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(Nearly-)Tight Bounds on the Linearity and Contiguity of Cographs

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Extended Abstract

Introduction. Linearity and contiguity are graph parameters introduced to obtain efficient codings of neighborhoods in graphs, by decomposing each neighborhood as a union of $p$ intervals chosen in one or several orders on the vertices [1]. Indeed, storing an order of the vertices as well as a pair of pointers for each of the $p$ intervals of this order (one pointer for the beginning of the interval and one for the end), with fixed $p$, allows to store the graph in $O(n)$ space (instead of $O(n + m)$ with adjacency lists) and access the neighborhood of any vertex $v$ in $O(d)$ time (instead of $O(n)$ with adjacency matrices), where $d$ is the degree of $v$.

More formally, a closed $p$-interval-model of a graph $G = (V,E)$ is a linear order $\sigma$ on $V$ such that $\forall v \in V, \exists (I_1, \ldots, I_p) \in (2^V)^p$ such that $\forall i \in \int 1, p, I_i$ is an interval of $\sigma$ and $N(x) = \bigcup_{1 \leq i \leq p} I_i$. The closed contiguity of $G$, denoted by $\text{cont}(G)$, is the minimum integer $p$ such that there exists a closed $p$-interval-model of $G$. A closed $p$-line-model of a graph $G = (V,E)$ is a tuple $(\sigma_1, \ldots, \sigma_p)$ of linear orders on $V$ such that $\forall v \in V, \exists (I_1, \ldots, I_p) \in (2^V)^p$ such that $\forall i \in \int 1, p, I_i$ is an interval of $\sigma_i$ and $N(x) = \bigcup_{1 \leq i \leq p} I_i$. The closed linearity of $G$, denoted by $\text{lin}(G)$, is the minimum $p$ such that there exists a closed $p$-line-model of $G$.

Not much is known about these parameters, which cannot be bounded by a constant even in very restricted graph classes, like interval or permutation graphs [1]. We focus here on the contiguity and linearity of cographs (graphs without induced $P_4$ subgraphs), whose very constrained structure can be represented by their cotree, a rooted tree with two kinds of nodes labeled by $P$ and $S$, giving a tight upper bound for the asymptotic contiguity of cographs and an upper bound for their linearity. To this aim, we first establish a min-max theorem on the link between the rank of rooted trees and their decompositions into paths.

A min-max theorem on the rank of a tree. The rank $[2, 3]$ of a tree $T$ is the maximal height of a complete binary tree obtained from $T$ by edge contractions, that is $\text{rank}(T) = \max \{ h(T') \mid T' \text{ complete binary tree, minor of } T \}$.

A path partition of a tree $T$ is a partition $\{P_1, \ldots, P_k\}$ of $V(T)$ such that for any $i$, the subgraph $T[P_i]$ of $T$ induced by $P_i$ is a path, as shown in Figure 1(a). The partition tree of a path partition $\mathcal{P}$, denoted by $T_p(\mathcal{P})$ and illustrated in Figure 1(b), is the tree whose nodes are $P_i$’s and where the node of $T_p(\mathcal{P})$ corresponding to $P_i$ is the parent of the node corresponding to $P_j$ iff some node of $P_i$ is the parent in $T$ of the root of $P_j$. The height of a path partition $\mathcal{P}$ of a tree $T$, denoted by $h(\mathcal{P})$, is the height $h(T_p(\mathcal{P}))$ of its partition tree. The path-height of $T$ is the minimal height of a path partition of $T$, that is $\text{ph}(T) = \min \{ h(\mathcal{P}) \mid \mathcal{P} \text{ path partition of } T \}$.

![Figure 1](image-url)
**Lemma 1** For a rooted complete binary tree $T$, $\text{rank}(T) = \text{ph}(T) = h(T)$.

**Theorem 2** For any rooted tree $T$, we have $\text{rank}(T) = \text{ph}(T)$.

Upper bounds for contiguity and linearity of cographs. We now combine the results of the previous section with a decomposition of the cotree of the input cograph into paths, in order to obtain a constructive proof that the contiguity of any cograph is at most $O(\log n)$. This decomposition is obtained recursively, using a root-path decomposition of the cotree, thanks to the Caterpillar Composition Lemma below.

A root-path decomposition (see Fig. 2) of a rooted tree $T$ is a set $\{T_1, \ldots, T_p\}$ of disjoint subtrees of $T$, with $p \geq 2$, such that every leaf of $T$ belongs to some $T_i$, with $i \in [1..p]$, and the sets of parents in $T$ of the roots of $T_i$'s is a path containing the root of $T$.

![Figure 2: The root-path decomposition $\{T_1, \ldots, T_p\}$ of a rooted tree $T$.](image)

**Lemma 3 (Caterpillar Composition Lemma)** Given a cograph $G = (V, E)$ and a root-path decomposition $\{T_i\}_{1 \leq i \leq p}$ of its cotree, where $X_i$ is the set of leaves of $T_i$, $\text{cont}(G) \leq 2 + \max_{i \in [1..p]} \text{cont}(G[X_i])$.

**Lemma 4** Given a rooted tree $T$ such that $\text{rank}(T) = k \geq 1$, there exists a root-path decomposition $\{T_1, \ldots, T_p\}$ of $T$ such that for each $i \in [1..p]$, $\text{rank}(T_i) \leq k - 1$.

**Lemma 5** Let $G$ be a cograph and $T$ its cotree. We have $\text{cont}(G) \leq 2 \text{rank}(T) + 1$.

**Theorem 6** The closed contiguity of a cograph is at most logarithmic in its number of vertices, or more formally, if $G = (V, E)$ is a cograph, then $\text{cont}(G) \leq 2 \log_2 |V| + 1$.

Lower bounds for contiguity and linearity of cographs. Finally, we focus on cographs whose cotrees are complete binary trees, and obtain a tight lower bound for their asymptotic contiguity as well as a lower bound for their asymptotic linearity.

**Theorem 7** Let $G$ be a cograph whose cotree is a complete binary tree. Then, $\text{cont}(G) = \Omega(\log n)$ and $\text{lin}(G) = \Omega(\log n / \log \log n)$.

**References**

