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(Nearly-)Tight Bounds on the Linearity and Contiguity of Cographs

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Extended Abstract

\textbf{Introduction.} Linearity and contiguity are graph parameters introduced to obtain efficient codings of neighborhoods in graphs, by decomposing each neighborhood as a union of $p$ intervals chosen in one or several orders on the vertices \cite{1}. Indeed, storing an order of the vertices as well as a pair of pointers for each of the $p$ intervals of this order (one pointer for the beginning of the interval and one for the end), with fixed $p$, allows to store the graph in $O(n)$ space (instead of $O(n+m)$ with adjacency lists) and access the neighborhood of any vertex $v$ in $O(d)$ time (instead of $O(n)$ with adjacency matrices), where $d$ is the degree of $v$.

More formally, a \textit{closed} $p$-interval-model of a graph $G = (V,E)$ is a linear order $\sigma$ on $V$ such that $\forall v \in V, \exists (I_1, \ldots, I_p) \in (2^V)^p$ such that $\forall i \in [1,p], I_i$ is an interval of $\sigma$ and $N[v] = \bigcup_{1 \leq i \leq p} I_i$. The \textit{closed contiguity} of $G$, denoted by $\text{cont}(G)$, is the minimum integer $p$ such that there exists a closed $p$-interval-model of $G$. A \textit{closed} $p$-line-model of a graph $G = (V,E)$ is a tuple $(\sigma_1, \ldots, \sigma_p)$ of linear orders on $V$ such that $\forall v \in V, \exists (I_1, \ldots, I_p) \in (2^V)^p$ such that $\forall i \in [1,p], I_i$ is an interval of $\sigma_i$ and $N[v] = \bigcup_{1 \leq i \leq p} I_i$. The \textit{closed linearity} of $G$, denoted by $\text{lin}(G)$, is the minimum $p$ such that there exists a closed $p$-line-model of $G$.

Not much is known about these parameters, which cannot be bounded by a constant even in very restricted graph classes, like interval or permutation graphs \cite{1}. We focus here on the contiguity and linearity of cographs (graphs without induced $P_4$ subgraphs), whose very constrained structure can be represented by their \textit{cotree}, a rooted tree with two kinds of nodes labeled by $P$ and $S$, giving a tight upper bound for the asymptotic contiguity of cographs and an upper bound for their linearity. To this aim, we first establish a min-max theorem on the link between the rank of rooted trees and their decompositions into paths.

A \textbf{min-max theorem on the rank of a tree.} The rank \cite{2, 3} of a tree $T$ is the maximal height of a complete binary tree obtained from $T$ by edge contractions, that is $\text{rank}(T) = \max \{ h(T') \mid T'$ complete binary tree, minor of $T \}$.

A \textit{path partition} of a tree $T$ is a partition $\{P_1, \ldots, P_k\}$ of $V(T)$ such that for any $i$, the subgraph $T[P_i]$ of $T$ induced by $P_i$ is a path, as shown in Figure 1(a). The \textit{partition tree} of a path partition $\mathcal{P}$, denoted by $T_p(\mathcal{P})$ and illustrated in Figure 1(b), is the tree whose nodes are $P_i$'s and where the node of $T_p(\mathcal{P})$ corresponding to $P_i$ is the parent of the node corresponding to $P_j$ iff some node of $P_i$ is the parent in $T$ of the root of $P_j$. The height of a path partition $\mathcal{P}$ of a tree $T$, denoted by $h(\mathcal{P})$, is the height $h(T_p(\mathcal{P}))$ of its partition tree. The \textit{path-height} of $T$ is the minimal height of a path partition of $T$, that is $\text{ph}(T) = \min \{ h(\mathcal{P}) \mid \mathcal{P} \text{ path partition of } T \}$.

\begin{figure}[h]
\begin{center}
\includegraphics[width=0.6\textwidth]{tree_partition.png}
\end{center}
\caption{A tree $T$ and a path partition $\mathcal{P} = \{P_1, P_2, P_3, P_4, P_5, P_6\}$ of $T$ (a), as well as the partition tree of $\mathcal{P}$ (b).}
\end{figure}
Lemma 1 For a rooted complete binary tree $T$, $\text{rank}(T) = \text{ph}(T) = h(T)$.

Theorem 2 For any rooted tree $T$, we have $\text{rank}(T) = \text{ph}(T)$.

Upper bounds for contiguity and linearity of cographs. We now combine the results of the previous section with a decomposition of the cotree of the input cograph into paths, in order to obtain a constructive proof that the contiguity of any cograph is at most $O(\log n)$. This decomposition is obtained recursively, using a root-path decomposition of the cotree, thanks to the Caterpillar Composition Lemma below.

A root-path decomposition (see Fig. 2) of a rooted tree $T$ is a set $\{T_1, \ldots, T_p\}$ of disjoint subtrees of $T$, with $p \geq 2$, such that every leaf of $T$ belongs to some $T_i$, with $i \in [1..p]$, and the sets of parents in $T$ of the roots of $T_i$’s is a path containing the root of $T$.

![Figure 2: The root-path decomposition $\{T_1, \ldots, T_p\}$ of a rooted tree $T$.](image)

Lemma 3 (Caterpillar Composition Lemma) Given a cograph $G = (V,E)$ and a root-path decomposition $\{T_i\}_{1 \leq i \leq p}$ of its cotree, where $X_i$ is the set of leaves of $T_i$, $\text{cont}(G) \leq 2 + \max_{i \in [1..p]} \text{cont}(G[X_i])$.

Lemma 4 Given a rooted tree $T$ such that $\text{rank}(T) = k \geq 1$, there exists a root-path decomposition $\{T_1, \ldots, T_p\}$ of $T$ such that for each $i \in [1..p]$, $\text{rank}(T_i) \leq k - 1$.

Lemma 5 Let $G$ be a cograph and $T$ its cotree. We have $\text{cont}(G) \leq 2 \text{rank}(T) + 1$.

Theorem 6 The closed contiguity of a cograph is at most logarithmic in its number of vertices, or more formally, if $G = (V,E)$ is a cograph, then $\text{cont}(G) \leq 2 \log_2 |V| + 1$.

Lower bounds for contiguity and linearity of cographs. Finally, we focus on cographs whose cotrees are complete binary trees, and obtain a tight lower bound for their asymptotic contiguity as well as a lower bound for their asymptotic linearity.

Theorem 7 Let $G$ be a cograph whose cotree is a complete binary tree. Then, $\text{cont}(G) = \Omega(\log n)$ and $\text{lin}(G) = \Omega(\log n / \log \log n)$.

References

