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(Nearly-)Tight Bounds on the Linearity and Contiguity of Cographs

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Extended Abstract

Introduction. Linearity and contiguity are graph parameters introduced to obtain efficient codings of neighborhoods in graphs, by decomposing each neighborhood as a union of $p$ intervals chosen in one or several orders on the vertices [1]. Indeed, storing an order of the vertices as well as a pair of pointers for each of the $p$ intervals of this order (one pointer for the beginning of the interval and one for the end), with fixed $p$, allows to store the graph in $O(n)$ space (instead of $O(n^2)$ with adjacency matrices) and access the neighborhood of any vertex $v$ in $O(d)$ time (instead of $O(n)$ with adjacency matrices), where $d$ is the degree of $v$.

More formally, a closed $p$-interval-model of a graph $G = (V,E)$ is a linear order $σ$ on $V$ such that $∀v ∈ V, ∃(I_1,\ldots,I_p) ∈ (2^V)^p$ such that $∀i ∈ [1,p], I_i$ is an interval of $σ$ and $N[v] = \bigcup_{1≤i≤p} I_i$. The closed contiguity of $G$, denoted by $cont(G)$, is the minimum integer $p$ such that there exists a closed $p$-interval-model of $G$. A closed $p$-line-model of a graph $G = (V,E)$ is a tuple $(σ_1,\ldots,σ_p)$ of linear orders on $V$ such that $∀v ∈ V, ∃(I_1,\ldots,I_p) ∈ (2^V)^p$ such that $∀i ∈ [1,p], I_i$ is an interval of $σ_i$ and $N[x] = \bigcup_{1≤i≤p} I_i$. The closed linearity of $G$, denoted by $lin(G)$, is the minimum $p$ such that there exists a closed $p$-line-model of $G$.

Not much is known about these parameters, which cannot be bounded by a constant even in very restricted graph classes, like interval or permutation graphs [1]. We focus here on the contiguity and linearity of cographs (graphs without induced $P_4$ subgraphs), whose very constrained structure can be represented by their cotree, a rooted tree with two kinds of nodes labeled by $P$ and $S$, giving a tight upper bound for the asymptotic contiguity of cographs and an upper bound for their linearity. To this aim, we first establish a min-max theorem on the link between the rank of rooted trees and their decompositions into paths.

A min-max theorem on the rank of a tree. The rank [2, 3] of a tree $T$ is the maximal height of a complete binary tree obtained from $T$ by edge contractions, that is $rank(T) = \max\{h(T') \mid T' \text{ complete binary tree, minor of } T\}$.

A path partition of a tree $T$ is a partition $\{P_1,\ldots,P_k\}$ of $V(T)$ such that for any $i$, the subgraph $T[P_i]$ of $T$ induced by $P_i$ is a path, as shown in Figure 1(a). The partition tree of a path partition $P$, denoted by $T_p(P)$ and illustrated in Figure 1(b), is the tree whose nodes are $P_i$’s and where the node of $T_p(P)$ corresponding to $P_i$ is the parent of the node corresponding to $P_j$ iff some node of $P_i$ is the parent in $T$ of the root of $P_j$. The height of a path partition $P$ of a tree $T$, denoted by $h(P)$, is the height $h(T_p(P))$ of its partition tree. The path-height of $T$ is the minimal height of a path partition of $T$, that is $ph(T) = \min\{h(P) \mid P \text{ path partition of } T\}$.

Figure 1: A tree $T$ and a path partition $P = \{P_1, P_2, P_3, P_4, P_5, P_6\}$ of $T$ (a), as well as the partition tree of $P$ (b).
**Lemma 1** For a rooted complete binary tree $T$, $\text{rank}(T) = \text{ph}(T) = h(T)$.

**Theorem 2** For any rooted tree $T$, we have $\text{rank}(T) = \text{ph}(T)$.

Upper bounds for contiguity and linearity of cographs. We now combine the results of the previous section with a decomposition of the cotree of the input cograph into paths, in order to obtain a constructive proof that the contiguity of any cograph is at most $O(\log n)$. This decomposition is obtained recursively, using a root-path decomposition of the cotree, thanks to the Caterpillar Composition Lemma below.

A **root-path decomposition** (see Fig. 2) of a rooted tree $T$ is a set $\{T_1, \ldots, T_p\}$ of disjoint subtrees of $T$, with $p \geq 2$, such that every leaf of $T$ belongs to some $T_i$, with $i \in [1..p]$, and the sets of parents in $T$ of the roots of $T_i$’s is a path containing the root of $T$.

![Figure 2: The root-path decomposition $\{T_1, \ldots, T_p\}$ of a rooted tree $T$.](image)

**Lemma 3** (Caterpillar Composition Lemma) Given a cograph $G = (V, E)$ and a root-path decomposition $\{T_i\}_{1 \leq i \leq p}$ of its cotree, where $X_i$ is the set of leaves of $T_i$, $\text{cont}(G) \leq 2 + \max_{i \in [1..p]} \text{cont}(G[X_i])$.

**Lemma 4** Given a rooted tree $T$ such that $\text{rank}(T) = k \geq 1$, there exists a root-path decomposition $\{T_1, \ldots, T_p\}$ of $T$ such that for each $i \in [1..p]$, $\text{rank}(T_i) \leq k - 1$.

**Lemma 5** Let $G$ be a cograph and $T$ its cotree. We have $\text{cont}(G) \leq 2 \text{rank}(T) + 1$.

**Theorem 6** The closed contiguity of a cograph is at most logarithmic in its number of vertices, or more formally, if $G = (V, E)$ is a cograph, then $\text{cont}(G) \leq 2 \log_2 |V| + 1$.

Lower bounds for contiguity and linearity of cographs. Finally, we focus on cographs whose cotrees are complete binary trees, and obtain a tight lower bound for their asymptotic contiguity as well as a lower bound for their asymptotic linearity.

**Theorem 7** Let $G$ be a cograph whose cotree is a complete binary tree. Then, $\text{cont}(G) = \Omega(\log n)$ and $\text{lin}(G) = \Omega(\log n / \log \log n)$.

**References**

