(Nearly-)Tight Bounds on the Linearity and Contiguity of Cographs

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EXTENDED ABSTRACT

Introduction. Linearity and contiguity are graph parameters introduced to obtain efficient codings of neighborhoods in graphs, by decomposing each neighborhood as a union of \( p \) intervals chosen in one or several orders on the vertices \([1]\). Indeed, storing an order of the vertices as well as a pair of pointers for each of the \( p \) intervals of this order (one pointer for the beginning of the interval and one for the end), with fixed \( p \), allows to store the graph in \( O(n) \) space (instead of \( O(n + m) \) with adjacency lists) and access the neighborhood of any vertex \( v \) in \( O(d) \) time (instead of \( O(n) \) with adjacency matrices), where \( d \) is the degree of \( v \).

More formally, a closed \( p \)-interval-model of a graph \( G = (V, E) \) is a linear order \( \sigma \) on \( V \) such that \( \forall v \in V, \exists (I_1, \ldots, I_p) \in (2^V)^p \) such that \( \forall i \in 1..p, I_i \) is an interval of \( \sigma \) and \( N[v] = \bigcup_{1 \leq i \leq p} I_i \). The closed contiguity of \( G \), denoted by \( \text{cont}(G) \), is the minimum integer \( p \) such that there exists a closed \( p \)-interval-model of \( G \). A closed \( p \)-line-model of a graph \( G = (V, E) \) is a tuple \((\sigma_1, \ldots, \sigma_p)\) of linear orders on \( V \) such that \( \forall v \in V, \exists (I_1, \ldots, I_p) \in (2^V)^p \) such that \( \forall i \in 1..p, I_i \) is an interval of \( \sigma_i \) and \( N[v] = \bigcup_{1 \leq i \leq p} I_i \). The closed linearity of \( G \), denoted by \( \text{lin}(G) \), is the minimum \( p \) such that there exists a closed \( p \)-line-model of \( G \).

Not much is known about these parameters, which cannot be bounded by a constant even in very restricted graph classes, like interval or permutation graphs \([1]\). We focus here on the contiguity and linearity of cographs (graphs without induced \( P_4 \) subgraphs), whose very constrained structure can be represented by their ctree, a rooted tree with two kinds of nodes labeled by \( P \) and \( S \), giving a tight upper bound for the asymptotic contiguity of cographs and an upper bound for their linearity. To this aim, we first establish a min-max theorem on the link between the rank of rooted trees and their decompositions into paths.

A min-max theorem on the rank of a tree. The rank \([2, 3]\) of a tree \( T \) is the maximal height of a complete binary tree obtained from \( T \) by edge contractions, that is \( \text{rank}(T) = \max\{h(T') \mid T' \text{ complete binary tree, minor of } T\} \).

A path partition of a tree \( T \) is a partition \( \{P_1, \ldots, P_k\} \) of \( V(T) \) such that for any \( i \), the subgraph \( T[P_i] \) of \( T \) induced by \( P_i \) is a path, as shown in Figure 1(a). The partition tree of a path partition \( \mathcal{P} \), denoted by \( T_{\mathcal{P}}(\mathcal{P}) \) and illustrated in Figure 1(b), is the tree whose nodes are \( P_i \)'s and where the node of \( T_{\mathcal{P}}(\mathcal{P}) \) corresponding to \( P_i \) is the parent of the node corresponding to \( P_j \) iff some node of \( P_i \) is the parent in \( T \) of the root of \( P_j \). The height of a path partition \( \mathcal{P} \) of a tree \( T \), denoted by \( h(\mathcal{P}) \), is the height \( h(T_{\mathcal{P}}(\mathcal{P})) \) of its partition tree. The path-height of \( T \) is the minimal height of a path partition of \( T \), that is \( ph(T) = \min\{h(\mathcal{P}) \mid \mathcal{P} \text{ path partition of } T\} \).

![Figure 1: A tree \( T \) and a path partition \( \mathcal{P} = \{P_1, P_2, P_3, P_4, P_5, P_6\} \) of \( T \) (a), as well as the partition tree of \( \mathcal{P} \) (b).](image-url)
Lemma 1 For a rooted complete binary tree $T$, $\text{rank}(T) = \text{ph}(T) = h(T)$.

Theorem 2 For any rooted tree $T$, we have $\text{rank}(T) = \text{ph}(T)$.

Upper bounds for contiguity and linearity of cographs. We now combine the results of the previous section with a decomposition of the cotree of the input cograph into paths, in order to obtain a constructive proof that the contiguity of any cograph is at most $O(\log n)$. This decomposition is obtained recursively, using a root-path decomposition of the cotree, thanks to the Caterpillar Composition Lemma below.

A root-path decomposition (see Fig. 2) of a rooted tree $T$ is a set $\{T_1, \ldots, T_p\}$ of disjoint subtrees of $T$, with $p \geq 2$, such that every leaf of $T$ belongs to some $T_i$, with $i \in [1..p]$, and the sets of parents in $T$ of the roots of $T_i$'s is a path containing the root of $T$.

![Figure 2: The root-path decomposition $\{T_1, \ldots, T_p\}$ of a rooted tree $T$.](image)

Lemma 3 (Caterpillar Composition Lemma) Given a cograph $G = (V,E)$ and a root-path decomposition $\{T_i\}_{1 \leq i \leq p}$ of its cotree, where $X_i$ is the set of leaves of $T_i$, $\text{cont}(G) \leq 2 + \max_{i \in [1..p]} \text{cont}(G[X_i])$.

Lemma 4 Given a rooted tree $T$ such that $\text{rank}(T) = k \geq 1$, there exists a root-path decomposition $\{T_1, \ldots, T_p\}$ of $T$ such that for each $i \in [1..p]$, $\text{rank}(T_i) \leq k - 1$.

Lemma 5 Let $G$ be a cograph and $T$ its cotree. We have $\text{cont}(G) \leq 2 \text{rank}(T) + 1$.

Theorem 6 The closed contiguity of a cograph is at most logarithmic in its number of vertices, or more formally, if $G = (V,E)$ is a cograph, then $\text{cont}(G) \leq 2 \log_2 |V| + 1$.

Lower bounds for contiguity and linearity of cographs. Finally, we focus on cographs whose cotrees are complete binary trees, and obtain a tight lower bound for their asymptotic contiguity as well as a lower bound for their asymptotic linearity.

Theorem 7 Let $G$ be a cograph whose cotree is a complete binary tree. Then, $\text{cont}(G) = \Omega(\log n)$ and $\text{lin}(G) = \Omega(\log n / \log \log n)$.

References

