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(Nearly-)Tight Bounds on the Linearity and Contiguity of Cographs

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EXTENDED ABSTRACT

Introduction. Linearity and contiguity are graph parameters introduced to obtain efficient codings of neighborhoods in graphs, by decomposing each neighborhood as a union of \( p \) intervals chosen in one or several orders on the vertices [1]. Indeed, storing an order of the vertices as well as a pair of pointers for each of the \( p \) intervals of this order (one pointer for the beginning of the interval and one for the end), with fixed \( p \), allows to store the graph in \( O(n) \) space (instead of \( O(n+m) \) with adjacency lists) and access the neighborhood of any vertex \( v \) in \( O(d) \) time (instead of \( O(n) \) with adjacency matrices), where \( d \) is the degree of \( v \).

More formally, a closed \( p \)-interval-model of a graph \( G = (V,E) \) is a linear order \( \sigma \) on \( V \) such that \( \forall v \in V, \exists (I_1, \ldots, I_p) \in (2^V)^p \) such that \( \forall i \in \{1, \ldots, p\} \), \( I_i \) is an interval of \( \sigma \) and \( N[x] = \bigcup_{1 \leq i \leq p} I_i \). The closed contiguity of \( G \), denoted by \( \text{cont}(G) \), is the minimum integer \( p \) such that there exists a closed \( p \)-interval-model of \( G \). A closed \( p \)-line-model of a graph \( G = (V,E) \) is a tuple \((\sigma_1, \ldots, \sigma_p)\) of linear orders on \( V \) such that \( \forall v \in V, \exists (I_1, \ldots, I_p) \in (2^V)^p \) such that \( \forall i \in \{1, \ldots, p\} \), \( I_i \) is an interval of \( \sigma_i \) and \( N[x] = \bigcup_{1 \leq i \leq p} I_i \). The closed linearity of \( G \), denoted by \( \text{lin}(G) \), is the minimum \( p \) such that there exists a closed \( p \)-line-model of \( G \).

Not much is known about these parameters, which cannot be bounded by a constant even in very restricted graph classes, like interval or permutation graphs [1]. We focus here on the contiguity and linearity of cographs (graphs without induced \( P_4 \) subgraphs), whose very constrained structure can be represented by their cotree, a rooted tree with two kinds of nodes labeled by \( P \) and \( S \), giving a tight upper bound for the asymptotic contiguity of cographs and an upper bound for their linearity. To this aim, we first establish a min-max theorem on the link between the rank of rooted trees and their decompositions into paths.

A min-max theorem on the rank of a tree. The rank [2, 3] of a tree \( T \) is the maximal height of a complete binary tree obtained from \( T \) by edge contractions, that is \( \text{rank}(T) = \max \{ h(T') \mid T' \text{ complete binary tree, minor of } T \} \).

A path partition of a tree \( T \) is a partition \( \{P_1, \ldots, P_k\} \) of \( V(T) \) such that for any \( i \), the subgraph \( T[P_i] \) of \( T \) induced by \( P_i \) is a path, as shown in Figure 1(a). The partition tree of a path partition \( \mathcal{P} \), denoted by \( T_\mathcal{P}(\mathcal{P}) \) and illustrated in Figure 1(b), is the tree whose nodes are \( P_i \)'s and where the node of \( T_\mathcal{P}(\mathcal{P}) \) corresponding to \( P_i \) is the parent of the node corresponding to \( P_j \) iff some node of \( P_i \) is the parent of \( P_j \) in \( T \) of the root of \( P_j \). The height of a path partition \( \mathcal{P} \) of a tree \( T \), denoted by \( h(\mathcal{P}) \), is the height \( h(T_\mathcal{P}(\mathcal{P})) \) of its partition tree. The path-height of \( T \) is the minimal height of a path partition of \( T \), that is \( \text{ph}(T) = \min \{ h(\mathcal{P}) \mid \mathcal{P} \text{ path partition of } T \} \).

![Figure 1: A tree \( T \) and a path partition \( \mathcal{P} = \{P_1, P_2, P_3, P_4, P_5, P_6\} \) of \( T \) (a), as well as the partition tree of \( \mathcal{P} \) (b).](image)
Lemma 1 For a rooted complete binary tree T, rank(T) = ph(T) = h(T).

Theorem 2 For any rooted tree T, we have rank(T) = ph(T).

Upper bounds for contiguity and linearity of cographs. We now combine the results of the previous section with a decomposition of the cotree of the input cograph into paths, in order to obtain a constructive proof that the contiguity of any cograph is at most \( O(\log n) \). This decomposition is obtained recursively, using a root-path decomposition of the cotree, thanks to the Caterpillar Composition Lemma below.

A root-path decomposition (see Fig. 2) of a rooted tree T is a set \( \{T_1, \ldots, T_p\} \) of disjoint subtrees of T, with \( p \geq 2 \), such that every leaf of T belongs to some \( T_i \), with \( i \in [1..p] \), and the sets of parents in T of the roots of \( T_i \)'s is a path containing the root of T.

Figure 2: The root-path decomposition \( \{T_1, \ldots, T_p\} \) of a rooted tree T.

Lemma 3 (Caterpillar Composition Lemma) Given a cograph \( G = (V, E) \) and a root-path decomposition \( \{T_i\}_{1 \leq i \leq p} \) of its cotree, where \( X_i \) is the set of leaves of \( T_i \), \( \operatorname{cont}(G) \leq 2 + \max_{i \in [1..p]} \operatorname{cont}(G[X_i]) \).

Lemma 4 Given a rooted tree T such that rank(T) = \( k \geq 1 \), there exists a root-path decomposition \( \{T_1, \ldots, T_p\} \) of T such that for each \( i \in [1..p] \), \( \operatorname{rank}(T_i) \leq k - 1 \).

Lemma 5 Let G be a cograph and T its cotree. We have \( \operatorname{cont}(G) \leq 2 \operatorname{rank}(T) + 1 \).

Theorem 6 The closed contiguity of a cograph is at most logarithmic in its number of vertices, or more formally, if \( G = (V, E) \) is a cograph, then \( \operatorname{cont}(G) \leq 2 \log_2 |V| + 1 \).

Lower bounds for contiguity and linearity of cographs. Finally, we focus on cographs whose cotrees are complete binary trees, and obtain a tight lower bound for their asymptotic contiguity as well as a lower bound for their asymptotic linearity.

Theorem 7 Let G be a cograph whose cotree is a complete binary tree. Then, \( \operatorname{cont}(G) = \Omega(\log n) \) and \( \operatorname{lin}(G) = \Omega(\log n / \log \log n) \).

References

