Time-domain formulation in computational dynamics for linear viscoelastic media with model uncertainties and stochastic excitation

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Abstract

The paper is devoted to the computational time-domain formulation of linear viscoelastic systems submitted to a nonstationary stochastic excitation and in presence of model uncertainties which are modeled in the framework of the probability theory. The objective is to introduce and to develop an adapted and complete formulation of such a problem in the context of computational mechanics. A reduced-order model in the time domain with stochastic excitation is derived from the computational model. For the reduced-order model, the stochastic modeling of both computational model-parameters uncertainties and modeling errors is carried out using the nonparametric probabilistic approach and the random matrix theory. We present a new formulation of model uncertainties to construct the random operators for viscoelastic media. We then obtained a linear Stochastic Integro-Differential Equation (SIDE) with random operators and with a stochastic nonhomogeneous part (stochastic excitation). A time discretization of this SIDE is proposed. In a first step, the SIDE is transformed to a linear Itô Stochastic Differential Equation (ISDE) with random operators. Then the ISDE is discretized using an extension of the Störmer-Verlet scheme which is a particularly well adapted algorithm for long-time good behavior of the numerical solution. Finally, for the stochastic solver and statistical estimations of the random responses, we propose to use the Monte Carlo simulation for Gaussian and non-Gaussian excitations.

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1. Introduction

It is well known that the computational models in linear viscoelastic dynamics can be analyzed in the frequency domain which constitutes an alternative approach to the time-domain formulation. Such a frequency-domain approach is nowadays very efficient considering the developments of massively parallel computers, because there are no data exchange between the calculations performed at each frequency. Nevertheless, the frequency approach is not really appropriate if additional local nonlinearities are added to the linear viscoelastic dynamical system yielding nonlinear dynamical systems. In such a case, the time formulation is better adapted.

In the present work, we are interested in the time-domain formulation of linear viscoelastic systems submitted to nonstationary stochastic excitation and in presence of uncertainties which are modeled in the framework of the probability theory. The time formulation is then proposed, developed and validated in this context of linear stochastic integro-differential equations with random operators and with a stochastic nonhomogeneous part (stochastic excitation). Although such systems could be analyzed in the frequency domain, it is interesting to present a complete analysis in the time domain in order to propose a general methodology which has the capability to analyze such systems with additional local nonlinearities, that is the case, for instance, of vibro-impact systems made up of a linear viscoelastic medium with stops. In addition, we are mainly interested in the response to transient excitation.

The time-domain formulation for dynamics of viscoelastic structures has been previously studied (see for instance [28, 43, 49, 60]), in particular for approximating the time-domain integral operator using either a sequence of linear differential operators acting on additional hidden variables (see for instance [19, 45]) or the fractional derivative operators (see for instance [4, 18, 20, 24, 63]). A lot of papers using different existing time-domain viscoelastic modeling for many applications have been published and we cannot refer here all these works. Many works have been published in the context of random vibration of viscoelastic systems submitted to stochastic excitation in the context of analytical approach of simple mechanical systems but there are no works in the context of computational mechanics for analyzing general complex mechanical systems.

Concerning uncertainties modeling in computational mechanics, the most popular method is the parametric probabilistic approach which consists in modeling the uncertain parameters of the computational model by random variables and then in constructing the probabilistic model of these random variables using the available information. Such an approach is very well adapted and very efficient to take into account the uncertainties on the computational model parameters as soon as the probability theory can be used. Many works have been published and a state-of-the-art can be found, for instance, in [17, 27, 44, 46, 64, 65, 73, 75]. Concerning model uncertainties induced by modeling errors, today, it is well understood that the prior and posterior probability models of the uncertain parameters of the computational model are not sufficient and do not have the capability to take into account model uncertainties in the context of computational mechanics as explained in [70]. The nonparametric probabilistic approach of both the computational model-parameters uncertainties and modeling errors has been proposed as a way to take into account modeling errors at the operators level by introducing random operators and not at the model output level by introducing an additive noise [70, 71, 72, 75]. A few works have been carried out on viscoelastic structures with uncertain parameters for simple mechanical systems (see for instance [23]) and no works can be found concerning the probabilistic approach of modeling errors for viscoelastic media.
At the knowledge of the authors, there is no work published that concerns methods in the time-domain formulation for analyzing general linear viscoelastic systems submitted to nonstationary stochastic excitations and in presence of both the computational model-parameters uncertainties and model uncertainties induced by modeling errors. In this paper, we present such an approach in the time domain for computational models of general linear viscoelastic systems submitted to stochastic excitation and in presence of model uncertainties. Concerning uncertainties modeling, we propose a new extension of the nonparametric probabilistic approach of uncertainties for viscoelastic media. The objective of this paper is to present an adapted and complete formulation of the problem in the context of computational mechanics.

Principal notations used for tensors and Fourier transform

(i) Let $S = \{S_{jk\ell m}\}_{jk\ell m}$ be a fourth-order real tensor and let $\boldsymbol{\eta} = \{\eta_{\ell m}\}_{\ell m}$ be a second-order real tensor. The contraction $S : \boldsymbol{\eta}$ of S with $\boldsymbol{\eta}$ is a second-order tensor such that $\{S : \boldsymbol{\eta}\}_{jk} = \sum_{\ell m} S_{jk\ell m}\eta_{\ell m}$.

(ii) Fourth-order tensor S is said to be symmetric if $S_{jk\ell m} = S_{kj\ell m} = S_{jkm\ell} = S_{\ell mjk}$ and is said to be positive (or positive definite) if, for all non zero second-order tensor $\boldsymbol{\eta}$, we have $\sum_{jk\ell m} S_{jk\ell m}\eta_{jk}\eta_{\ell m} \geq 0$ (or $\sum_{jk\ell m} S_{jk\ell m}\eta_{jk}\eta_{\ell m} > 0$). If tensor S is positive definite, then there exists a positive constant c_S (the smallest positive eigenvalue of S) such that $\sum_{jk\ell m} S_{jk\ell m}\eta_{jk}\eta_{\ell m} \geq c_S \sum_{jk} \eta_{jk}^2$.

(iii) Let $\tau \mapsto g(\tau)$ be an integrable real function defined on $\mathbb{R} =]-\infty\,, +\infty[$. The Fourier transform of g is defined as the complex continuous function $\omega \mapsto \widehat{g}(\omega)$ on $\mathbb{R}$ such that $\widehat{g}(\omega) = \int_{\mathbb{R}} e^{-i\omega\tau} g(\tau)\, d\tau$ and which is such that $\lim_{|\omega|\to+\infty} \widehat{g}(\omega) = 0$.

(iv) Let X be any set and let B be any subset of X. The indicator function $x \mapsto \mathbb{1}_B(x)$ from X into $\mathbb{R}$ is such that $\mathbb{1}_B(x) = 1$ if $x \in B$ and $\mathbb{1}_B(x) = 0$ if $x \notin B$.

2. Boundary value problem in time domain for computational dynamics of linear viscoelastic media

We consider a linear viscoelastic medium occupying an open bounded domain Ω of $\mathbb{R}^3$, with boundary $\partial\Omega = \Gamma_0 \cup \Gamma$, in a Cartesian frame $(Ox_1x_2x_3)$. Let $\mathbf{x} = (x_1, x_2, x_3)$ be any point in Ω and let $d\mathbf{x} = dx_1\, dx_2\, dx_3$. The external unit normal to $\partial\Omega$ is denoted by $\mathbf{n} = (n_1, n_2, n_3)$. Let $\mathbf{u}(\mathbf{x}, t) = (u_1(\mathbf{x}, t), u_2(\mathbf{x}, t), u_3(\mathbf{x}, t))$ be the displacement field defined on Ω. On part Γ_0 of the boundary, there is a Dirichlet condition $\mathbf{u} = 0$. Consequently, there will be no rigid body displacements. This hypothesis can be released in the developments presented in this paper. A surface force field, $\mathbf{f}^{\text{surf}}(\mathbf{x}, t) = (f_1^{\text{surf}}(\mathbf{x}, t), f_2^{\text{surf}}(\mathbf{x}, t), f_3^{\text{surf}}(\mathbf{x}, t))$, is applied to part Γ on the boundary and will be random. In addition, there is a volume force field, $\mathbf{f}^{\text{vol}}(\mathbf{x}, t) = (f_1^{\text{vol}}(\mathbf{x}, t), f_2^{\text{vol}}(\mathbf{x}, t), f_3^{\text{vol}}(\mathbf{x}, t))$ applied in Ω and will be random. We are interested in the linear transient response of this viscoelastic medium around a static equilibrium considered as the reference configuration defined by Ω. It is assumed that there is no prestress. The boundary value problem in the time domain is written, for all t in $]0\,, T]$, with T a finite positive real number, as

$$\begin{aligned} &\rho\,\frac{\partial^2 \mathbf{u}}{\partial t^2} - \operatorname{div}\boldsymbol{\sigma} = \mathbf{f}^{\text{vol}} && \text{in } \Omega\,, \\ &\mathbf{u} = \mathbf{0} && \text{in } \Gamma_0\,, \\ &\boldsymbol{\sigma}\,\mathbf{n} = \mathbf{f}^{\text{surf}} && \text{in } \Gamma\,, \end{aligned} \tag{1}$$

where $\rho(\mathbf{x})$ is the mass density which is assumed to be a positive-valued bounded function on Ω, $\boldsymbol{\sigma}(\mathbf{x}, t)$ is the second-order stress tensor, in which $\{\operatorname{div}\boldsymbol{\sigma}(\mathbf{x}, t)\}_j =$

$\sum_{k=1}^{3} \partial\sigma_{jk}(\mathbf{x},t)/\partial x_k$ and where $\{\boldsymbol{\sigma}(\mathbf{x},t)\,\mathbf{n}\}_j = \sum_{k=1}^{3} \sigma_{jk}(\mathbf{x},t) n_k(\mathbf{x})$. Let $\boldsymbol{\varepsilon}(\mathbf{x},t)$ be the strain tensor defined by

$$\varepsilon_{jk}(\mathbf{x},t) = \frac{1}{2}\left(\frac{\partial u_j(\mathbf{x},t)}{\partial x_k} + \frac{\partial u_k(\mathbf{x},t)}{\partial x_j}\right). \tag{2}$$

The stress tensor $\boldsymbol{\sigma}(\mathbf{x},t)$ will be related to the strain tensor $\boldsymbol{\varepsilon}(\mathbf{x},t)$ by a constitutive equation which is nonlocal in time for a linear viscoelastic material and which will be detailed in Section 3. Finally, for the time-evolution problem, the initial conditions are chosen such that, for all $\mathbf{x}$ in Ω,

$$\mathbf{u}(\mathbf{x},0) = \mathbf{0}\,, \quad \frac{\partial \mathbf{u}}{\partial t}(\mathbf{x},0) = \mathbf{0}\,. \tag{3}$$

3. Viscoelastic constitutive equation

In this Section, we use the *linear viscoelastic theory* presented in [77] and consider *strongly dissipative materials*. This assumption will imply that the operators of the problem, which belong to the set of all the positive linear operators, will be, in fact, in the subset of all the positive-definite operators. Others presentations can be found, for instance, in [15, 35, 51, 56]. In order to simplify the notation, $\mathbf{x}$ is removed from the equations appearing in this Section. Consequently, $\boldsymbol{\sigma}(\mathbf{x},t)$ and $\boldsymbol{\varepsilon}(\mathbf{x},t)$ are rewritten as $\boldsymbol{\sigma}(t)$ and $\boldsymbol{\varepsilon}(t)$. It is assumed that

$$\boldsymbol{\sigma}(t) = \mathbf{0}\,, \quad \boldsymbol{\varepsilon}(t) = \mathbf{0}\,, \quad \text{for} \quad t \leq 0\,. \tag{4}$$

In linear viscoelasticity, the constitutive equation is written, for all $t > 0$, as $\boldsymbol{\sigma}(t) = \int_0^t C(t-\tau) : \dot{\boldsymbol{\varepsilon}}(\tau)\, d\tau$, in which $\tau \mapsto C(\tau)$ is the fourth-order tensor-valued *relaxation function* defined on $\mathbb{R}^+ = [0\,, +\infty[$. Performing an integration by parts and since $\boldsymbol{\varepsilon}(0) = \mathbf{0}$, this constitutive equation can be rewritten as

$$\begin{aligned}\boldsymbol{\sigma}(t) &= C(0) : \boldsymbol{\varepsilon}(t) + \int_0^t \dot{C}(t-\tau) : \boldsymbol{\varepsilon}(\tau)\, d\tau \\ &= C(0) : \boldsymbol{\varepsilon}(t) + \int_0^t \dot{C}(\tau) : \boldsymbol{\varepsilon}(t-\tau)\, d\tau\,,\end{aligned} \tag{5}$$

where the function $\tau \mapsto \dot{C}(\tau)$ is defined on $\mathbb{R}^{*+} =]0\,, +\infty[$ as the first derivative of C with respect to τ and such that $\dot{C}(0) = \lim_{\tau \downarrow 0} \dot{C}(\tau) = \dot{C}(0^+)$. The fourth-order tensor $C(0)$ is called the *initial elasticity* and in the right-hand side of Eq. (5), the second term is called the *hereditary response*. We introduce the

fourth-order tensor-valued function $\tau \mapsto H(\tau)$ defined on $\mathbb{R} =]-\infty\,,+\infty[$ such that

$$H(\tau) = 0 \text{ for } \tau < 0\,, \quad H(0) = \dot{C}(0^+)\,, \quad H(\tau) = \dot{C}(\tau) \text{ for } \tau > 0\,. \tag{6}$$

Consequently, H is a causal function because the support of function H is $\mathbb{R}^+$. In addition, it is assumed that function H is integrable on $\mathbb{R}$ (that is to say on $\mathbb{R}^+$). Taking into account Eq. (6), Eq. (5) can be rewritten as

$$\boldsymbol{\sigma}(t) = C(0) : \boldsymbol{\varepsilon}(t) + \int_0^t H(\tau) : \boldsymbol{\varepsilon}(t-\tau)\, d\tau\,. \tag{7}$$

Using Eqs. (4) and (6), Eq. (7) can also be written as

$$\boldsymbol{\sigma}(t) = C(0) : \boldsymbol{\varepsilon}(t) + \int_{-\infty}^{+\infty} H(\tau) : \boldsymbol{\varepsilon}(t-\tau)\, d\tau\,. \tag{8}$$

Let H^∞ be the fourth-order tensor defined by

$$H^\infty = \int_0^{+\infty} H(\tau)\, d\tau\,. \tag{9}$$

Since $C(t) = C(0) + \int_0^t \dot{C}(\tau)\, d\tau$, the *equilibrium modulus* is introduced as the fourth-order tensor $C^\infty = \lim_{t\to+\infty} C(t)$ and is such that

$$C^\infty = C(0) + H^\infty\,. \tag{10}$$

The viscoelastic material is dissipative which means that, for all $t \geq 0$, we have $\int_0^t \boldsymbol{\sigma}(\tau) : \dot{\boldsymbol{\varepsilon}}(\tau)\, d\tau > 0$. Using this inequality for sufficiently smooth $\boldsymbol{\varepsilon}$ with $\boldsymbol{\varepsilon}(0) = 0$, Gurtin and Herrera [29] prove that initial elasticity $C(0)$ and equilibrium modulus C^∞ are positive-definite symmetric fourth-order tensors. Using thermodynamics arguments, Coleman [13] has shown that $C(0) - C^\infty$ is a positive-definite symmetric fourth-order tensor which means that $H^\infty = C^\infty - C(0)$ is a negative-definite symmetric fourth-order tensor. Using a time-reversal argument, Day [16] has shown that, for all t in $\mathbb{R}^+$, the fourth-order tensor $C(t)$ is symmetric.

In the following, we will assume that $C(0)$, C^∞ and $-H^\infty$ are positive-definite symmetric fourth-order tensors (strongly dissipative material) and that $C(t)$ is symmetric for all t in $\mathbb{R}^{*+}$. Thus, for all t in $\mathbb{R}^+$, the fourth-order tensor $H(t)$ is symmetric.

Assuming that $\boldsymbol{\varepsilon}$ is an integrable function on $\mathbb{R}^+$, taking the Fourier transform of Eq. (8) yields

$$\widehat{\boldsymbol{\sigma}}(\omega) = (C(0) + \widehat{H}(\omega)) : \widehat{\boldsymbol{\varepsilon}}(\omega)\,. \tag{11}$$

Let

$$\widehat{H}^R(\omega) = \Re e\{\widehat{H}(\omega)\}\,, \quad \widehat{H}^I(\omega) = \Im m\{\widehat{H}(\omega)\} \tag{12}$$

be the real part of $\widehat{H}(\omega)$ and its imaginary part.

For all ω in $\mathbb{R}$, we define the fourth-order real *elasticity tensor* $A(\omega)$ and the fourth-order real *damping tensor* $B(\omega)$, such that

$$\widehat{\boldsymbol{\sigma}}(\omega) = (A(\omega) + i\,\omega\,B(\omega)) : \widehat{\boldsymbol{\varepsilon}}(\omega)\,. \tag{13}$$

From Eqs. (11) and (13), we deduce that $A(\omega) + i\,\omega\,B(\omega) = C(0) + \widehat{H}(\omega)$. We then obtain

$$A(\omega) = C(0) + \widehat{H}^R(\omega)\,, \quad \omega\,B(\omega) = \widehat{H}^I(\omega)\,. \tag{14}$$

For all $\mathbf{x}$ fixed in Ω and for all ω fixed in $\mathbb{R}$, we have the following important properties for tensors $A(\mathbf{x},\omega)$ and $B(\mathbf{x},\omega)$ (which are denoted by $A(\omega)$ and $B(\omega)$, as previously):

(i) Since H is a real tensor-valued function, we have $\widehat{H}^R(-\omega) = \widehat{H}^R(\omega)$ and $\widehat{H}^I(-\omega) = -\widehat{H}^I(\omega)$. Taking into account Eq. (14), it can be deduced that

$$A(-\omega) = A(\omega)\,, \quad B(-\omega) = B(\omega)\,. \tag{15}$$

(ii) Due to Eqs. (9) and (14), and since $\widehat{H}^I(0) = 0$, the negative-definite symmetric real fourth-order tensor H^∞ can be written as $H^\infty = A(0) - C(0)$. Consequently, $A(0) = C^\infty$, which appears as the equilibrium modulus tensor, is a positive-definite symmetric real fourth-order tensor and corresponds to usual elasticity coefficients of the material for a static deformation ($t \to +\infty$).

(iii) As explained above, it should be noted that, for all fixed τ in $\mathbb{R}^+$, the fourth-order real tensor $H(\tau)$ is symmetric. We can then deduce that the fourth-order real tensors $A(\omega)$ and $B(\omega)$ are symmetric,

$$A_{jk\ell m}(\omega) = A_{kj\ell m}(\omega) = A_{jkm\ell}(\omega) = A_{\ell mjk}(\omega)\,, \tag{16}$$

$$B_{jk\ell m}(\omega) = B_{kj\ell m}(\omega) = B_{jkm\ell}(\omega) = B_{\ell mjk}(\omega)\,. \tag{17}$$

(iv) The elasticity tensor $A(\omega)$ and the damping tensor $B(\omega)$ are positive-definite symmetric real tensors and, for all second-order real tensor $\boldsymbol{\eta}$, there are positive constants $c_{A(\omega)}$ and $c_{B(\omega)}$ such that

$$\sum_{jk\ell m} A_{jk\ell m}(\omega)\eta_{jk}\eta_{\ell m} \geq c_{A(\omega)} \sum_{jk} \eta_{jk}^2\,, \tag{18}$$

$$\sum_{jk\ell m} B_{jk\ell m}(\omega)\eta_{jk}\eta_{\ell m} \geq c_{B(\omega)} \sum_{jk} \eta_{jk}^2\,. \tag{19}$$

(v) Since H is an integrable function, $\widehat{H}$ is a continuous function on $\mathbb{R}$ and $\widehat{H}(\omega)$ goes to zero when $|\omega|$ goes to infinity. Using Eqs. (12) and (14) yields

$$\lim_{|\omega|\to+\infty} A(\omega) = C(0)\,, \qquad \lim_{|\omega|\to+\infty} \omega B(\omega) = 0\,. \tag{20}$$

Therefore, for high frequencies ($\omega \to +\infty$), the viscoelastic material becomes an elastic material with elasticity coefficients defined by initial elasticity tensor $C(0)$ which differs from C^∞.

(vi) Since $\tau \mapsto H(\tau)$ is a causal and an integrable function on $\mathbb{R}^+$, for all real ω, the real part $[\widehat{H}^R(\omega)]$ and the imaginary part $[\widehat{H}^I(\omega)]$ of the Fourier transform $[\widehat{H}(\omega)]$ must satisfy relations involving the Hilbert transform (see [30, 52]), which are also called the Kramers-Kronig relations [38, 40]). Consequently, taking into account Eq. (14), tensor-valued functions A and B are not algebraically independent but are linked by the Kramers-Kronig relations involving the Hilbert transform. This means that, if B is given, then A is deduced from B and conversely. In this section, we do not give these integral relations because they are not directly used but it is assumed that the data related to the constitutive equation are such that these integral relations are satisfied. Nevertheless, in Section 9, we will introduce the Kramers-Kronig relations for the operators of the reduced-order model for establishing the equations that will be useful in Section 10 for implementing the nonparametric probabilistic approach of uncertainties at the operator level.

4. Weak formulation of the boundary value problem in the time domain

In this section, we construct the weak formulation of the boundary value problem defined, for t in $]0\,,T]$, by Eq. (1) for which the constitutive equation is given by

Eq. (7), with strain tensor $\boldsymbol{\varepsilon}$ given by Eq. (2), and with the initial conditions at time $t = 0$, defined by Eq. (3).

We then introduce the admissible function space $\mathcal{C}_{\text{ad}}$ of $\mathbb{R}^3$-valued functions $\mathbf{x} \mapsto \mathbf{v}(\mathbf{x}) = (v_1(\mathbf{x}), v_2(\mathbf{x}), v_3(\mathbf{x}))$ defined on Ω, such that

$$\mathcal{C}_{\text{ad}} = \{\, \mathbf{v} \in (\mathcal{H}^1(\Omega))^3\,, \quad \mathbf{v} = 0 \text{ on } \Gamma_0\,\}\,, \tag{21}$$

in which $\mathcal{H}^1(\Omega)$ is the Sobolev space of square integrable real functions on Ω for which each partial derivative with respect to x_j, for $j = 1, 2, 3$, is a square integrable real function on Ω. The weak formulation is then written as follows. For all fixed t in $]0\,, T]$, find the function, $\mathbf{u}(\cdot, t) = \{\mathbf{x} \mapsto \mathbf{u}(\mathbf{x}, t)\}$, in $\mathcal{C}_{\text{ad}}$ such that, for all $\mathbf{v}$ in $\mathcal{C}_{\text{ad}}$, we have

$$m(\partial_t^2 \mathbf{u}(\cdot, t), \mathbf{v}) + k_0(\mathbf{u}(\cdot, t), \mathbf{v}) + \int_0^t h(\mathbf{u}(\cdot, t - \tau), \mathbf{v}; \tau)\, d\tau = f(\mathbf{v}; t)\,, \tag{22}$$

in which the mass bilinear form $(\mathbf{u}(\cdot, t), \mathbf{v}) \mapsto m(\mathbf{u}(\cdot, t), \mathbf{v})$ is defined by

$$m(\mathbf{u}(\cdot, t), \mathbf{v}) = \int_\Omega \rho(\mathbf{x})\, \mathbf{u}(\mathbf{x}, t) \cdot \mathbf{v}(\mathbf{x})\, d\mathbf{x}\,. \tag{23}$$

Reintroducing the full notation $C(\mathbf{x}, t)$ instead of the abbreviate notation $C(t)$, the initial elasticity bilinear form $(\mathbf{u}(\cdot, t), \mathbf{v}) \mapsto k_0(\mathbf{u}(\cdot, t), \mathbf{v})$, associated with initial elasticity $C(\mathbf{x}, 0)$, is defined by

$$k_0(\mathbf{u}(\cdot, t), \mathbf{v}) = \int_\Omega \{C(\mathbf{x}, 0) : \boldsymbol{\varepsilon}(\mathbf{u}(\mathbf{x}, t))\} : \boldsymbol{\varepsilon}(\mathbf{v}(\mathbf{x}))\, d\mathbf{x}\,. \tag{24}$$

Again, reintroducing the full notation $H(\mathbf{x}, \tau)$ instead of the abbreviate notation $H(\tau)$, the hereditary bilinear form $(\{\mathbf{u}(\cdot, \tau), \tau \in [0, t]\}, \mathbf{v}) \mapsto \int_0^t h(\mathbf{u}(\cdot, t - \tau), \mathbf{v}; \tau)\, d\tau$, associated with relaxation function $\tau \mapsto C(\mathbf{x}, \tau)$ through Eq. (6), is such that

$$h(\mathbf{u}(\cdot, t - \tau), \mathbf{v}; \tau) = \int_\Omega \{H(\mathbf{x}, \tau) : \boldsymbol{\varepsilon}(\mathbf{u}(\mathbf{x}, t - \tau))\} : \boldsymbol{\varepsilon}(\mathbf{v}(\mathbf{x}))\, d\mathbf{x}\,. \tag{25}$$

The linear form $\mathbf{v} \mapsto f(\mathbf{v}; t)$ is defined by

$$f(\mathbf{v}; t) = \int_\Omega \mathbf{f}^{\text{vol}}(\mathbf{x}, t) \cdot \mathbf{v}(\mathbf{x})\, d\mathbf{x} + \int_\Gamma \mathbf{f}^{\text{surf}}(\mathbf{x}, t) \cdot \mathbf{v}(\mathbf{x})\, ds(\mathbf{x})\,, \tag{26}$$

in which $ds(\mathbf{x})$ is the measure on Γ.

Under the hypotheses introduced in Sections 2 and 3, and if, for all $\mathbf{v}$, the function $t \mapsto f(\mathbf{v}; t)$ is bounded on $[0\,, T]$, then (see for instance [15, 62, 77]), Eq. (23) for t in $]0\,, T]$ with the initial conditions defined by Eq. (3), has a unique solution $t \mapsto \mathbf{u}(\cdot, t)$ square integrable from $[0\,, T]$ in $\mathcal{C}_{\text{ad}}$.

5. Computational model as an integro-differential equation in the time domain

For a given deterministic excitation represented by $f(\mathbf{v}, \cdot)$, the weak formulation of the boundary value problem defined by Eqs. (22) with the initial conditions defined by Eq. (3), is discretized by the usual finite element method. Let $\mathbf{u}(t) = (u_1(t), \ldots, u_n(t))$ be the deterministic vector in $\mathbb{R}^n$ of the n degrees of freedom related to the spatial discretization of the deterministic field $\mathbf{x} \mapsto \mathbf{u}(\mathbf{x}, t)$ (note that the same notation has been used to represent the field and its finite spatial discretization but no confusion can be induced with such a choice). Let $\mathbf{f}(t) = (f_1(t), \ldots, f_n(t))$ be the corresponding finite element discretization of the linear form $\mathbf{v} \mapsto f(\mathbf{v}, t)$ (with the same abuse of notation). We then obtained,

$$[\mathbb{M}]\, \ddot{\mathbf{u}}(t) + [\mathbb{K}_0]\, \mathbf{u}(t) + \int_0^t [\mathbb{N}(\tau)]\, \mathbf{u}(t-\tau)\, d\tau = \mathbf{f}(t)\,, \quad t \in]0\,, T]\,, \tag{27}$$

$$\mathbf{u}(0) = \dot{\mathbf{u}}(0) = 0\,. \tag{28}$$

Taking into account the results presented in Section 3, it can easily be proven that the matrices $[\mathbb{M}]$ and $[\mathbb{K}_0]$, associated with the bilinear forms m and k_0, are positive-definite symmetric $(n \times n)$ real matrices. The function $\tau \mapsto [\mathbb{N}(\tau)]$ from $\mathbb{R}$ into the set of all the symmetric $(n \times n)$ real matrices, such that $\int_0^t [\mathbb{N}(\tau)]\, \mathbf{u}(t - \tau)\, d\tau$ is associated with the hereditary bilinear form, has a support which is $\mathbb{R}^+$ (that implies $[\mathbb{N}(\tau)] = [0]$ for $\tau < 0$) and is integrable on $\mathbb{R}$.

Let

$$[\widehat{\mathbb{N}}(\omega)] = \int_0^{+\infty} e^{-i\omega t} [\mathbb{N}(\tau)]\, d\tau \tag{29}$$

be the Fourier transform of $[\mathbb{N}]$ which is written as

$$[\widehat{\mathbb{N}}(\omega)] = [\widehat{\mathbb{N}}^R(\omega)] + i\, [\widehat{\mathbb{N}}^I(\omega)]\,, \tag{30}$$

$$[\widehat{\mathbb{N}}^R(\omega)] = \Re e\{[\widehat{\mathbb{N}}(\omega)]\}\,,\ [\widehat{\mathbb{N}}^I(\omega)] = \Im m\{[\widehat{\mathbb{N}}(\omega)]\}, \tag{31}$$

in which $[\widehat{\mathbb{N}}^R(\omega)]$ and $[\widehat{\mathbb{N}}^I(\omega)]$ are the real part and the imaginary part of $[\widehat{\mathbb{N}}(\omega)]$. Let $[\mathbb{K}(\omega)]$ and $[\mathbb{D}(\omega)]$ be the $(n \times n)$ real matrices defined by

$$[\mathbb{K}(\omega)] = [\mathbb{K}_0] + [\widehat{\mathbb{N}}^R(\omega)]\,, \quad \omega\,[\mathbb{D}(\omega)] = [\widehat{\mathbb{N}}^I(\omega)]\,. \tag{32}$$

Taking into account the results given in Section 3, it can be proven that $[\mathbb{D}(\omega)]$ and $[\mathbb{K}(\omega)]$ are positive-definite symmetric matrices and such that

$$[\mathbb{K}(-\omega)] = [\mathbb{K}(\omega)]\,, \quad [\mathbb{D}(-\omega)] = [\mathbb{D}(\omega)]\,. \tag{33}$$

In addition, for $\omega = 0$, the symmetric real matrix $[\mathbb{K}(0)]$, constructed with the equilibrium tensor C^∞ defined by Eq. (10), is positive definite and can be written as

$$[\mathbb{K}(0)] = [\mathbb{K}_0] + [\mathbb{N}^\infty]\,, \quad [\mathbb{N}^\infty] = [\widehat{\mathbb{N}}^R(0)]\,. \tag{34}$$

in which the positive-definite symmetric real matrix $[\mathbb{K}_0]$ is constructed with the initial elasticity tensor $C(0)$ and where $[\mathbb{N}^\infty]$ is a negative-definite symmetric real matrix. Consequently, we have

$$[\mathbb{K}_0] = \lim_{|\omega|\to+\infty} [\mathbb{K}(\omega)]\,. \tag{35}$$

It should be noted that Eq. (27) corresponds to the most general formulation in the time domain within the framework of the linear theory of viscoelasticity. An approximation of the integral operator in the right-hand side of Eq. (27) can be constructed in the time domain using a sequence of linear differential operators acting on additional hidden variables (see for instance Chapter XII of [39] for a general mathematical construction and see [19, 45] for developments in the framework of viscoelasticity). This type of approximation can efficiently be described using fractional derivative operators (see for instance [4, 18]).

6. Nonstationary stochastic model of transient excitation

It is now assumed that the excitation $t \mapsto \mathbf{f}(t)$ from $[0\,,T]$ into $\mathbb{R}^n$ is modeled by a Gaussian nonstationary second-order centered $\mathbb{R}^n$-valued stochastic process $\{\mathbf{F}(t), t \in [0\,,T]\}$ defined on a probability space $(\Theta', \mathcal{T}', \mathcal{P}')$. In this work, we are interested in the time-domain formulation of linear viscoelastic media which is particularly efficient for analyzing transient responses induced by a transient excitation. Consequently, we will propose to represent stochastic process $\mathbf{F}$ as the product of a deterministic matrix-valued continuous function $t \mapsto [\mathbb{O}(t)]$ (which

allows the time duration and the signal envelope to be controlled) with a Gaussian stationary second-order centered $\mathbb{R}^{n_V}$-valued stochastic process $\{\mathbf{V}(t), t \in \mathbb{R}\}$ defined as a vector-valued diffusion process. We can then write,

$$\mathbf{F}(t) = [\mathbb{O}(t)]\,\mathbf{V}(t)\,, \quad \forall t \in [0\,, T]\,, \tag{36}$$

$$d\mathbf{V}(t) = [\mathbb{S}]\,\mathbf{V}(t)\,dt + [\mathbb{J}]\,d\mathbf{W}(t)\,, \quad \forall t \in]0\,, T]\,, \tag{37}$$

with the initial condition

$$\mathbf{V}(0) = \mathbf{V}^S \quad \text{a.s}\,. \tag{38}$$

In Eq. (37), $\mathbf{W}$ is the normalized $\mathbb{R}^{n_W}$-valued Wiener stochastic process and this equation must be read as an Itô linear stochastic differential equation for the $\mathbb{R}^{n_V}$-valued stochastic process $\mathbf{V}$. The matrix $[\mathbb{S}]$ is given and is a $(n_V \times n_V)$ real matrix and $[\mathbb{J}]$ is a given $(n_V \times n_W)$ real matrix. It is assumed that matrix $[\mathbb{S}]$ is such that Eq. (37) has a second-order solution $\mathbf{V}$ (stable system). The initial condition $\mathbf{V}^S$ is chosen in order that the stochastic solution $\mathbf{V}$ of Eqs. (37) and (38) be a stationary stochastic process. Consequently, the probability distribution of random vector $\mathbf{V}^S$ is the invariant measure, that is to say, $\mathbf{V}^S$ is a Gaussian second-order centered vector with a covariance matrix corresponding to the Gaussian stationary second-order centered solution. In Eq. (36), it is assumed that $t \mapsto [\mathbb{O}(t)]$ is a continuous function from $[0\,, T]$ into the set of all the $(n \times n_V)$ real matrices. Therefore, $\mathbf{F}$ is a Gaussian nonstationary second-order centered stochastic process indexed by $[0\,, T]$ with values in $\mathbb{R}^n$ and having almost surely continuous trajectories.

Let E be the mathematical expectation and let $[R_V(\tau)] = E\{\mathbf{V}(t+\tau)\,\mathbf{V}(t)^T\}$ be the matrix-valued autocorrelation function of the stationary stochastic process $\mathbf{V}$ for which the matrix-valued spectral measure admits a matrix-valued spectral density function $[S_V(\omega)]$ with respect to $d\omega$ which is such that

$$[R_V(\tau)] = \int_{\mathbb{R}} e^{i\omega t}\,[S_V(\omega)]\,d\omega\,. \tag{39}$$

It can easily be proven that

$$[S_V(\omega)] = \frac{1}{2\pi}[\widehat{h}_V(\omega)]\,[\widehat{h}_V(\omega)]^*\,, \tag{40}$$

in which $*$ means the conjugate and the transpose operations and where $[\widehat{h}_V(\omega)]$ is the $(n_V \times n_V)$ complex matrix such that

$$[\widehat{h}_V(\omega)] = (i\omega\,[I_{n_V}] - [\mathbb{S}])^{-1}\,[\mathbb{J}]\,. \tag{41}$$

The matrix-valued autocorrelation function $[R_F(t,t')] = E\{\mathbf{F}(t)\,\mathbf{F}(t')^T\}$ of the nonstationary stochastic process $\mathbf{F}$ is such that

$$[R_F(t,t')] = [\mathbb{O}(t)]\,[R_V(t-t')]\,[\mathbb{O}(t')]^T\,, \tag{42}$$

and can be written as

$$[R_F(t,t')] = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{i\omega t - i\omega' t'}\,[S_F(\omega,\omega')]\,d\omega\,d\omega'\,. \tag{43}$$

It can easily be proven that the matrix-valued spectral density function $[S_F(\omega,\omega')]$ is written as

$$[S_F(\omega,\omega')] = \frac{1}{(2\pi)^2}\int_{\mathbb{R}} [\widehat{\mathbb{O}}(\omega-\Omega)]\,[S_V(\Omega)]\,[\widehat{\mathbb{O}}(\omega'-\Omega)]^*\,d\Omega\,, \tag{44}$$

in which $[\widehat{\mathbb{O}}(\omega)]$ is the Fourier transform of $[\mathbb{O}(t)]$ such that

$$[\widehat{\mathbb{O}}(\omega)] = \int_{\mathbb{R}} e^{-i\omega t}\,[\mathbb{O}(t)]\,dt\,. \tag{45}$$

7. Computational model in the time domain with nonstationary stochastic transient excitation

In Eq. (27), the deterministic excitation represented by function $\mathbf{f}$ is replaced by the stochastic process $\mathbf{F}$ defined in Section 6. Therefore, the deterministic response $\mathbf{u}$ becomes a $\mathbb{R}^n$-valued stochastic process $\mathbf{U}_c$ indexed by $[0\,,T]$ and defined on probability space $(\Theta',\mathcal{T}',\mathcal{P}')$. Taking account Eqs. (27) and (28), the computational model in the time domain with nonstationary stochastic transient excitation is written as

$$[\mathbb{M}]\,\ddot{\mathbf{U}}_c(t) + [\mathbb{K}_0]\,\mathbf{U}_c(t) + \int_0^t [\mathbb{N}(\tau)]\,\mathbf{U}_c(t-\tau)\,d\tau = \mathbf{F}(t)\,, \quad t\in]0\,,T]\,, \tag{46}$$

$$\mathbf{U}_c(0) = \dot{\mathbf{U}}_c(0) = 0 \quad \text{a.s}\,, \tag{47}$$

in which the Gaussian nonstationary second-order centered stochastic process $\mathbf{F}$ is defined by Eqs. (36) to (38). It can be proven that $\{(\mathbf{U}_c(t),\dot{\mathbf{U}}_c(t)), t\in[0\,,T]\}$ is a Gaussian nonstationary second-order centered $\mathbb{R}^n\times\mathbb{R}^n$-valued stochastic process. Since stochastic process $\mathbf{F}$ has almost surely continuous trajectories, it can be deduced that stochastic process $\dot{\mathbf{U}}_c$ has also almost surely continuous trajectories.

8. Vector basis for constructing the reduced-order model

This section deals with the construction of an adapted vector basis $\{\boldsymbol{\varphi}^1, \ldots, \boldsymbol{\varphi}^n\}$ of $\mathbb{R}^n$ in order to construct a reduced-order model. We then introduce the subspace V^m of $\mathbb{R}^n$, spanned by $\{\boldsymbol{\varphi}^1, \ldots, \boldsymbol{\varphi}^m\}$ with $m \ll n$. The reduced-order model is obtained in projecting Eq. (46) on V^m (see Section 9). There are several approaches to construct such a vector basis. The most common approach consists in using the elastic modes of an associated elastic conservative system. In the context of the viscoelastic material, such elastic modes are not clearly defined. This difficulty can be viewed in rewriting Eq. (27) in the frequency domain, that yields

$$(-\omega^2 [\mathbb{M}] + i\omega [\mathbb{D}(\omega)] + [\mathbb{K}(\omega)])\, \widehat{\mathbf{u}}(\omega) = \widehat{\mathbf{f}}(\omega)\,, \tag{48}$$

in which $[\mathbb{D}(\omega)]$ and $[\mathbb{K}(\omega)]$ are defined by Eq. (32). A non usual eigenvalue problem related to the conservative part of homogeneous Eq. (48) would be to find ω and the assoicated vertor $\boldsymbol{\varphi}$ such that $[\mathbb{K}(\omega)]\, \boldsymbol{\varphi} = \omega^2 [\mathbb{M}]\, \boldsymbol{\varphi}$. This problem would become a usual generalized eigenvalue problem if $[\mathbb{K}(\omega)]$ was independent of ω or if $[\mathbb{K}(\omega)]$ was a polynomial in ω with matrix coefficients (that is not the case in the general framework of viscoelasticity). A way would be to construct the vector basis with the Proper Orthogonal Decomposition (POD) method related to stochastic Eqs. (46) and (47) in which the stochastic process $\mathbf{F}$ is defined in Section 6 for the Gaussian case, but which could be defined in another way for a non-Gaussian case. Concerning the POD method, many works have been published (see for instance [33]). Nevertheless, if the POD reduction seems very efficient for the nonlinear static case or for the nonlinear dynamic case for which only the first resonances are dominant in the nonlinear dynamical response, the efficiency does not seem so clear in presence of a large number of resonances (see for instance [61]). Concerning the use of the POD method for linear viscoelastic dynamical system formulated in the frequency domain, only a few works have been published (see for instance [9] in which only the first resonances are dominant in the frequency responses). In the present work, we are interested in the time-domain formulation of linear viscoelastic system for which the response to transient excitation is studied (it should be noted that the extension of the linear equations which are presented to the cases of dynamical systems with local nonlinearities, such as systems with stops, is straightforward; such nonlinear dynamical systems with linear viscoelastic constitutive equation requires the use of a time formulation). In addition, the POD basis strongly depends on the excitation. If the excitation is modified, in general, the POD basis has to be recomputed to preserve a good speed of convergence for the reduced-order model. Finally,

the use of a POD basis for the time formulation proposed in this work would require to solve the stochastic computational model defined by Eq. (46). In practice, n can be very large (several millions) and therefore, the numerical effort would be very important to construct a vector basis for which the potential gain is not clearly identified. Nevertheless, the use of a POD basis could straightforwardly be used in the methodology proposed in this work which is not devoted to the POD method.

Case 1. For the *low- and medium-frequency ranges*, we then propose to use the m first elastic modes of the elastic system corresponding to the viscoelastic system for zero frequency (static case), solving the following generalized eigenvalue problem

$$[\mathbb{K}(0)]\,\boldsymbol{\varphi} = \lambda\,[\mathbb{M}]\,\boldsymbol{\varphi}\,, \tag{49}$$

in which $[\mathbb{K}(0)]$, defined by Eq. (34), is the positive-definite stiffness matrix constructed with the equilibrium tensor C^{∞} defined by Eq. (10), and where $[\mathbb{M}]$ is the positive-definite mass matrix.

Case 2. For the *high-frequency range*, we propose to use the m first elastic modes of the elastic system corresponding to the viscoelastic system for frequency $\omega = +\infty$, solving the following generalized eigenvalue problem

$$[\mathbb{K}(+\infty)]\,\boldsymbol{\varphi} = \lambda\,[\mathbb{M}]\,\boldsymbol{\varphi}\,, \tag{50}$$

in which $[\mathbb{K}(+\infty)] = [\mathbb{K}_0]$ is the positive-definite stiffness matrix constructed with the initial elasticity tensor $C(0)$ defined in Section 3.

For these two cases, we will denote the m first eigenvalues by $0 < \lambda_1 \leq \ldots \leq \lambda_m$ and the associated eigenvectors by $\{\boldsymbol{\varphi}^1, \ldots, \boldsymbol{\varphi}^m\}$. We then have [51]

$$< [\mathbb{M}]\,\boldsymbol{\varphi}^{\alpha}\,, \boldsymbol{\varphi}^{\beta} >= \mu_{\alpha}\delta_{\alpha\beta}\,, \quad < [\mathbb{K}']\,\boldsymbol{\varphi}^{\alpha}\,, \boldsymbol{\varphi}^{\beta} >= \lambda_{\alpha}\delta_{\alpha\beta}\,, \tag{51}$$

in which $[\mathbb{K}']$ is $[\mathbb{K}(0)]$ or $[\mathbb{K}(+\infty)]$, with $\delta_{\alpha\beta}$ the Kronecker symbol and where $\mu_1, \ldots, \mu_m$ are the positive generalized masses defining the normalization of the eigenvectors with respect to the mass matrix.

9. Reduced-order model in the time domain with stochastic excitation

As explained in Section 8, Eq. (46) is projected on V^m which is the subspace of $\mathbb{R}^n$, spanned by the m vectors $\{\boldsymbol{\varphi}^1, \ldots, \boldsymbol{\varphi}^m\}$. In order to simplify the presentation, we will consider only Case 1 of Section 8 (the developments for case 2 is

straightforward). Consequently, $[\mathbb{K}(0)]\,\boldsymbol{\varphi} = \lambda\,[\mathbb{M}]\,\boldsymbol{\varphi}$ and Eq. (51) is rewritten as $<[\mathbb{K}(0)]\,\boldsymbol{\varphi}^\alpha\,,\boldsymbol{\varphi}^\beta>=\lambda_\alpha\delta_{\alpha\beta}$ and corresponds to Eq. (49). The $(n\times m)$ real matrix $[\boldsymbol{\varphi}^1\ldots\boldsymbol{\varphi}^m]$ will be noted $[\Phi]$. We then have

$$[\Phi]^T\,[\mathbb{M}]\,[\Phi] = [M]\,,\quad [\Phi]^T\,[\mathbb{K}(0)]\,[\Phi] = [K]\,, \tag{52}$$

in which $[M]$ is the positive-definite diagonal $(m\times m)$ real matrix such that $[M]_{\alpha\alpha}=\mu_\alpha$ and where $[K]$ is the positive-definite diagonal $(m\times m)$ real matrix such that $[K]_{\alpha\alpha}=\lambda_\alpha$. Therefore, the $\mathbb{R}^n$-valued stochastic process indexed by $[0\,,T]$ is given by the following reduced-order model with stochastic excitation,

$$\mathbf{U}_c(t) = [\Phi]\,\mathbf{Q}_c(t)\,,\quad \dot{\mathbf{U}}_c(t) = [\Phi]\,\dot{\mathbf{Q}}_c(t)\,,\quad t\in[0\,,T]\,, \tag{53}$$

$$[M]\,\ddot{\mathbf{Q}}_c(t)+[K_0]\,\mathbf{Q}_c(t)+\int_0^t[N(\tau)]\,\mathbf{Q}_c(t-\tau)\,d\tau = [\Phi]^T\,\mathbf{F}(t)\,,\quad t\in]0\,,T]\,, \tag{54}$$

$$\mathbf{Q}_c(0)=\dot{\mathbf{Q}}_c(0)=0\quad \text{a.s}\,, \tag{55}$$

in which

$$[K_0] = [\Phi]^T\,[\mathbb{K}_0]\,[\Phi]\,,\quad [N(\tau)] = [\Phi]^T\,[\mathbb{N}(\tau)]\,[\Phi]\,. \tag{56}$$

The matrix $[K_0]$ is a full positive-definite symmetric $(m\times m)$ real matrix which can be written as $[K_0]=[K]-[N^\infty]$ in which $[N^\infty]=[\Phi]^T\,[\mathbb{N}^\infty]\,[\Phi]$ is a full negative-definite symmetric $(m\times m)$ real matrix. The symmetric $(m\times m)$ real matrix $[N(\tau)]$ is written as $[N(\tau)]=[\Phi]^T\,[\mathbb{N}(\tau)]\,[\Phi]$. The function $\tau\mapsto[N(\tau)]$ is such that $[N(\tau)]=[0]$ for $\tau<0$ (causal function) and is integrable on $\mathbb{R}^+$. It can be deduced that $\{(\mathbf{Q}_c(t),\dot{\mathbf{Q}}_c(t)),t\in[0\,,T]\}$ is a Gaussian nonstationary second-order centered $\mathbb{R}^m\times\mathbb{R}^m$-valued stochastic process. Since stochastic process $\mathbf{F}$ has almost surely continuous trajectories, it can be deduced that stochastic process $\ddot{\mathbf{Q}}_c$ has also almost surely continuous trajectories.

From Eqs. (30) to (32), it can be deduced that the Fourier transform

$$[\widehat{N}(\omega)] = \int_0^{+\infty} e^{-i\omega t}[N(\tau)]\,d\tau \tag{57}$$

of $[N]$ is written as

$$[\widehat{N}(\omega)] = [\Phi]^T\,[\widehat{\mathbb{N}}(\omega)]\,[\Phi] = [\widehat{N}^R(\omega)] + i\,[\widehat{N}^I(\omega)]\,, \tag{58}$$

in which $[\widehat{N}^R(\omega)]$ and $[\widehat{N}^I(\omega)]$ are the real part and the imaginary part of $[\widehat{N}(\omega)]$. The $(m\times m)$ real matrices

$$[D(\omega)] = [\Phi]^T\,[\mathbb{D}(\omega)]\,[\Phi]\,,\quad [K(\omega)] = [\Phi]^T\,[\mathbb{K}(\omega)]\,[\Phi] \tag{59}$$

are then written as

$$[K(\omega)] = [K_0] + [\widehat{N}^R(\omega)]\,, \quad \omega\,[D(\omega)] = [\widehat{N}^I(\omega)]\,. \tag{60}$$

For all fixed ω, $[D(\omega)]$ and $[K(\omega)]$ are then positive-definite symmetric $(m \times m)$ real matrices and, taking into account Eqs. (59) and (33), are such that

$$[K(-\omega)] = [K(\omega)]\,, \quad [D(-\omega)] = [D(\omega)]\,. \tag{61}$$

Since $\tau \mapsto [N(\tau)]$ is a causal and an integrable function on $\mathbb{R}^+$, for all real ω, the real part $[\widehat{N}^R(\omega)]$ and the imaginary part $[\widehat{N}^I(\omega)]$ of the Fourier transform $[\widehat{N}(\omega)]$ must satisfy the following integral relations involving the Hilbert transform (see [30, 52]),

$$[\widehat{N}^R(\omega)] = \frac{1}{\pi}\,\text{p.v}\int_{\mathbb{R}} \frac{1}{\omega - \omega'}\,[\widehat{N}^I(\omega')]\,d\omega'\,, \tag{62}$$

$$[\widehat{N}^I(\omega)] = -\frac{1}{\pi}\,\text{p.v}\int_{\mathbb{R}} \frac{1}{\omega - \omega'}\,[\widehat{N}^R(\omega')]\,d\omega'\,, \tag{63}$$

in which p.v denotes the Cauchy principal value defined, for a singulatity in $y = 0$, as

$$\text{p.v}\int_{\mathbb{R}} h(y)\,dy = \lim_{\chi\to+\infty,\eta\to 0^+} \{\int_{-\chi}^{-\eta} h(y)\,dy + \int_{\eta}^{\chi} h(y)\,dy\}\,. \tag{64}$$

The projection of Eqs. (34) and (35) yields

$$[K(0)] = [K_0] + [N^\infty]\,, \quad [N^\infty] = [\widehat{N}^R(0)]\,, \tag{65}$$

$$[K_0] = \lim_{|\omega|\to+\infty} [K(\omega)]\,, \tag{66}$$

in which $[K_0]$ and $[K(0)]$ are positive-definite symmetric real matrices and where $[N^\infty] = [K(0)] - [K_0]$ is a negative-definite symmetric real matrix. Therefore, as soon as the function $[K(\omega)]$ is known, $[K_0]$ can be calculated with Eq. (66) and then, $[N^\infty] = [K(0)] - [K_0]$ can be deduced from Eq. (65). From Eqs. (60) and (62), it can be deduced that, for all ω in $\mathbb{R}$,

$$[K(\omega)] = [K_0] + \frac{1}{\pi}\,\text{p.v}\int_{\mathbb{R}} \frac{\omega'}{\omega - \omega'}\,[D(\omega')]\,d\omega' \quad \in\ \mathcal{M}_m^+(\mathbb{R})\,. \tag{67}$$

in which $\mathcal{M}_m^+(\mathbb{R})$ is the set of all the positive-definite symmetric $(m \times m)$ real matrices. In particular, for $\omega = 0$, and since $[D(-\omega')] = [D(\omega')]$, we obtain

$$[K(0)] = [K_0] - \frac{2}{\pi}\int_0^{+\infty} [D(\omega)]\,d\omega \quad \in\ \mathcal{M}_m^+(\mathbb{R})\,. \tag{68}$$

Since $[D(\omega)]$ is a positive-definite symmetric real matrix for all ω, Eq. (68) shows that

$$[N^{\infty}] = [\widehat{N}^{R}(0)] = -\frac{2}{\pi} \int_0^{+\infty} [D(\omega')]\, d\omega' = [K(0)] - [K_0] \qquad (69)$$

is effectively a negative-definite symmetric real random matrix. It should be noted that $\omega' \mapsto [D(\omega')]$ is an integrable function on $\mathbb{R}^+$ because $\omega' \mapsto \omega' [D(\omega')] = [\widehat{N}^{I}(\omega')]$ is, as the imaginary part of the Fourier transform of an integrable function, a continuous function such that $[\widehat{N}^{I}(\omega')]$ goes to zero as ω' goes to $+\infty$.

Comments about the gain obtained by the use of a reduced-order model. In computational structural dynamics and for a large computational model of a complex structure, if elastic modes are used as basis functions, then the ratio of the number of degrees-of-freedom divided by the number of basis vectors is currently larger than $1,000$ for a dynamic analysis in the low- and medium-frequency ranges (see for instance [22]) and can be $10,000$ for a dynamic analysis in the low-frequency range.

10. Stochastic modeling of both computational model-parameters uncertainties and modeling errors using the nonparametric probabilistic approach and the random matrix theory

This Section is devoted to the construction of the stochastic model of both computational model-parameters uncertainties and modeling errors using the nonparametric probabilistic approach and the random matrix theory. The bases of this approach can be found in [70, 71, 72], some theoretical extensions in [50, 75] and, experimental validations and uncertainty quantification in [6, 10, 12, 21, 22, 25, 74]. This approach (which allows both the computational model-parameters uncertainties and the model uncertainties induced by modeling errors to be taken into account in computational model) consists in replacing the operators of the reduced-order computational model by random operators. Applying this methodology to Eq. (54), the matrices $[M]$, $[K_0]$ and $[N(\tau)]$ of the family of matrices $\{[N(\tau)], \tau \geq 0\}$ are then modeled by random matrices $[\mathbf{M}]$, $[\mathbf{K}_0]$ and $[\mathbf{N}(\tau)]$. In the framework of the nonparametric probabilistic approach of uncertainties, the probability distributions and the generators of independent realizations of such random matrices are constructed using the random matrix theory [48] and the Maximum Entropy principle [36, 73] from the Information Theory [67]. The Maximum Entropy principle consists in maximizing the entropy under the constraints defined by the available information. Consequently, it is important to

define the algebraic properties of the random matrices for which the probability distributions have to be constructed. In the present case, each random matrix $[\mathbf{M}]$ or $[\mathbf{K}_0]$ has a given mean value such that $E\{[\mathbf{M}]\} = [M]$ or $E\{[\mathbf{K}_0]\} = [K_0]$, is with values in the set of all the positive-definite symmetric real matrices. Concerning the family of random matrices $\{[\mathbf{N}(\tau)], \tau \geq 0\}$, each random matrix $[\mathbf{N}(\tau)]$ has a given mean value such that $E\{[\mathbf{N}(\tau)]\} = [N(\tau)]$, is with values in the set of all the symmetric real matrices and the family is made up of statistically dependent random matrices. In addition, we will see that random matrix $[\mathbf{K}_0]$ will be statistically dependent of the family $\{[\mathbf{N}(\tau)], \tau \geq 0\}$ of random matrices. In this framework of the linear viscoelasticity, the construction cannot directly be made with the basic known theory. The difficulty looks like to the one analyzed in [14] for which a strict extension of the basic theory has been performed. For the linear viscoelastic theory, we then propose a new extension based on the use of the Hilbert transform [52] in the frequency domain (also called the Kramers-Kroning relations [38, 40]) and on the transformation of the stochastic modeling carried out in the frequency domain to the time domain using an inverse Fourier transform. It should be noted that the additional constraints related to the positivity of certain operators must be taken into account as available information in order to assure that the stochastic integro-differential equation with random operators and with stochastic excitation has a unique second-order stochastic solution. In a first Subsection, we will recall some main useful results concerning the nonparametric probabilistic approach of uncertainties. In a second Subsection, we will present the stochastic modeling of uncertainties in the context of linear viscoelasticity.

10.1. Basic ensembles of random matrices for the nonparametric probabilistic approach of uncertainties

The Gaussian orthogonal ensemble (GOE) of random matrices [48] cannot be used when positiveness property and integrability of the inverse are required. Consequently, we need new ensembles of random matrices which will be used to develop the nonparametric probabilistic approach of uncertainties in computational solid mechanics, and which differ from the GOE and from the other known ensembles of the random matrix theory. Below, we summarize the construction [70, 71] of the ensemble SG_0^+ of random matrices $[\mathbf{G}_0]$ defined on the probability space $(\Theta, \mathcal{T}, \mathcal{P})$, with values in $\mathcal{M}_m^+(\mathbb{R})$ and such that

$$E\{[\mathbf{G}_0]\} = [I_m] \quad , \quad E\{\log(\det[\mathbf{G}_0])\} = \chi\,, \quad |\chi| < +\infty\,, \tag{70}$$

with $[I_m]$ the $(m \times m)$ identity matrix, $\log$ the Neperian logarithm and $\det$ the determinant. The probability distribution $P_{[\mathbf{G}_0]} = p_{[\mathbf{G}_0]}([G])\ \widetilde{dG}$ is defined by a

probability density function $[G\,] \mapsto p_{[\mathbf{G}_0]}([G\,])$ from $\mathcal{M}_m^+(\mathbb{R})$ into $\mathbb{R}^+$ with respect to the volume element $\widetilde{d}G$ on the set $\mathcal{M}_m^S(\mathbb{R})$ of all the symmetric $(m \times m)$ real matrices, which is such that $\widetilde{d}G = 2^{m(m-1)/4} \, \Pi_{1 \leq j \leq k \leq m} \, dG_{jk}$. This probability density function can then verify the normalization condition,

$$\int_{\mathcal{M}_m^+(\mathbb{R})} p_{[\mathbf{G}_0]}([G\,]) \; \widetilde{d}G = 1 \, . \tag{71}$$

Let δ be the positive real number defined by

$$\delta = \left\{ \frac{E\{\| \, [\mathbf{G}_0] - E\{[\mathbf{G}_0]\} \, \|_F^2\}}{\| \, E\{[\mathbf{G}_0]\} \, \|_F^2} \right\}^{1/2} = \left\{ \frac{1}{m} E\{\| \, [\mathbf{G}_0] - [I_m] \, \|_F^2\} \right\}^{1/2} , \tag{72}$$

which will allow the dispersion of the probability model of random matrix $[\mathbf{G}_0]$ to be controlled and where $\| \, \cdot \, \|_F$ is the Frobenius matrix norm. For δ such that $0 < \delta < (m+1)^{1/2}(m+5)^{-1/2}$, the use of the maximum entropy principle under the constraints defined by the above available information yields the following algebraic expression of the probability density function of random matrix $[\mathbf{G}_0]$,

$$p_{[\mathbf{G}_0]}([G\,]) = \mathbb{1}_{\mathcal{M}_m^+(\mathbb{R})}([G\,]) \times C_{\mathbf{G}_0} \times \big(\det [G\,]\big)^{(m+1)\frac{(1-\delta^2)}{2\delta^2}} \times e^{-\frac{(m+1)}{2\delta^2} \, \mathrm{tr}[G\,]} , \tag{73}$$

in which $\mathrm{tr}\,[G\,]$ is the trace of matrix $[G\,]$ and with $C_{\mathbf{G}_0}$ a positive constant such that

$$C_{\mathbf{G}_0} = (2\pi)^{-m(m-1)/4} \left(\frac{m+1}{2\delta^2} \right)^{m(m+1)(2\delta^2)^{-1}} \left\{ \Pi_{j=1}^m \Gamma\big(\frac{m+1}{2\delta^2} + \frac{1-j}{2}\big) \right\}^{-1} , \tag{74}$$

and where, for all $z > 0$, $\Gamma(z) = \int_0^{+\infty} t^{z-1} \, e^{-t} \, dt$. Note that $\{[\mathbf{G}_0]_{jk}, 1 \leq j \leq k \leq m\}$ are dependent random variables. If $(m+1)/\delta^2$ is an integer, then this probability density function coincides with the Wishart probability distribution [1, 26]. If $(n+1)/\delta^2$ is not an integer, then this probability density function can be viewed as a particular case of the Wishart distribution, in infinite dimension, for stochastic processes [68]. It can be proven that $E\{\|[\mathbf{G}_0]^{-1}\|_F^2\} < +\infty$. The generator of independent realizations (which is required to solve the random equations with the Monte Carlo method) is constructed using the following algebraic representation. Random matrix $[\mathbf{G}_0]$ is written (Cholesky decomposition) as $[\mathbf{G}_0] = [\mathbf{L}]^T \, [\mathbf{L}]$ in which $[\mathbf{L}]$ is an upper triangular $(m \times m)$ random matrix such that:

(1) random variables $\{[\mathbf{L}]_{jj'}, j \leq j'\}$ are independent;

(2) for $j < j'$, the real-valued random variable $[\mathbf{L}]_{jj'}$ is written as $[\mathbf{L}]_{jj'} = \sigma_m U_{jj'}$ in which $\sigma_m = \delta(m+1)^{-1/2}$ and where $U_{jj'}$ is a real-valued Gaussian random variable with zero mean and variance equal to 1;

(3) for $j = j'$, the positive-valued random variable $[\mathbf{L}]_{jj}$ is written as $[\mathbf{L}]_{jj} = \sigma_m \sqrt{2V_j}$ in which V_j is a positive-valued gamma random variable whose probability density function is $p_{V_j}(v) = \mathbb{1}_{\mathbb{R}^+}(v) \frac{1}{\Gamma(a_j)}\ v^{a_j - 1}\ e^{-v}$, in which $a_j = \frac{m+1}{2\delta^2} + \frac{1-j}{2}$. It should be noted that the probability density function of each diagonal element $[\mathbf{L}]_{jj}$ of the random matrix $[\mathbf{L}]$ depends on the rank j of the element.

Let $0 \leq \varepsilon \ll 1$ be a positive number as small as one wants. The ensemble $\mathrm{SG}_\varepsilon^+$ is defined as the ensemble of all the random matrices such that

$$[\mathbf{G}] = \frac{1}{1+\varepsilon}\{[\mathbf{G}_0] + \varepsilon\,[I_m]\}\,, \tag{75}$$

in which $[\mathbf{G}_0]$ is a random matrix which belongs to ensemble SG_0^+. Let $[\mathbf{G}]$ be in $\mathrm{SG}_\varepsilon^+$ with $\varepsilon \geq 0$ fixed as small as one wants (possibly, ε can be equal to zero and in such a case, $\mathrm{SG}_\varepsilon^+ = \mathrm{SG}_0^+$ and then, $[\mathbf{G}] = [\mathbf{G}_0]$). It can easily be seen that $E\{[\mathbf{G}]\} = [I_m]$, for all second-order random vector $\mathbf{X}$ with values in $\mathbb{R}^m$, $E\{\mathbf{X}^T\,[\mathbf{G}]\,\mathbf{X}\} \geq c_\varepsilon E\{\|\mathbf{X}\|^2\}$, in which $c_\varepsilon = \varepsilon/(1+\varepsilon)$ and, for all $\varepsilon \geq 0$, $E\big\{\|[\mathbf{G}]^{-1}\|_F^2\big\} < +\infty$.

10.2. Stochastic modeling of random matrix $[\boldsymbol{M}]$

The random matrix $[\mathbf{M}]$ is defined on probability space $(\Theta, \mathcal{T}, \mathcal{P})$. There is no available information concerning the statistical dependence of random matrix $[\mathbf{M}]$ with random matrices $[\mathbf{K}_0]$ and $\{[\mathbf{N}(\tau)], \tau \geq 0\}$. Therefore, the Maximum Entropy principle shows that $[\mathbf{M}]$ is independent of $[\mathbf{K}_0]$ and $\{[\mathbf{N}(\tau)], \tau \geq 0\}$(see [70]). The deterministic matrix $[M]$, introduced in Section 9, is positive definite and consequently, can be written as $[M] = [L_M]^T\,[L_M]$ in which $[L_M]$ is an upper triangular $(m \times m)$ real matrix. Using the nonparametric probabilistic approach of uncertainties, the stochastic model of the positive-definite symmetric random matrix $[\mathbf{M}]$ is then defined by

$$[\mathbf{M}] = [L_M]^T\,[\mathbf{G}_M]\,[L_M]\,, \tag{76}$$

where $[\mathbf{G}_M]$ is a random matrix belonging to ensemble $\mathrm{SG}_\varepsilon^+$ defined in Section 10.1 and whose probability distribution and generator of independent realizations depend only on dimension m and on the dispersion parameter δ_M.

10.3. Stochastic modeling of the family of random matrices $\{[\boldsymbol{N}(\tau)], \tau \geq 0\}$

The family of random matrices $\{[\mathbf{N}(\tau)], \tau \geq 0\}$ is defined on probability space $(\Theta, \mathcal{T}, \mathcal{P})$. The nonparametric probabilistic approach of uncertainties then consists in modeling the positive-definite symmetric $(m \times m)$ real matrices $[D(\omega)]$ and $[K(\omega)]$ defined by Eq. (60) by random matrices $[\mathbf{D}(\omega)]$ and $[\mathbf{K}(\omega)]$ defined on probability space $(\Theta, \mathcal{T}, \mathcal{P})$, with values in $\mathcal{M}_m^+(\mathbb{R})$ and such that,

$$E\{[\mathbf{D}(\omega)]\} = [D(\omega)] \ , \quad E\{[\mathbf{K}(\omega)]\} = [K(\omega)] \, , \tag{77}$$

$$[\mathbf{D}(-\omega)] = [\mathbf{D}(\omega)] \ , \quad [\mathbf{K}(-\omega)] = [\mathbf{K}(\omega)] \, . \tag{78}$$

The methodology proposed for such a construction is the following. In the first step, the stochastic model of $[\mathbf{D}(\omega)]$ is constructed in the context of the nonparametric probabilistic approach. In the second step, we deduce the stochastic model of $\{[\mathbf{N}(\tau)], \tau \geq 0\}$ from the stochastic model of $[\mathbf{D}(\omega)]$.

First step. For all ω, the deterministic matrix $[D(\omega)]$ introduced in Section 9 is positive definite and consequently, can be written as $[D(\omega)] = [L_D(\omega)]^T [L_D(\omega)]$ in which $[L_D(\omega)]$ is an upper triangular $(m \times m)$ real matrix. Using the nonparametric probabilistic approach of uncertainties, the stochastic model of the family $\{[\mathbf{D}(\omega)], \omega \in \mathbb{R}\}$ of positive-definite symmetric random matrices, is defined as follows. For all ω in $\mathbb{R}$, we write

$$[\mathbf{D}(\omega)] = [L_D(\omega)]^T [\mathbf{G}_D] [L_D(\omega)] \, , \tag{79}$$

where $[\mathbf{G}_D]$ is a random matrix belonging to ensemble $\mathrm{SG}_\varepsilon^+$ defined in Section 10.1 and whose probability distribution and generator of independent realizations depend only on dimension m and on the dispersion parameter δ_D. Random matrix $[\mathbf{G}_D]$ does not depend on ω and is statistically independent of $[\mathbf{G}_M]$. It should be noted that the stochastic modeling of $[\mathbf{D}(\omega)]$ could be improved in taking a frequency dependence for the positive-definite random matrix $[\mathbf{G}_D]$ whose mean function must be independent of the frequency and must be equal to the unity matrix. This stochastic modeling which has been chosen corresponds to a compromise between the stochastic complexity of the model and the available information deduced from a physical support. The stochastic model defined by Eq. (79) is carried out at the level of the damping operator of the dynamical system in order to take globally into account the model uncertainties induced by modeling errors (nonparametric probabilistic approach of model uncertainties). It is not constructed at the level of the constitutive equation (parametric probabilistic

approach of system-parameter uncertainties). In this framework, the construction proposed is chosen to minimize the stochastic dimension of the stochastic model of the damping operator that is to say, the statistical fluctuation of the normalized damping operator $[L_D(\omega)]^{-T}\,[\mathbf{D}(\omega)]\,[L_D(\omega)]^{-1}$ is taken as a normalized positive-definite random matrix $[\mathbf{G}_D]$ independent of ω. The construction of a more sophisticated stochastic modeling of the damping operator could be introduced if some information based on a physical support were available concerning the modeling errors.

Second step. Using the nonparametric probabilistic approach of uncertainties, the stochastic model of the family $\{[N(\tau)], \tau \geq 0\}$ of deterministic matrices is the family $\{[\mathbf{N}(\tau)], \tau \geq 0\}$ of random matrices defined on probability space $(\Theta, \mathcal{T}, \mathcal{P})$. The random function $\tau \mapsto [\mathbf{N}(\tau)]$ must be almost surely a causal and an integrable function on $\mathbb{R}$. This means that, for all θ in Θ, the realization $\tau \mapsto [\mathbf{N}(\tau, \theta)]$ is a deterministic causal function on $\mathbb{R}$ (that is to say $[\mathbf{N}(\tau, \theta)] = [0]$ for all $\tau < 0$) and is integrable (that is to say, for all j and k, $\int_0^{+\infty} |[\mathbf{N}(\tau, \theta)]_{jk}|\, d\tau < +\infty$. By construction of the nonparametric stochastic model, for all τ, we have $E\{[\mathbf{N}(\tau)]\} = [N(\tau)]$. Let $\{[\widehat{\mathbf{N}}(\omega)], \omega \in \mathbb{R}\}$ be the family of random $(m \times m)$ complex matrices such that $[\widehat{\mathbf{N}}(\omega)] = \int_0^{+\infty} e^{-i\omega\tau}\,[\mathbf{N}(\tau)]\, d\tau$. Therefore, $E\{[\widehat{\mathbf{N}}(\omega)]\} = [\widehat{N}(\omega)]$ in which $[\widehat{N}(\omega)]$ is defined by Eq. (57). Taking into account Eq. (60), the random imaginary part $[\widehat{\mathbf{N}}^I(\omega)]$ of the Fourier transform $[\widehat{\mathbf{N}}(\omega)]$ must be such that

$$[\widehat{\mathbf{N}}^I(\omega)] = \omega\,[\mathbf{D}(\omega)]\,. \tag{80}$$

Since $\tau \mapsto [\mathbf{N}(\tau)]$ is almost surely a causal and an integrable function on $\mathbb{R}$, for all real ω, the real part $[\widehat{\mathbf{N}}^R(\omega)]$ and the imaginary part $[\widehat{\mathbf{N}}^I(\omega)]$ of the Fourier transform $[\widehat{\mathbf{N}}(\omega)]$ must satisfy, almost surely, the following integral relations involving the Hilbert transform (see [30, 52]),

$$[\widehat{\mathbf{N}}^R(\omega)] = \frac{1}{\pi}\,\text{p.v}\int_{\mathbb{R}} \frac{1}{\omega - \omega'}\,[\widehat{\mathbf{N}}^I(\omega')]\, d\omega' \quad \text{a.s}\,, \tag{81}$$

in which p.v is the Cauchy principal value defined by Eq. (64). Since $[\widehat{\mathbf{N}}^R(-\omega)] = [\widehat{\mathbf{N}}^R(\omega)]$, we will consider Eq. (81) for $\omega \geq 0$. From Eqs. (80) and (78), it can be deduced that,

$$\omega \geq 0\ , \quad [\widehat{\mathbf{N}}^R(\omega)] = \frac{2}{\pi}\,\text{p.v}\int_0^{+\infty} \frac{\omega'^{\,2}}{\omega^2 - \omega'^{\,2}}\,[\mathbf{D}(\omega')]\, d\omega'\,. \tag{82}$$

We now give another expression of Eq. (82) useful for computation. For $\omega = 0$, Eq. (82) yields

$$[\widehat{\mathbf{N}}^R(0)] = -\frac{2}{\pi}\int_0^{+\infty} [\mathbf{D}(\omega)]\, d\omega\,. \tag{83}$$

As explained at the end of Section 9, $\omega \mapsto [\mathbf{D}(\omega)]$ is almost surely integrable on $\mathbb{R}^+$. For $\omega > 0$, Eq. (82) can be rewritten as

$$[\widehat{\mathbf{N}}^R(\omega)] = \frac{2}{\pi}\,\text{p.v}\int_0^{+\infty} \frac{u^2}{1-u^2}\,\omega\,[\mathbf{D}(\omega u)]\,du = \frac{2}{\pi}\lim_{\eta\to 0^+}\{\int_0^{1-\eta} + \int_{1+\eta}^{+\infty}\}\,. \tag{84}$$

10.4. Stochastic modeling of random matrix $[\boldsymbol{K}_0]$

Considering Eq. (60), that is to say $[K(\omega)] = [K_0] + [\widehat{N}^R(\omega)]$, the nonparametric stochastic modeling of the positive-definite matrix $[K_0]$ leads us to introduce the positive-definite random matrix $[\mathbf{K}_0]$ defined on probability space $(\Theta, \mathcal{T}, \mathcal{P})$, such that

$$\omega \geq 0\ , \quad [\mathbf{K}(\omega)] = [\mathbf{K}_0] + [\widehat{\mathbf{N}}^R(\omega)]\,. \tag{85}$$

Eq. (85) defines a constraint for the construction of the random matrix $[\mathbf{K}_0]$. Let $[\mathbf{D}^+]$ be the positive-definite random matrix defined by

$$[\mathbf{D}^+] = -[\widehat{\mathbf{N}}^R(0)] = \frac{2}{\pi}\int_0^{+\infty} [\mathbf{D}(\omega)]\, d\omega\,. \tag{86}$$

From Eqs. (83), (85) and (86), it can be deduced that

$$[\mathbf{K}(0)] = [\mathbf{K}_0] - [\mathbf{D}^+]\,. \tag{87}$$

Since $[\widehat{\mathbf{N}}^R(\omega)]$ goes to $[0]$ as ω goes to $+\infty$, we obtain

$$\lim_{\omega\to+\infty}[\mathbf{K}(\omega)] = [\mathbf{K}_0] = [\mathbf{K}(0)] + [\mathbf{D}^+]\,. \tag{88}$$

In the context of the linear viscoelasticity, for all $\omega \geq 0$, the random matrix $[\mathbf{K}(\omega)]$ must be positive definite almost surely (a.s). Consequently, $[\mathbf{K}(0)]$ must be constructed as a positive-definite random matrix and Eq. (87) allows the random matrix $[\mathbf{K}_0]$ to be constructed as $[\mathbf{K}_0] = [\mathbf{K}(0)] + [\mathbf{D}^+]$, which shows that $[\mathbf{K}_0]$ is positive definite a.s and is statistically dependent of $[\mathbf{D}^+]$, that is to say of $[\mathbf{G}_D]$.

The mean value of the positive-definite random matrix $[\mathbf{K}(0)]$ is the positive-definite matrix which written as

$$E\{[\mathbf{K}(0)]\} = [K_0] - \frac{2}{\pi}\int_0^{+\infty} [D(\omega)]\, d\omega\,. \tag{89}$$

Taking into account Eq. (68) yields

$$E\{[\mathbf{K}(0)]\} = [K(0)] \tag{90}$$

and thus, $[K(0)]$ is a positive-definite symmetric real matrix which can be written as $[K(0)] = [L_{K(0)}]^T [L_{K(0)}]$ in which $[L_{K(0)}]$ is an upper triangular $(m \times m)$ real matrix. Using the nonparametric probabilistic approach of uncertainties, the stochastic model of the positive-definite symmetric random matrix $[\mathbf{K}(0)]$ is then defined by

$$[\mathbf{K}(0)] = [L_{K(0)}]^T [\mathbf{G}_{K(0)}] [L_{K(0)}] , \tag{91}$$

where $[\mathbf{G}_{K(0)}]$ is a random matrix belonging to ensemble $\mathrm{SG}_\varepsilon^+$ defined in Section 10.1 and whose probability distribution and generator of independent realizations depend only on dimension m and on the dispersion parameter $\delta_{K(0)}$. The random matrix $[\mathbf{K}_0]$ is then defined by

$$[\mathbf{K}_0] = [\mathbf{K}(0)] + [\mathbf{D}^+] . \tag{92}$$

10.5. Summarizing the construction procedure

We can now summarize the construction of the stochastic model of the family of random matrices $\{[\mathbf{N}(\tau)], \tau \geq 0\}$ and $[\mathbf{K}_0]$.

- For $\omega \geq 0$, constructing the family $[\mathbf{D}(\omega)]$ of random matrices such that $[\mathbf{D}(\omega)] = [L_D(\omega)]^T [\mathbf{G}_D] [L_D(\omega)]]$ in which the level of uncertainties is controlled by the dispersion parameter δ_D and where $[L_D(\omega)]$ is such that $[D(\omega)] = [L_D(\omega)]^T [L_D(\omega)]$.

- For $\omega \geq 0$, defining the family $[\widehat{\mathbf{N}}^I(\omega)]$ of random matrices such that $[\widehat{\mathbf{N}}^I(\omega)] = \omega [\mathbf{D}(\omega)]$.

- Constructing the family $\{[\widehat{\mathbf{N}}^R(\omega)], \omega \geq 0\}$ of random matrices using Eq. (82) (or equivalently, Eqs. (83) and (84)).

- Defining the family $\{[\widehat{\mathbf{N}}(\omega)], \omega \geq 0\}$ of random matrices such that $[\widehat{\mathbf{N}}(\omega)] = [\widehat{\mathbf{N}}^R(\omega)] + i [\widehat{\mathbf{N}}^I(\omega)]$.

- Computing the family $\{[\mathbf{N}(\tau)], \tau \geq 0\}$ of random matrices such that $[\mathbf{N}(\tau)] = (2\pi)^{-1} \int_{\mathbb{R}} e^{+i\omega\,t} [\widehat{\mathbf{N}}(\omega)]\, d\omega$ in which $[\widehat{\mathbf{N}}(-\omega)] = [\widehat{\mathbf{N}}^R(\omega)] - i [\widehat{\mathbf{N}}^I(\omega)]$. It should be noted that the inverse Fourier transform can be rewritten as

$$[\mathbf{N}(\tau)] = \frac{1}{\pi} \int_0^{+\infty} \{\cos(\omega\tau) [\widehat{\mathbf{N}}^R(\omega)] - \sin(\omega\tau) [\widehat{\mathbf{N}}^I(\omega)]\}\, d\omega . \tag{93}$$

- Constructing the random matrix $[\mathbf{K}(0)] = [L_{K(0)}]^T [\mathbf{G}_{K(0)}] [L_{K(0)}]$ in which the level of uncertainties is controlled by the dispersion parameter $\delta_{K(0)}$ and where $[L_{K(0)}]$ is such that $[K(0)] = [L_{K(0)}]^T [L_{K(0)}]$.
- Computing the random matrix $[\mathbf{D}^+] = -[\widehat{\mathbf{N}}^R(0)] = \frac{2}{\pi} \int_0^{+\infty} [\mathbf{D}(\omega)] \, d\omega$.
- Defining the random matrix $[\mathbf{K}_0] = [\mathbf{K}(0)] + [\mathbf{D}^+]$.
- Verifying that, for all $\omega \geq 0$, $[\mathbf{K}(\omega)] = [\mathbf{K}_0] + [\widehat{\mathbf{N}}^R(\omega)]$ is effectively positive definite a.s.

10.6. Remark concerning the positivity of random matrix $[\boldsymbol{K}(\omega)]$

As explained in Section 10.5, the last step of the probabilistic construction consists in verifying that, for all $\omega \geq 0$, $[\mathbf{K}(\omega)]$ is a positive-definite random matrix. In the theory presented in Sections 3 to 5 and in Sections 7 to 9, such a property holds for the mean value of $[\mathbf{K}(\omega)]$. This means that, for all $\omega \geq 0$, $[K(\omega)] = E\{[\mathbf{K}(\omega)]\}$ is a positive-definite matrix. Using the nonparametric probabilistic modeling presented in Sections 10.1 to 10.5, it is not self-evident that the proposed probabilistic construction yields, for all $\omega \geq 0$, a random matrix $[\mathbf{K}(\omega)]$ which is positive definite a.s. For the general case, such a property seems difficult enough to be proven under relatively simple hypotheses, because this depends on the frequency variations of the stochastic process $\{[\mathbf{D}(\omega)], \omega \geq 0\}$ which, taking into account Eq. (79), directly depends on the frequency variations of the mean value, $\{[D(\omega)], \omega \geq 0\}$. Nevertheless, in this section, we present a complete mathematical proof of such a property under a reasonable hypothesis which is often verified in the applications.

Since $\tau \mapsto [\mathbf{N}(\tau)]$ is an integrable function on $\mathbb{R}^+$ a.s, then $[\widehat{\mathbf{N}}^I(\omega)]$ goes to 0 a.s as ω goes to infinity. From Eq. (80), it can be deduced that $[\mathbf{D}(\omega)]$ is a decreasing function in ω, at least in $1/\omega$, as ω goes to infinity. So, we have the following result:

If for all $\mathbf{y}$ in $\mathbb{R}^m$, the random function $\omega \mapsto D_{\mathbf{y}}(\omega) =< [\mathbf{D}(\omega)] \, \mathbf{y} \, , \mathbf{y} >$ is decreasing a.s on $\mathbb{R}^+$, then, for all $\omega \geq 0$, $[\mathbf{K}(\omega)]$ is positive definite a.s.

The proof is the following. From Eqs. (84), (85), (86) and (88), it can be deduced that $[\mathbf{K}(\omega)] = [\mathbf{K}(0)] + [\mathbf{T}(\omega)]$ in which

$$[\mathbf{T}(\omega)] = \frac{2}{\pi} \, \text{p.v} \int_0^{+\infty} \frac{1}{1-u^2} \, \omega \, [\mathbf{D}(\omega u)] \, du \, .$$

Since $[\mathbf{K}(0)]$ is a positive-definite random matrix, it is sufficient to prove, for all $\omega \geq 0$, $[\mathbf{T}(\omega)]$ is a positive random matrix, or equivalently, to prove that, for all non zero $\mathbf{y}$ in $\mathbb{R}^m$, $T_{\mathbf{y}}(\omega) =< [\mathbf{T}(\omega)]\,\mathbf{y}\,,\mathbf{y} >$ is a positive random variable. It can be seen that $T_{\mathbf{y}}(\omega)$ can be written as

$$T_{\mathbf{y}}(\omega) = \frac{2}{\pi}\,\text{p.v}\int_0^{+\infty} \frac{1}{1-u^2}\;\omega\,D_{\mathbf{y}}(\omega u)\,du\,.$$

Since $\omega \mapsto D_{\mathbf{y}}(\omega)$ is assumed to be a decreasing function on $\mathbb{R}^+$, it can be seen that, for all $\omega \geq 0$, the following two inequalities hold: if $u \geq 1$, then $D_{\mathbf{y}}(\omega) \geq D_{\mathbf{y}}(\omega u)$, and if $0 \leq u \leq 1$, then $D_{\mathbf{y}}(\omega u) \geq D_{\mathbf{y}}(\omega)$. Using the first inequality yields

$$T_{\mathbf{y}}(\omega) \geq \frac{2}{\pi}\,\omega \lim_{\eta\to 0^+}\left\{\int_0^{1-\eta} \frac{1}{1-u^2}\;D_{\mathbf{y}}(\omega)\,du + \int_{1+\eta}^{+\infty} \frac{1}{1-u^2}\,D_{\mathbf{y}}(\omega u)\,du\right\}.$$

In an other hand, it can easily be proven that

$$\lim_{\eta\to 0^+}\int_0^{1-\eta} \frac{du}{1-u^2} = \lim_{\eta\to 0^+}\int_{1+\eta}^{+\infty} \frac{du}{u^2-1} = \lim_{\eta\to 0^+}\frac{1}{2}\,\log\frac{2}{\eta}\,,$$

which means that $\text{p.v}\int_0^{+\infty} \frac{du}{1-u^2} = 0$. It can then be deduced that

$$T_{\mathbf{y}}(\omega) \geq \frac{2}{\pi}\,\omega \lim_{\eta\to 0^+}\int_{1+\eta}^{+\infty} \frac{1}{u^2-1}\,(D_{\mathbf{y}}(\omega) - D_{\mathbf{y}}(\omega u))\,du\,.$$

Since for $u \geq 1+\eta > 1$, we have $u^2 - 1 > 0$ and using the second inequality allow us to deduce that, for all $\omega \geq 0$, $T_{\mathbf{y}}(\omega) \geq 0$ and the proof is complete.

11. Stochastic reduced-order model in the time domain with stochastic excitation and uncertainties

Taking into account the nonparametric stochastic modeling presented in Section 10, for both the computational model-parameters uncertainties and the model uncertainties induced by modeling errors, and from the reduced-order model in the time domain with stochastic excitation presented in Section 9, we deduced the following formulation for the stochastic reduced-order model in the time domain with stochastic excitation and uncertainties.

The $\mathbb{R}^n$-valued stochastic process $\mathbf{U}$ indexed by $[0\,,T]$ is such that

$$\mathbf{U}(t) = [\Phi]\,\mathbf{Q}(t)\,, \quad \dot{\mathbf{U}}(t) = [\Phi]\,\dot{\mathbf{Q}}(t)\,, \quad t \in [0\,,T]\,, \tag{94}$$

$$[\mathbf{M}]\,\ddot{\mathbf{Q}}(t) + [\mathbf{K}_0]\,\mathbf{Q}(t) + \int_0^t [\mathbf{N}(\tau)]\,\mathbf{Q}(t-\tau)\,d\tau = [\Phi]^T\,\mathbf{F}(t)\,, \quad t \in]0\,,T]\,, \tag{95}$$

$$\mathbf{Q}(0) = \dot{\mathbf{Q}}(0) = 0 \quad \text{a.s}\,, \tag{96}$$

in which the stochastic process $\mathbf{F}$ is defined by Eqs. (36) to (38). It can be deduced that $\{(\mathbf{Q}(t), \dot{\mathbf{Q}}(t)), t \in [0\,,T]\}$ is a non-Gaussian nonstationary second-order centered $\mathbb{R}^m \times \mathbb{R}^m$-valued stochastic process. Since the stochastic modeling of uncertainties is such that $(\mathbf{Q}, \dot{\mathbf{Q}})$ is a second-order stochastic process and since stochastic process $\mathbf{F}$ has almost surely continuous trajectories, it can be deduced that stochastic process $\ddot{\mathbf{Q}}$ has also almost surely continuous trajectories.

Let $\mathbb{G}$ be the random quantity defined by $\mathbb{G} = \{[\mathbf{G}_M], [\mathbf{G}_D], [\mathbf{G}_{K(0)}], \}$. We introduced the conditional stochastic solution $\{(\mathbf{Q}(t \,|\, \mathbb{G}), \dot{\mathbf{Q}}(t \,|\, \mathbb{G})), t \in [0\,,T]\}$ of Eq. (95), given $\mathbb{G}$. Taking into account Section 9, it can easily be deduced that the stochastic process $\{(\mathbf{Q}(t \,|\, \mathbb{G}), \dot{\mathbf{Q}}(t \,|\, \mathbb{G})), t \in [0\,,T]\}$ is Gaussian. In particular, if $\mathbb{G} = \mathbb{I}_m$ with $\mathbb{I}_m = \{[I_m], [I_m], [I_m]\}$, then $\{(\mathbf{Q}(t \,|\, \mathbb{I}_m), \dot{\mathbf{Q}}(t \,|\, \mathbb{I}_m)), t \in [0\,,T]\}$ $= \{(\mathbf{Q}_c(t), \dot{\mathbf{Q}}_c(t)), t \in [0\,,T]\}$, in which the stochastic process $(\mathbf{Q}_c, \dot{\mathbf{Q}}_c)$ is defined in Section 9.

The level of uncertainties is controlled by the three dispersion parameters δ_M, δ_D and $\delta_{K(0)}$.

12. Time discretization of the stochastic integro-differential equation

It is well-known (see for instance [11]) that the solution of integro-differential equations is a very difficult problem even for the deterministic case. Troubles still more arise in the linear and the nonlinear stochastic cases. But in spite of the existence of a few results concerning solvers for stochastic integro-differential equations (see for instance [34, 42, 53, 76]), it is very useful to transform such a stochastic integro-differential equation to a stochastic differential equation because there are many existing methods to study such a system (see for instance [37, 41, 47]). Once the stochastic integro-differential equation is transformed in a stochastic differential equation, we have then to introduce an integration scheme to solve it.

12.1. Transforming the stochastic integro-differential equation to a stochastic differential equation

We construct such transformation for Eq. (95) on the base of the scheme presented in [54, 55]. For ease the further developments, we introduce the change $s = t - \tau$ of integration variable τ that transforms Eq. (95) into the following equation,

$$[\mathbf{M}]\,\ddot{\mathbf{Q}}(t) + [\mathbf{K}_0]\,\mathbf{Q}(t) + \int_0^t [\mathbf{N}(t-s)]\,\mathbf{Q}(s)\,ds = [\Phi]^T\,\mathbf{F}(t)\,, \quad t \in]0\,,T]\,. \quad (97)$$

We introduce the mesh $\{t_k\}$ with $t_k = k\,h$ for $k = 0, 1, \ldots k_T$ and $h = T/k_T$, where the positive integer number k_T is selected to obtain a sufficient accuracy. For all k fixed in $\{1, \ldots, k_T\}$ and for all t fixed in the interval $[t_{k-1}, t_k]$, the function $s \mapsto [\mathbf{N}(t-s)]$ is approximated by the $(m \times m)$ real random matrix $[\underline{\mathbf{N}}_k(t)]$ on the interval $[t_{k-1}, t_k]$, which is such that

$$[\underline{\mathbf{N}}_k(t)] = \frac{1}{h}\int_{t_{k-1}}^{t_k} [\mathbf{N}(t-s)]\,ds\,. \quad (98)$$

The following approximation of the right-hand side of Eq. (98) is introduced using the Simpson rule,

$$\begin{aligned}[\underline{\mathbf{N}}_k(t)] &= \frac{1}{h}\,\frac{\Delta s}{3}\sum_{r=0}^{2L} c_r [\mathbf{N}(t-s_{kr})] \\ &= \frac{1}{6L}\sum_{r=0}^{2L} c_r [\mathbf{N}(t-s_{kr})]\,, \end{aligned} \quad (99)$$

where the error of approximation is in $\mathcal{O}\{(\Delta s)^4\}$. The positive integer number L is chosen sufficiently large to get a good accuracy. The time step Δs is defined by $\Delta s = h/(2L)$. The nodes s_{kr} are such that $s_{kr} = t_{k-1} + r\Delta s$ with $r = 0, 1, \ldots, 2L$. Finally, the constant c_r is defined by $c_r = 1$ if $r = 0$ or $r = 2L$, $c_r = 4$ if r is odd and $c_r = 2$ otherwise. Therefore, function $s \mapsto [\mathbf{N}(t-s)]$ is approximated by a piecewise constant random matrix-valued function. As the result, Eq. (97) is written, for $t_\nu < t \leq t_{\nu+1}$ with $\nu = 0, 1, \ldots, k_T - 1$, as

$$\begin{aligned}[\mathbf{M}]\,\ddot{\mathbf{Q}}(t) + [\mathbf{K}_0]\,\mathbf{Q}(t) + \sum_{k=1}^{\nu} [\underline{\mathbf{N}}_k(t)] \int_{t_{k-1}}^{t_k} \mathbf{Q}(s)\,ds \\ + [\underline{\mathbf{N}}_{\nu+1}(t)] \int_{t_\nu}^{t} \mathbf{Q}(s)\,ds = [\Phi]^T\,\mathbf{F}(t)\,. \end{aligned} \quad (100)$$

We introduce the family of stochastic processes $\{\mathbf{Z}_k\}_k$ indexed in t, such that

$$\mathbf{Z}_k(t) = \int_{t_{k-1}}^{t} \mathbf{Q}(s)\, ds\,, \quad t_{k-1} < t \leq t_k\,. \tag{101}$$

Taking into account Eq. (101), Eqs. (100) and (101) can be rewritten, for $t_\nu < t \leq t_{\nu+1}$, as

$$[\mathbf{M}]\,\ddot{\mathbf{Q}}(t) + [\mathbf{K}_0]\,\mathbf{Q}(t) + \sum_{k=1}^{\nu} [\underline{\mathbf{N}}_k(t)]\,\mathbf{Z}_k(t_k) + [\underline{\mathbf{N}}_{\nu+1}(t)]\,\mathbf{Z}_{\nu+1}(t) = [\Phi]^T\,\mathbf{F}(t)\,, \tag{102}$$

$$\dot{\mathbf{Z}}_{\nu+1}(t) = \mathbf{Q}(t)\,, \qquad \mathbf{Z}_{\nu+1}(t_\nu) = 0\,. \tag{103}$$

Introducing the family of stochastic processes $\{\mathbf{Q}_k\}_k$ indexed by t such that $\mathbf{Q}_k(t) = \mathbf{Q}(t)$ for $t_{k-1} < t \leq t_k$, Eqs. (102) and (103) can be rewritten, for $t_\nu < t \leq t_{\nu+1}$, as

$$[\mathbf{M}]\ddot{\mathbf{Q}}_{\nu+1}(t)+[\mathbf{K}_0]\mathbf{Q}_{\nu+1}(t)+\sum_{k=1}^{\nu}[\underline{\mathbf{N}}_k(t)]\mathbf{Z}_k(t_k)+[\underline{\mathbf{N}}_{\nu+1}(t)]\mathbf{Z}_{\nu+1}(t)=[\Phi]^T\mathbf{F}(t), \tag{104}$$

$$\dot{\mathbf{Z}}_{\nu+1}(t) = \mathbf{Q}_{\nu+1}(t)\,, \tag{105}$$

$$\mathbf{Q}_{\nu+1}(t_\nu) = \mathbf{Q}_\nu(t_\nu)\,, \quad \mathbf{Z}_{\nu+1}(t_\nu) = 0\,. \tag{106}$$

The linear stochastic differential equations defined by Eqs. (104) and (105), with initial conditions defined by Eq. (106), represent the time approximation of the integro-differential equation defined by Eq. (97), for $t_\nu < t \leq t_{\nu+1}$. It should be noted that the time-approximation scheme which is proposed does not require any additional features for the stochastic process $\mathbf{F}$ under consideration. Moreover, at this point, the nonstationary second-order centered vector-valued stochastic process $\mathbf{F}$ is not obligatorily a Gaussian one. As explained in Section 11, stochastic process $\ddot{\mathbf{Q}}$ has almost surely continuous trajectories.

12.2. Time discretization of the stochastic integro-differential equation

We have now to solve Eqs. (104) and (105), with Eqs. (36) and (37), for $t_\nu < t \leq t_{\nu+1}$, with the initial conditions defined, for $t = t_\nu$, by Eqs. (106) and (38). The second-order differential equation defined by Eq. (102) is transformed to a first-order differential equation in introducing the vector $\mathbf{P}_{\nu+1}(t)$ such that $\mathbf{P}_{\nu+1}(t) =$

$[\mathbf{M}]\,\dot{\mathbf{Q}}_{\nu+1}(t)$. For ν fixed in $\{0, 1, \ldots, k_T - 1\}$, for $t_\nu \leq t \leq t_{\nu+1}$, we introduce the vectors $\mathbf{R}(t)$ and $\mathbf{S}(t)$ such that

$$\mathbf{R}(t) = \begin{bmatrix} \mathbf{Q}_{\nu+1}(t) \\ \mathbf{Z}_{\nu+1}(t) \end{bmatrix} \ , \ \mathbf{S}(t) = \begin{bmatrix} \mathbf{P}_{\nu+1}(t) \\ \mathbf{V}_{\nu+1}(t) \end{bmatrix} \ . \tag{107}$$

For all t in $]t_\nu\,, t_{\nu+1}]$, the stochastic differential equations defined by Eqs. (104) and (105), with Eqs. (36) and (37), can be rewritten as

$$\begin{array}{rl} d\mathbf{R}(t) = & \mathcal{F}(\mathbf{S}(t), \mathbf{R}(t))\, dt\,, \\ d\mathbf{S}(t) = & \mathcal{G}(\mathbf{S}(t), \mathbf{R}(t), t)\, dt + d\mathbf{X}(t)\,. \end{array} \tag{108}$$

Using Eq. (106) and the fact that

$$\mathbf{P}_{\nu+1}(t_\nu) = \mathbf{P}_\nu(t_\nu) \ , \ \mathbf{V}_{\nu+1}(t_\nu) = \mathbf{V}_\nu(t_\nu)\,, \tag{109}$$

with $\mathbf{V}_0(0) = \mathbf{V}^S$ (see Eq. (38)), for $t = t_\nu$, the initial conditions are rewritten as

$$\mathbf{R}(t_\nu) = \begin{bmatrix} \mathbf{Q}_\nu(t_\nu) \\ \mathbf{0} \end{bmatrix} \ , \ \mathbf{S}(t_\nu) = \begin{bmatrix} \mathbf{P}_\nu(t_\nu) \\ \mathbf{V}_\nu(t_\nu) \end{bmatrix} \ . \tag{110}$$

In Eq. (108), the vector $\mathbf{X}(t)$ and the vector-valued functions $\mathcal{F}$ and $\mathcal{G}$, are such that

$$\mathbf{X}(t) = \begin{bmatrix} \mathbf{0} \\ [\mathbb{J}]\,\mathbf{W}(t) \end{bmatrix} \quad , \quad \mathcal{F}(\mathbf{S}(t), \mathbf{R}(t)) = \begin{bmatrix} [\mathbf{M}]^{-1}\mathbf{P}_{\nu+1}(t) \\ \mathbf{Q}_{\nu+1}(t) \end{bmatrix} \ , \tag{111}$$

$$\mathcal{G}(\mathbf{S}(t), \mathbf{R}(t), t) = \tag{112}$$

$$\begin{bmatrix} -[\mathbf{K}_0]\mathbf{Q}_{\nu+1}(t) - [\underline{\mathbf{N}}_{\nu+1}(t)]\mathbf{Z}_{\nu+1}(t) - \sum_{k=1}^{\nu}[\underline{\mathbf{N}}_k(t)]\mathbf{Z}_k(t_k) + [\Phi]^T[\mathbb{O}(t)]\mathbf{V}_{\nu+1}(t) \\ [\mathbb{S}]\mathbf{V}_{\nu+1}(t) \end{bmatrix}.$$

To solve Eq. (108) for $t \in]t_\nu\,, t_{\nu+1}]$, with the initial conditions defined by Eq. (110) for $t = t_\nu$, we propose to use an extension of the Störmer-Verlet scheme [31]. It should be noted that such an algorithm is particularly well adapted to conservative system, such as Hamiltonian system, because it allows long-time energy conservation and long-time good behavior of the numerical solution to be obtained. In our case, the system is dissipative with a nonhomogeneous excitation term. Nevertheless, among all the possible classical integration time scheme such as explicit schemes (Euler-Maruyama, Milstein, etc.) (see for instance [37, 41]) or implicit schemes (central difference scheme, Newmark, θ-Wilson, etc.) (see for instance

[5, 69]), the extension of the Störmer-Verlet scheme is a very interesting algorithm.

The time step h is such that $h = t_{\nu+1} - t_\nu$. Let ℓ_h be the number of time steps of interval $[t_\nu\,, t_{\nu+1}]$ and let $\delta t = h/\ell_h$ be the time step. For $\ell = 0, 1, \ldots, \ell_h$, the time sampling of interval $[t_\nu\,, t_{\nu+1}]$ is then defined by $\tau_\ell = t_\nu + \ell\,\delta t$ and we use the notation: $\mathbf{R}_\ell = \mathbf{R}(\tau_\ell)$, $\mathbf{S}_\ell = \mathbf{S}(\tau_\ell)$ and $\mathbf{X}_\ell = \mathbf{X}(\tau_\ell)$. We then propose the following natural extension of the Störmer-Verlet scheme adapted to the Itô stochastic differential equation defined by Eq. (108), which is written, for all $\ell = 0, 1, \ldots, \ell_h$, as

$$\mathbf{R}_{\ell+1/2} = \mathbf{R}_\ell + \frac{\delta t}{2}\mathcal{F}(\mathbf{S}_\ell, \mathbf{R}_{\ell+1/2})\,, \tag{113}$$

$$\mathbf{S}_{\ell+1} = \mathbf{S}_\ell + \frac{\delta t}{2}\{\mathcal{G}(\mathbf{S}_\ell, \mathbf{R}_{\ell+1/2}, \tau_\ell) + \mathcal{G}(\mathbf{S}_{\ell+1}, \mathbf{R}_{\ell+1/2}, \tau_{\ell+1})\} + \delta\mathbf{X}_{\ell+1}\,, \tag{114}$$

$$\mathbf{R}_{\ell+1} = \mathbf{R}_{\ell+1/2} + \frac{\delta t}{2}\mathcal{F}(\mathbf{S}_{\ell+1}, \mathbf{R}_{\ell+1/2})\,, \tag{115}$$

in which

$$\delta\mathbf{X}_{\ell+1} = \begin{bmatrix} \mathbf{0} \\ [\mathbb{J}]\,\delta\mathbf{W}_{\ell+1} \end{bmatrix}\,, \tag{116}$$

where $\delta\mathbf{W}_{\ell+1} = \mathbf{W}(\tau_{\ell+1}) - \mathbf{W}(\tau_\ell)$. For $\nu = 0, 1, \ldots, k_T - 1$ and for $\ell = 0, 1, \ldots, \ell_h$, the family $\{\delta\mathbf{W}_{\ell+1}\}_{\nu,\ell}$ is made up of independent random vectors $\delta\mathbf{W}_{\ell+1}$ which are Gaussian random vectors with zero mean and covariance matrix $\delta t\,[I]$. For ν fixed, we introduce the following notations,

$$\mathbf{Q}^\ell = \mathbf{Q}_{\nu+1}(\tau_\ell)\ ,\ \mathbf{Z}^\ell = \mathbf{Z}_{\nu+1}(\tau_\ell)\ ,\ \mathbf{P}^\ell = \mathbf{P}_{\nu+1}(\tau_\ell)\ ,\ \mathbf{V}^\ell = \mathbf{V}_{\nu+1}(\tau_\ell)\,. \tag{117}$$

From Eq. (113), it can be deduced that

$$\mathbf{Q}^{\ell+1/2} = \mathbf{Q}^\ell + \frac{\delta t}{2}\,[\mathbf{M}]^{-1}\mathbf{P}^\ell\,, \tag{118}$$

$$\mathbf{Z}^{\ell+1/2} = \mathbf{Z}^\ell + \frac{\delta t}{2}\,\mathbf{Q}^\ell + \frac{(\delta t)^2}{4}\,[\mathbf{M}]^{-1}\mathbf{P}^\ell\,. \tag{119}$$

Eq. (114) can be rewritten as

$$\mathbf{P}_{\ell+1} = \mathbf{P}_\ell + \frac{\delta t}{2}\,\{-2\,[\mathbf{K}_0]\,\mathbf{Q}^{\ell+1/2} - ([\underline{\mathbf{N}}_{\nu+1}(\tau_\ell)] + [\underline{\mathbf{N}}_{\nu+1}(\tau_{\ell+1})])\mathbf{Z}^{\ell+1/2}$$
$$- \sum_{k=1}^{\nu}([\underline{\mathbf{N}}_k(\tau_\ell)] + [\underline{\mathbf{N}}_k(\tau_{\ell+1})])\mathbf{Z}_k(t_k) + [\Phi]^T([\mathbb{O}(\tau_\ell)]\mathbf{V}^\ell + [\mathbb{O}(\tau_{\ell+1})]\mathbf{V}^{\ell+1})\}, \tag{120}$$

$$([I_{n_V}] - \frac{\delta t}{2}[\mathbb{S}])\mathbf{V}^{\ell+1} = ([I_{n_V}] + \frac{\delta t}{2}[\mathbb{S}])\mathbf{V}^{\ell} + [\mathbb{J}]\,\delta\mathbf{W}_{\ell+1}\,. \tag{121}$$

Finally, Eq. (115) yields

$$\mathbf{Q}^{\ell+1} = \mathbf{Q}^{\ell+1/2} + \frac{\delta t}{2}[\mathbf{M}]^{-1}\mathbf{P}^{\ell+1}\,, \tag{122}$$

$$\mathbf{Z}^{\ell+1} = \mathbf{Z}^{\ell+1/2} + \frac{\delta t}{2}\mathbf{Q}^{\ell+1/2}\,. \tag{123}$$

We can then summarize the different steps for the computation of $\mathbf{Q}^{\ell+1}$, $\mathbf{Z}^{\ell+1}$, $\mathbf{P}^{\ell+1}$ and $\mathbf{V}^{\ell+1}$ from $\mathbf{Q}^{\ell}$, $\mathbf{Z}^{\ell}$, $\mathbf{P}^{\ell}$ and $\mathbf{V}^{\ell}$, with for $\ell = 0$, $\mathbf{Q}^0 = \mathbf{Q}_\nu(t_\nu)$, $\mathbf{Z}^0 = \mathbf{0}$, $\mathbf{P}^0 = \mathbf{P}_\nu(t_\nu)$ and $\mathbf{V}^0 = \mathbf{V}_\nu(t_\nu)$,

- Eqs. (118) and (119) allow $\mathbf{Q}^{\ell+1/2}$ and $\mathbf{Z}^{\ell+1/2}$ to be computed as functions of $\mathbf{Q}^{\ell}$, $\mathbf{Z}^{\ell}$ and $\mathbf{P}^{\ell}$.
- Eq. (121) allows $\mathbf{V}^{\ell+1}$ to be computed as a function of $\mathbf{V}^{\ell}$.
- Eq. (120) allows $\mathbf{P}^{\ell+1}$ to be computed as a function of $\mathbf{P}^{\ell}$, $\mathbf{Q}^{\ell+1/2}$, $\mathbf{Z}^{\ell+1/2}$, $\mathbf{V}^{\ell}$ and $\mathbf{V}^{\ell+1}$.
- Eq. (122) allows $\mathbf{Q}^{\ell+1}$ to be computed as a function of $\mathbf{Q}^{\ell+1/2}$ and $\mathbf{P}^{\ell+1}$.
- Eq. (123) allows $\mathbf{Z}^{\ell+1}$ to be computed as a function of $\mathbf{Z}^{\ell+1/2}$ and $\mathbf{Q}^{\ell+1/2}$.

12.3. Comments about the estimation of the time steps values

The time discretization of the stochastic equations is controlled by three time steps which are Δs, h and δt.

(i) Time steps Δs and h are related to the approximation that we have introduced for the integral $\int_0^t [\mathbf{N}(t-s)]\,\mathbf{Q}(s)\,ds$ (see Eqs. (97) to (101)) and strongly depends on the random function $\omega \mapsto [\mathbf{D}(\omega)]$ (see Section 10.3). Therefore, no estimation of Δs and h can be done without knowing an explicit description of deterministic function $\omega \mapsto [D(\omega)]$. In practice, the quality of the approximation must be checked in performing a convergence analysis with respect to these two parameters.

(ii) Concerning the estimation of time step δt, Eqs. (118) to (123) show that the scheme used in conditionally stable. We then have to estimate a value of δt to

preserve the stability. Fortunately, since the scheme is applied to the stochastic reduced-order dynamical system, the highest random eigenfrequency of the associated conservative system is finite and not too large for the practical applications. Let us consider Case 1 introduced in Section 8 and let λ_m be the largest eigenvalue used for constructing the reduced-order model. The reduced-order dynamical system without uncertainties is then a deterministic linear filter for which the bandwidth can be approximated by $[0\,,\omega_f]$ in which $\omega_f \sim \sqrt{\lambda_m}$. This linear filter decreases in ω^{-2} for ω larger than the highest resonance (of the order of ω_f) contained in the deterministic reduced-order model. Since the stiffness matrix of the stochastic reduced-order system is random and depends on ω, we introduce a factor $a_f > 1$ such that $\omega_{\max} = a_f\,\omega_f$ where $[0\,,\omega_{\max}]$ is a reasonable approximation of the upper bound of the bandwidth of the random linear filter corresponding to the stochastic reduced-order model. Concerning the stochastic excitation, since $t \mapsto [\mathbb{O}(t)]$ is the signal envelope with a slow time variation, a cutoff frequency ω_V can be defined such that $\int_0^{\omega_V} \text{tr}[S_V(\omega)]\,d\omega \geq (1-\varepsilon_V)\int_0^{+\infty} \text{tr}[S_V(\omega)]\,d\omega$ in which ε_V is the relative error in terms of the mean spectral power. Consequently, the mean spectral power of the random responses is mainly concentrated in the frequency band $[0\,,\omega_r]$ with $\omega_r = \min\{\omega_{\max}, \omega_V\}$. The value of time step δt can then be estimated by $\delta t \sim \pi/\omega_r$. Nevertheless, a convergence analysis must be performed to control the quality of the approximation constructed.

12.4. Comments about the stochastic solver

In this paper, we are interested in the time-domain formulation in computational dynamics for linear viscoelastic media with model uncertainties and stochastic excitation. Clearly, for time-invariant linear dynamical system an alternative to the time formulation is the frequency formulation. Nevertheless, such an alternative is not really appropriate for nonlinear dynamical systems and a time formulation is then required. The developments presented in this work is relevant of this case and can be applied without difficulties for nonlinear dynamical systems constituted of the linear viscoelastic system defined by Eqs. (94) to (96), in which local nonlinearities are added. For instance, it would be the case of a vibro-impact system made up of a linear viscoelastic medium with stops. In this framework, and taking into account the presence of stochastic excitation and random operators, an efficient stochastic solver is the Monte Carlo method. The direct Monte Carlo numerical simulation method (see for instance [32, 59]) is a very effective and efficient method because this method (i) is non-intrusive with respect to software, (ii) is adapted to massively parallel computation without any software developments, (iii) is such that its convergence can be controlled during the computation,

and (iv) the speed of convergence is independent of the dimension. The speed of convergence of the Monte Carlo method can be improved using advanced Monte Carlo simulation procedures [57, 66], subset simulation techniques [3], important sampling for high dimension problems [2], local domain Monte Carlo Simulation [58], recent variance reduction technique based on a reduced basis of control variates [7, 8].

13. Conclusions

For the first time, a complete time-domain formulation is proposed, including new results, in computational mechanics for studying the transient response of linear viscoelastic systems, submitted to a nonstationary stochastic excitation and in presence of model uncertainties which are modeled using the nonparametric probabilistic approach and the random matrix theory. A time discretization of the obtained linear Stochastic Integro-Differential Equation (SIDE) with random operators and with a stochastic nonhomogeneous part (stochastic excitation) is proposed to transform the SIDE to a linear Itô Stochastic Differential Equation (ISDE) with random operators. Then we have proposed an extension of the Störmer-Verlet scheme to solve this ISDE. The general methodology which has been presented has the capability to analyze such systems with additional local nonlinearities, such as vibro-impact systems made up of a linear viscoelastic medium with stops.

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