_{jj})^{-1/2}$  of estimated matrix  $[S_{\mathcal{X}}(\omega)]$  are shown in Figs. 2 to 4.

### 5.2 Model Identification

The procedure presented in Section 3 is used to identify the parameters  $\xi_i$  of the model. The results obtained are shown in Table 1.

$i$	1	2	3	4	5
$\alpha_{\Lambda_{i,0}}$	$3.13 \cdot 10^3$	$2.38 \cdot 10^3$	$7.46 \cdot 10^2$	$9.37 \cdot 10^2$	$7.19 \cdot 10^2$
$\beta_{\Lambda_{i,0}}$	$1.55 \cdot 10^3$	$1.17 \cdot 10^3$	$3.67 \cdot 10^2$	$4.61 \cdot 10^2$	$3.53 \cdot 10^2$
$\lambda_{i,0}^{(1)}$	$-3.56 \cdot 10^{-2}$	$-4.09 \cdot 10^{-2}$	$-6.22 \cdot 10^{-2}$	$-5.81 \cdot 10^{-2}$	$-6.53 \cdot 10^{-2}$

Table 1. Parameters of the model resulting from the identification procedure

### 5.3 Results on Matrix-Valued Spectral Density Functions

(1)- Figs. 2 and 3 are related to the comparisons between the matrix-valued spectral density functions obtained by "experiments" and by identification of the model with constant coefficients. Fig. 2 shows the comparison between  $[S_{\mathcal{X}}]_{ii}$  and  $[S_{\mathbf{X}}]_{ii}$  for  $i$  in  $\{1, 2, 3, 4, 5\}$ . It is recalled that that matrix  $[S_{\mathbf{X}}]$  is the spectral density function of the stationary response of Eq. (3). Fig. 3 shows the comparison between  $[\gamma_{\mathcal{X}}]_{ij}$  and  $[\gamma_{\mathbf{X}}]_{ij}$  for some  $i \neq j$ . These results correspond to those obtained by Fillâtre [3]. It should be noted that this first identification already yields a good identification (taking into account the intrinsic difficulties of the example considered), but as mentioned in Section 1, this kind of results can be improved using a more advanced model for identification (see below).

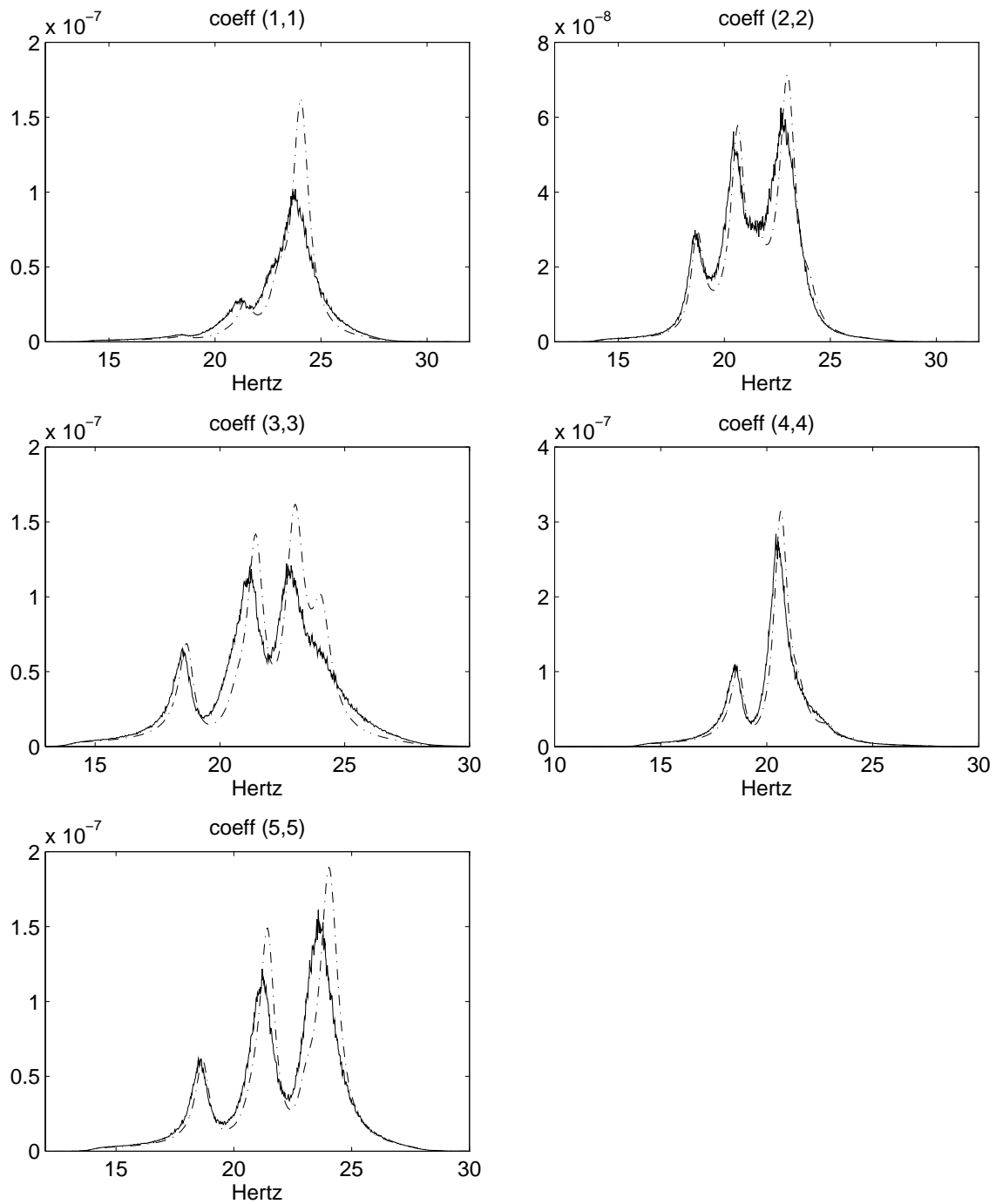


Fig. 2. Power spectral density functions  $[S_{\mathcal{X}}]_{ii}$  and  $[S_{\mathbf{X}}]_{ii}$ :  
 — experiments  
 - - - - identification with stochastic linearization method with constant parameters

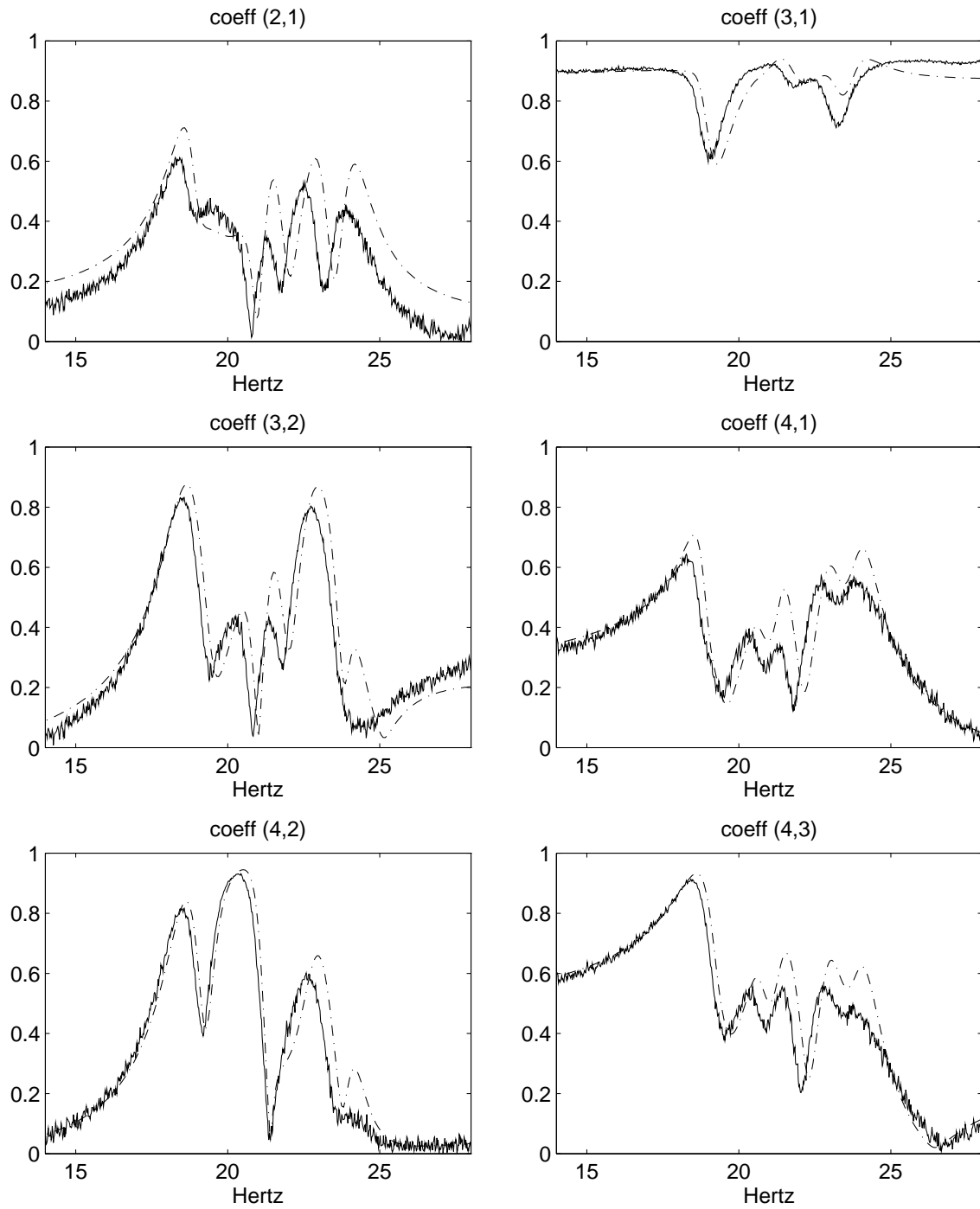


Fig. 3. Coherence functions  $[\gamma_{\mathcal{X}}]_{ij}$  and  $[\gamma_{\mathbf{x}}]_{ij}$ :  
 — experiments  
 - - - identification with stochastic linearization method with constant parameters

(2)- Figs. 4 and 5 are related to the comparisons between the matrix-valued spectral density functions obtained by "experiments" and by identification of the model with random coefficients for which the procedure was described in Sections 2 to 4. Fig. 4 shows the comparison between  $[S_{\mathcal{X}}]_{ii}$  and  $[S_{\mathcal{Z}}]_{ii}$  for  $i$  in  $\{1, 2, 3, 4, 5\}$  and Fig. 5 shows the comparison between  $[\gamma_{\mathcal{X}}]_{ij}$  and  $[\gamma_{\mathcal{Z}}]_{ij}$  for some  $i \neq j$ . It can be seen that the results obtained are much better than above.

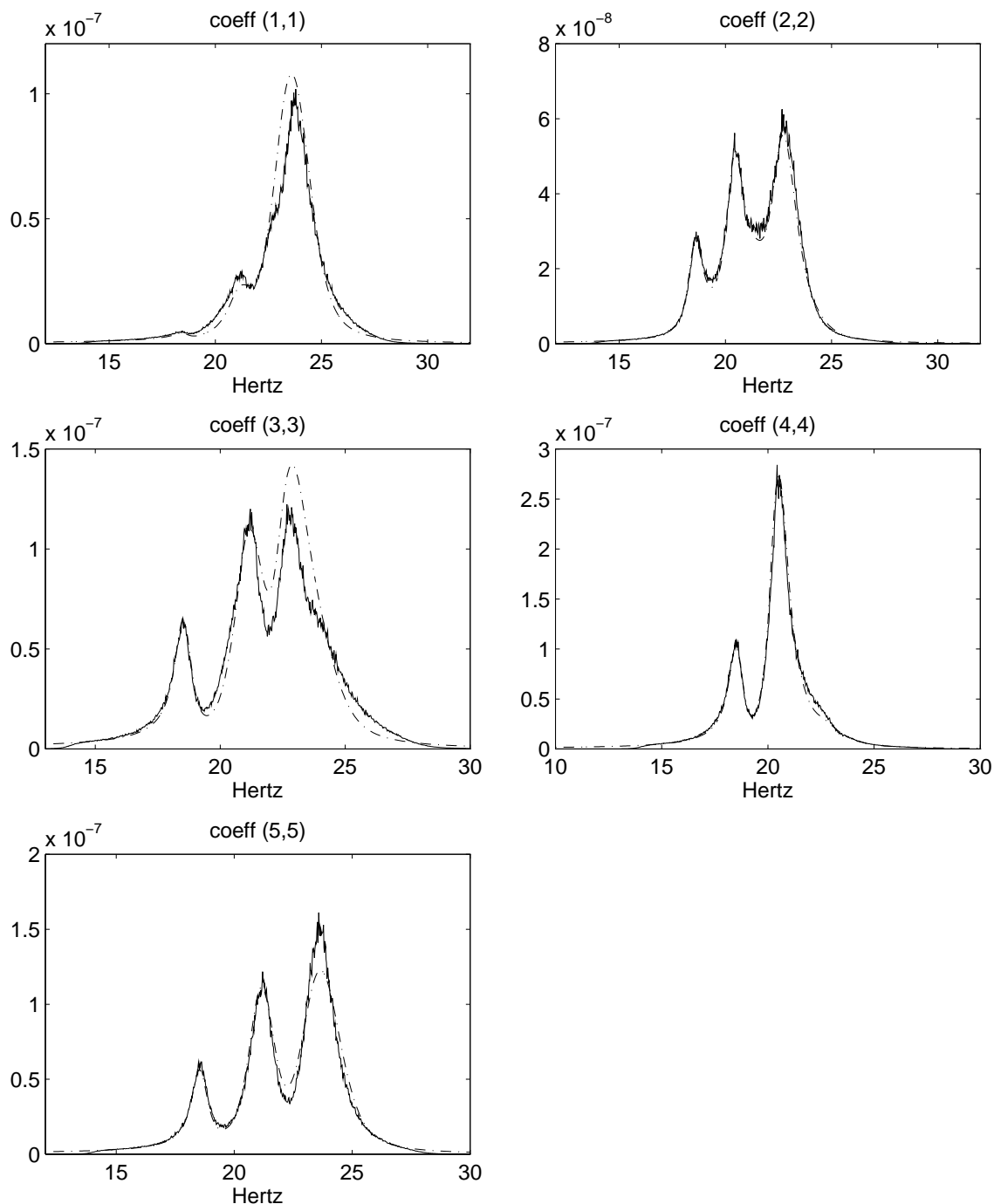


Fig. 4. Power spectral density functions  $[S_{\mathcal{X}}]_{ii}$  and  $[S_{\mathcal{Z}}]_{ii}$ :

— experiments

- - - - identification with stochastic linearization method with random parameters

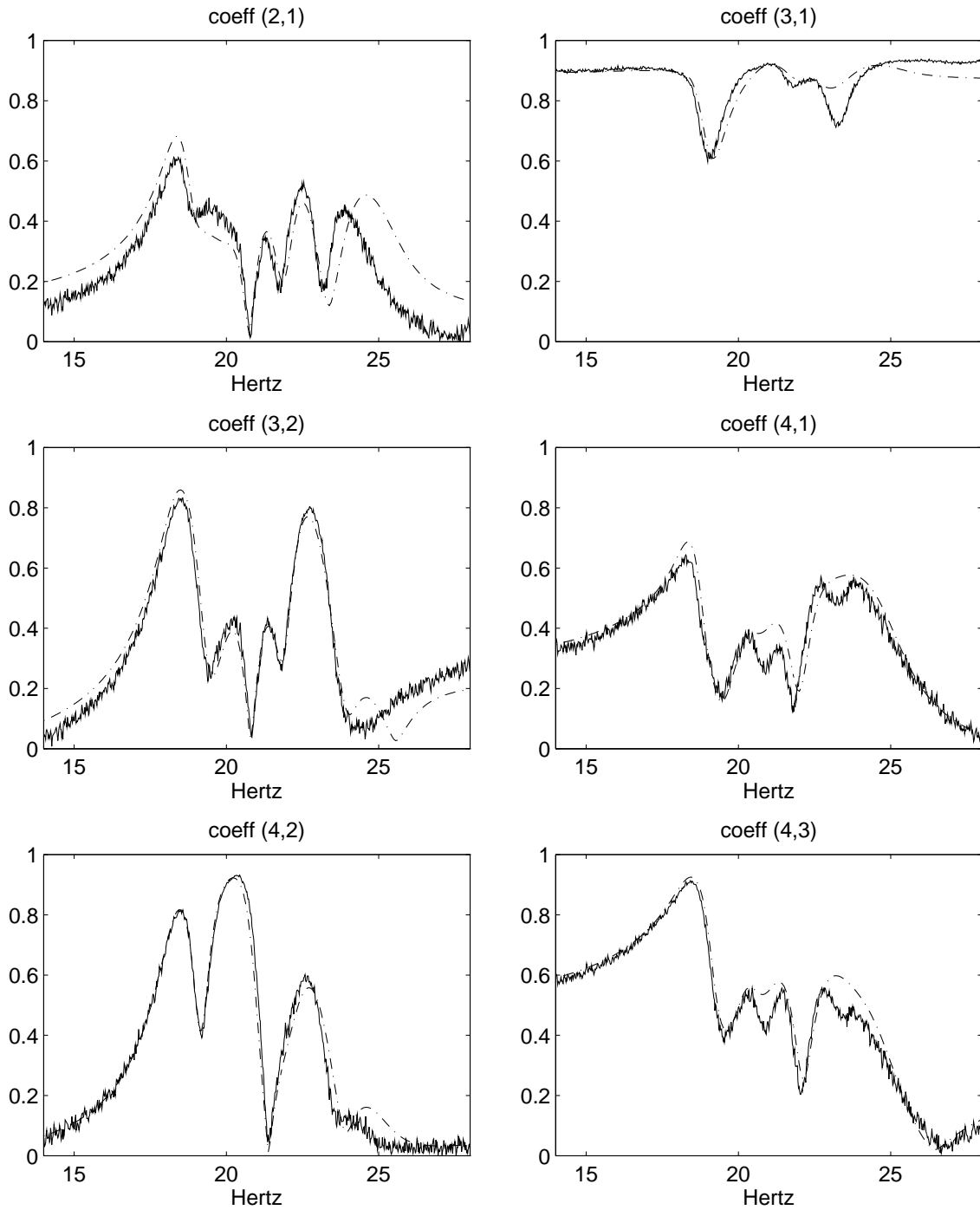


Fig. 5. Coherence functions  $[\gamma_{\mathcal{X}}]_{ij}$  and  $[\gamma_{\mathcal{Z}}]_{ij}$ :  
 — experiments  
 - - - - identification with stochastic linearization method with random parameters

## 6. CONCLUSION

This work started from previous research on modal identification of weakly nonlinear multidimensional second-order dynamical systems (based on the use of the equivalent stochastic linearization with

constant coefficients). In some cases, this method has difficulty identifying the matrix-valued spectral density function of the stationary responses. In the present work, we have used the previous work to identify the mean part of the model. In order to improve the identification of spectral quantities, we developed a method based on stochastic linearization with random coefficients. This new identification procedure seems to be very efficient and can be implemented easily. The results show that this method yields better results than the previous one. Nevertheless, this method could be improved by introducing some statistical dependence between the components of the random coefficients expressed in the modal coordinates (or possibly by introducing extra-diagonal terms). It should be noted that in this last case, the optimization problem introduced in the method could not be split into several optimization problems with a smaller size. This being the case, the efficiency of such a procedure would have to be studied with great care.

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