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(Nearly-)Tight Bounds on the Linearity and Contiguity of Cographs

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EXTENDED ABSTRACT

Introduction. Linearity and contiguity are graph parameters introduced to obtain efficient codings of neighborhoods in graphs, by decomposing each neighborhood as a union of \(p\) intervals chosen in one or several orders on the vertices [1]. Indeed, storing an order of the vertices as well as a pair of pointers for each of the \(p\) intervals of this order (one pointer for the beginning of the interval and one for the end), with fixed \(p\), allows to store the graph in \(O(n)\) space (instead of \(O(n+m)\) with adjacency lists) and access the neighborhood of any vertex \(v\) in \(O(d)\) time (instead of \(O(n)\) with adjacency matrices), where \(d\) is the degree of \(v\).

More formally, a **closed \(p\)-interval-model** of a graph \(G = (V,E)\) is a linear order \(\sigma\) on \(V\) such that \(\forall v \in V, \exists (I_1,\ldots,I_p) \in (2^V)^p\) such that \(\forall i \in \{1,p\}, I_i\) is an interval of \(\sigma\) and \(N[x] = \bigcup_{1 \leq i \leq p} I_i\). The **closed contiguity** of \(G\), denoted by \(\text{cont}(G)\), is the minimum integer \(p\) such that there exists a closed \(p\)-interval-model of \(G\). A **closed \(p\)-line-model** of a graph \(G = (V,E)\) is a tuple \((\sigma_1,\ldots,\sigma_p)\) of linear orders on \(V\) such that \(\forall v \in V, \exists (I_1,\ldots,I_p) \in (2^V)^p\) such that \(\forall i \in \{1,p\}, I_i\) is an interval of \(\sigma_i\) and \(N[x] = \bigcup_{1 \leq i \leq p} I_i\). The **closed linearity** of \(G\), denoted by \(\text{lin}(G)\), is the minimum \(p\) such that there exists a closed \(p\)-line-model of \(G\).

Not much is known about these parameters, which cannot be bounded by a constant even in very restricted graph classes, like interval or permutation graphs [1]. We focus here on the contiguity and linearity of cographs (graphs without induced \(P_4\) subgraphs), whose very constrained structure can be represented by their **co-tree**, a rooted tree with two kinds of nodes labeled by \(P\) and \(S\), giving a tight upper bound for the asymptotic contiguity of cographs and an upper bound for their linearity. To this aim, we first establish a min-max theorem on the link between the rank of rooted trees and their decompositions into paths.

A **min-max theorem on the rank of a tree**. The **rank** \([2,3]\) of a tree \(T\) is the maximal height of a complete binary tree obtained from \(T\) by edge contractions, that is \(\text{rank}(T) = \max\{h(T') \mid T' \text{ complete binary tree, minor of } T\}\).

A **path partition** of a tree \(T\) is a partition \(\{P_1,\ldots,P_k\}\) of \(V(T)\) such that for any \(i\), the subgraph \(T[P_i]\) of \(T\) induced by \(P_i\) is a path, as shown in Figure 1(a). The **partition tree** of a path partition \(\mathcal{P}\), denoted by \(T_{\mathcal{P}}(\mathcal{P})\) and illustrated in Figure 1(b), is the tree whose nodes are \(P_i\)'s and where the node of \(T_{\mathcal{P}}(\mathcal{P})\) corresponding to \(P_i\) is the parent of the node corresponding to \(P_j\) iff some node of \(P_i\) is the parent in \(T\) of the root of \(P_j\). The height of a path partition \(\mathcal{P}\) of a tree \(T\), denoted by \(h(\mathcal{P})\), is the height \(h(T_{\mathcal{P}}(\mathcal{P}))\) of its partition tree. The **path-height** of \(T\) is the minimal height of a path partition of \(T\), that is \(ph(T) = \min\{h(\mathcal{P}) \mid \mathcal{P} \text{ path partition of } T\}\).

\begin{figure}[h]
\centering
\begin{subfigure}[b]{0.45\textwidth}
\centering
\includegraphics[width=\textwidth]{tree}
\caption{(a)}
\end{subfigure} \hspace{0.5cm}
\begin{subfigure}[b]{0.45\textwidth}
\centering
\includegraphics[width=\textwidth]{partition}
\caption{(b)}
\end{subfigure}
\caption{A tree \(T\) and a path partition \(\mathcal{P} = \{P_1, P_2, P_3, P_4, P_5, P_6\}\) of \(T\) (a), as well as the partition tree of \(\mathcal{P}\) (b).}
\end{figure}
Lemma 1 For a rooted complete binary tree $T$, $\text{rank}(T) = \text{ph}(T) = h(T)$.

Theorem 2 For any rooted tree $T$, we have $\text{rank}(T) = \text{ph}(T)$.

Upper bounds for contiguity and linearity of cographs. We now combine the results of the previous section with a decomposition of the cotree of the input cograph into paths, in order to obtain a constructive proof that the contiguity of any cograph is at most $O(\log n)$. This decomposition is obtained recursively, using a root-path decomposition of the cotree, thanks to the Caterpillar Composition Lemma below.

A root-path decomposition (see Fig. 2) of a rooted tree $T$ is a set $\{T_1, \ldots, T_p\}$ of disjoint subtrees of $T$, with $p \geq 2$, such that every leaf of $T$ belongs to some $T_i$, with $i \in [1..p]$, and the sets of parents in $T$ of the roots of $T_i$'s is a path containing the root of $T$.

Figure 2: The root-path decomposition $\{T_1, \ldots, T_p\}$ of a rooted tree $T$.

Lemma 3 (Caterpillar Composition Lemma) Given a cograph $G = (V,E)$ and a root-path decomposition $\{T_i\}_{1 \leq i \leq p}$ of its cotree, where $X_i$ is the set of leaves of $T_i$, $\text{cont}(G) \leq 2 + \max_{i \in [1..p]} \text{cont}(G[X_i])$.

Lemma 4 Given a rooted tree $T$ such that $\text{rank}(T) = k \geq 1$, there exists a root-path decomposition $\{T_1, \ldots, T_p\}$ of $T$ such that for each $i \in [1..p]$, $\text{rank}(T_i) \leq k - 1$.

Lemma 5 Let $G$ be a cograph and $T$ its cotree. We have $\text{cont}(G) \leq 2 \text{rank}(T) + 1$.

Theorem 6 The closed contiguity of a cograph is at most logarithmic in its number of vertices, or more formally, if $G = (V,E)$ is a cograph, then $\text{cont}(G) \leq 2 \log_2 |V| + 1$.

Lower bounds for contiguity and linearity of cographs. Finally, we focus on cographs whose cotrees are complete binary trees, and obtain a tight lower bound for their asymptotic contiguity as well as a lower bound for their asymptotic linearity.

Theorem 7 Let $G$ be a cograph whose cotree is a complete binary tree. Then, $\text{cont}(G) = \Omega(\log n)$ and $\text{lin}(G) = \Omega(\log n / \log \log n)$.

References

