Stochastic modeling of uncertainties in computational structural dynamics - Recent theoretical advances

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Abstract

This paper deals with a short overview on stochastic modeling of uncertainties. We introduce the types of uncertainties, the variability of real systems, the types of probabilistic approaches, the representations for the stochastic models of uncertainties, the construction of the stochastic models using the maximum entropy principle, the propagation of uncertainties, the methods to solve the stochastic dynamical equations, the identification of the prior and the posterior stochastic models, the robust updating of the computational models and the robust design with uncertain computational models. We present recent theoretical advances in this field concerning the parametric and the nonparametric probabilistic approaches of uncertainties in computational structural dynamics for the construction of the prior stochastic models of both the uncertainties on the computational model parameters and on the modeling uncertainties, and for their identification with experimental data. We also present the construction of the posterior stochastic model of uncertainties using the Bayesian method when experimental data are available.

Keywords: Uncertainties, stochastic modeling, structural dynamics, computational model.

1. Introduction

This paper is devoted to a short overview on stochastic modeling of uncertainties and on related topics, including recent theoretical advances. Many references are given in order to give an idea of the large number of works published in this field. However, this area is too vast to being exhaustive. We have then limited the references to those adapted to the guideline of the synthesis proposed in this paper. In Section 3, an overview on stochastic modeling of uncertainties is presented, introducing the types of uncertainties, the variability of real systems, the types of probabilistic approaches and the representations for the stochastic models of uncertainties. The construction of the prior stochastic models using the maximum entropy principle is recalled. The

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different problems related to the propagation of uncertainties and the methods to solve the stochastic dynamical equations are briefly presented. The fundamental problem relative to the identification of the prior and the posterior stochastic models is developed. All these tools allow the robust updating of the computational models and the robust design with uncertain computational models to be performed. Sections 4 and 5 deal with recent theoretical advances concerning the parametric and the nonparametric probabilistic approaches of uncertainties in computational structural dynamics for the construction of the prior stochastic models of both the uncertainties on the computational model parameters and on the modeling uncertainties. Finally, Section 6 is devoted to the construction of the posterior stochastic model of uncertainties using the Bayesian method when experimental data are available.

2. Comments concerning notation used

In this paper, the following notations are used:
(1) A lower case letter is a real deterministic variable (e.g. x).
(2) A boldface lower case letter is a real deterministic vector (e.g. $\mathbf{x} = (x_1, \ldots, x_N)$.
(3) An upper case letter is a real random variable (e.g. X).
(4) A boldface upper case letter is a real random vector (e.g. $\mathbf{X} = (X_1, \ldots, X_N)$).
(5) An upper case letter between brackets is a real deterministic matrix (e.g. $[A\,]$).
(6) A boldface upper case letter between brackets is a real random matrix (e.g. $[\mathbf{A}]$).
(7) Any deterministic quantities above (e.g. $x, \mathbf{x}, [A\,]$) with an underline (e.g. $\underline{x}, \underline{\mathbf{x}}, [\underline{A}]$) means that these deterministic quantities are related to the mean model (or to the nominal model).

3. Short overview on stochastic modeling of uncertainties and on related topics

3.1. Uncertainties and variability

The *designed system* is used to manufacture the *real system* and to construct the nominal computational model (also called the *mean computational model* or sometimes, the mean model) using a mathematical-mechanical modeling process for which the main objective is the prediction of the responses of the real system in its environment. The real system, submitted to a given environment, can exhibit a variability in its responses due to fluctuations in the manufacturing process and due to small variations of the configuration around a nominal configuration associated with the designed system. The mean computational model which results from a mathematical-mechanical modeling process of the design system, has parameters which can be uncertain. In this case, there are *uncertainties on the computational model parameters*. In an other hand, the modeling process induces some modeling errors defined as the *modeling uncertainties*. It is important to take into account both the uncertainties on the computational model parameters and the modeling uncertainties to improve the predictions of computational models in order to use such a computational model to carry out robust optimization, robust design and robust updating with respect to uncertainties. Today, it is well understood that, as soon as the probability theory can be used, then the probabilistic approach of uncertainties is certainly the most powerful, efficient and effective tool for modeling

and for solving direct and inverse problem. The developments presented in this paper are limited to the probabilistic approaches.

3.2. Types of approach for stochastic modeling of uncertainties

The *parametric probabilistic approach* consists in modeling the *uncertain parameters of the computational model* by random variables and then in constructing the stochastic model of these random variables using the available information. Such an approach is very well adapted and very efficient to take into account the uncertainties on the computational model parameters as soon as the probability theory can be used. Many works have been published in this field and a state-of-the-art can be found, for instance, in [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. Nevertheless, the parametric probabilistic approach does not allow the modeling uncertainties to be taken into account (see for instance [12, 13]).

Concerning *modeling uncertainties* induced by *modeling errors*, it is today well understood that the prior and posterior stochastic models of the uncertain parameters of the computational model are not sufficient and do not have the capability to take into account modeling uncertainties in the context of computational mechanics (see for instance [12, 13, 14, 15]). Two main methods can be used to take into account modeling uncertainties (modeling errors).

(i) The first one consists in introducing a stochastic model of the *output-prediction-error*, considered as a noise, which is the difference between the real system output and the computational model output [12]. In this approach, the modeling errors and the measurements errors are simultaneously taken into account and cannot really be separately identified. Note that if no experimental data are available, such a method cannot be used because, generally, there are no information for constructing the stochastic model of such a noise. Such an approach is simple and efficient but requires experimental data. It should be noted that a lot of experimental data are required in high dimension. However, with such an approach, the posterior stochastic model of the uncertain parameters of the computational model strongly depends on the stochastic model of the noise which is added to the model output and which is often unknown. In addition, for many problems, it can be necessary to take into account the modeling errors at the operators level of the mean computational model (for instance, to take into account the modeling errors on the mass and stiffness operators in order to analyze the generalized eigenvalue problem related to a dynamical system, or, if the design parameters of the computational model are not fixed but run through an admissible set of values in the context of the robust design optimization).

(ii) The second one is based on the *nonparametric probabilistic approach* of modeling uncertainties (modeling errors) which has been proposed in [13] as an alternative method to the output-prediction-error method in order to take into account modeling errors at the operators level and not at the model output level by the introduction of an additive noise. It should be noted that such an approach allows a prior stochastic model of modeling uncertainties to be constructed even if no experimental data are available. The nonparametric probabilistic approach is based on the use of a reduced-order model and the random matrix theory. It consists in directly constructing the stochastic modeling of the operators of the mean computational model. The random matrix theory

[16] (see also [17, 18, 19] for such a theory in the context of linear acoustics) is used to construct the prior probability distribution of the random matrices modeling the uncertain operators of the mean computational model. This prior probability distribution is constructed by using the Maximum Entropy Principle [20] (from Information Theory [21]) for which the constraints are defined by the available information [13, 14, 22, 15]. Since the paper [13], many works have been published in order to validate the nonparametric probabilistic approach of model uncertainties with experimental results (see for instance [23, 24, 25, 26, 27, 28, 15, 29]), to extend the applicability of the theory to other areas [30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41], to extend the theory to new ensembles of positive-definite random matrices yielding a more flexible description of the dispersion levels [42], to apply the theory for the analysis of complex dynamical systems in the medium-frequency range, including vibroacoustic systems, [43, 44, 23, 45, 25, 26, 27, 28, 46, 47, 48, 39], to analyze nonlinear dynamical systems (i) for local nonlinear elements [49, 50, 37, 51, 52, 53, 54] and (ii) for distributed nonlinear elements or nonlinear geometrical effects [55].

Concerning the coupling of the parametric probabilistic approach of the uncertainties on the computational model parameters with the nonparametric probabilistic approach of modeling uncertainties (modeling errors), a methodology has recently been proposed in computational mechanics [56]. This *generalized probabilistic approach* of uncertainties in computational dynamics uses the random matrix theory. The proposed approach allows the prior stochastic model of each type of uncertainties (uncertainties on the computational model parameters and modeling uncertainties) to be separately constructed and identified. The modeling errors are not taken into account with the usual output-prediction-error method but with the nonparametric probabilistic approach.

3.3. Types of representation for the stochastic modeling of uncertainties

A fundamental question is the construction of the prior stochastic model of uncertainties on the computational model parameters and also the prior stochastic model of modeling uncertainties. Such a prior stochastic model can then be used to study the propagation of uncertainties through the mechanical system which is analyzed. If experimental data are available for the mechanical system, then they can be used (1) to identify the parameters of the prior stochastic model [29] using, for instance, the maximum likelihood method [57, 58] or (2) to construct a posterior stochastic model [12] using, for instance, the Bayesian method (see for instance [59, 60, 61, 62, 58]).

Two main methods are available to construct the prior stochastic model of a random vector $\theta \mapsto \mathbf{X}(\theta) = (X_1(\theta), \ldots, X_N(\theta))$ defined on a probability space $(\Theta, \mathcal{T}, \mathcal{P})$, with values in $\mathbb{R}^N$ and whose probability distribution on $\mathbb{R}^N$ is denoted by $P_{\mathbf{X}}(\mathrm{d}\mathbf{x})$ (this random vector can be the approximation in finite dimension of a stochastic process or a random field). It should be noted the two following methods can also be used for random matrices, random fields, etc. We introduce the space $\mathcal{L}^2_N$ of all the second-order random variables such that

$$E\{\|\mathbf{X}\|^2\} = \int_{\mathbb{R}^N} \|\mathbf{x}\|^2 \, P_{\mathbf{X}}(\mathrm{d}\mathbf{x}) < +\infty \quad , \tag{1}$$

in which E is the mathematical expectation, where $\mathbf{x} = (x_1, \ldots, x_N)$ and $\|\mathbf{x}\|^2 = x_1^2 + \ldots + x_N^2$ is the square of the Euclidean norm of vector $\mathbf{x}$.

(i)- The first method is a direct approach which consists in directly constructing the probability distribution $P_{\mathbf{X}}(\mathrm{d}\mathbf{x})$ on $\mathbb{R}^N$ in using, for instance, the maximum entropy principle (see Section 3.4).

(ii)- The second one is an indirect approach which consists in introducing a representation $\mathbf{X} = \mathbf{h}(\boldsymbol{\Xi})$ for which $\mathbf{X}$ is the transformation by a deterministic nonlinear (measurable) mapping $\mathbf{h}$ (which has to be constructed) of a $\mathbb{R}^{n_p}$-valued random variable $\boldsymbol{\Xi} = (\Xi_1, \ldots, \Xi_{n_p})$ whose probability distribution $P_{\boldsymbol{\Xi}}(\mathrm{d}\boldsymbol{\xi})$ is given and then is known. Then $P_{\mathbf{X}}$ is the transformation of $P_{\boldsymbol{\Xi}}$ by mapping $\mathbf{h}$. Two main types of methods can be used.

(ii.1) - The first one corresponds to the spectral methods such as the Polynomial Chaos representations (see [63, 64] and also [65, 66, 67, 68, 69, 70, 71, 72, 73]) which can be applied in infinite dimension for stochastic processes and random fields, which allow the effective construction of mapping $\mathbf{h}$ to be carried out and which allow any random variable $\mathbf{X}$ in $\mathcal{L}^2_N$, to be written as

$$\mathbf{X} = \Sigma_{j_1=0}^{+\infty} \ldots \Sigma_{j_{n_p}=0}^{+\infty} \, \mathbf{a}_{j_1,\ldots,j_{n_p}} \, \psi_{j_1}(\Xi_1) \times \ldots \times \psi_{j_{n_p}}(\Xi_{n_p}) \quad , \tag{2}$$

in which ψ_{j_k} are given real polynomials related to the probability distribution of $\boldsymbol{\Xi}$ and where $\mathbf{a}_{j_1,\ldots,j_{n_p}}$ are deterministic coefficients in $\mathbb{R}^N$. Introducing the multi-index $\boldsymbol{\alpha} = (j_1, \ldots, j_{n_p})$ and the multi-dimensional polynomials $\psi_{\boldsymbol{\alpha}}(\boldsymbol{\Xi}) = \psi_{j_1}(\Xi_1) \times \ldots \times \psi_{j_{n_p}}(\Xi_{n_p})$ of random vector $\boldsymbol{\Xi}$, the previous polynomial chaos decomposition can be rewritten as

$$\mathbf{X} = \Sigma_{\boldsymbol{\alpha}} \, \mathbf{a}_{\boldsymbol{\alpha}} \, \psi_{\boldsymbol{\alpha}}(\boldsymbol{\Xi}) \quad . \tag{3}$$

The multi-dimensional polynomials $\psi_{\boldsymbol{\alpha}}(\boldsymbol{\Xi})$ are orthogonal and normalized,

$$E\{\psi_{\boldsymbol{\alpha}}(\boldsymbol{\Xi}) \, \psi_{\boldsymbol{\beta}}(\boldsymbol{\Xi})\} = \int_{\mathbb{R}^{n_p}} \psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi}) \, \psi_{\boldsymbol{\beta}}(\boldsymbol{\xi}) \, P_{\boldsymbol{\Xi}}(\mathrm{d}\boldsymbol{\xi}) = \delta_{\boldsymbol{\alpha}\boldsymbol{\beta}} \quad , \tag{4}$$

in which $\delta_{\boldsymbol{\alpha}\boldsymbol{\alpha}} = 1$ and $\delta_{\boldsymbol{\alpha}\boldsymbol{\beta}} = 0$. The coefficients $\mathbf{a}_{\boldsymbol{\alpha}} = \mathbf{a}_{j_1,\ldots,j_{n_p}}$ are vectors in $\mathbb{R}^N$ which completely define mapping $\mathbf{g}$ and which are given by

$$\mathbf{a}_{\boldsymbol{\alpha}} = E\{\mathbf{X} \, \psi_{\boldsymbol{\alpha}}(\boldsymbol{\Xi})\} \quad . \tag{5}$$

The construction of $\mathbf{h}$ then consists in identifying the vector-valued coefficients $\mathbf{a}_{\boldsymbol{\alpha}}$. If $\boldsymbol{\Xi}$ is a normalized Gaussian random vector, then the polynomials are the normalized Hermite polynomials. Today, many applications of such an approach have been carried out for direct and inverse problems. We refer the reader to [65, 74, 75, 76, 77, 78, 79, 80, 81] and in particular, to Section 3.7 for a short overview concerning the identification and inverse stochastic problems related to the parametric and nonparametric probabilistic approaches of uncertainties.

(ii.2) - The second one consists in introducing a prior algebraic representation $\mathbf{X} =$

$\mathbf{h}(\boldsymbol{\Xi}, \mathbf{s})$ in which $\mathbf{s}$ is a vector parameter with small dimension (which must be identified), where $\boldsymbol{\Xi}$ is a vector-valued random variable with a given probability distribution $P_{\boldsymbol{\Xi}}$ and where $\mathbf{h}$ is a given nonlinear mapping. For instance, tensor-valued random fields representations constructed with such an approach can be found in [82, 83, 84].

In theory, method (ii.1) allows any random vector $\mathbf{X}$ in $\mathcal{L}_N^2$ to be represented by a Polynomial Chaos expansion. In practice, the representation can require a very large number of coefficients to get convergence yielding very difficult problems in high dimension for the identification problem and then requires adapted methodologies and adapted methods [85, 86]. In general, method (ii.2) does not allow any random vector $\mathbf{X}$ in $\mathcal{L}_N^2$ to be represented but allows a family of representations to be constructed in a subspace of $\mathcal{L}_N^2$ when $\mathbf{s}$ runs through all the admissible space of $\mathbf{s}$ (but, in counter part, the identification of $\mathbf{s}$ is effective and efficient).

3.4. Construction of the stochastic models using the maximum entropy principle under the constraints defined by the available information

The measure of uncertainties using the entropy has been introduced by Shannon [21] in the framework of Information Theory. The maximum entropy principle (that is to say the maximization of the level of uncertainties) has been introduced by Jaynes [20] to construct the prior stochastic model of a random variable under the constraints defined by the available information. This principle appears as a major tool to construct the prior stochastic model (1) of the uncertain parameters of the computational model using the parametric probabilistic approach, (2) of both the uncertainties on the computational model parameters and modeling uncertainties, using the nonparametric probabilistic approach and (3) of the generalized approach of uncertainties corresponding to a full coupling of the parametric and nonparametric probabilistic approaches.

Let $\mathbf{x} = (x_1, \ldots, x_N)$ be a real vector and let $\mathbf{X} = (X_1, \ldots, X_N)$ be a second-order random variable with values in $\mathbb{R}^N$ whose probability distribution $P_{\mathbf{X}}$ is defined by a probability density function $\mathbf{x} \mapsto p_{\mathbf{X}}(\mathbf{x})$ on $\mathbb{R}^N$ with respect to $\mathrm{d}\mathbf{x} = \mathrm{d}x_1 \ldots \mathrm{d}x_N$ and which verifies the normalization condition $\int_{\mathbb{R}^N} p_{\mathbf{X}}(\mathbf{x})\, \mathrm{d}\mathbf{x} = 1$. In fact, it is assumed that $\mathbf{X}$ is with values in any bounded or unbounded part $\mathcal{X}$ of $\mathbb{R}^N$ and consequently, the support of $p_{\mathbf{X}}$ is $\mathcal{X}$. The available information defines a constraint equation on $\mathbb{R}^\mu$ written as $E\{\mathbf{g}(\mathbf{X})\} = \mathbf{f}$ in which E is the mathematical expectation, $\mathbf{f}$ is a given vector in $\mathbb{R}^\mu$ and where $\mathbf{x} \mapsto \mathbf{g}(\mathbf{x})$ is a given function from $\mathbb{R}^N$ into $\mathbb{R}^\mu$. Let $\mathcal{C}$ be the set of all the probability density functions $\mathbf{x} \mapsto p_{\mathbf{X}}(\mathbf{x})$ defined on $\mathbb{R}^N$, with values in $\mathbb{R}^+$, with support $\mathcal{X}$, verifying the normalization condition and the constraint equation $E\{\mathbf{g}(\mathbf{X})\} = \mathbf{f}$. The maximum entropy principle consists in finding $p_{\mathbf{X}}$ in $\mathcal{C}$ which maximizes the entropy (that is to say which maximizes the uncertainties),

$$p_{\mathbf{X}} = \arg\max_{p \in \mathcal{C}} S(p) \quad , \quad S(p) = -\int_{\mathbb{R}^N} p(\mathbf{x})\, \log(p(\mathbf{x}))\, \mathrm{d}\mathbf{x} \tag{6}$$

in which $S(p)$ is the entropy of the probability density function p. Introducing the Lagrange multiplier $\boldsymbol{\lambda} \in \mathcal{L}_\mu \subset \mathbb{R}^\mu$ (associated with the constraint) where $\mathcal{L}_\mu$ is the subset of $\mathbb{R}^\mu$ of all the admissible values for $\boldsymbol{\lambda}$, it can easily be seen that the solution of the optimization problem can be written as

$$p_{\mathbf{X}}(\mathbf{x}) = c_0\, 1\!\!1_{\mathcal{X}}(\mathbf{x})\, \exp(- < \boldsymbol{\lambda}, \mathbf{g}(\mathbf{x}) >) \quad , \quad \forall \mathbf{x} \in \mathbb{R}^N \tag{7}$$

in which $< \mathbf{x}\,, \mathbf{y} >= x_1 y_1 + \ldots + x_\mu y_\mu$ and where $1\!\!1_{\mathcal{X}}$ is the indicatrix function of set $\mathcal{X}$. The normalization constant c_0 and the Lagrange multiplier $\boldsymbol{\lambda}$ are calculated in solving a nonlinear vectorial algebraic equation deduced from the normalization condition and from the constraint equation. This algebraic equation can be solved using appropriated algorithms. Then, it is necessary to construct a generator of independent realizations of random variable $\mathbf{X}$ whose probability density function is that which has been built. In small dimension (N is a few units), there is no difficulty. In high dimension (N hundreds or thousands), there are two major difficulties. The first one is related to the calculation of an integral in high dimension of the type $c_0 \int_{\mathcal{X}} \mathbf{g}(\mathbf{x}) \, \exp(- < \boldsymbol{\lambda}, \mathbf{g}(\mathbf{x}) >) \, \mathrm{d}\mathbf{x}$. Such a calculation is necessary to implement the algorithm for computing c_0 and $\boldsymbol{\lambda}$. The second one is the construction of the generator once c_0 et $\boldsymbol{\lambda}$ have been calculated. These two aspects can be solved using the Markov Chain Monte Carlo methods (MCMC) (see for instance [62, 87, 88, 58]). The transition kernel of the homogeneous (stationary) Markov chain of the MCMC method can be constructed using the Metropolis-Hastings algorithm [89, 90, 91, 92] or the Gibbs algorithm [93] which is a slightly different algorithm for which the kernel is directly derived from the transition probability density function and for which the Gibbs realizations are always accepted. These two algorithms construct the transition kernel for which the invariant measure is $P_{\mathbf{X}}$. In general, these algorithms are effective but can not be when there are regions of attraction that do not correspond to the invariant measure. These situations can not be easily detected and are time consuming. Recently, an approach [41] of the class of the Gibbs method has been proposed to avoid these difficulties and is based on the introduction of an Itô stochastic differential equation whose unique invariant measure is $P_{\mathbf{X}}$ and is the explicit solution of a Fokker-Planck equation [94]. The algorithm is then obtained by the discretization of the Itô equation.

3.5. Random Matrix Theory

The random matrix theory were introduced and developed in mathematical statistics by Wishart and others in the 1930s and was intensively studied by physicists and mathematicians in the context of nuclear physics. These works began with Wigner in the 1950s and received an important effort in the 1960s by Wigner, Dyson, Mehta and others. In 1965, Poter published a volume of important papers in this field, followed, in 1967 by the first edition of the Mehta book whose second edition [16] published in 1991 is an excellent synthesis of the random matrix theory. We refer the reader to [19] and [18] for an introduction to random matrix theory presented in the context of Mechanics. Concerning multivariate statistical analysis and statistics of random matrices, the reader will find additional developments in [95] and [96].

3.5.1. Why the Gaussian orthogonal ensemble cannot be used if positiveness property is required

The Gaussian Orthogonal Ensemble (GOE), for which the mean value is the unity matrix $[I_n]$ (see [22]), is the set of random matrices $[\mathbf{G}^{\mathrm{GOE}}]$) which can be written as $[\mathbf{G}^{\mathrm{GOE}}] = [I_n] + [\mathbf{B}^{\mathrm{GOE}}]$ in which $[\mathbf{B}^{\mathrm{GOE}}]$ belongs to the GOE (see [16]), for which the mean value is the zero matrix $[0]$, and consequently, would be a second-order centered random matrix with values in $\mathbb{M}_n^S(\mathbb{R})$ such that $E\{[\mathbf{B}^{\mathrm{GOE}}]\} = [\,0\,]$ and $E\{\|[\mathbf{B}^{\mathrm{GOE}}]\|_F^2\} <$

$+\infty$, and for which the probability density function, with respect to $\widetilde{d}[B] = 2^{n(n-1)/4}\, \Pi_{1\leq i\leq j\leq n}\, d[B]_{ij}$, is written as

$$p_{[\mathbf{B}^{\mathrm{GOE}}]}([B]) = C_n \times \exp\left\{-\frac{(n+1)}{4\delta^2}\mathrm{tr}\{[B]^2\}\right\} \quad . \tag{8}$$

The constant C_n of normalization can easily be calculated and δ is the coefficient of variation of the random matrix $[\mathbf{G}^{\mathrm{GOE}}]$ which is such that $\delta^2 = n^{-1}E\{\| [\mathbf{G}^{\mathrm{GOE}}] - [I_n] \|_F^2\}$ because $\| [I_n] \|_F^2 = n$. The real-valued random variables $\{[\mathbf{B}^{\mathrm{GOE}}]_{jk}, j \leq k\}$ are statistically independent, second order, centered and Gaussian. It can be seen that $[\mathbf{G}^{\mathrm{GOE}}]$ is with values in $\mathbb{M}_n^S(\mathbb{R})$ but is not positive definite. In addition,

$$E\{\|[\mathbf{G}^{\mathrm{GOE}}]^{-1}\|^2\} = +\infty \quad . \tag{9}$$

Consequently, $[\mathbf{G}^{\mathrm{GOE}}]$ is not acceptable if positiveness property and integrability of the inverse are required.

3.5.2. Ensemble SG_0^+ of random matrices

The GOE cannot be used when positiveness property and integrability of the inverse are required. Consequently, we need new ensembles of random matrices which will be used to develop the nonparametric probabilistic approach of uncertainties in computational solid and fluid mechanics and which differ from the GOE and from the other known ensembles of the random matrix theory.

The objective of this section is then to summarize the construction given in [13, 14, 22, 15] of the ensemble SG_0^+ of random matrices $[\mathbf{G}_0]$ defined on the probability space $(\Theta, \mathcal{T}, \mathcal{P})$, with values in the set $\mathbb{M}_n^+(\mathbb{R})$ of all the positive definite symmetric $(n \times n)$ real matrices and such that

$$E\{[\mathbf{G}_0]\} = [I_n] \quad , \quad E\{\log(\det[\mathbf{G}_0])\} = C \quad , \quad |C| < +\infty \quad . \tag{10}$$

The probability distribution $P_{[\mathbf{G}_0]} = p_{[\mathbf{G}_0]}([G])\ \widetilde{d}G$ is defined by a probability density function $[G] \mapsto p_{[\mathbf{G}_0]}([G])$ from $\mathbb{M}_n^+(\mathbb{R})$ into $\mathbb{R}^+$ with respect to the volume element $\widetilde{d}G$ on the set $\mathbb{M}_n^S(\mathbb{R})$ of all the symmetric $(n \times n)$ real matrices, which is such that $\widetilde{d}G = 2^{n(n-1)/4}\, \Pi_{1\leq j\leq k\leq n}\, dG_{jk}$. This probability density function can then verify the normalization condition,

$$\int_{\mathbb{M}_n^+(\mathbb{R})} p_{[\mathbf{G}_0]}([G])\ \widetilde{\mathrm{d}}G = 1 \quad . \tag{11}$$

Let $\|A\|_F^2 = \sum_{j=1}^n \sum_{k=1}^n [A]_{jk}^2$ be the square of the Frobenius norm of symmetric real matrix $[A]$. Let δ be the positive real number defined by

$$\delta = \left\{\frac{E\{\| [\mathbf{G}_0] - E\{[\mathbf{G}_0]\} \|_F^2\}}{\| E\{[\mathbf{G}_0]\} \|_F^2}\right\}^{1/2} = \left\{\frac{1}{n}E\{\| [\mathbf{G}_0] - [I_n] \|_F^2\}\right\}^{1/2} \quad , \tag{12}$$

which will allow the dispersion of the stochastic model of random matrix $[\mathbf{G}_0]$ to be controlled. For δ such that $0 < \delta < (n+1)^{1/2}(n+5)^{-1/2}$, the use of the maximum entropy principle under the constraints defined by the above available information yields

the following algebraic expression of the probability density function of random matrix $[\mathbf{G}_0]$,

$$p_{[\mathbf{G}_0]}([G]) = 1_{\mathbb{M}_n^+(\mathbb{R})}([G]) \times C_{\mathbf{G}_0} \times \big(\det[G]\big)^{(n+1)\frac{(1-\delta^2)}{2\delta^2}} \times \mathrm{e}^{-\frac{(n+1)}{2\delta^2}\,\mathrm{tr}[G]} \quad , \quad (13)$$

in which $1_{\mathbb{M}_n^+(\mathbb{R})}([G])$ is equal to 1 if $[G] \in \mathbb{M}_n^+(\mathbb{R})$ and is equal to zero if $[G] \notin \mathbb{M}_n^+(\mathbb{R})$, where tr$[G]$ is the trace of matrix $[G]$, where $\det[G]$ is the determinant of matrix $[G]$ and where the positive constant $C_{\mathbf{G}_0}$ is such that

$$C_{\mathbf{G}_0} = (2\pi)^{-n(n-1)/4} \left(\frac{n+1}{2\delta^2}\right)^{n(n+1)(2\delta^2)^{-1}} \left\{\Pi_{j=1}^n \Gamma\big(\frac{n+1}{2\delta^2} + \frac{1-j}{2}\big)\right\}^{-1} \quad , \quad (14)$$

and where, for all $z > 0$, $\Gamma(z) = \int_0^{+\infty} t^{z-1}\,\mathrm{e}^{-t}\,\mathrm{d}t$. Note that $\{[\mathbf{G}_0]_{jk}, 1 \leq j \leq k \leq n\}$ are dependent random variables.

Let $\|G\| = \sup_{\|\mathbf{x}\|\leq 1} \|[G]\,\mathbf{x}\|$ be the operator norm of matrix $[G]$ which is such that $\|[G]\,\mathbf{x}\| \leq \|G\|\,\|\mathbf{x}\|$ for all $\mathbf{x}$ in $\mathbb{R}^n$. Let $\|G\|_F$ be the Frobenius norm of $[G]$ which is defined by $\|G\|_F^2 = \mathrm{tr}\{[G]^T[G]\} = \sum_{j=1}^n \sum_{k=1}^n [G]_{jk}^2$ and which is such that $\|G\| \leq \|G\|_F \leq \sqrt{n}\,\|G\|$. It is proven [14] that

$$E\{\|[\mathbf{G}_0]^{-1}\|^2\} \leq E\{\|[\mathbf{G}_0]^{-1}\|_F^2\} < +\infty \quad . \quad (15)$$

In general, the above equation does not imply that $n \mapsto E\{\|[\mathbf{G}_0]^{-1}\|^2\}$ is a bounded function with respect to n, but, in the present case, we have the following fundamental property,

$$\forall n \geq 2 \quad , \quad E\{\|[\mathbf{G}_0]^{-1}\|^2\} \leq C_\delta < +\infty \quad , \quad (16)$$

in which C_δ is a positive finite constant which is independent of n but which depends on δ. The above equation means that $n \mapsto E\{\|[\mathbf{G}_0]^{-1}\|^2\}$ is a bounded function from $\{n \geq 2\}$ into $\mathbb{R}^+$.

The generator of independent realizations (which is required to solve the random equations with the Monte Carlo method) can easily be constructed using the following algebraic representation. Random matrix $[\mathbf{G}_0]$ can be written (Cholesky decomposition) as $[\mathbf{G}_0] = [\mathbf{L}]^T\,[\mathbf{L}]$ in which $[\mathbf{L}]$ is an upper triangular $(n \times n)$ random matrix such that:

(1) random variables $\{[\mathbf{L}]_{jj'}, j \leq j'\}$ are independent;

(2) for $j < j'$, the real-valued random variable $[\mathbf{L}]_{jj'}$ is written as $[\mathbf{L}]_{jj'} = \sigma_n U_{jj'}$ in which $\sigma_n = \delta(n+1)^{-1/2}$ and where $U_{jj'}$ is a real-valued Gaussian random variable with zero mean and variance equal to 1;

(3) for $j = j'$, the positive-valued random variable $[\mathbf{L}]_{jj}$ is written as $[\mathbf{L}]_{jj} = \sigma_n\sqrt{2V_j}$ in which σ_n is defined above and where V_j is a positive-valued gamma random variable whose probability density function is $p_{V_j}(v) = 1_{\mathbb{R}^+}(v)\frac{1}{\Gamma(a_j)}\,v^{a_j-1} \times \mathrm{e}^{-v}$, in which $a_j = \frac{n+1}{2\delta^2} + \frac{1-j}{2}$. It should be noted that the probability density function of each diagonal element $[\mathbf{L}]_{jj}$ of the random matrix $[\mathbf{L}]$ depends on the rank j of the element.

3.5.3. Ensemble SG_ε^+ of random matrices

Let $\mathcal{L}_n^2$ be the set of all the second-order random variables, defined on probability space $(\Theta, \mathcal{T}, P)$, with values in $\mathbb{R}^n$, equipped with the inner product $\ll \mathbf{X}, \mathbf{Y} \gg = E\{< \mathbf{X}, \mathbf{Y} >\}$ and with the associated norm $|||\mathbf{X}||| = \ll \mathbf{X}, \mathbf{Y} \gg^{1/2}$.

Let $0 \leq \varepsilon \ll 1$ be a positive number as small as one wants. The ensemble $\mathrm{SG}_\varepsilon^+$ is defined as the ensemble of all the random matrices which are written as

$$[\mathbf{G}] = \frac{1}{1+\varepsilon}\{[\mathbf{G}_0] + \varepsilon\,[I_n]\} \quad , \tag{17}$$

in which $[\mathbf{G}_0]$ is a random matrix which belongs to ensemble SG_0^+.

Let $[\mathbf{G}]$ be in $\mathrm{SG}_\varepsilon^+$ with $\varepsilon \geq 0$ fixed as small as one wants (possibly, ε can be equal to zero and in such a case, $\mathrm{SG}_\varepsilon^+ = \mathrm{SG}_0^+$ and then, $[\mathbf{G}] = [\mathbf{G}_0]$). It can easily be seen that

$$E\{[\mathbf{G}]\} = [I_n] \quad . \tag{18}$$

Let $\mathtt{b}(\mathbf{X}, \mathbf{Y})$ be the bilinear form on $\mathcal{L}_n^2 \times \mathcal{L}_n^2$ such that $\mathtt{b}(\mathbf{X}, \mathbf{Y}) = \ll [\mathbf{G}]\,\mathbf{X}, \mathbf{Y} \gg$. For all $\mathbf{X}$ in $\mathcal{L}_n^2$, we have

$$\mathtt{b}(\mathbf{X}, \mathbf{X}) \geq c_\varepsilon |||\mathbf{X}|||^2 \tag{19}$$

in which $c_\varepsilon = \varepsilon/(1+\varepsilon)$. The proof can easily be obtained. We have $\mathtt{b}(\mathbf{X}, \mathbf{X}) = 1/(1+\varepsilon) \ll [\mathbf{G}_0]\,\mathbf{X}, \mathbf{X} \gg + \varepsilon/(1+\varepsilon)|||\mathbf{X}|||^2 \geq c_\varepsilon |||\mathbf{X}|||^2$, because, for all $[\mathbf{G}_0]$ in SG_0^+, and for all $\mathbf{X}$ in $\mathcal{L}_n^2$, we have $\ll [\mathbf{G}_0]\,\mathbf{X}, \mathbf{X} \gg \geq 0$.

Finally, for all $\varepsilon \geq 0$, it can be proven that

$$E\big\{\|[\mathbf{G}]^{-1}\|_F^2\big\} < +\infty \quad . \tag{20}$$

3.6. Propagation of uncertainties or what are the methods to solve the stochastic dynamical equations

Concerning the methods and formulations to analyze the propagation of uncertainties in the computational model, the choice of a specific method depends on the desired accuracy on the model output and on the nature of the expected probabilistic information. These last two decades, a growing interest has been devoted to spectral stochastic methods, which provide an explicit representation of the random model output as a function of the basic random parameters modeling the input uncertainties [63, 64, 97, 98, 99]. An approximation of the random model output is sought on suitable functional approximation bases. There are two distinct classes of techniques for the definition and computation of this approximation. The first class is composed by direct simulation methods [100, 74, 101, 102, 103, 104]. These methods are often called non-intrusive since they offer the advantage of only requiring the availability of classical deterministic codes. The second class of techniques rely on Galerkin-type projections of weak solutions of models involving differential or partial differential equations [**?** 106, 107, 68, 108, 109, 103, 110, 111, 112]. However, for applications with high complexity, alternative solution techniques must be provided

in order to drastically reduce computational requirements. Recently, some alternative methods, based on the construction of optimal separated representations of the solution, have been proposed. They consist in representing the solution on optimal reduced bases of deterministic functions and stochastic functions (scalar-valued random variables). Several strategies have been proposed for the extraction of reduced bases from approximate Karhunen-Loeve expansions (or classical spectral decompositions) of the solution [113, 103]. Another method, called Generalized Spectral Decomposition method, has been recently proposed for the construction of such representations without knowing *a priori* the solution nor an approximation of it [69, 114, 70]. A major advantage of these algorithms is that they allow a separation of deterministic problems for the computation of deterministic functions and stochastic algebraic equations for the computation of stochastic functions. In that sense, they can be considered as partly non-intrusive techniques for computing Galerkin-type projections. However, it does not circumvent the curse of dimensionality associated with the dramatic increase in the dimension of stochastic approximation spaces when dealing with high stochastic dimension and high approximation resolution along each stochastic dimension. Recently, multidimensional extensions of separated representations techniques have been proposed [115, 116]. These methods exploit the tensor product structure of the solution function space, resulting from the product structure of the probability space defined by input random parameters.
Finally, the direct Monte Carlo numerical simulation method (see for instance [117, 87, 88] is a very effective and efficient method because this method (1) is non-intrusive, (2) is adapted to massively parallel computation without any software developments and (3) is such that its convergence can be controlled during the computation and (4) the speed of convergence is independent of the dimension. The speed of convergence of the Monte Carlo method can be improved using advanced Monte Carlo simulation procedures [118, 119, 120, 121], subset simulation technics [122], important sampling for high dimension problems [123], local domain Monte Carlo Simulation [124].

3.7. Identification of the prior and posterior stochastic models of uncertainties

The identification of the parameters of the stochastic model of uncertainties (parametric and nonparametric probabilistic approach) is of the class of the statistical inverse problems (see for instance [62]).

3.7.1. Identification of the stochastic model of random variables in the context of the parametric probabilistic approach of uncertainties on the computational model parameters

Let us assume that experimental data are available for observations related to the random computational model output. The experimental identification of the parameters of the prior probability distribution of the random variable modeling uncertainties of the computational model parameters can be performed, either in minimizing a distance between the experimental observations and the random model observations (least-square method for instance), or in using the maximum likelihood method (see for instance [57],[58]). Such an approach is described in Sections 4.5 and 5.3, and many applications can be found in the literature. In the domain of structural dynamics, vibrations

and vibroacoustics, we refer, for instance, the reader to [125] for the stochastic system identification for operational modal analysis, [77] for the stochastic inversion in acoustic scattering, [126] for the stochastic modeling of a nonlinear dynamical system used for producing voice and [127] for the computational model for long-range non-linear propagation over urban cities. With such an identification, we then obtained an optimal prior probability distribution. The Bayesian method then allows a posterior probability distribution to be constructed from the optimal prior probability distribution and the experimental data. Many works have been published in the literature (see for instance textbooks on the Bayesian method such as [59, 60, 61, 62] and papers devoted to the use of the Bayesian method in the context of uncertain mechanical and dynamical systems such as [12, 128, 129, 130, 131, 132, 133]. We will use such a Bayesian approach in Sections 4.6 and 6.

3.7.2. Identification of the stochastic model of random matrices in the context of the nonparametric probabilistic approach of both the uncertainties on the computational model parameters and modeling uncertainties

General developments concerning the experimental identification of the parameters of the prior probability distribution of random matrices used for modeling uncertainties in computational mechanics can be found in [15]. Many works have been published in this field, such as [79] in micromechanics, [78] for the identification and prediction of stochastic dynamical systems in a polynomial chaos basis, [23] for the experimental identification of the nonparametric stochastic model of uncertainties in structural dynamics [134, 24, 29], in structural acoustics for the low- and medium-frequency ranges [26, 28], in nonlinear structural dynamics [49, 50]. The identification of the generalized probabilistic approach of uncertainties can be found in [56] and also below in Sections 5.3 and 6.

4. Parametric probabilistic approach of uncertainties in computational structural dynamics

4.1. Introduction of the mean computational model in computational structural dynamics

The developments are presented for computational models in structural dynamics. The dynamical system is a damped fixed structure around a static equilibrium configuration considered as a natural state without prestresses and subjected to an external load. For given nominal values of the parameters of the dynamical system, the basic finite element model is called the *mean computational model*. In addition, it is assumed that a set of model parameters has been identified as uncertain model parameters. These model parameters are the components of a vector $\mathbf{x} = (x_1, \ldots, x_{n_p})$ belonging to an admissible set $\mathcal{C}_{\text{par}}$ which is a subset of $\mathbb{R}^{n_p}$. The dynamical equation of the mean computational model is then written as

$$[\mathbb{M}(\mathbf{x})]\,\ddot{\mathbb{y}}(t) + [\mathbb{D}(\mathbf{x})]\,\dot{\mathbb{y}}(t) + [\mathbb{K}(\mathbf{x})]\,\mathbb{y}(t) + \mathbb{f}_{\text{NL}}(\mathbb{y}(t), \dot{\mathbb{y}}(t); \mathbf{x}) = \mathbb{f}(t; \mathbf{x}) \quad , \qquad (21)$$

in which $\mathbb{y}(t)$ is the unknown time response vector of the m degrees of freedom (DOF) (displacements and/or rotations); $\dot{\mathbb{y}}(t)$ and $\ddot{\mathbb{y}}(t)$ are the velocity and acceleration vectors

respectively; $\mathtt{f}(t;\mathbf{x})$ is the known external load vector of the m inputs (forces and/or moments); $[\mathtt{M}(\mathbf{x})]$, $[\mathtt{D}(\mathbf{x})]$ and $[\mathtt{K}(\mathbf{x})]$ are the mass, damping and stiffness matrices of the mean computational model, which belong to the set $\mathtt{M}^{+}_{m}(\mathtt{R})$ of all the positive-definite symmetric $(m \times m)$ real matrices; $(\mathbf{y},\mathbf{z}) \mapsto \mathtt{f}_{\mathrm{NL}}(\mathbf{y},\mathbf{z};\mathbf{x})$ is the nonlinear mapping modeling local nonlinear forces.

4.2. Introduction of the reduced mean computational model

In the context of the parametric probabilistic approach of model-parameter uncertainties, the parameter $\mathbf{x}$ is modeled by a random variable $\mathbf{X}$. The mean value of $\mathbf{X}$ will be the nominal value $\underline{\mathbf{x}} = (\underline{x}_1, \ldots, \underline{x}_{n_p})$ of the uncertain model parameter $\mathbf{x}$ and the support of its probability distribution on $\mathtt{R}^{n_p}$ is $\mathcal{C}_{\mathrm{par}}$.

For all $\mathbf{x}$ fixed in $\mathcal{C}_{\mathrm{par}} \in \mathtt{R}^{n_p}$, let $\{\boldsymbol{\phi}_1(\mathbf{x}), \ldots, \boldsymbol{\phi}_m(\mathbf{x})\}$ be an algebraic basis of $\mathtt{R}^m$ constructed, for instance, either with the elastic modes of the linearized system or of the underlying linear system, or with the POD (Proper Orthogonal Decomposition) modes of the nonlinear system). Below, it is assumed that the elastic modes of the underlying linear system are used. In such a framework, there are two main possibilities to construct the reduced mean computational model.

(1) The first one consists in solving the generalized eigenvalue problem associated with the mean mass and stiffness matrices for $\mathbf{x}$ fixed to its nominal value $\underline{\mathbf{x}}$. We then obtain the elastic modes of the nominal mean computational model which are independent of $\mathbf{x}$ and which depend only on $\underline{\mathbf{x}}$ which is fixed. In this case, when $\mathbf{x}$ runs through $\mathcal{C}_{\mathrm{par}}$, matrices $[\mathtt{M}(\mathbf{x})]$ and $[\mathtt{K}(\mathbf{x})]$ have to be projected on the subspace spanned by the elastic modes of the nominal mean computational model. For very large computational model (m can be several tens of millions) such an operation is not easy to perform with usual commercial softwares which often are black boxes.
(2) The second one consists in solving the generalized eigenvalue problem associated with the mean mass and stiffness matrices for each required $\mathbf{x}$ belonging to $\mathcal{C}_{\mathrm{par}}$. In this case, the elastic modes of the mean computational model depend on $\mathbf{x}$. In the context of the parametric probabilistic approach of model-parameter uncertainties, we then have to solve a random generalized eigenvalue problem and such an approach is better adapted to usual commercial softwares and allows a fast convergence to be obtained with respect to the reduced order dimension. In addition, some algorithms have been developed in this context for random eigenvalue problems of large systems [135, 136]. In order to limit the developments, we will focus the presentation using this second approach. The extension to the first approach is straightforward from a theoretical point of view (see for instance [13]). Finally, it should be noted that the random generalized eigenvalue problem can also be considered in a polynomial chaos decomposition for which an efficient approach has been proposed [137]. Such an ingredient can be added without difficulty in the developments presented below but would induce additional difficulties in the understanding which could mask the ideas of the method proposed.

For each value of $\mathbf{x}$ given in $\mathcal{C}_{\mathrm{par}}$, the generalized eigenvalue problem associated with

the mean mass and stiffness matrices is written as

$$[\,\mathtt{K}(\mathbf{x})]\,\boldsymbol{\varphi}(\mathbf{x}) = \lambda(\mathbf{x})\,[\,\mathtt{M}(\mathbf{x})]\,\boldsymbol{\varphi}(\mathbf{x}) \quad , \tag{22}$$

for which the eigenvalues $0 < \lambda_1(\mathbf{x}) \leq \lambda_2(\mathbf{x}) \leq \ldots \leq \lambda_m(\mathbf{x})$ and the associated elastic modes $\{\boldsymbol{\varphi}_1(\mathbf{x}), \boldsymbol{\varphi}_2(\mathbf{x}), \ldots\}$ are such that

$$<[\,\mathtt{M}(\mathbf{x})]\,\boldsymbol{\varphi}_\alpha(\mathbf{x})\,,\boldsymbol{\varphi}_\beta(\mathbf{x})>= \mu_\alpha(\mathbf{x})\,\delta_{\alpha\beta} \quad , \tag{23}$$

$$<[\,\mathtt{K}(\mathbf{x})]\,\boldsymbol{\varphi}_\alpha(\mathbf{x})\,,\boldsymbol{\varphi}_\beta(\mathbf{x})>= \mu_\alpha(\mathbf{x})\,\omega_\alpha(\mathbf{x})^2\,\delta_{\alpha\beta} \quad , \tag{24}$$

in which $\omega_\alpha(\mathbf{x}) = \sqrt{\lambda_\alpha(\mathbf{x})}$ is the eigenfrequency of elastic mode $\boldsymbol{\varphi}_\alpha(\mathbf{x})$ whose normalization is defined by the generalized mass $\mu_\alpha(\mathbf{x})$ and where $<\mathbf{u}\,,\mathbf{v}>= \sum_j u_j v_j$ is the Euclidean inner product of the vectors $\mathbf{u}$ and $\mathbf{v}$. For each value of $\mathbf{x}$ given in $\mathcal{C}_{\text{par}}$, the reduced mean computational model of the dynamical system is obtained in constructing the projection of the mean computational model on the subspace $\mathtt{H}_n$ of $\mathtt{R}^m$ spanned by $\{\boldsymbol{\varphi}_1(\mathbf{x}), \ldots, \boldsymbol{\varphi}_n(\mathbf{x})\}$ with $n \ll m$. Let $[\,\phi(\mathbf{x})]$ be the $(m \times n)$ real matrix whose columns are vectors $\{\boldsymbol{\varphi}_1(\mathbf{x}), \ldots, \boldsymbol{\varphi}_n(\mathbf{x})\}$. The generalized force $\mathbf{f}(t;\mathbf{x})$ is an $\mathtt{R}^n$-vector such that $\mathbf{f}(t;\mathbf{x}) = [\,\phi(\mathbf{x})]^T\,\mathtt{f}(t;\mathbf{x})$. For all $\mathbf{x}$ in $\mathcal{C}_{\text{par}}$, the generalized mass, damping and stiffness matrices $[M(\mathbf{x})]$, $[D(\mathbf{x})]$ and $[K(\mathbf{x})]$ belong to the set $\mathtt{M}_n^+(\mathtt{R})$ of all the positive-definite symmetric $(n \times n)$ real matrices, and are defined by

$$[M(\mathbf{x})]_{\alpha\beta} = \mu_\alpha(\mathbf{x})\,\delta_{\alpha\beta} \quad , \quad [D(\mathbf{x})]_{\alpha\beta} =<[\,\mathtt{D}(\mathbf{x})]\,\boldsymbol{\varphi}_\beta(\mathbf{x})\,,\boldsymbol{\varphi}_\alpha(\mathbf{x})> \quad , \tag{25}$$

$$[K(\mathbf{x})]_{\alpha\beta} = \mu_\alpha(\mathbf{x})\,\omega_\alpha(\mathbf{x})^2\,\delta_{\alpha\beta} \quad , \tag{26}$$

in which, generally, $[D(\mathbf{x})]$ is a full matrix. Consequently, for all $\mathbf{x}$ in $\mathcal{C}_{\text{par}}$ and for all fixed t, the reduced mean computational model of the dynamical system is written as the projection $\mathbf{y}^n(t)$ of $\mathtt{y}(t)$ on $\mathtt{H}_n$ which yields

$$\begin{aligned} &\mathbf{y}^n(t) = [\,\phi(\mathbf{x})]\,\mathbf{q}(t) \quad , \\ &[M(\mathbf{x})]\,\ddot{\mathbf{q}}(t) + [D(\mathbf{x})]\,\dot{\mathbf{q}}(t) + [K(\mathbf{x})]\,\mathbf{q}(t) + \mathbf{F}_{\text{NL}}(\mathbf{q}(t),\dot{\mathbf{q}}(t);\mathbf{x}) = \mathbf{f}(t;\mathbf{x}) \quad , \end{aligned} \tag{27}$$

in which $\mathbf{F}_{\text{NL}}(\mathbf{q}(t),\dot{\mathbf{q}}(t);\mathbf{x}) = [\,\phi(\mathbf{x})]^T\,\mathtt{f}_{\text{NL}}([\,\phi(\mathbf{x})]\,\mathbf{q}(t),[\,\phi(\mathbf{x})]\,\dot{\mathbf{q}}(t);\mathbf{x})$ and where $\mathbf{q}(t) \in \mathtt{R}^n$ is the vector of the generalized coordinates. In the particular case for which $\mathtt{f}_{\text{NL}} = 0$, the corresponding equation, in the frequency domain, is written as

$$-\omega^2[M(\mathbf{x})]\,\mathbf{q}(\omega) + i\omega[D(\mathbf{x})]\,\mathbf{q}(\omega) + [K(\mathbf{x})]\,\mathbf{q}(\omega) = \mathbf{f}(\omega;\mathbf{x}) \quad . \tag{28}$$

in which $\mathbf{q}(\omega) = \int_{\mathtt{R}} \mathrm{e}^{-i\omega t}\mathbf{q}(t)\,dt$ and $\mathbf{f}(\omega;\mathbf{x}) = \int_{\mathtt{R}} \mathrm{e}^{-i\omega t}\mathbf{f}(t;\mathbf{x})\,\mathrm{d}t$

Below, we will denote by n_0 the value of n for which the response $\mathbf{y}^n$ is converged to $\mathtt{y}$, with a given accuracy, for all values of $\mathbf{x}$ in $\mathcal{C}_{\text{par}}$.

4.3. Methodology of the parametric probabilistic approach of model-parameter uncertainties

The methodology of the parametric probabilistic approach of model-parameter uncertainties consists in modeling the uncertain model parameter $\mathbf{x}$ (whose nominal value is

$\underline{\mathbf{x}}$) by a random variable $\mathbf{X}$ defined on a probability space $(\Theta, \mathcal{T}, \mathcal{P})$, with values in $\mathbb{R}^{n_p}$. Consequently, the generalized matrices in Eq. (27) become random matrices $[M(\mathbf{X})]$, $[D(\mathbf{X})]$ and $[K(\mathbf{X})]$ and, for all fixed t, the generalized external force $\mathbf{f}(t;\mathbf{x})$ becomes a random vector $\mathbf{f}(t;\mathbf{X})$. The mean values of these random matrices are denoted by $[\underline{M}]$, $[\underline{D}]$ and $[\underline{K}]$. We then have

$$E\{[M(\mathbf{X})]\} = [\underline{M}] \ , \ E\{[D(\mathbf{X})]\} = [\underline{D}] \ , \ E\{[K(\mathbf{X})]\} = [\underline{K}] \ , \tag{29}$$

in which E is the mathematical expectation. It should be noted that the mean matrices $[\underline{M}]$, $[\underline{D}]$ and $[\underline{K}]$ are different from the matrices $[M(\underline{\mathbf{x}})]$, $[D(\underline{\mathbf{x}})]$ and $[K(\underline{\mathbf{x}})]$ of the mean (nominal) computational model. The parametric probabilistic approach of uncertainties then consists in replacing the mean computational model by the following stochastic reduced computational model,

$$\mathbf{Y}(t) = [\,\phi(\mathbf{X})]\,\mathbf{Q}(t) \quad , \tag{30}$$

$$[M(\mathbf{X})]\,\ddot{\mathbf{Q}}(t) + [D(\mathbf{X})]\,\dot{\mathbf{Q}}(t) + [K(\mathbf{X})]\,\mathbf{Q}(t) + \mathbf{F}_{\text{NL}}(\mathbf{Q}(t), \dot{\mathbf{Q}}(t); \mathbf{X}) = \mathbf{f}(t;\mathbf{X}) \quad , \tag{31}$$

in which for all fixed t, $\mathbf{Y}(t)$ is an $\mathbb{R}^m$-valued random vector and $\mathbf{Q}(t)$ is an $\mathbb{R}^n$-valued random vector. As soon as the stochastic model of random vector $\mathbf{X}$ is constructed, the stochastic computational model defined by Eqs. (30) and (31) can be solved using the methods presented in Section 3.6, in particular the direct Monte Carlo method, taking into account that n_p can be very large and that we have to solve a reduced-order stochastic computational model.

4.4. Construction of the prior stochastic model of model-parameter uncertainties

The unknown probability distribution of $\mathbf{X}$ is assumed to be defined by a probability density function $\mathbf{x} \mapsto p_{\mathbf{X}}(\mathbf{x})$ from $\mathbb{R}^{n_p}$ into $\mathbb{R}^+ = [0\,, +\infty[$. Under the assumption that no experimental data are available to construct $p_{\mathbf{X}}$, the prior model can be constructed using the maximum entropy principle (see Section 3.4). For such a construction, the available information has to be defined. Since $\mathbf{x}$ belongs to $\mathcal{C}_{\text{par}}$, the support of $p_{\mathbf{X}}$ must be $\mathcal{C}_{\text{par}}$ and the normalization condition must be verified. We then have,

$$\text{supp } p_{\mathbf{X}} = \mathcal{C}_{\text{par}} \subset \mathbb{R}^{n_p} \quad , \quad \int_{\mathbb{R}^{n_p}} p_{\mathbf{X}}(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_{\mathcal{C}_{\text{par}}} p_{\mathbf{X}}(\mathbf{x})\,\mathrm{d}\mathbf{x} = 1 \quad . \tag{32}$$

Since the nominal value of $\mathbf{x}$ is $\underline{\mathbf{x}} \in \mathcal{C}_{\text{par}}$, an additional available information consists in writing that the mean value $E\{\mathbf{X}\}$ of $\mathbf{X}$ is equal to $\underline{\mathbf{x}}$ which yields the following constraint equation,

$$\int_{\mathbb{R}^{n_p}} \mathbf{x}\; p_{\mathbf{X}}(\mathbf{x})\,\mathrm{d}\mathbf{x} = \underline{\mathbf{x}} \quad . \tag{33}$$

In general, an additional available information can be deduced from the analysis of the mathematical properties of the solution of the stochastic computational model under construction. The random solution $\mathbf{Q}$ of the stochastic computational model defined by Eq. (31) must be a second-order vector-valued stochastic process (because the dynamical system is stable) which means that, for all t, we must have $E\{\|\mathbf{Q}(t)\|^2\} < +\infty$. In order that such a property be verified, it is necessary to introduce a constraint which can

always be written as the equation $E\{\mathbf{g}(\mathbf{X})\} = \boldsymbol{\gamma}$ on $\mathbb{R}^{\mu_X}$, in which $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_{\mu_X})$ is a given vector in $\mathbb{R}^{\mu_X}$ with $\mu_X \geq 1$ and where $\mathbf{x} \mapsto \mathbf{g}(\mathbf{x}) = (g_1(\mathbf{x}), \ldots, g_{\mu_X}(\mathbf{x}))$ is a given mapping from $\mathbb{R}^{n_p}$ into $\mathbb{R}^{\mu_X}$. Consequently, the additional available information defines the following constraint equation,

$$\int_{\mathbb{R}^{n_p}} \mathbf{g}(\mathbf{x})\, p_{\mathbf{X}}(\mathbf{x})\, \mathrm{d}\mathbf{x} = \boldsymbol{\gamma} \quad . \tag{34}$$

Let $\mathcal{C}$ be the set of all the probability density functions $p_{\mathbf{X}}$ defined on $\mathbb{R}^{n_p}$ with values in $\mathbb{R}^+$ such that Eqs. (32) to (34) hold. The prior model $p_{\mathbf{X}} \in \mathcal{C}$ can then be constructed using the maximum entropy principle (see Section 3.4). Since Eq. (34) is a vectorial equation of dimension μ_X, the solution $p_{\mathbf{X}}$ of the maximum entropy principle depends on the free $\mathbb{R}^{\mu_X}$-valued parameter $\boldsymbol{\gamma}$. However, parameter $\boldsymbol{\gamma}$ has generally no physical meaning and it is better to express $\boldsymbol{\gamma}$ in terms of an $\mathbb{R}^{\mu_X}$-valued parameter $\boldsymbol{\delta}_X$ which corresponds to a well defined statistical quantity for random variable $\mathbf{X}$. In general, $\boldsymbol{\delta}_X$ does not run through $\mathbb{R}^{\mu_X}$ but must belong to an admissible set $\mathcal{C}_X$ which is a subset of $\mathbb{R}^{\mu_X}$. Consequently, $p_{\mathbf{X}}$ depends on $\underline{\mathbf{x}}$ and $\boldsymbol{\delta}_X$ and is rewritten as

$$\mathbf{x} \mapsto p_{\mathbf{X}}(\mathbf{x}; \underline{\mathbf{x}}, \boldsymbol{\delta}_X) \quad \text{with} \quad (\underline{\mathbf{x}}, \boldsymbol{\delta}_X) \in \mathcal{C}_{\text{par}} \times \mathcal{C}_X \subset \mathbb{R}^{n_p} \times \mathbb{R}^{\mu_X} \quad . \tag{35}$$

4.5. Estimation of the parameters of the prior stochastic model of the uncertain model parameter

The value of n is fixed to the value n_0 for which, for all values of $\mathbf{x}$ in $\mathcal{C}_{\text{par}}$, the response $\mathbf{y}^n$ is converged to $\mathtt{y}$ with a given accuracy. The prior stochastic model $p_{\mathbf{X}}(\mathbf{x}; \underline{\mathbf{x}}, \boldsymbol{\delta}_X)$ of random variable $\mathbf{X}$ relative to the model-parameter uncertainties, depends on parameters $\underline{\mathbf{x}}$ and $\boldsymbol{\delta}_X$ belonging to admissible sets $\mathcal{C}_{\text{par}}$ and $\mathcal{C}_X$. If no experimental data are available, then $\underline{\mathbf{x}}$ is fixed to the nominal value and $\boldsymbol{\delta}_X$ must be considered as a parameter to perform a sensitivity analysis of the stochastic solution. Such a prior stochastic model of model-parameter uncertainties then allows the robustness of the solution to be analyzed in function of the level of model-parameter uncertainties controlled by $\boldsymbol{\delta}_X$.

For the particular case for which a few experimental data exist, $\underline{\mathbf{x}}$ can be updated and $\boldsymbol{\delta}_X$ can be estimated with the experimental data. The updating of $\underline{\mathbf{x}}$ and the estimation of $\boldsymbol{\delta}_X$ must then be performed with observations of the systems for which experimental data are available. There are several possibilities in the choice of such observations satisfying these criteria (for instance, the first eigenfrequencies and the associated structural elastic modes if an experimental modal analysis is available (see for instance [138]), the frequency response functions on a given frequency band, etc). Let $\mathbf{W}$ be the random vector which is observed. For all $(\underline{\mathbf{x}}, \boldsymbol{\delta}_X) \in \mathcal{C}_{\text{par}} \times \mathcal{C}_X \subset \mathbb{R}^{n_p} \times \mathbb{R}^{\mu_X}$, and for all $\mathbf{w}$ fixed, the probability density function of $\mathbf{W}$ is denoted by $p_{\mathbf{W}}(\mathbf{w}; \underline{\mathbf{x}}, \boldsymbol{\delta}_X)$. On the other hand, it is assumed that ν_{exp} independent experimental values $\mathbf{w}_1^{\text{exp}}, \ldots, \mathbf{w}_{\nu_{\text{exp}}}^{\text{exp}}$ are measured. The optimal value $(\underline{\mathbf{x}}^{\text{opt}}, \boldsymbol{\delta}_X^{\text{opt}})$ of $(\underline{\mathbf{x}}, \boldsymbol{\delta}_X)$ can be estimated by maximizing the logarithm of the likelihood function (maximum likelihood method [57],[58]),

$$(\underline{\mathbf{x}}^{\text{opt}}, \boldsymbol{\delta}_X^{\text{opt}}) = \arg \max_{(\underline{\mathbf{x}}, \boldsymbol{\delta}_X) \in \mathcal{C}_{\text{par}} \times \mathcal{C}_X} \{ \sum_{r=1}^{\nu_{\text{exp}}} \log\, p_{\mathbf{W}}(\mathbf{w}_r^{\text{exp}}; \underline{\mathbf{x}}, \boldsymbol{\delta}_X) \} \quad . \tag{36}$$

The quantities $p_{\mathbf{W}}(\mathbf{w}_r^{\text{exp}}; \underline{\mathbf{x}}, \boldsymbol{\delta}_X)$ are estimated using the independent realizations of $\mathbf{W}$ calculated with the stochastic computational model (using the Monte Carlo method) and the multivariate Gaussian kernel density estimation method (see for instance [139, 140]).

4.6. Posterior probability model of uncertainties using output-prediction-error method and the bayesian method

Let $p_{\mathbf{X}}^{\text{prior}}(\mathbf{x}) = p_{\mathbf{X}}(\mathbf{x}; \underline{\mathbf{x}}^{\text{opt}}, \boldsymbol{\delta}_X^{\text{opt}})$ be the optimal prior probability density function of $\mathbf{X}$, constructed with the optimal value $(\underline{\mathbf{x}}^{\text{opt}}, \boldsymbol{\delta}_X^{\text{opt}})$ of $(\underline{\mathbf{x}}, \boldsymbol{\delta}_X)$ which has been determined in Section 4.5, solving the optimization problem defined by Eq. (36). As we have explained, the parametric probabilistic approach does not have the capability to take into account the modeling errors (modeling uncertainties). A possible method to take into account the modeling errors is to use the output-prediction-error method (with the difficulties that we have indicated in Section 3.2-(i)). With such an approach the posterior probability density function $p_{\mathbf{X}}^{\text{post}}(\mathbf{x})$ of $\mathbf{X}$ can be estimated using the experimental data associated with observation $\mathbf{W}$ introduced in Section 4.5 and the Bayesian method.

In order to apply the Bayesian method [59, 58, 61, 60] to estimate the posterior probability density function $p_{\mathbf{X}}^{\text{post}}(\mathbf{x})$ of random vector $\mathbf{X}$, the output-prediction-error method [141, 12, 62] is used and then, an additive noise $\mathbf{B}$ is added to the observation in order to simultaneously take into account the modeling errors and the measurements errors as explained in Section 3.2. The random observed output $\mathbf{W}^{\text{out}}$ with values in $\mathbb{R}^{m_{\text{obs}}}$, for which experimental data $\mathbf{w}_1^{\text{exp}}, \ldots, \mathbf{w}_{\nu_{\text{exp}}}^{\text{exp}}$ are available (see Section 4.5), is then written as

$$\mathbf{W}^{\text{out}} = \mathbf{W} + \mathbf{B} \quad , \tag{37}$$

in which $\mathbf{W}$ is the computational model output with values in $\mathbb{R}^{m_{\text{obs}}}$. The noise $\mathbf{B}$ is a $\mathbb{R}^{m_{\text{obs}}}$-valued random vector, defined on probability space $(\Theta, \mathcal{T}, \mathcal{P})$, which is assumed to be independent of $\mathbf{X}$, and consequently, which is independent of $\mathbf{W}$. It should be noted that this hypothesis concerning the independence of $\mathbf{W}$ and $\mathbf{B}$ could easily be removed. It is also assumed that the probability density function $p_{\mathbf{B}}(\mathbf{b})$ of random vector $\mathbf{B}$, with respect to $d\mathbf{b}$, is known. For instance, it is often assumed that $\mathbf{B}$ is a centered Gaussian random vector for which the covariance matrix $[C_{\mathbf{B}}] = E\{\mathbf{B}\mathbf{B}^T\}$ is assumed to be invertible. In such a case, we would have

$$p_{\mathbf{B}}(\mathbf{b}) = (2\pi)^{-m_{\text{obs}}/2} \, (\det[C_{\mathbf{B}}])^{-1/2} \, \exp(-\frac{1}{2} < [C_{\mathbf{B}}]^{-1}\mathbf{b}\,, \mathbf{b} >) \quad . \tag{38}$$

The posterior probability density function $p_{\mathbf{X}}^{\text{post}}(\mathbf{x})$ is then estimated by

$$p_{\mathbf{X}}^{\text{post}}(\mathbf{x}) = \mathcal{L}(\mathbf{x}) \, p_{\mathbf{x}}^{\text{prior}}(\mathbf{x}) \quad , \tag{39}$$

in which $\mathbf{x} \mapsto \mathcal{L}(\mathbf{x})$ is the likelihood function defined on $\mathbb{R}^{n_p}$, with values in $\mathbb{R}^+$, such that

$$\mathcal{L}(\mathbf{x}) = \frac{\Pi_{r=1}^{\nu_{\text{exp}}} \, p_{\mathbf{W}^{\text{out}}|\mathbf{X}}(\mathbf{w}_r^{\text{exp}}|\mathbf{x})}{E\{\Pi_{r=1}^{\nu_{\text{exp}}} \, p_{\mathbf{W}^{\text{out}}|\mathbf{X}}(\mathbf{w}_r^{\text{exp}}|\mathbf{X}^{\text{prior}})\}} \quad . \tag{40}$$

In Eq. (40), $p_{\mathbf{W}^{\text{out}}|\mathbf{X}}(\mathbf{w}_\ell^{\text{exp}}|\mathbf{x})$ is the values, for the experimental data, of the conditional probability density function $\mathbf{w} \mapsto p_{\mathbf{W}^{\text{out}}|\mathbf{X}}(\mathbf{w}|\mathbf{x})$ of the random observed output $\mathbf{W}^{\text{out}}$ given $\mathbf{X} = \mathbf{x}$ in $\mathcal{C}_{\text{par}}$. Taking into account Eq. (37) and as $\mathbf{B}$ and $\mathbf{W}$ are assumed to be independent, it can easily be deduced that

$$p_{\mathbf{W}^{\text{out}}|\mathbf{X}}(\mathbf{w}_r^{\text{exp}}|\mathbf{x}) = p_{\mathbf{B}}(\mathbf{w}_r^{\text{exp}} - \mathtt{h}(\mathbf{x})) \quad , \tag{41}$$

in which $\mathbf{w} = \mathtt{h}(\mathbf{x})$ is the model observation depending on $\{\mathbf{y}^n(t), t \in \mathcal{J}\}$ which is computed using Eq. (27). It should be noted that the posterior probability density function strongly depends on the choice of the stochastic model of the output additive noise $\mathbf{B}$.

5. Nonparametric probabilistic approach of uncertainties in computational structural dynamics. The generalized probabilistic approach

The nonparametric probabilistic approach of uncertainties has been introduced in [13, 14, 15] to take into account both the model-parameter uncertainties and the modeling errors inducing model uncertainties without separating the contribution of each one of these two types of uncertainties. Recently, in [56], an improvement of the nonparametric approach of uncertainties, called the generalized probabilistic approach of uncertainties, has been proposed and allows the prior stochastic model of each type of uncertainties (model-parameter uncertainties and modeling errors) to be separately constructed. Below, we summarized this approach which allows modeling errors to be taken into account.

5.1. Methodology to take into account model uncertainties (modeling errors)

Let $(\Theta', \mathcal{T}', \mathcal{P}')$ be another probability space. The probability space $(\Theta, \mathcal{T}, \mathcal{P})$ is devoted to the stochastic model of model-parameter uncertainties using the parametric probabilistic approach (see Section 4), while $(\Theta', \mathcal{T}', \mathcal{P}')$ is devoted to the stochastic model of model uncertainties (modeling errors) using the nonparametric probabilistic approach. Therefore, for all $\mathbf{x}$ fixed in $\mathcal{C}_{\text{par}}$, the matrices $[M(\mathbf{x})]$, $[D(\mathbf{x})]$ and $[K(\mathbf{x})]$ are replaced by independent random matrices $[\mathbf{M}(\mathbf{x})] = \{\theta' \mapsto [\mathbf{M}(\theta'; \mathbf{x})]\}$, $[\mathbf{D}(\mathbf{x})] = \{\theta' \mapsto [\mathbf{D}(\theta'; \mathbf{x})]\}$ and $[\mathbf{K}(\mathbf{x})] = \{\theta' \mapsto [\mathbf{K}(\theta'; \mathbf{x})]\}$ on probability space $(\Theta', \mathcal{T}', \mathcal{P}')$ and defined, below, in Section 5.2. The nonparametric probabilistic approach of uncertainties then consists in replacing in Eq. (31) the dependent random matrices $[M(\mathbf{X})]$, $[D(\mathbf{X})]$ and $[K(\mathbf{X})]$ by the dependent random matrices $[\mathbf{M}(\mathbf{X})] = \{(\theta, \theta') \mapsto [\mathbf{M}(\theta'; \mathbf{X}(\theta))]\}$, $[\mathbf{D}(\mathbf{X})] = \{(\theta, \theta') \mapsto [\mathbf{D}(\theta'; \mathbf{X}(\theta))]\}$ and $[\mathbf{K}(\mathbf{X})] = \{(\theta, \theta') \mapsto [\mathbf{K}(\theta'; \mathbf{X}(\theta))]\}$, defined on $\Theta \times \Theta'$. It can easily be deduced that

$$E\{[\mathbf{M}(\mathbf{X})]\} = [\underline{M}] \ , \ E\{[\mathbf{D}(\mathbf{X})]\} = [\underline{D}] \ , \ E\{[\mathbf{K}(\mathbf{X})]\} = [\underline{K}] \ . \tag{42}$$

Consequently, the stochastic reduced computational model, defined by Eqs. (30) and (31), is replaced by the following stochastic reduced computational model,

$$\mathbf{Y}(t) = [\,\phi(\mathbf{X})]\,\mathbf{Q}(t) \quad , \tag{43}$$

$$[\mathbf{M}(\mathbf{X})]\,\ddot{\mathbf{Q}}(t) + [\mathbf{D}(\mathbf{X})]\,\dot{\mathbf{Q}}(t) + [\mathbf{K}(\mathbf{X})]\,\mathbf{Q}(t) + \mathbf{F}_{\text{NL}}(\mathbf{Q}(t), \dot{\mathbf{Q}}(t); \mathbf{X}) = \mathbf{f}(t; \mathbf{X}) \quad , \tag{44}$$

in which for all fixed t, $\mathbf{Y}(t) = \{(\theta,\theta') \mapsto \mathbf{Y}(\theta,\theta';t)\}$ is an $\mathbb{R}^m$-valued random vector and $\mathbf{Q}(t) = \{(\theta,\theta') \mapsto \mathbf{Q}(\theta,\theta';t)\}$ is an $\mathbb{R}^n$-valued random vector defined on $\Theta \times \Theta'$. Let $\mathbf{X}(\theta)$ be any realization of random variable $\mathbf{X}$ for θ in Θ. For all $\mathbf{x}$ in $\mathcal{C}_{\text{par}}$, let $[\mathbf{M}(\theta';\mathbf{x})]$, $[\mathbf{D}(\theta';\mathbf{x})]$ and $[\mathbf{K}(\theta';\mathbf{x})]$ be any realizations of random matrices $[\mathbf{M}(\mathbf{x})]$, $[\mathbf{D}(\mathbf{x})]$, $[\mathbf{K}(\mathbf{x})]$ for θ' in Θ'. The realization $\mathbf{Y}(\theta,\theta';t)$ of the random variable $\mathbf{Y}(t)$ and the realization $\mathbf{Q}(\theta,\theta';t)$ of the random variable $\mathbf{Q}(t)$ verify the deterministic equations

$$\mathbf{Y}(\theta,\theta';t) = [\,\phi(\mathbf{X}(\theta))]\,\mathbf{Q}(\theta,\theta';t) \quad , \tag{45}$$

$$\begin{aligned}[\mathbf{M}(\theta';\mathbf{X}(\theta))]\,\ddot{\mathbf{Q}}(\theta,\theta';t) + [\mathbf{D}(\theta';\mathbf{X}(\theta))]\,\dot{\mathbf{Q}}(\theta,\theta';t)\\ +[\mathbf{K}(\theta';\mathbf{X}(\theta))]\,\mathbf{Q}(\theta,\theta';t) + \mathbf{F}_{\text{NL}}(\mathbf{Q}(\theta,\theta';t),\dot{\mathbf{Q}}(\theta,\theta';t);\mathbf{X}(\theta))\\ = \mathbf{f}(t;\mathbf{X}(\theta)) \quad .\end{aligned} \tag{46}$$

5.2. Construction of the prior stochastic model of the random matrices

As explained in [56], the dependent random matrices $[\mathbf{M}(\mathbf{X})]$, $[\mathbf{D}(\mathbf{X})]$ and $[\mathbf{K}(\mathbf{X})]$, introduced in Eq. (44), are written as

$$\begin{aligned}[\mathbf{M}(\mathbf{X})] &= [L_M(\mathbf{X})]^T\,[\mathbf{G}_M]\,[L_M(\mathbf{X})] \quad ,\\ [\mathbf{D}(\mathbf{X})] &= [L_D(\mathbf{X})]^T\,[\mathbf{G}_D]\,[L_D(\mathbf{X})] \quad ,\\ [\mathbf{K}(\mathbf{X})] &= [L_K(\mathbf{X})]^T\,[\mathbf{G}_K]\,[L_K(\mathbf{X})] \quad ,\end{aligned} \tag{47}$$

in which, for all $\mathbf{x}$ in $\mathcal{C}_{\text{par}}$, $[L_M(\mathbf{x})]$, $[L_D(\mathbf{x})]$ and $[L_K(\mathbf{x})]$ are the upper triangular matrices such that $[M(\mathbf{x})] = [L_M(\mathbf{x})]^T[L_M(\mathbf{x})]$, $[D(\mathbf{x})] = [L_D(\mathbf{x})]^T[L_D(\mathbf{x})]$ and $[K(\mathbf{x})] = [L_K(\mathbf{x})]^T[L_K(\mathbf{x})]$. In Eq. (47), the random matrices $[\mathbf{G}_M]$, $[\mathbf{G}_D]$ and $[\mathbf{G}_K]$ are random matrices which are defined on probability space $(\Theta',\mathcal{T}',\mathcal{P}')$ with values in $\mathbb{M}_n^+(\mathbb{R})$. The joint probability density function of these random matrices $[\mathbf{G}_M]$, $[\mathbf{G}_D]$ and $[\mathbf{G}_K]$ is constructed using the maximum entropy principle under the constraints defined by the available information. Taking into account the available information (see [13, 14, 15, 56]), it is proven that these three random matrices are statistically independent, belong to ensemble SG_ε^+ defined in Section 3.5.3, and then depend on free positive real dispersion parameters δ_M, δ_D and δ_K which allow the level of the statistical fluctuations, that is to say the level of model uncertainties, to be controlled. Let $\boldsymbol{\delta}_G = (\delta_M,\delta_D,\delta_K)$ be the vector of the dispersion parameters, which belongs to an admissible set $\mathcal{C}_G \subset \mathbb{R}^3$. Consequently, the joint probability density function of the random matrices $[\mathbf{G}_M]$, $[\mathbf{G}_D]$ and $[\mathbf{G}_K]$ is written as

$$\begin{aligned}([G_M],[G_D],[G_K]) \mapsto p_{\mathbf{G}}([G_M],[G_D],[G_K];\boldsymbol{\delta}_G) = p_{[\mathbf{G}_M]]}([G_M];\delta_M)\\ \times\, p_{[\mathbf{G}_D]]}([G_D];\delta_D)\;\times\; p_{[\mathbf{G}_K]]}([G_K];\delta_K) \quad , \quad \boldsymbol{\delta}_G \in \mathcal{C}_G \subset \mathbb{R}^3 \quad .\end{aligned} \tag{48}$$

The probability distributions of random matrices $[\mathbf{G}_M]$, $[\mathbf{G}_D]$ and $[\mathbf{G}_K]$ and their algebraic representations useful for generating independent realizations $[\mathbf{G}_M(\theta')]$, $[\mathbf{G}_D(\theta')]$ and $[\mathbf{G}_K(\theta')]$ are explicitly defined in Section 3.5.3 using Section 3.5.2. From Eq. (47) and for (θ,θ') in $\Theta \times \Theta'$, it can be deduced that

$$[\mathbf{M}(\theta';\mathbf{X}(\theta))] = [L_M(\mathbf{X}(\theta))]^T\,[\mathbf{G}_M(\theta')]\,[L_M(\mathbf{X}(\theta))] \quad , \tag{49}$$

$$[\mathbf{D}(\theta';\mathbf{X}(\theta))] = [L_D(\mathbf{X}(\theta))]^T\,[\mathbf{G}_D(\theta')]\,[L_D(\mathbf{X}(\theta))] \quad , \tag{50}$$

$$[\mathbf{K}(\theta';\mathbf{X}(\theta))] = [L_K(\mathbf{X}(\theta))]^T\,[\mathbf{G}_K(\theta')]\,[L_K(\mathbf{X}(\theta))] \quad . \tag{51}$$

5.3. Estimation of the parameters of the prior stochastic model of uncertainties

The prior stochastic model of uncertainties then depends on parameters $\underline{\mathbf{x}}$, $\boldsymbol{\delta}_X$ and $\boldsymbol{\delta}_G$ belonging to the admissible sets $\mathcal{C}_{\text{par}}$, $\mathcal{C}_X$ and $\mathcal{C}_G$. If no experimental data are available, then $\underline{\mathbf{x}}$ is fixed to the nominal value and $\boldsymbol{\delta}_X$ and $\boldsymbol{\delta}_G$ must be considered as parameters to perform a sensitivity analysis of the stochastic solution. Such a prior stochastic model of both model-parameter uncertainties and model uncertainties then allows the robustness of the solution to be analyzed in function of the level of model-parameter uncertainties controlled by $\boldsymbol{\delta}_X$ and in function of model uncertainties controlled by $\boldsymbol{\delta}_G$.

If a few experimental data are available, the methodology presented in Section 4.5 can then be used to update $\underline{\mathbf{x}}$ and to estimate $\boldsymbol{\delta}_X$ and $\boldsymbol{\delta}_G$. As previously, let $\mathbf{W}$ be the random vector which is observed and which is deduced from the random solution $\mathbf{Y}$ given by Eq. (43) with Eq. (44). For all $(\underline{\mathbf{x}}, \boldsymbol{\delta}_X, \boldsymbol{\delta}_G) \in \mathcal{C}_{\text{par}} \times \mathcal{C}_X \times \mathcal{C}_G$, the probability density function of $\mathbf{W}$ is denoted by $p_{\mathbf{W}}(\mathbf{w}; \underline{\mathbf{x}}, \boldsymbol{\delta}_X, \boldsymbol{\delta}_G)$. The optimal value $(\underline{\mathbf{x}}^{\text{opt}}, \boldsymbol{\delta}_X^{\text{opt}}, \boldsymbol{\delta}_G^{\text{opt}})$ of $(\underline{\mathbf{x}}, \boldsymbol{\delta}_X, \boldsymbol{\delta}_G)$ can be estimated by maximizing the logarithm of the likelihood function,

$$(\underline{\mathbf{x}}^{\text{opt}}, \boldsymbol{\delta}_X^{\text{opt}}, , \boldsymbol{\delta}_G^{\text{opt}}) = \arg \max_{(\underline{\mathbf{x}}, \boldsymbol{\delta}_X, , \boldsymbol{\delta}_G) \in \mathcal{C}_{\text{par}} \times \mathcal{C}_X \times \mathcal{C}_G} \{ \sum_{r=1}^{\nu_{\text{exp}}} \log p_{\mathbf{W}}(\mathbf{w}_r^{\text{exp}}; \underline{\mathbf{x}}, \boldsymbol{\delta}_X, \boldsymbol{\delta}_G) \} . \tag{52}$$

Similarly to Section 4.5, the quantities $p_{\mathbf{W}}(\mathbf{w}_r^{\text{exp}}; \underline{\mathbf{x}}, \boldsymbol{\delta}_X, \boldsymbol{\delta}_G)$ are estimated using the independent realizations of $\mathbf{W}$ calculated with the stochastic computational model defined by Eqs. (43) and (44) (using the Monte Carlo method) and the multivariate Gaussian kernel density estimation method.

6. Posterior stochastic model of uncertainties using the Bayesian method

Let $p_{\mathbf{X}}^{\text{prior}}(\mathbf{x}) = p_{\mathbf{X}}(\mathbf{x}; \underline{\mathbf{x}}^{\text{opt}}, \boldsymbol{\delta}_X^{\text{opt}})$ and $p_{\mathbf{G}}([G_M], [G_D], [G_K]; \boldsymbol{\delta}_G^{\text{opt}})$ be the optimal prior probability density functions of random vector $\mathbf{X}$ and random matrices $[\mathbf{G}_M]$, $[\mathbf{G}_D]$, $[\mathbf{G}_K]$, constructed with the optimal value $(\underline{\mathbf{x}}^{\text{opt}}, \boldsymbol{\delta}_X^{\text{opt}}, \boldsymbol{\delta}_G^{\text{opt}})$ of $(\underline{\mathbf{x}}, \boldsymbol{\delta}_X, \boldsymbol{\delta}_G)$ which has been calculated in Section 5.3 solving the optimization problem defined by Eq. (52). The objective of this section is the following. For this given optimal prior probability model $p_{\mathbf{G}}([G_M], [G_D], [G_K]; \boldsymbol{\delta}_G^{\text{opt}})$ of model uncertainties induced by modeling errors, the posterior probability density function $p_{\mathbf{X}}^{\text{post}}(\mathbf{x})$ of model-parameter uncertainties is estimated using the experimental data associated with observation $\mathbf{W}$ and using the stochastic computational model, the optimal prior probability density function $p_{\mathbf{X}}^{\text{prior}}(\mathbf{x})$ and the Bayesian method.

6.1. Posterior probability density function of the model-parameter uncertainties

Let $\mathbf{w}_1^{\text{exp}}, \ldots, \mathbf{w}_{\nu_{\text{exp}}}^{\text{exp}}$ be the ν_{exp} independent experimental data corresponding to observation $\mathbf{W}$, introduced in Section 4.5 and used in Section 5.3. The Bayesian method (see for instance [12, 59, 60, 61, 57, 58, 142, 141]) allows the posterior probability density function $p_{\mathbf{X}}^{\text{post}}(\mathbf{x})$ to be calculated by

$$p_{\mathbf{X}}^{\text{post}}(\mathbf{x}) = \mathcal{L}(\mathbf{x})\, p_{\mathbf{x}}^{\text{prior}}(\mathbf{x}) \quad , \tag{53}$$

in which $\mathbf{x} \mapsto \mathcal{L}(\mathbf{x})$ is the likelihood function defined on $\mathbb{R}^{n_p}$, with values in $\mathbb{R}^+$, such that

$$\mathcal{L}(\mathbf{x}) = \frac{\Pi_{r=1}^{\nu_{\text{exp}}} p_{\mathbf{W}|\mathbf{X}}(\mathbf{w}_r^{\text{exp}}|\mathbf{x})}{E\{\Pi_{r=1}^{\nu_{\text{exp}}} p_{\mathbf{W}|\mathbf{X}}(\mathbf{w}_r^{\text{exp}}|\mathbf{X}^{\text{prior}})\}} \quad . \tag{54}$$

In Eq. (54), $p_{\mathbf{W}|\mathbf{X}}(\mathbf{w}_\ell^{\text{exp}}|\mathbf{x})$ is the value, for the experimental data, of the conditional probability density function $\mathbf{w} \mapsto p_{\mathbf{W}|\mathbf{X}}(\mathbf{w}|\mathbf{x})$ of the random observation $\mathbf{W}$ given $\mathbf{X} = \mathbf{x}$ in $\mathcal{C}_{\text{par}}$. Equation (54) shows that the likelihood function $\mathcal{L}$ must verify the following equation,

$$E\{\mathcal{L}(\mathbf{X}^{\text{prior}})\} = \int_{\mathbb{R}^{n_p}} \mathcal{L}(\mathbf{x})\, p_{\mathbf{X}}^{\text{prior}}(\mathbf{x})\, \mathrm{d}\mathbf{x} = 1 \quad . \tag{55}$$

For a high dimension uncertain model parameter, the usual Bayesian method presented above can be not efficient for partial experimental data. In such a case a novel approach derived from Baye's approach has been proposed in [143].

6.2. Posterior probability density functions of the responses

Let $\mathbf{U} = (\mathbf{V}, \mathbf{W})$ be the random response vector in which $\mathbf{W}$ is experimentally observed and where $\mathbf{V}$ is not experimentally observed. Random response vector $\mathbf{U}$ is constructed from the random solution $\mathbf{Y}$ of Eqs. (43) and (44). The probability density function $\mathbf{u} \mapsto p_{\mathbf{U}^{\text{post}}}(\mathbf{u})$ of the posterior random response vector $\mathbf{U}^{\text{post}}$ is then given by

$$p_{\mathbf{U}^{\text{post}}}(\mathbf{u}) = \int_{\mathbb{R}^{n_p}} p_{\mathbf{U}|\mathbf{X}}(\mathbf{u}|\mathbf{x})\, p_{\mathbf{X}}^{\text{post}}(\mathbf{x})\, \mathrm{d}\mathbf{x} \quad , \tag{56}$$

in which the conditional probability density function $p_{\mathbf{U}|\mathbf{X}}(\mathbf{u}|\mathbf{x})$ of $\mathbf{U}$, given $\mathbf{X} = \mathbf{x}$, is constructed using the stochastic computational model defined in Sections 5.1 and 5.2. From Eqs. (53) and (56), it can be deduced that the posterior probability density function $p_{\mathbf{U}^{\text{post}}}$ can be written as $p_{\mathbf{U}^{\text{post}}}(\mathbf{u}) = E\{\mathcal{L}(\mathbf{X}^{\text{prior}})\, p_{\mathbf{U}|\mathbf{X}}(\mathbf{u}|\mathbf{X}^{\text{prior}})\}$. Let U_k^{post} be any component of random vector $\mathbf{U}^{\text{post}}$. The probability density function $u_k \mapsto p_{U_k^{\text{post}}}(u_k)$ on $\mathbb{R}$ of the posterior random variable U_k^{post} is then given by

$$p_{U_k^{\text{post}}}(u_k) = E\{\mathcal{L}(\mathbf{X}^{\text{prior}})\, p_{U_k|\mathbf{X}}(u_k|\mathbf{X}^{\text{prior}})\} \quad , \tag{57}$$

in which $u_k \mapsto p_{U_k|\mathbf{X}}(u_k|\mathbf{x})$ is the conditional probability density function of the real valued random variable U_k given $\mathbf{X} = \mathbf{x}$ and which is constructed using the computational model defined in Sections 5.1 and 5.2.

6.3. Computational aspects

We use the notation introduced in Sections 5.1 and 5.2 concerning the realizations. Let $\mathbf{X}^{\text{prior}}(\theta_1), \ldots, \mathbf{X}^{\text{prior}}(\theta_\nu)$ be ν independent realizations of $\mathbf{X}^{\text{prior}}$ whose probability density function is $\mathbf{x} \mapsto p_{\mathbf{X}}^{\text{prior}}(\mathbf{x})$. For ν sufficiently large, the right-hand side of Eq. (57) can be estimated by

$$p_{U_k^{\text{post}}}(u_k) \simeq \frac{1}{\nu} \sum_{\ell=1}^{\nu} \mathcal{L}(\mathbf{X}^{\text{prior}}(\theta_\ell))\, p_{U_k|\mathbf{X}}(u_k|\mathbf{X}^{\text{prior}}(\theta_\ell)) \quad . \tag{58}$$

Let $([\mathbf{G}_M(\theta'_1)], [\mathbf{G}_D(\theta'_1)], [\mathbf{G}_K(\theta'_1)]), \ldots, ([\mathbf{G}_M(\theta'_{\nu'})], [\mathbf{G}_D(\theta'_{\nu'})], [\mathbf{G}_K(\theta'_{\nu'})])$ be ν' independent realizations of $([\mathbf{G}_M], [\mathbf{G}_D], [\mathbf{G}_K])$ whose probability density function is $([G_M], [G_D], [G_K]) \mapsto p_{\mathbf{G}}([G_M], [G_D], [G_K]; \boldsymbol{\delta}_G^{\text{opt}})$. For fixed θ_ℓ, the computational model defined in Sections 5.1 and 5.2 is used to calculate the ν' independent realizations $\mathbf{U}(\theta'_1|\mathbf{x}), \ldots, \mathbf{U}(\theta'_{\nu'}|\mathbf{x})$ for $\mathbf{x} = \mathbf{X}^{\text{prior}}(\theta_\ell)$. We can then deduce $\mathbf{W}(\theta'_1|\mathbf{X}^{\text{prior}}(\theta_\ell))$, $\ldots$, $\mathbf{W}(\theta'_{\nu'}|\mathbf{X}^{\text{prior}}(\theta_\ell))$ and, for all fixed k, we can deduce $U_k(\theta'_1|\mathbf{X}^{\text{prior}}(\theta_\ell)), \ldots,$ $U_k(\theta'_{\nu'}|\mathbf{X}^{\text{prior}}(\theta_\ell))$.

(1) Using the independent realizations $\mathbf{W}(\theta'_1|\mathbf{X}^{\text{prior}}(\theta_\ell)), \ldots, \mathbf{W}(\theta'_{\nu'}|\mathbf{X}^{\text{prior}}(\theta_\ell))$ and using the multivariate Gaussian kernel density estimation method, for $r = 1, \ldots, \nu_{\text{exp}}$, we can estimate $p_{\mathbf{W}|\mathbf{X}}(\mathbf{w}_r^{\text{exp}}|\mathbf{X}^{\text{prior}}(\theta_\ell))$ and then, for $\ell = 1, \ldots, \nu$, we can deduce $\mathcal{L}(\mathbf{X}^{\text{prior}}(\theta_\ell))$ using Eq. (54).

(2) For all fixed k, $p_{U_k|\mathbf{X}}(u_k|\mathbf{X}^{\text{prior}}(\theta_\ell))$ is estimated using the independent realizations $U_k(\theta'_1|\mathbf{X}^{\text{prior}}(\theta_\ell)), \ldots, U_k(\theta'_{\nu'}|\mathbf{X}^{\text{prior}}(\theta_\ell))$ and using the kernel estimation method. From Eq. (58), it can then be deduced $p_{U_k^{\text{post}}}(u_k)$.

7. Conclusion

We have presented a brief survey concerning the state-of-the-art and the recent advances in the field of stochastic modeling of uncertainties in computational structural dynamics, their propagation and their quantification when experimental data are available or not. Many interesting and important works have been published and are in progress concerning the methodologies which allow the propagation of uncertainties to be analyzed. Nevertheless, an excellent stochastic solver will compute a bad stochastic solution if the stochastic model of the input (the uncertainties) is bad. In particular, if no experimental data are available, the quality of the random responses directly depends on the quality of the prior stochastic model of uncertainties. For the complex dynamical systems, the major source of uncertainties is due to modeling errors and is not due to model-parameter uncertainties. This is the reason why the overview presented in this paper is mainly focussed on the methodologies and tools, such as the maximum entropy principle, useful to construct the prior stochastic models of uncertainties and in particular, the prior stochastic model of modeling errors using the nonparametric probabilistic approach. This latter approach requires the use of adapted ensembles of random matrices and has been proposed as an alternative method to the output-prediction-error method in order to take into account modeling errors at the operators level and not at the model output level by the introduction of an additive noise. We have then detailed the methodologies for the construction of the prior stochastic model of model-parameter uncertainties (using the parametric probabilistic approach) and the prior stochastic model of modeling errors (using the nonparametric probabilistic approach) in computational structural dynamics. In addition, when experimental data are available for some observations of the dynamical system, we have presented methodologies to perform the identification of the stochastic model of uncertainties, that is to say, the uncertainties quantification. When a few experimental data are available

for some observations of the dynamical system, these experimental data can be used to identify, with the maximum likelihood method, an optimal prior stochastic model of uncertainties. If a large experimental data basis is available the posterior stochastic model of system-parameter uncertainties, in presence of modeling errors in the computational model, can be estimated with the bayesian method.

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