Topology on digital label images
Loïc Mazo, Nicolas Passat, Michel Couprie, Christian Ronse

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Response to Editor:

**Editor:** We have received the reports from our advisors on your manuscript, "Topology on digital label images", which you submitted to Journal of Mathematical Imaging and Vision.

Based on the advice received, the Editor feels that your manuscript could be accepted for publication should you be prepared to incorporate minor revisions. When preparing your revised manuscript, you are asked to carefully consider the reviewer comments, which are attached, and submit a list of responses to the comments. Your list of responses should be uploaded as a file in addition to your revised manuscript.

**Authors:** We would like to thank the Editor for the management of the review process of this manuscript. It has been revised by taking into account the suggestions of the reviewer.

Response to Reviewer #2:

**Authors:** We would like to thank the Reviewer for having taken time to evaluate our manuscript. A point-by-point response to each of the issues raised by the Reviewer is given below.

**Referee:** The authors have made different corrections that significantly improve their article. However, I would greatly appreciate that some efforts will be made to give an easier access to the paper content for novice readers.

For example, add some references at the beginning of section 2.1, the books of Munkres (Elements of Algebraic Topology), Hatcher (Algebraic Topology) or Giblin (Graphs, Surfaces and Homology) are valuable references for novice reader. When adding references pay particular attention to the availability of it and be sure that the style and the way of the concepts are presented are adapted to the "modern" readers. For example, page 4 line 36: of course Whitehead as defined elementary transformations on complexes but I am not that this reference is easy to find and it would really help the reader, again a book like Giblin’s book is more adapted.
**Authors:** We have added these references together with Maunder *Algebraic Topology* and May *A Concise Course in Algebraic Topology*.

Page 4 line 36: we have given two references, the original one (which is freely available on the web) and Giblin’s book.

**Referee:** Page 4, line 9, second column: You talk about topological spaces with base points without introducing them. After lines 12 and following: The tentative explanation about weak homotopy equivalence is very hard to follow if you don’t know what is the homotopy class of a map.

**Authors:** A subsection about homotopy have been added in order to introduce the minimal knowledge of algebraic topology that is needed in the paper.

**Referee:** page 5 line 2: exemple 24 is very far from this text.

**Authors:** A new figure has been added just after the text.

**Referee:** page 5 section 2.3: in the first paragraph you talk about A-space and the about Alexandroff space this is puzzling.

**Authors:** This has been corrected.

**Referee:** page 6, lines 45-50. You use property 3 that is given below and you refer to property 6 that is on page 6 in section 2.4 after the introduction of unipolar points. Please reconsider the redaction of this paragraph.

**Authors:** The paragraph has been rewritten (and the references to properties 3 and 6 have been removed).
Topology on digital label images

Loïc Mazo · Nicolas Passat · Michel Couprie · Christian Ronse

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Abstract In digital imaging, after several decades devoted to the study of topological properties of binary images, there is an increasing need of new methods enabling to take into (topological) consideration n-ary images (also called label images). Indeed, while binary images enable to handle one object of interest, label images authorize to simultaneously deal with a plurality of objects, which is a frequent requirement in several application fields. In this context, one of the main purposes is to propose topology-preserving transformation procedures for such label images, thus extending the ones (e.g., growing, reduction, skeletonisation) existing for binary images. In this article, we propose, for a wide range of digital images, a new approach that permits to locally modify a label image, while preserving not only the topology of each label set, but also the topology of any arrangement of the labels understood as the topology of any union of label sets. This approach enables in particular to unify and extend some previous attempts devoted to the same purpose.

Keywords digital imaging · topology · label images · homotopy · simple points

1 Introduction

In a digital image, when performing processes such as registration, deformation or thinning, the preservation of the topological properties of the objects contained in the image (e.g., connected components, tunnels, cavities, etc.) is an important requirement. For 50 years, several tools enabling the analysis (adjacency graphs, digital fundamental groups, homology groups –see, e.g., [1, 2, 3]) and the modification under topological constraints (simple points, P-simple points, simple sets –see, e.g., [4, 5, 6, 7]) of binary images have been proposed and used. Nevertheless, in many fields (e.g., medical imaging, remote sensing, computer vision), an image is generally composed of several objects, and it is often important to understand or maintain their topological properties all together, that is the topology of each and the topology of the scene. In such images, the objects are characterised by specific labels on which there generally exists no meaningful order relation (unlike grey-level images for instance).

1.1 Previous works

To the best of our knowledge, the literature about topology in label images is quite limited and generally motivated by practical considerations. The most common approach is to consider only one label at a time, the other labels being momentarily considered as a part of the background. However, except in the most simple cases where the label configuration leads to a binary modelling (see, e.g., [8, 9]), one cannot directly deal with the relations between the labels but only with the topology of each label and of its associated background [10, 11, 12] (if necessary, one uses in addition an adjacency tree between labels in order to control their topological relations). These methods are often used with a cost function, which depends on the applicative context, whose...
Fig. 1 An image with two labels (in grey and black). If we consider the grey label as the object of the picture using the (8,4)-adjacency pair (8-adjacency for the object and 4-adjacency for the background), the object is a ring. The black pixels together then form the inner component of the background, while the white pixels form the outer component. However, if we now consider the black pixels as the object (still in 8-adjacency), rejecting grey pixels to the background, these latters must be understood with the 4-adjacency and they appear to have two connected components, one inside the black torus and one outside.

Fig. 2 Forbidden configurations in (binary) well composed images. (a) In $Z^2$, (b,c) In $Z^3$ (configuration (b) shall not appear neither in the object nor in the background). A label image is well composed if each binary image obtained by isolating a particular label is well composed.

Purpose is to assign a given label, or not, to a point of the image. Thereby points go from background to a label or vice versa but not from a label to another. Note that some points may sometimes take an undetermined status since they cannot be assigned a label without breaking a topology defined by an a priori knowledge or to avoid object crossings when the objects are seen under the filter of the 8-adjacency in the plane or 26-adjacency in the space (see Figure 1). The question of the adjacencies to be used in a digital label image is a recurrent issue. Indeed, in digital topology, in the framework developed by Rosenfeld [13], the object and the background of an image are understood with different (dual) adjacencies [14]. So, when objects in a label image are processed one at a time, being alternately the object and part of the background, they are inevitably seen under two distinct adjacencies. For instance, an object can have one connected component at one step of the process and two components at the next step though no change did occur on the image (see Figure 1).

To overcome this problem, a class of “well composed” images has been defined in which the same adjacency relation can be used for the object and the background. This adjacency relation is necessarily the 4-adjacency in 2D images and the 6-adjacency in 3D images [17]. This class of images is obtained by excluding all the images in which at least one of the three configurations depicted on Figure 2 appears. In other words, it is assumed in these images that the boundaries of the objects (viewed as an union of n-cubes) are $(n-1)$-manifolds. In the case where label images present forbidden configurations, an algorithm has been proposed to dispose of them [18]. However, since the objects identified by the labels are sequentially “repaired”, one needs first to determine an order on the labels, and this order biases the result.

Another approach [19] takes further the specificity of label images into account. A notion of “homotopy set” is defined, which is the set of the labels that can be assigned to a point without modification on the topology of each label and of its complement in the image. A local criterion is provided to decide whether a particular label belongs to the homotopy set of a point or not. Thereby, a point can move from a label to another and not solely from the background to a label or vice versa.

In [20], the authors go further and require, before any change of label at a point, the guarantee that not only the topology of each label will be preserved but also the topology of the unions of two labels in 2D images and of three labels in 3D images (see Figure 3). Nevertheless, this request is not sufficient. Figure 3 (c) provides a counterexample in 2D where there is the need to consider the union of three labels.

In [15], the authors study 3D label images with a frontier approach. The 3D image is divided into regions which are 6-connected (hence, the configurations of Figure 2 cannot occur) and in which the voxels share the same label. Moreover, they only take into account the 6-adjacency between regions. To move a voxel $x$ from a region $A$ to another region $B$, the authors make requirements on surfaces between $x$ and $A \setminus \{x\}$ (resp. between $x$ and $B \setminus \{x\}$): they have to be homeomorphic to a 2-disk. Furthermore, for each region $C$ 6-adjacent to $x$, the frontier between the regions $A$ and $C$ before the move (resp. between $B$ and $C$ after the move), must collapse onto the corresponding frontier after the move (resp. before the move).
The authors observe that, if we look at the picture with the characteristics of the sets of a partition of which amounts to require topologically sound procedures ages, following the idea to preserve any union of labels, (a) An image with four labels. (b) The label of a single pixel has changed. Neither the topologies of the labels nor of their complements in the image are modified. However, the topology of the partition is not preserved in the sense that the union 1 + 2 becomes contractible, 1 + 3 is split into two components in 4-adjacency, 3 + 4 loses a component, 2 + 3 + 4 loses a component in 4-adjacency. (c) This example is from 20. The authors observe that, if we look at the picture with the (8, 4)-adjacency pair, the central pixel can move from 3 to 2 without altering the topologies of the four labels and of the six pairs of labels but they do not take into consideration the union 1 + 3 + 4 though it passes from a ball to a ring. Observe also that the well-composedness of this image is destroyed by the move of the central pixel from 3 to 2.

1.2 Purpose

The aim of this article is to study the topology of label images, following the idea to preserve any union of labels, which amounts to require topologically sound procedures on digital label images not to change the topological characteristics of the sets of a partition of \( \mathbb{Z}^n \) and of any coarser partition of the initial one. In other words, one could say that the actual set of objects in a digital label image is the power set of the partition. We have adopted a theoretical standpoint with the will to cover a wide range of situations. In our framework, we do not make any assumption on the topologies of individual objects (we do not use a priori knowledge) and there is no forbidden configurations. Weak homotopy equivalence in finite spaces (which corresponds to homotopy equivalence in continuous ones) is used to perform topological comparisons. To avoid the pitfall of distinct adjacency pairs on the same object described above, we embed the digital space of the image into a richer space equipped with a genuine topology, that is a poset whose minimal points are the points of the digital image. This enrichment of the space leads us to embed also the label set into a richer one, namely an atomistic lattice whose atoms are the labels of the digital image. Thereby, we can extend the digital image on its poset, assigning extended labels to new points, and we can define gradual modifications of the images more adapted to topology preservation.

1.3 Contribution and structure of the article

The remainder of this article is organised as follows.

Section 2 gathers results on binary images on which relies our work. It is intended to make the article self-contained and to introduce our notations. The last subsection of Section 2 establishes, in particular, two new results whose proofs are provided in Appendix B and C.

In Section 3 we introduce our framework for the topological understanding of label images. We describe a first tool to locally modify such a label image while keeping unchanged all homotopy groups of the objects and their unions (to be more precise, we have weak homotopy equivalences). When the poset is the space \( \mathbb{F}^n \) of cubical complexes defined in Section 2, our tool keeps also unchanged the homotopy groups of the complements. Furthermore, the changes can be processed in parallel under certain conditions, thus leading to well-balanced algorithms.

In Section 4 we are interested in images in which the sets of points that share a label (we say the support of the label) are closed sets, as in (26, 6) digital images. In this case, we define an elementary modification, named cut, inspired by collapses. It has the same (good) topological properties as the one defined in Section 5 while the supports of the labels remain closed sets.

In Section 5 we study regular images in which the label of a point in the poset is defined by the labels of the minimal points beneath it. Regular images can be built from digital images defined on \( \mathbb{Z}^n \) and we have proved in [22] that, when the poset is the space of cubical complexes, this construction puts in one-to-one correspondence the connected components of the regular image with the ones of the digital image. Moreover, it induces isomorphisms between the fundamental groups of the regular image and the digital fundamental groups of the digital image (as defined in [4]). In regular images, we give conditions for cuts to preserve regularity allowing thereby to modify a regular image in a topologically sound manner, the result being also a regular image (allowing to go back to \( \mathbb{Z}^n \)).

Section 6 concludes this paper and describes further works in preparation.

2 Simplicity in sets

The aim of this section is to gather notions and results on which relies this work, and also to present our notations. Note that in Section 2.3 we establish (new) results which are specific to complexes. Operations and relations on functions (in particular, on images) will always be implicitly pointwise ones.

2.1 Homotopy

Two continuous maps \( f, g : X \to Y \) are homotopic if there exists a continuous map, called a homotopy, \( h : X \times [0, 1] \to Y \) such that \( h(x, 0) = f(x) \) and \( h(x, 1) = g(x) \) for all \( x \in X \). The spaces \( X \) and \( Y \) are homotopy equivalent (or have the same homotopy type) if there exist two continuous maps.
Two cubical 3-complexes X and Y such that Y ⊂ X. Their geometric realisations have the same homotopy type and, therefore, are weakly homotopy equivalent. Nevertheless, it is clear that the inclusion i : Y → X is not a weak homotopy equivalence for it associates non-contractible loops to contractible loops. Likely, in image processing, we would reject such a thinning. So, the nature of the weak homotopy equivalence is an important information.

Fig. 4 (From [29]) (a) A cubical 3-complex X. (b) A subcomplex Y. Their geometric realisations have the same homotopy type. However, the inclusion i : Y → X is not a weak homotopy equivalence.

There is a case in which the weak homotopy equivalence reduces to the knowledge of the homotopy groups. When a set is weakly homotopy equivalent to a point, then it is connected and all its homotopy groups are trivial. Thus, obviously, any constant map is a weak homotopy equivalence. Such a space is said to be homotopically trivial. There are spaces that are homotopically trivial and that are not contractible as shown on Figure 4.

Fig. 5 A set of points (in red), closed lines (in yellow) and closed squares (in green) of \( \mathbb{R}^3 \) whose union forms a hollow cube with a fence. Equipped with the inclusion, this set is a finite topological space (see below Subsection 2.4) that is homotopically trivial but not contractible (the reader will be able to establish the proofs of these two assertions after the reading of Subsections 2.3 and 2.6).

2.2 Complexes

We do not recall definitions about simplicial complexes which are generally well known. The reader who wishes to recollect such a notion, or any one rapidly exposed below, is invited to find complementary information in a lecture book on algebraic topology, e.g. [30, 31, 32, 33, 34]. In digital images, grids are often cubic ones. It is then convenient, in image analysis, to replace simplices in complexes by n-cubes.
As cubical complexes are not commonly used, we recall hereafter the main basic definitions (see also [23]). We set $P^1_0 = \{(a) \mid a \in \mathbb{Z}\}$ and $P^1_1 = \{(a, a+1) \mid a \in \mathbb{Z}\}$. A subset $f$ of $\mathbb{Z}^n$ which is the Cartesian product of $m$ elements of $P^1_i$ and $n - m$ elements of $P^1_0$ is a face or an $m$-face (of $\mathbb{Z}^n$), $m$ is the dimension of $f$, and we write $\dim(f) = m$. We denote by $P^n_m$ the set composed of all $m$-faces of $\mathbb{Z}^n$ and by $P^n$ the set composed of all faces of $\mathbb{Z}^n$. Let $f \in P^n$ be a face. The set $\{g \in P^n \mid g \subseteq f\}$ is a cell and any union of cells is an abstract cubical complex. The geometric cubical complexes are defined in the same manner, except that we change the definition of $P^1_0$ by setting $P^1_0 = \{(a, a+1) \mid a \in \mathbb{Z}\} \subset \mathbb{R}^n$. The geometric realisation $|K|$ of a geometric cubical complex $K$ is the union of its faces. Figure 6 illustrates these definitions.

![Fig. 6](image)

(a) Four points in $\mathbb{Z}^2$, $a = (i, j)$, $b = (i+1, j)$, $c = (i+1, j+1)$, $d = (i, j+1)$. The faces $f = \{a\}$, $g = \{b, c\} = (i+1) \times (j, j+1)$ and $h = \{a, b, c, d\} = (i+1) \times (j, j+1)$ are symbolically depicted with ellipses. (b) Another (more semantic) symbolic representation, often used in this article. In black, the 0-face $f$. In dark grey, the 1-face $g$. In light grey, the 2-face $h$.

Whitehead [21] (an easier reference for modern readers is [34]) has defined elementary transformations on complexes as follows. Let $X$ be a complex (simplicial or cubical) and $(x, y, z)$ a pair of faces in $X$ such that $x$ is the only face of $X$ including $y$ (i.e., $X \setminus \{x, y\}$ is still a complex). Then, $(x, y, z)$ is a free pair, and the set $Y = X \setminus \{x, y\}$ is an elementary collapse of $X$, or $X$ is an elementary expansion of $Y$. If a set $Y$ is obtained from $X$ by a sequence of elementary collapses (a sequence of elementary collapses and expansions), then $X$ is a collapse of $Y$ and $X$ and $Y$ have the same simple-homotopy type and one writes $X \smallsetminus_Y Y$ ($X \smallsetminus_Y Y$). A set is collapsible if it collapses onto a singleton.

If $Y$ is a collapse of $X$ then $|Y|$ is a strong deformation retract of $|X|$ and thus $|X|$ and $|Y|$ are homotopy equivalent [21]. Figure 7 illustrates this property. In particular, if the complex is collapsible, its geometric realisation is contractible. The converse is not true as shown by the thin version of Bing’s house with two rooms [26] or by Zeeman’s dunce hat [27].

![Fig. 7](image)

(a) A complex $X$. (d) A complex $Y$ which is an elementary collapse of $X$. (b-c) Two steps in a strong deformation retraction of $|X|$ onto $|Y|$.

2.3 Partially ordered sets

The motivation for considering partially ordered sets (or posets) comes from (i) the observation that digital images are essentially finite (even when they are defined on $\mathbb{Z}^n$ to avoid difficulties on boundaries), (ii) that finite topological spaces of interest have the $T_0$-separation property but not the $T_1$-separation property (otherwise either some points could not be distinguished from a topological viewpoint or the space is totally disconnected), and (iii) that $T_0$-spaces in which any intersection of open sets is an open set (as in finite spaces) are posets [33, 36] (this point is developed in Section 2.4).

Let $X$ be a set. A binary relation on $X$ is a partial order if it is reflexive, antisymmetric, and transitive. A partially ordered set, or poset, is a couple $(X, \leq)$ where the relation $\leq$ is a partial order on $X$. The relation $\geq$, defined on $X$ by $x \geq y$ if $y \leq x$, is a partial order on $X$ called the dual order. We say that two points $x, y$ in $X$ are comparable if $x \leq y$ or $y \leq x$. If, for all pairs $(x, y)$ of elements of $X$, $x$ and $y$ are comparable, the relation $\leq$ is a total order on $X$. We write $x < y$ when $x \leq y$ and $x \neq y$ and we set:

- $x^\uparrow = \{y \in X \mid x \leq y\}$ and $x^{\uparrow\ast} = x^\uparrow \setminus \{x\} = \{y \in X \mid x < y\}$;
- $x^\downarrow = \{y \in X \mid y \leq x\}$ and $x^{\downarrow\ast} = x^\downarrow \setminus \{x\} = \{y \in X \mid y < x\}$.

If $x$ and $y$ are comparable, we write $x \approx y$; otherwise, we write $x \neq y$. The set of points comparable with a given point $x$ is denoted $x^{\uparrow\ast}$ ($x^\uparrow \cup x^\downarrow$), and we set $x^{\uparrow\ast\ast} = x^\uparrow \setminus \{x\} = x^\uparrow \setminus x^{\downarrow\ast}$. A point $x$ in $X$ is minimal if $x^\uparrow = \{x\}$ and maximal if $x^\downarrow = \{x\}$. A point $x$ in $X$ is the minimum of $X$ if $x^\downarrow = X$ and is the maximum of $X$ if $x^\uparrow = X$. We say that a poset is locally finite if for each point $x$ in $X$, there are finitely many points comparable with $x$. A chain in $X$ is a totally ordered subset of $X$. The length of a chain is its cardinality minus one. The length of a poset $X$ is the maximal length of a chain in $X$ if such a maximum exists. The height of a point $x$ in $X$, denoted $\text{ht}(x)$, is the length of $x^\downarrow$. If $x < y$ and there is no

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5 A space has the $T_0$-separation property if for any pair of distinct points there exists an open set that contains one of them and not the other.

4 A space has the $T_1$-separation property if for any ordered pair of distinct points there exists an open set that contains the first of them and not the other. Now, let $x$ be a point in a finite $T_1$-space $X$. For each $y \in X, y \neq x$, there exists an open neigbourhood of $x, U$, not containing $y$. Hence, $x = \bigcap U$, is open, that is to say, the topology on $X$ is the discrete topology in which all subsets are both open and closed and the only connected sets are the singletons.

3 Some authors define the length of a chain as its cardinality and the maximal length of a chain in $X$ is also called the height of $X$. 
where such that \( x < z < y \), we say that \( y \) covers \( x \) and we write \( x \prec y \). The Hasse diagram of the relation \( \preceq \) is the oriented graph of the relation \( < \). When orienting all arcs from bottom to top, this diagram offers a good visual representation of (small) posets (see Figure 8).

Fig. 8 The Hasse diagram of a poset defined by the set \([a, b, c, d, e]\) equipped with the order \([x, a, (a), b, (a, e), (b, b), (b, d), (b, e), (c, e), (d, e), (d, e)]\). Between parentheses, we give the height of the points. The length of this poset is 3.

Indeed, the choice to set \( x \preceq y \) if \( x \in U_i \) is purely arbitrary. We could set \( x \preceq y \) if \( y \in U_i \) and in literature both settings can be found (for instance, the choice \( x \preceq y \) if \( y \in U_i \) is made by McCord [35] and the other choice by [37, 39, 40, 41]).

If \( Y \) is a subset of \( X \), the topology associated to the poset \((Y, \preceq)\) is the topology induced by the one associated to the poset \((X, \preceq)\). The dual topology of the topology associated to the poset \((X, \preceq)\) is the topology associated to the dual order \(\succeq\).

From now on, posets will always be equipped with the topology \(U\) described in Theorem 1. This topology leads to a nice characterisation of continuous maps.

Property 2 ([39]) Let \( X, Y \) be posets. A function \( f : X \to Y \) is continuous if it is non-decreasing.

In the sequel, we will often have to test if a poset is contractible. Remember that a space is contractible if it has the homotopy type of a point, that is, if there exists a continuous map \( H : X \times [0, 1] \to X \) such that \( H(x, 0) = x \) for any \( x \in X \) and \( x \mapsto H(x, 1) \) is a constant map. Intuitively, a set is contractible if it can be continuously shrunk to a point. Nevertheless, this intuition is of little help in a finite space. For instance, consider a geometric cubical complex \( X \) composed of a closed unit square of \( \mathbb{R}^2 \), together with all its faces. Say, it is the one depicted on Figure 9(a). This complex is collapsible by \( X \setminus \{a, b\} \setminus \{d, e, f, h, i\} \setminus \{e, f, i\} \setminus \{i\} \). Since each elementary collapse is associated to a strong deformation retract in the Euclidean space \( \mathbb{R}^n \), the realisation of this unit square is contractible and one can actually continuously shrink the square following the above sequence of collapses (which first step is the one illustrated on Figure 7). Now, this complex, equipped with the inclusion, is also a poset (the Hasse diagram of which is depicted on Figure 9(b)). Hence, \( X \) is not only a combinatorial structure but also a topological space. However, we cannot follow the same steps to continuously shrink \( X \) as before. For instance, we cannot remove continuously the face \([a]\) from \( X \setminus \{b\} \) for there does not exist a non-decreasing function \( X \setminus \{b\} \to X \setminus \{a, b\} \). Furthermore, in [33], we have shown that if \( X \) is a two faces in \( \mathbb{R}^n \) such that \( y \preceq x \) the poset \( \{z \preceq x \mid z \neq y, \} \subseteq \), which looks like a sphere with a hole, is not contractible when \( \dim(y) \leq \dim(x) - 2 \). This is clearly counterintuitive.

Hopefully, even if we have to build a new intuition to deal with finite spaces, there exist very easy properties like the following one which provides a sufficient condition to
Fig. 9 (a) An abstract cubical cell $a^4$ which models a digital point of $Z^2$. (b) The Hasse diagram of $X(a^4)$. (c) The simplicial complex $\mathcal{X}(X(a^4))$. (d) The geometric realisation of $\mathcal{X}(X(a^4))$.

As the complex $\mathcal{X}(X)$ does not change if we consider the dual order on $X$, Theorem 4 implies that $(X, \leq)$ is weakly homotopy equivalent to $(X, \geq)$.

In the sequel of this section we direct our interest towards minimal deformations of the posets which do not alter their topology. To better understand the differences between the notions described below, we will take the same example all along the three next subsections. Consider the space $\Bbb{R}^3$ as defined in Subsection 2.2. The set $\Bbb{R}^3$ together with inclusion is obviously a poset. Let $x_0$ be a 3-face in $\Bbb{R}^3$ and $x_1$ be a face in $x_0^\ast$. We set $X_0 = \Bbb{R}^3 \setminus \{x_0\}$ and $X_1 = X_0 \setminus \{x_1\}$. Our goal is to shrink $X_0$ onto $X_1$.

2.5 Unipolar points

The significance of unipolar points in posets was discovered by Stong [39] in the 60’s and later rediscovered by Bertrand [59]. Most results in this subsection were first established in Stong’s article for finite spaces but can be easily adapted to any posets.

**Definition 5 (Unipolar point)** Let $X$ be a poset. A point $x \in X$ is:

- down unipolar if $x^\ast$ has a maximum;
- up unipolar if $x^\ast$ has a minimum;
- unipolar if it is either down unipolar or up unipolar.

**Property 6** (Proof of Theorem 2) and (25 Proposition 4) Let $(X, \leq)$ be a poset. A point $x \in X$ is unipolar iff $X \setminus \{x\}$ is a strong deformation retract of $X$.

**Definition 7 (Core)** Let $(X, \leq)$ be a poset. Let $Y \subseteq X$ be a subset of $X$. We say that $Y$ is a core of $X$ if the poset $(Y, \leq)$ has no unipolar point and it is a strong deformation retract of $X$.

**Property 8** (39 Theorems 2, 4)

1. Any finite poset has a core and two cores of the same poset are homeomorphic.
2. Two finite posets are homotopy equivalent iff they have homeomorphic cores.

Observe in particular that Property 8 implies that one can greedily remove the unipolar points of a finite poset in order to obtain a core which will be homeomorphic to any other core of the same poset. In particular, when the poset is contractible, we have the following corollary.

**Corollary 9** (25 Corollary 4) If $X$ is finite and contractible, there is a sequence $(x_i)_{i=0}^r (r \geq 0)$ of points in $X$ such that $X = \{x_i\}_{i=0}^r$ and, for all $j \in [1, r]$, $x_j$ is unipolar in $\{x_i\}_{i=0}^r$. Furthermore, if $x \in X$ is unipolar, we can choose $x_r = x$. 
As an unipolar point in a poset \((X, \leq)\) is, clearly, also an unipolar point in the poset \((X, \geq)\), one can easily deduce from Corollary \(^{[9]}\) and Property \(^{[6]}\) the following corollary.

**Corollary 10** Let \((X, \leq)\) be a finite poset. Then, \((X, \leq)\) is contractible iff \((X, \geq)\) is contractible.

Thanks to the next Property, one can build well balanced shrinking algorithms by deleting unipolar points with same heights in parallel.

**Property 11** (\(^{[28]}\) Property 3 and \(^{[25]}\) Proposition 5) If \(x \neq y\) are unipolar points, then either (a) \(y\) is unipolar in \(X\setminus\{x\}\), or (b) for one order on \(X\) (\(\leq\) or \(\geq\)), \(x\) is down-unipolar and covers \(y\), for the other order \(y\) is down-unipolar and covers \(x\) and the map \(\varphi : X \setminus \{x\} \rightarrow X \setminus \{y\}\) defined by \(\varphi(z) = z\) if \(z \neq y\) and \(\varphi(y) = x\) is an homeomorphism.

**Example 12** Let us consider the test set \(X_0\), described at the end of Subsection 2.4. It is plain that the 2-faces of \(x_0\) are unipolar in \(X_0\). Thus, if \(\dim(x_1) = 2\), the set \(X_1\) is a strong deformation retract of \(X_0\). If \(\dim(x_1) \leq 1\), \(x_1\) is not unipolar so \(X_1\) is not a strong deformation retract of \(X_0\).

This example shows us that unipolar points are not enough “powerful” to be used in thinning or growing procedures. This is the reason why we introduce now \(\beta\)-simple points.

### 2.6 \(\beta\)-simple points

The notion of \(\beta\)-simple points was first introduced by Bertrand\(^{[4]}\) in \(^{[38]}\) in order to define topologically sound thinning algorithms in posets. In his article, Bertrand uses a specific definition for the homotopy type. On the other hand, Baranek and Minian\(^{[41]}\) gives the same definition in the classical framework in order to perform a collapse operation in posets which actually corresponds to the collapse operation in complexes associated to posets.

**Definition 13** (\(\beta\)-simple point) Let \((X, \leq)\) be a poset. A point \(x \in X\) is:

- down \(\beta\)-simple (in \(X\)) if \(x^{1*}\) is contractible;
- up \(\beta\)-simple (in \(X\)) if \(x^{1*}\) is contractible;
- \(\beta\)-simple (in \(X\)) if it is either down \(\beta\)-simple or up \(\beta\)-simple.

From this definition and Corollary \(^{[10]}\) we straightforwardly infer the next proposition.

**Proposition 14** Let \((X, \leq)\) be a poset. Let \(x\) be a \(\beta\)-simple point in \(X\). Then \(x\) is \(\beta\)-simple in \(X\) equipped with the reverse order and the dual topology.

Unipolar points are \(\beta\)-simple points since if \(x \in X\) is a down (resp. up) unipolar point, \(x^{1*}\) (resp. \(x^{1*}\)) has a maximum (resp. minimum) and is therefore contractible (Property \(^{[3]}\)). We saw previously (Property \(^{[6]}\) that the removal of a unipolar point is a strong deformation retraction. It is no longer true for \(\beta\)-simple points (see our test set \(X_0\) of Example \(^{[12]}\) with \(\dim(x_1) \leq 1\) for a counterexample). Nevertheless, the next property states that homotopy groups are not changed by such a deletion and, furthermore, the following theorem ensures that this deletion corresponds to a strong deformation retraction on the continuous analogue.

**Property 15** (\(^{[41]}\) Proposition 3.3) Let \(X\) be a finite poset. Let \(x \in X\) be a \(\beta\)-simple point. Then, the inclusion map \(i : X \setminus \{x\} \rightarrow X\) is a weak homotopy equivalence.

**Theorem 16** (\(^{[41]}\) Theorem 3.10) Let \(X\) be a finite poset. Let \(x \in X\) be a \(\beta\)-simple point and \(\mathcal{K}(X), \mathcal{K}(X \setminus \{x\})\) the simplicial complexes associated to \(X\) and \(X \setminus \{x\}\), respectively. Then, \(\mathcal{K}(X)\) collapses onto \(\mathcal{K}(X \setminus \{x\})\).

From an algorithmic point of view, like unipolar points, \(\beta\)-simple points have good properties since they can be deleted in parallel. Obviously, if \(x, y\) are two points in \(X\) with \(ht(x) = ht(y)\), there is no need to know whether \(x\) has been deleted from \(X\) or not to decide if \(y^{1*}\), or \(y^{1*}\) is contractible. Moreover, as we have seen above, the decision on the contractibility can be greedily performed. Thus, a topology-preserving thinning procedure in a poset \(X\) of finite length \(\ell\) consists of repeating until stability the removal of the \(\beta\)-simple points of height \(k\) for \(k = 0\) to \(\ell\).

**Example 17** Let us consider once again the test set \(X_0\). If \(\dim(x_1) = 2\), we have already seen that \(x_1\) is unipolar, so it is also \(\beta\)-simple. If \(\dim(x_1) = 1\), the Hasse diagram of \(x^{1*}\) in the poset \(X_0\) is an acyclic graph composed of the four 2-faces of \(\mathbb{P}^3\) including \(x_1\) and the three 3-faces of \(\mathbb{P}^3\) including \(y\) and distinct from \(x_0\). Thus, it is contractible and \(x_1\) is up \(\beta\)-simple. The inclusion map \(i_1 : X_1 \rightarrow X_0\) is therefore a weak homotopy equivalence. If \(\dim(x_1) = 0\), let \(y_0, y_1, y_2\) be the three 2-faces including \(x_1\) and included in \(x_0\). The reader can check in Figure \(^{[17]}\) that these three faces are up-unipolar in \(x^{1*}\) and that \(x^{1*} \setminus \{y_0, y_1, y_2\}\) is a core of \(x^{1*}\). Hence, \(x^{1*}\) is not contractible and \(x_1\) is not \(\beta\)-simple.

### 2.7 \(\gamma\)-simple points

The example set \(X_0\) highlights the need for a weaker condition on points to be deleted when processing a thinning in a
Property 21 ([42] Proposition 3.17) Let X be a finite poset and x a point in X. Then $x^{\gamma^*}$ is homotopically trivial if $x^{\beta^*}$ or $x^{\beta^*}$ is homotopically trivial.

If the length of the space is less than or equal to 3, and x is neither a maximum nor a minimum, the height of $x^{\gamma^*}$ and $x^{\gamma^*}$ is less than or equal to 1. Hence, if $x^{\gamma^*}$ or $x^{\gamma^*}$ is homotopically trivial, it is contractible. Thanks to Property 18 we deduce that $x^{\gamma^*}$ is contractible and therefore homotopically trivial.

The next property ensures that the deletion of a $\gamma$-simple point does not modify the homotopy groups.

Property 22 ([42] Proposition 3.10) Let X be a finite poset. Let $x \in X$ be a $\gamma$-simple point. Then, the inclusion $i : X \setminus \{x\} \rightarrow X$ is a weak homotopy equivalence.

Finally, the following theorem states that, when deleting a $\gamma$-point in a finite poset, the homotopy type of the continuous analogue is unchanged.

Theorem 23 ([42] Theorem 3.15) Let X be a finite poset and let $x \in X$ be a $\gamma$-simple point. Then $|\mathcal{K}(X \setminus \{x\})|$ and $|\mathcal{K}(X)|$ are simple-homotopy equivalent.

In a 3D image X, the cost to decide whether the set $x^{\gamma^*}$ is homotopically trivial is not expensive. Indeed, $\mathcal{K}(x^{\gamma^*})$ is a 2-dimensional simplicial complex and it is enough to compute its connected components and its Euler characteristic. An alternative to look at $\gamma$-simple points, in any dimension, is to remove $\beta$-simple points in $x^{\gamma^*}$ until stability. If the result is a singleton, by Property 15 $x^{\gamma^*}$ is homotopy equivalent to a point and therefore homotopically trivial. Moreover, the scheme proposed for the deletion of simple points is still valid ($\gamma$-simple points with same height can be removed in parallel).

Example 24 Let us consider the test set $X_0$. We have seen that $x_1$ is a $\beta$-simple point iff $\dim(x_1) \geq 1$. Suppose now that $\dim(x_1) = 0$. The chain complex $\mathcal{K}(x^{\gamma^*})$ (see Subsection 2.4) is depicted in Figure 11 in a 2D-space and in a 3D-space. It is clearly contractible, so $x^{\gamma^*}$ is homotopically trivial (Theorem 4). Thus, $x_1$ is a $\gamma$-point and the injection $i : X_1 \rightarrow X_0$ is a weak homotopy equivalence.

2.8 Complexes and simplicity

In this subsection, we establish some specific properties of spaces of cubical or simplicial complexes. The proofs of these new results are provided in Appendices B and C.

In Section 4 the proof of Theorem 47 needs the space to have a property that can be understood in the framework of complexes as asking the boundary of a cell with a “large hole” to be homotopically trivial. So, we introduce the following definition.

Definition 25 A poset X has the pierced sphere property if, for any $x, y \in X$ such that y covers x, the set $x^{\gamma^*} \setminus \{y\}$ is homotopically trivial.
Fig. 11  (a) The pure simplicial 2-complex $\mathcal{K}(x_1^*)$ in a 2D space. The large/middle/small circles are vertices associated to 3/2/1-faces of $x_1^*$. (b) The complex $\mathcal{K}(x_1^*)$ in a 3D space. The seven vertices associated to the 3-faces of $x_1^*$ are in corner position and the vertices associated to the 1-faces are in centre position.

The next proposition states that this pierced sphere property is satisfied by the spaces of cubical or simplicial complexes. In Appendix B, we actually prove an extended version of this statement (Proposition 58).

Proposition 26 Let $X$ be a cubical or a simplicial complex equipped with the order $\supseteq$. Then, $X$ has the pierced sphere property.

In digital topology, the usual requirement for a point $y$ to be simple for an object $Y$ in a space $X$ (that is a point which can be removed from $Y$ in a topologically sound thinning procedure) is that (i) the inclusion $i : Y \setminus \{y\} \to Y$ induces a one-to-one correspondence between the connected components of the object before and after the removal (i.e., $Y$ and $Y \setminus \{y\}$), (ii) the inclusion $i' : X \setminus Y \to (X \setminus Y) \cup \{y\}$ induces a one-to-one correspondence between the connected components of the background before and after the removal (i.e., $X \setminus Y$ and $X \setminus Y \cup \{y\}$), (iii) the inclusion $i$ induces isomorphisms between the fundamental groups of the connected components of the object before and after the removal, (iv) the inclusion $i'$ induces isomorphisms between the fundamental groups of the connected components of the background before and after the removal [43]. In [29], it has been proved, thanks to the linking number borrowed to knots theory, that for 3D digital images interpreted with the (6,26) or the (26,6) pair of adjacencies, there is no need to consider the fundamental groups of the background since their preservation is implied by the three first conditions. The following theorem generalises, in our framework, this property to spaces of any dimension (and, in a certain sense, defined in [22], for any pair of adjacencies).

Theorem 27 Let $X$ be a cubical complex equipped with the order $\supseteq$ which is also a cubical complex for the dual order $\subseteq$. Let $Y$ be a proper subset of $X$ and $y$ be a $\beta$-simple point in $Y$. Then $y$ is $\gamma$-simple in $(X \setminus Y) \cup \{y\}$.

Remark 28 We do not know if this theorem remains true in any dimension if we replace the hypothesis “$y$ is a $\beta$-simple point” by “$y$ is a $\gamma$-simple point”. Nevertheless, if the dimension of $X$ is 2, $\gamma$-simple points are $\beta$-simple points, so it is obviously true in this case. Moreover, we have proved, by checking all configurations with the help of a computer program, that it is also true in $\mathbb{R}^3$, the space of 3-dimensional cubical complexes. In Appendix D, Counterexample 61 provides a case where Theorem 27 is false when the space $X$ is not a complex for the dual order.

3 Label images

Let $L$ be a finite poset with a minimal element, denoted $\bot$, and such that two distinct elements in $L \setminus \{\bot\}$ are not comparable. We set $L^\star = L \setminus \{\bot\}$ and we write $\ell$ for the cardinality of $L^\star$. The elements of $L^\star$ are called proto-labels. The Hasse diagram of the poset $L$ is depicted in Figure 12. A label digital image is a function defined on $\mathbb{Z}^n$, with values in $L$, and equal to $\bot$ everywhere except on a finite set of points of $\mathbb{Z}^n$.

Let $l \in L, l \neq \bot$ be a proto-label and $\lambda$ a label digital image. The set $\lambda^{-1}((l))$ is the support of the proto-label $l$ (in the label digital image $\lambda$). The union of the supports of all proto-labels is the domain of the image $\lambda$. (This domain is finite by definition.) The set $\lambda^{-1}(\{\bot\})$ is the background of the image $\lambda$. The background and the supports define a partition of $\mathbb{Z}^n$.

In order to equip the discrete grid on $\mathbb{Z}^n$ with a topology, we enrich this grid by adding low dimensional points between the xels of $\mathbb{Z}^n$ (for instance, in $\mathbb{Z}^3$, we add surfels, linels and pointels) whose purpose is to link the distinct adjacent xels and to confer a poset structure to the discrete space. Typically, such a space is the space of cubical complexes, $\mathbb{P}^n$, or any poset associated to a cellular decomposition of the space $\mathbb{R}^n$. Thereby, the label digital images considered in this article are defined on a locally finite poset $(X, \leq)$: we wish to link points of $\mathbb{Z}^n$ to finitely many neighbours. Indeed, all sets $x^\star$ and $x_i^\star$ which appear in the definitions of $\beta/\gamma$-simple points will be finite. This will allow us to use the results of Section 2.

Furthermore, we suppose that the embedding of $\mathbb{Z}^n$ in $X$ puts in one-to-one correspondence the points of $\mathbb{Z}^n$ with the minimal points of $X$. The reader must be aware that this is counterintuitive. For instance, if the poset is the space of cubical complexes, $\mathbb{P}^n$, this one must be ordered by the

Fig. 12  Hasse diagram of the label set $L = \{l\}_{\ell=1} \cup \{\bot\}$.
dual of the inclusion, \( \sqsupseteq \), i.e., the height of a face is its codimension. The reason to do so is to put the xels of \( \mathbb{Z}^n \), which contain all the information of the original image, at the same height in the poset, namely “on the floor”. Then, we can add, above those minimal points, the topological “glue” that is needed to interpret the image. Most of the time, the labels of the minimal points will be proto-labels, or \( \perp \), that is minimal labels in \( T \) and the image will be non-decreasing. In the sequel, we identify the points of \( \mathbb{Z}^n \) with their images in \( X \) so the xels are the minimal points of \( X \).

Since we enrich the initial space with low dimensional faces in order to get both a topological space and an algebraic structure, we are led to do so with the label set to extend the digital label image on these supplementary low dimensional faces. That is why we embed the label set in an atomistic lattice \( (T, \leq) \) whose minimum is the embedding of \( \perp \) and atoms are the embeddings of the proto-labels of \( L \) (a few definitions and properties about lattices can be found in Appendix A). In the sequel, we identify the elements of \( L \) with their images in \( T \). We denote by \( \top \) the maximum of \( T \). A label is an element of \( T \). Given a (proto-)label set \( L^* \) the smallest lattice \( T \) including \( L = L^* \cup \{ \top \} \). This is the lattice used by Ronse and Agnus in [49, 50] to define morphological operators on label images. The largest atomistic lattice in which we can embed \( L \) is the power set \( 2^{L^*} \) (with the natural embedding which associates \( \emptyset \) to \( \perp \) and the singleton \( \{ l \} \) to any proto-label \( l \)).

Some ways to associate labels to points in \( X \) that are not xels are discussed in [44, 51]. We have proposed, in [22], our own modus operandi to embed a binary digital image defined on \( \mathbb{Z}^n \) in a binary image defined on \( \mathbb{Z}^p \). It can be straightforwardly extended to label images and we use it, in a particular case, in Section 5 but we do not develop more this issue in this article. This is why we actually just set the following definition for label images.

**Definition 29 (Label images)** Let \( X \) be a locally finite poset and \( T \) an atomistic lattice. A label image is a function \( \mu : X \to T \).

Figure 13 provides various examples of label images.

We have seen that when we start from a label digital image \( L : \mathbb{Z}^n \to L \) and we construct a label image \( \mu : X \to T \), the labels of the minimal points of \( X \) (i.e., the xels) are the atoms of \( T \) (i.e., the proto-labels). When a label image has this property, we say that this image is pure.

A label image can be seen as a superposition of binary layers. Indeed, if \( \mu \) is a label image, and \( l \in L^* \) is a proto-label, the image \( \mu_l = \mu \wedge l \) is a binary image whose codomain is \( \{ \perp, l \} \) (remember that we denote by \( \wedge \) and \( \vee \) the infimum and supremum operations of the lattice \( T \) : see Appendix A). The next proposition establishes that \( \mu \) is the supremum of all the binary images \( \mu_l, l \in L^* \).

**Proposition 30** Let \( \mu : X \to T \) be a label image. Let \( L^* \) be the set of atoms of \( T \). Then, \( \mu = \bigvee_{l \in L^*} \mu_l \) where, for all \( l \in L^* \), \( \mu_l = \mu \wedge l \).

**Proof** We set \( L^* = \{ l_i \}_{i=1}^n \). Let \( x \) be a point in \( X \). Let \( A \subseteq L^* \) be the set of atoms in \( T \) which are less than or equal to \( \mu(x) \). Then, \( \mu(x) = \bigvee_{a \in A} a \) for \( T \) is atomistic. Let \( l \in L^* \), be an atom in \( T \). We have \( (\mu \wedge l)(x) = \mu(x) \wedge l = (\bigvee_{a \in A} a) \wedge l \). It is plain that \( (\mu \wedge l)(x) = l \) iff \( l \in A \) and \( (\mu \wedge l)(x) = \perp \) iff \( l \notin A \). Thus, \( \mu(x) = \bigvee_{a \in A} a \vee \{ (\mu(x) \wedge l) = \bigvee_{l \in L^*} (\mu(x) \wedge l) \). \( \square \)

Let \( \mu : X \to T \) be a label image and \( t \) be a label. The support of \( t \) in \( \mu \) is the subset \( (t)_\mu \) of \( X \) equal to \( \{ x \in X \mid \mu(x) \wedge t \neq \perp \} \). When there is no ambiguity, we also say the support of \( t \) instead of the support of \( t \) in \( \mu \) and we write

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**Fig. 13** Label images. The proto-labels are \( r, g, b \) (depicted in red, green and blue). The other labels are obtained by using the additive colour model (e.g., \( [r, b] \) is depicted in magenta) except \( \perp \) which is depicted in black (\( \perp \) is depicted in white). (a) \( X \) is a subset of \( \mathbb{Z}^2 \). \( T \) is the power set \( 2^{[0,1]} \). Observe that in this image, there are points of height 0 that have distinct dimensions. (b) \( X \) is built from an hexagonal tessellation. \( T \) is the power set \( 2^{[\pm 1]} \). The labels of the points of height greater than 0 are assigned according to the rule which will be used in Section 5 a label is the supremum of the labels of the minimal points in the neighbourhood. (c) \( X \) is built from a semi-regular tessellation. The labels are given according to the same rule as in (b) but \( T \) is not a power set: \( T = \{ \perp, r, g, b, \top \} \).

**Fig. 14** (a) A label image whose domain is \( \mathbb{Z}^2 \) and whose codomain is the power set \( T = 2^{[0,1]} = \{ \emptyset, [b], [r], [b, r] \} \) equipped with the inclusion. The points with label \( [b] \) are depicted in blue, those with label \( [r] \) in red and those with label \( [b, r] \) in magenta. The points of the background (label \( \perp = \emptyset \)) are depicted in white with a black border or are not depicted. (b) In blue, the support of the label \( [b] \). (c) In red, the support of the label \( [r] \). (d) In magenta, the support of \( \top = [b, r] \).
Fig. 15 (a) A label image $\mu$ whose domain is $\mathbb{R}^2$ and whose codomain is the power set $T = 2^{[0,1]}$ equipped with the inclusion. The labels are depicted as in Figure 14 plus the labels $\{g,b\}, \{r,g\}$ which are depicted respectively in cyan and yellow. The point $x$ is the 0-face at the centre of the figure. We have $\mu(x) = \{b\}$. We want to test if the point $x$ is simple for the label $t = \langle r \rangle$ (that is, we want to know if we can label the point $x$ with $t$ while keeping unchanged the topology of the supports of the $2^3 - 1$ non-minimal labels in $T$). There are two labels $u$ such that $u \wedge \mu(x) = \perp$ and $u \wedge t = \perp$: $\{b\}$ and $\{g,b\}$. (b) In blue, the set $x^* \cap \langle \{b\} \rangle$ which is contractible. (c) In cyan, the set $x^* \cap \langle \{g,b\} \rangle$ which is contractible. Hence, the first half of the test (namely, condition (i), in Definition 31) succeeds. Now, let us consider the labels $u$ for which $u \wedge \mu(x) = \perp$ and $u \wedge t \neq \perp$: $\{r\}$ and $\{r,g\}$. (d) In red, the set $x^* \cap \langle \{r\} \rangle$ which is contractible. (e) In yellow, the set $x^* \cap \langle \{r,g\} \rangle$ which is not contractible (it has two connected components). The second half of the test (namely, condition (ii), in Definition 31) then fails. Therefore, the point $x$ is not simple for the label $t$. Giving the label $\{r\}$ to $x$ would connect two distinct components of the label $\{r,g\}$.

(1) instead of $\langle t \rangle_{\mu}$. The support of a proto-label (in a label image) is the subset of $X$ whose points have a label greater or equal to this proto-label. The support of a label $t \neq \perp$ is the union of the supports of the proto-labels in $T$: $x \in \langle t \rangle \Leftrightarrow \mu(x) \wedge t \neq \perp \Rightarrow \exists \text{ atom } t_i \leq \mu(x) \wedge t \Leftrightarrow \exists t_i \leq t \wedge x \in \langle t_i \rangle$. The support of the label $\perp$ is the empty set. The cosupport of $t$ in $\mu$ (or the cosupport of $t$) is the complement in $X$ of the support of $t$ in $\mu$. We denote it by $\langle t \rangle_{\mu}^c$ or $\langle t \rangle^c$. Figure 14 illustrates these definitions.

We have seen in Section 1 that in a label digital image in which one wants to preserve the topological properties inside the supports of the proto-labels and between these supports, it is important, when performing a change of label on a point, to maintain the topology of any union of supports of proto-labels. In other words, we have to preserve the topology of any set identified by a proto-label in the partition of the space associated to the initial digital label image but also of any set defined by any coarser partition of the space. In the proposed framework, the supports (of the labels) are subsets of $X$ that are exactly the unions of the supports of the proto-labels. Hence, the supports of the labels in $T$ are the sets for which we must ensure the topological invariance. This has lead us to the following definition, exemplified in Figure 15.

**Definition 31 (Simple point for a label)** Let $\mu : X \to T$ be a label image. Let $t \in T$ be a label. A point $x \in X$ is a simple point for (the label) $t$ if the following two conditions are verified:

(i) for all labels $u \in T$ such that $u \wedge \mu(x) \neq \perp$ and $u \wedge t = \perp$, $x$ is $\beta$-simple for the set $\langle u \rangle$;

(ii) for all labels $u \in T$ such that $u \wedge \mu(x) = \perp$ and $u \wedge t \neq \perp$, $x$ is $\beta$-simple for the set $\langle u \rangle \cup \{x\}$.

In this definition, the first condition, $u \wedge \mu(x) \neq \perp$ and $u \wedge t = \perp$, means that $x$ is in the support of $u$ in $\mu$ but it will no more be in it if the image is modified by giving the label $t$ to $x$. Conversely, the second condition, $u \wedge \mu(x) = \perp$ and $u \wedge t \neq \perp$, means that $x$ is not in the support of $u$ in $\mu$ but it will be in it if the image is modified by giving the label $t$ to $x$. In each case, by requiring $x$ to be $\beta$-simple for the sets $\langle u \rangle_{\mu}$, we ensure that there exists a weak homotopy equivalence between each support before and after the modification of the image $\mu$ and, if $X = \mathbb{R}^n$, the cosupports will also be weakly homotopic (see Proposition 34). Remember that this implies also that the operation corresponds to strong deformation retractions on the realisations of the simplicial complexes associated to these supports (Theorem 16). In a binary image (i.e., with $T = \langle \perp, T \rangle$), Definition 31 comes down to require $x$ to be $\beta$-simple in $\langle T \rangle$ or $x$ to be $\beta$-simple in $\langle T \rangle \cup \{x\}$, depending on $x$ is in the object or in the background. Observe also that any point in $\mu^{-1}(t)$ is simple for the label $t$.

Since the poset $X$ is locally finite, the sets $x^*$ and $x^*$ are finite. Therefore, according to Corollary 9, one can test the simplicity of a point $x \in X$ by greedily removing unipolar points in the sets $x^* \cap Y$ and $x^* \cap Y$ where $Y = \langle u \rangle$ or $Y = \langle u \rangle \cup \{x\}$, for all $u \in T$. When the lattice $T$ is distributive, the following proposition allows us to speed up this test by reducing temporarily the size of $T$ by identifying the atoms of $T$ not “involved” in $\mu(x^*)$ with the label $\perp$. Observe that if the lattice $T$ is distributive and $\mu$ is defined from a label digital image $\lambda : \mathbb{Z}^n \to L$ as suggested in the introduction of Section 3, then $T$ is a finite, atomistic and distributive lattice whose atoms are identified with the elements of $L^*$, that is $T$ is the power set $2^{L^*}$ (see Appendix A).

**Proposition 32** Let $\mu : X \to T$ be a label image. Let $t$ be a label and $x$ be a point in $X$. Let $L^*$ be the set of atoms in $T$ and $L_x$ be the subset of $L^*$ whose elements are less than or equal to an element of $\mu(x^*)$. Let $\varphi : T \to T$ be the function that maps any label $u$ onto the label $\varphi(u) = \bigvee \{a \in L_x \mid a \leq u\}$.

(i) If the point $x$ is simple for $t \in \mu$ then $t \in \varphi(T)$ and $x$ is simple for $t$ in the image $\varphi \circ \mu : X \to \varphi(T)$.

(ii) Conversely, if the lattice $T$ is distributive, $t \in \varphi(T)$, and $x$ is simple for $t$ in the image $\varphi \circ \mu$, then the point $x$ is simple for $t \in \mu$.

**Proof** (i) We assume that $x$ is simple for $t \in \mu$. Let us suppose that $t \notin \varphi(T)$. Then it must exist an atom $a \notin L_x$ such
that $a \leq t$ (otherwise $t = \varphi(t) \in \varphi(T)$). This label $a$ is such that $a \land \mu(x) = \perp$ (by definition of $a$ and $L_n$) and $a \land t \neq \perp$.

But $x$ cannot be $\beta$-simple for the set $(a)_{\land x} \cup \{x\}$ since $x^* \cap (a)_{\land x}$ is empty (by definition of $a$ and $L_n$). Thus, we have a contradiction with the simplicity of $x$ for $t$. So $t \in \varphi(T)$. Let $u$ be a label in $\varphi(T)$ such that $u \land \varphi(u(x)) \neq \perp$ and $u \land t = \perp$.

Since, trivially, $\varphi$ reduces to identity on $\mu(x)$, we have that $u \land \mu(x) = u \land \varphi(u(x)) \neq \perp$. As $x$ is simple for $t$, $u \land \mu(x) \neq \perp$ and $u \land t = \perp$, it comes that $x$ is $\beta$-simple for the set $(u)_{\land x}$. We have already observed that the images $\mu$ and $\varphi \circ \mu$ are equal on $x$. Note, moreover, that the $\beta$-simplicity only involves a subset of $x^2$. Hence, $x$ is $\beta$-simple for the set $(u)_{\land x}$. Similarly, when $u$ is a label in $\varphi(T)$ such that $u \land \varphi(u(x)) = \perp$ and $u \land t \neq \perp$, we deduce as above that $x$ is $\beta$-simple for the set $(u)_{\land x} \cup \{x\}$. We can then conclude that $x$ is simple for $t$ in $\varphi \circ \mu$.

(ii) We now assume that $x$ is simple for $t$ in $\varphi \circ \mu$ with $t \in \varphi(T)$. Let $u$ be a label in $T$ such that $u \land \mu(x) \neq \perp$ and $u \land t = \perp$. By the very definition of $L_n$, we have $u \land \mu(x) = \varphi(u \land \mu(x))$. Thus $\varphi(u \land \mu(x)) \leq u$ and $\varphi(u \land \mu(x)) \leq \mu(x)$. Then, since $x$ is an opening (see Appendix A), we have $\varphi(u \land \mu(x)) \leq \varphi(u)$ and $\varphi(u \land \mu(x)) \leq \varphi(u(x))$. Thereafter, $u \land \mu(x) = \varphi(u \land \mu(x)) \leq \varphi(u) \land \varphi(u(x))$. Thus, $\varphi(u) \land \varphi(u(x)) \neq \perp$. We have also $\varphi(u) \land t = \varphi(u) \land \varphi(t)$, since an opening is idempotent and $t \in \varphi(T)$, and $\varphi(u) \land \varphi(t) \leq u \land t$, since an opening is anti-extensive. Thus, we get $\varphi(u \land t) = \perp$. As $x$ is simple for $t$ in $\varphi \circ \mu$, $\varphi(u \land \mu(x)) \neq \perp$ and $\varphi(u) \land \varphi(t) = \perp$, we derive that $x$ is $\beta$-simple for the set $(u)_{\land x}$, which implies that $x$ is $\beta$-simple for the set $(u)_{\land x}$ (for $x^* \cap (u)_{\land x} = x^* \cap (\varphi(u))_{\land x}$). When $u$ is a label in $T$ such that $u \land \mu(x) = \perp$ and $u \land t \neq \perp$, we derive that $\varphi(u) \land \varphi(u(x)) = \perp$ (with similar arguments as above). Let us now assume that $T$ is distributive. It can easily be seen that $t \in \varphi(T)$ implies that $u \land t \in \varphi(T)$ (any atom less than or equal to $t$ is in $L_n$). Thus, $\perp \neq u \land t = \varphi(u \land t) \leq \varphi(u) \land \varphi(t) = \varphi(u) \land t$. We conclude, as previously, that $x$ is $\beta$-simple for $(u)_{\land x} \cup \{x\}$. Hence, $x$ is simple for $t$ in $\mu$.


Let us now define the topological properties we want to preserve when processing a label image.

**Definition 33** Let $\mu, \nu : X \to T$ be two label images.
- If, for all labels $t \in T$, $(\mu)_t$ and $(\nu)_t$ are weak homotopy equivalent, we say that these images are equivalent and we write $\mu \approx \nu$.
- If, furthermore, $(\mu)_t^{\ominus}$ and $(\nu)_t^{\ominus}$ are weak homotopy equivalent for all labels $t$, we say that the images are strongly equivalent.

We write $\mu + (x, t)$ for the image equal to $\mu$ except in $x$, where its value is $t$.

Based on these definitions, we have the following result.

**Proposition 34** Let $\mu : X \to T$ be a label image. Let $x$ be a simple point for the label $t$. Then, $\mu$ and $\mu + (x, t)$ are equivalent, strongly equivalent if $X = \mathbb{P}^n$.

**Proof** Let $v$ be the image $\mu + (x, t)$. Let $u$ be a label. By definition of the image $v$, the supports $(u)_t$ and $(v)_t$ are equal, except possibly on $x$. Therefore, if $(u \land \mu(x) = \perp$ and $u \land t = \perp$) or $(u \land \mu(x) \neq \perp$ and $u \land t \neq \perp$), then $(u)_t = (u)_t$.

In the other cases, from Definition [3] $x$ is $\beta$-simple for $(u)_u$ (if $x \in (u)_\mu$) or $x$ is $\beta$-simple for $(u)_\mu \cup \{x\}$ (if $x \notin (u)_\mu$). Hence, $(u)_t$ is weak homotopy equivalent to $(u)_u$. If $X = \mathbb{P}^n$, we derive from Theorem [17] that $x$ is $\gamma$-simple for $(u)_\mu$ (if $x \in (u)_\mu$) or $x$ is $\gamma$-simple for $(u)_\mu \cup \{x\}$ (if $x \notin (u)_\mu$). Thus, $(u)_t$ and $(u)_\mu$ are weak homotopy equivalent (Property [22]).

The next proposition is an easy consequence of Definitions [13] and [11]. From a practical point of view, it is quite important since it allows us to define parallel thinning (or growing) algorithms in label image by simultaneously modifying the label of simple points with the same height.

**Proposition 35** Let $\mu_1 : X \to T$ be a label image. Let $t \in T$ be a label. Let $Y = \{y\}_{i=1}^k (k \geq 1)$ be a set of points with the same height, simple for the label $t$. For all $i \in [1, k]$, we set $\mu_i = \mu_{i-1} + (y_{i-1}, t)$. Then, for all $i \in [0, k]$, $y_i$ is a simple point for the label $t$ in $\mu_i$.

Figure [16] provides an example of label thinning/growing by giving the label $t$ to simple points for $t$, processing points with same height during the same pass on the image.

### 4 Closed support images

In this section, we focus on digital images that could be associated to digital images considered with the $(3^n - 1, 2n)$-adjacency pair in nD cubic grids (namely $\mathbb{Z}^n$). This adjacency pair corresponds to the adjacency of closed objects of the continuous space [22] and has therefore led us to investigate label images in which the support of any label is a closed set. Hence, we define a closed support (label) image as a label image whose supports are closed sets (for any label). The following proposition establishes that the closed support label images are the non-decreasing maps from $X$ onto $T$ (that is the continuous maps from $X$ to $T$ (Property [2])).

**Proposition 36** Let $\mu : X \to T$ be a label image. The supports of the labels in $\mu$ are closed sets if $\mu$ is a non-decreasing function from $(X, \leq)$ to $(T, \leq)$. 
we defined the supports, we have established that
\[ \langle \text{atom less than or equal to } \bigcup x \rangle \] has a label and
\[ \text{x} \text{ is simple for the label } [r]. \] (b) The image \[ \mu + (y, [r]) \] is no longer a closed support image.

In the poset \( X \), a set \( F \) is closed iff for any point \( x \in F \)
the points greater than \( x \) are also in \( F \). This is like in a simplicial complex, where any subset of a face of the complex is also a face of the complex. It is well known that the set of simplicial complexes is closed under the collapse operation, which furthermore “preserves topology” \( \text{(21)} \). So, we have adapted this notion to label images in order to maintain both the closedness and the topology of any label support. Roughly speaking, we have found that this goal can be achieved if we require the supports of some labels in the sub-poset \( x^2 \) to be contractible (where \( x \) is the point whose label has to be modified).

**Proposition 37** Let \( \mu : X \to T \) be a closed support image. Let \( x < y \) be two points in \( X \). The following statements are equivalent.

(i) For all \( u \in T \) such that \( y \in \langle u \rangle \), \( x^2 \cap \langle u \rangle \) is contractible.

(ii) For all \( u \in T \) such that \( y \in \langle u \rangle \) and \( x \not\in \langle u \rangle \), \( x \) is \( \beta \)-simple for \( \langle u \rangle \cup \{x\} \).

(iii) The point \( x \) is simple for the label \( \mu(y) \).

**Proof** (i) \( \Rightarrow \) (ii) Let \( u \) be a label such that \( y \in \langle u \rangle \) and \( x \not\in \langle u \rangle \). Then, \( x^2 \cap (\langle u \rangle \cup \{x\}) = x^2 \cap \langle u \rangle \) is contractible by hypothesis. So, \( x \) is \( \beta \)-simple for the set \( \langle u \rangle \cup \{x\} \) (Property \( \text{(13)} \) and Definition \( \text{(15)} \)).

(ii) \( \Rightarrow \) (iii) First, we observe that, since \( \mu \) is a closed support image, \( \mu \) is non-decreasing (Proposition \( \text{(16)} \)). Thus \( \mu(x) \leq \mu(y) \) and, therefore, \( u \cap \mu(x) \leq u \cap \mu(y) \) for all \( u \in T \). Thereafter, there does not exist any label \( u \) such that \( u \cap \mu(x) \neq \{\} \) and \( u \cap \mu(y) = \{\} \). If \( u \) is a label such that \( u \cap \mu(x) = \{\} \) and \( u \cap \mu(y) \neq \{\} \), by hypothesis, \( x \) is \( \beta \)-simple for \( (u) \cup \{x\} \). Hence, \( x \) is simple for \( \mu(y) \).

(iii) \( \Rightarrow \) (i) Let \( u \) be a label such that \( y \in \langle u \rangle \). Then either \( x \in \langle u \rangle \) and, since the set \( x^2 \) is contractible in any poset (Property \( \text{(3)} \), in particular in the poset \( \langle u \rangle \), \( x^2 \cap \langle u \rangle \) is contractible, or \( x \notin \langle u \rangle \) and, by the hypothesis, \( x \) is \( \beta \)-simple for \( \langle u \rangle \cup \{x\} \), that is, \( x^2 \cap \langle u \rangle = x^2 \cap (\langle u \rangle \cup \{x\}) \) is contractible.

\[ \square \]
Before beginning the proof, note that the definition $L^* \rightarrow \mu$ from Proposition 32, we derive that an element $\mu$ such that $y \in (\mu)$ and $x \notin (\mu)$ is clearly contractible. (b) $(x, y)$ is not a free pair for $(t)$ since $(t) \cap (x^*)$ is not connected (this set contains $y$, the 0-face in magenta and the two 0 faces in black). (c) $(x, y)$ is free for the label $(b)$ (since here, there is no label $u$ such that $y \in (u)$ and $x \notin (u)$, Definition 38 reduces to the classical definition of a free pair in complexes).

**Definition 38 (Free pair)** Let $\mu : X \rightarrow T$ be a closed support image and $t \in T$ be a label. A pair $(x, y)$ of points in $(t)$ is a free pair for the label $t$ if $x$ is the only point in $(t)$ such that $x < y$ and the statements of Proposition 37 are satisfied by the pair $(x, y)$.

The label $t$ involved in Definition 38 cannot be the label $\perp$ since $(t)$ contains at least two points of the free pair and $\langle \perp \rangle = \emptyset$. We exemplify in Figure 15 the definition of free pairs.

The following proposition is the analogue of Proposition 32 for free pairs.

**Proposition 39** Let $\mu : X \rightarrow T$ be a label image. Let $t$ be a label and $x, y$ be two points in $(t)$. Let $L^*$ be the set of atoms in $T$ and $L_x$ be the subset of $L^*$ whose elements are less than or equal to an element of $\mu(x^*)$. Let $\varphi : T \rightarrow T$ be the function that maps the label $u$ onto the label $\varphi(u) = \bigvee \{a \in L_x \mid a \leq u\}$. If the pair $(x, y)$ is free for $t$ in $\mu$ then $(x, y)$ is free for $t$ in the closed support image $\varphi \circ \mu : X \rightarrow \varphi(T)$. Conversely, if the lattice $T$ is distributive and $(x, y)$ is a free pair for $t$ in the image $\varphi \circ \mu$, then the pair $(x, y)$ is free for $t$ in $\mu$.

**Proof** Before beginning the proof, note that the definition of $L_x$ is the same as in Proposition 32 though we have set $L_x = \{a \in L^* \mid 3y \in x^*, a \leq \mu(y)\}$ instead of $L_x = \{a \in L^* \mid 3y \in x^*, a \leq \mu(y)\}$. Indeed, any atom $a$ less than or equal to an element $\mu(z)$, $z \leq x$, is less than or equal to $\mu(x)$ since here $\mu$ is non-decreasing (Proposition 36). Thus, the two definitions coincide.

Now, suppose that $(x, y)$ is free for $t$ in $\mu$. Since $\varphi$ is an opening (see Appendix A) and an opening is order-preserving, $\varphi \circ \mu$ is non-decreasing and is thus a closed support image (Proposition 36). Moreover, $\varphi$ is anti-extensive so $\mu(z) \land t = \perp \Rightarrow \varphi \circ \mu(z) \land t = \perp$ and $\varphi$ reduces to identity on $\mu(x^*)$ so $\varphi(\mu(x)) = (\mu(x))$. Thus, $y^* \cap (t)_{\varphi \circ \mu} = y^* \cap (t)_{\mu} = \{x\}$. Now, from Proposition 32, we derive that $x$ is simple for the label $\mu(y)$ in the image $\varphi \circ \mu$. We conclude that $(x, y)$ is a free pair for $t$ in the image $\varphi \circ \mu$.

Conversely, suppose that the lattice $T$ is distributive and that $(x, y)$ is free for $t$ in the image $\varphi \circ \mu$. Since $\varphi(\mu(y)) = \mu(y)$, we derive from Proposition 32 that $x$ is simple for the label $\mu(y)$ in the image $\mu$. Furthermore, let $z$ be a point in $y^* \setminus x$ such that $\mu(z) \land t \neq \perp$. As $\mu$ is non-decreasing, any atom of $T$ less than or equal to $\mu(z)$ is less than or equal to $\mu(y)$. Thus, $\varphi(\mu(y)) = \mu(y)$ and $\varphi(\mu(z)) \land t \neq \perp$. From the hypothesis we derive that $z = x$. Thereby, $y^* \cap (t)_y = \{x\}$ and $(x, y)$ is a free pair for the label $t$ in $\mu$.

The definition of free pairs in a label image is an extension of the notion of free pair in complexes: if $X$ is a simplicial or cubical complex, $\mu : X \rightarrow T$ a label image and $(x, y)$ a free pair for the label $t$ in $\mu$, then $(x, y)$ is a free pair for the complex $(t)_y$. The following proposition shows that Definition 38 reduces to the classical definition of a free pair when the two points in the pair share the same label.

**Proposition 40** Let $\mu$ be a closed support image, $t \in T$ a label and $(x, y)$ a pair of points in $(t)$. If $\mu(x) = \mu(y)$ and $y^* \cap (t) = \{x\}$, then $(x, y)$ is a free pair for the label $t$.

**Proof** Since $\mu(x) = \mu(y)$, there is no label $u \in T$ such that $y \in (u)$ and $x \notin (u)$ so the statement (ii) of Proposition 37 is satisfied.

**Proposition 41** Let $\mu$ be a closed support image, $t \in T$ a label and $(x, y)$ a free pair for $t$. Then, $x$ is a minimal element in $(t)$, $y$ is down unipolar in $(t)$ and $x < y$ in $X$.

**Proof** The point $x$ is a minimal element of $(t)$ for $x$ is the only point in $(t)$ such that $x < y$. The point $y$ is down unipolar for the same reason. Finally, as $(t)$ is a closed set, $x^*$ is included in $(t)$ and there does not exist any point in $(t)$ between $x$ and $y$ for $y^* \cap (t) = \{x\}$. Thereafter $x < y$.

The next definition introduces the notion of cut. Broadly speaking, a cut of the label $t$ in a closed support image $\mu$ consists of removing $t$ from a free pair $(x, y)$ for $t$. Indeed, in order to maintain the boundaries between supports, the label of $y$ must move towards the other points of $x^* +$ and the labels “behind the boundary”, i.e., the labels of $y^* \setminus \{x\}$, must replace $t$ on $(x, y)$. Figure 19 exemplifies this definition.

**Definition 42 (Cut)** Let $\mu : X \rightarrow T$ be a closed support image, $t \in T$ a label and $(x, y)$ a free pair for the label $t$. The label image $\mu_{\gamma,t} : X \rightarrow T$ defined by:

$$
\mu_{\gamma,t}(x) = \begin{cases} 
\bigvee_{z \in y^* \setminus \{x\}} \mu(z) & \text{if } z \in [x, y] \\
\mu(z) \lor \mu(y) & \text{if } z \in x^* \setminus \{y\} \\
\mu(z) & \text{otherwise}
\end{cases}
$$

is a cut of $t$ at $y$ in $\mu$ (if $y^* \setminus \{x\} = \emptyset$, we set $\mu(x) = \mu(y) = \perp$).
Figure 20 shows that cuts are of no interest in non-distributive lattices since it may happen that the label to be removed from a free pair is still present in the cut.

The notion of cut is an extension to label images of the notion of collapse for complexes. When $X$ is a simplicial or cubical complex and $T$ is distributive, the following proposition states that a cut for the label $t$ is a collapse for the support of $t$ and in particular, if $T = \{\perp, \top\}$, that is when $µ$ is a binary image, a cut is nothing but a collapse.

**Proposition 43** Let $µ_{0}, µ_{1}$ be two closed support images from the complex $X$ to the distributive lattice $T$ and $t \in T$ be a label. If $µ_{1}$ is a cut of $µ_{0}$ for $t$, then $(t)_{µ_{1}}$ is a collapse of $(t)_{µ_{0}}$.

**Proof** Let $µ_{0}$ be a closed support image, $(x, y)$ a free pair of $µ_{0}$ for the label $t$ and $µ_{1}$ the cut $µ_{t}$. From Definition 38, the pair $(x, y)$ is free for the set $(t)_{µ_{0}}$ and from Definition 33, the supports of $t$ in $µ_{0}$ and $µ_{1}$ are equal except possibly in $x^1$. As $µ_{0}$ is a closed support image, $x^1$ is included in $(t)_{µ_{0}}$. As $µ_{1}(z) = µ_{0}(z) \lor µ_{0}(y)$ for all $z \in x^1 \setminus \{x, y\}$, the set $x^1 \setminus \{x, y\}$ is also included in $(t)_{µ_{1}}$. The label of the points $x, y$ in the image $µ_{1}$ is $\forall z \in x^1 \setminus \{x, y\} µ_{0}(z)$. Since $x$ is the only point in $(t)_{µ_{0}} \cap y^1$ and we assume $T$ to be distributive, we have $µ_{1}(x) \land t = µ_{0}(x) \land t = \forall z \in x^1 \setminus \{x, y\} µ_{0}(z) \land t = \perp$. Thus, neither $x$ nor $y$ is in $(t)_{µ_{0}}$ and $(t)_{µ_{1}} = (t)_{µ_{0}} \setminus \{x, y\}$. We conclude that either $(t)_{µ_{1}}$ is a collapse of $(t)_{µ_{0}}$.

When the lattice $T$ is distributive, the following proposition enables to specify which supports are modified by a cut. If the lattice $T$ is not distributive, this proposition fails (see Counterexample 43 in Appendix D).

**Proposition 44** Let $T$ be a distributive lattice and $µ : X \rightarrow T$ be a closed support image. Let $(x, y)$ be a free pair for the label $t$ in $µ$. For any label $u \in T$ whose support does not contain $y$, we have $µ_{t} = (u)_{µ_{t}} = (u)_{µ_{0}}$.

**Proof** Let $u \in T$ be a label such that $y \notin (u)_{µ_{0}}$ and, since $µ$ is a closed support image, $x \notin (u)_{µ_{0}}$. From Definition 32, $µ_{t}(z) = µ(z)$ for any point $z \in (u)_{µ_{0}}$ not in $x^1$. Since $µ$ is non-decreasing, $µ_{t}(z) \leq µ(y)$ for all $z < y$. Hence, $\forall z \in x^1 \setminus \{x\} µ_{t}(z) \leq µ(y)$. Thus, $µ_{t}(x) \land u = µ_{t}(x) \land u = (\forall z \in x^1 \setminus \{x\} µ(z)) \land u \leq µ(y) \land u = \perp$, that is $x, y \notin (u)_{µ_{t}}$. Finally, for any point $z \in x^1 \setminus \{y\}$, $µ_{t}(z) = µ(z) \lor µ(y)$ thus $z \in (u)_{µ_{t}}$ (i.e., $µ_{t}(y) \land u \neq \perp$ iff $z \in (u)_{µ}$ (for $T$ is distributive). □

As stated at the beginning of this subsection, the main advantage of free pairs and cuts on simple points for labels is to enable to remain inside the set of closed support images when we modify a label image with topological constraints.

**Proposition 45** Let $µ : X \rightarrow T$ be a closed support image, $t \in T$ a label and $(x, y)$ a free pair for the label $t$. Then the cut $µ_{t}$ is a closed support image.

**Proof** Let $(x, y)$ be a free pair for a label $t$ in a closed support image $µ$. By hypothesis, $µ$ is non-decreasing (Proposition 36). Let us prove that $µ_{t}$ is also non-decreasing. Let $a, b$ be two points in $X$ such that $a < b$ and, thereafter, such that $µ(a) \leq µ(b)$. The proof is made by exhaustion.

- If $b \notin x^1$ then $a \notin x^1$. Then $µ_{t}(b) = µ(b)$ and $µ_{t}(a) = µ(a)$. In this case, obviously, we have $µ_{t}(a) \leq µ_{t}(b)$.
- If $b \in \{x, y\} = a \notin x^1$ then $a \notin y^1 \setminus \{y\}$ and $µ_{t}(a) = µ(a) \leq \forall z \in y^1 \setminus \{y\} µ(z) = µ_{t}(b)$. Thus, $µ_{t}(a) \leq µ_{t}(b)$.
- If $a, b \in \{x, y\}$ then $µ_{t}(a) = µ_{t}(b)$.

Note that it is impossible to have $b \in \{x, y\}$ and $a \notin x^1$ then $µ_{t}(a) = µ(a) \leq µ(b) \lor µ(y) = µ_{t}(b)$. In each possible case, we have $µ_{t}(a) \leq µ_{t}(b)$. Hence, $µ_{t}$ is non-decreasing. □

When a label image $µ$ is obtained from a label digital image (defined on $\mathbb{Z}^n$) by the procedure we have described at the beginning of Section 4 this image is pure ($µ(x)$ is an atom, or $\perp$, for any $x \in X$). Cuts preserve purity under an hypothesis which is satisfied, for example, by pseudo-manifolds (see, e.g., [83]).

**Proposition 46** Let $µ : X \rightarrow T$ be a pure, closed support image, $t \in T$ a label and $(x, y)$ a free pair for the label $t$. If
any point in \( X \) that covers a xel (a minimal point) covers at most one other xel and no other points, then \( \mu_y \) is pure.

Proof Let \( \mu : X \to T \) be a pure, closed support image. We assume that any point in \( X \) that covers a xel covers at most one other xel and no other points. Let \((x, y)\) be a free pair for the label \( t \). If \( \text{ht}(x) \geq 1 \) then the xels of \( X \) have the same label in \( \mu_x \) as in \( \mu \). As \( y \) covers \( x \) (Proposition 41), if \( x \) is a xel, then we derive from the hypothesis that \( y^* \setminus \{x\} \neq \emptyset \) or \( y^* \setminus \{y\} = \emptyset \) for some xel \( z \in X \). Then, \( \mu_y(x) = \perp \) or \( \mu_y(x) = \mu(z) \) and \( \mu(z) \) is an atom, or \( \mu(z) = \perp \), since \( \mu \) is pure. \( \Box \)

Note that the condition “any point in \( X \) that covers a xel (a minimal point) covers at most one other xel and no other points” could be stated in a complex as “any point of height 1 covers at most two xels” but this is generally not equivalent (in a poset, the height of a point that covers a minimal point need not be one). Figure 21 shows some posets, included in \( \mathbb{P}^3 \), for which this condition is, or is not, satisfied.

4.2 Homotopy

Theorem 47 establishes that connected components and homotopy groups are preserved by cuts provided that the domain of the image has the pierced sphere property (see Subsection 2.8) and the codomain is distributive. Figure 22 illustrates the sequence of changes described in the proof. In Appendix D, some counterexamples show that this preservation is no longer guaranteed when \( T \) is not distributive (Figure 50) or when \( X \) has not the pierced sphere property (Counterexample 64).

Theorem 47 Let \( \mu : X \to T \) be a closed support image and \((x, y)\) a free pair for the label \( t \in T \). If \( X \) has the pierced sphere property and if the lattice \( T \) is distributive, the cut \( \mu_x \) is equivalent to \( \mu \) and, if \( X = \mathbb{P}^n \), \( \mu_x \) is strongly equivalent to \( \mu \).

Proof 1. By Definition 38 \( x \) is simple for the label \( \mu(y) \) in the image \( \mu \). Therefore, \( \mu' = \mu + (x, \mu(y)) \) is equivalent to \( \mu \) (strongly equivalent if \( X = \mathbb{P}^n \), according to Proposition 53).

2. Let \( y \) be the smallest closed support image greater than or equal to \( \mu' \). Since \( \mu \) is a closed support image, \( y \) is defined by \( \nu(z) = \mu(z) \lor \mu(y) \) if \( z \succ x \) and \( \nu(z) = \mu(z) \) otherwise. We shall prove that \( y \) is equivalent to \( \mu' \). To do so, thanks to Proposition 55, it suffices to establish that the points \( z \in x^* \) with same height \( k \), \( k \geq 1 \), are simple for the label \( \nu(z) \) in the image \( \mu' \) defined by \( \mu_k(a) = \nu(a) \) if \( a \in x^* \) and \( \text{ht}(a) < k \) and \( \mu_k(a) = \mu(a) \) otherwise. Thereby, according to Definition 51 we consider a point \( z \in x^* \) such that \( \nu(z) \neq \mu(z) \), i.e., \( \mu(y) \neq \mu(z) \), and let \( k \) be the height of \( z \). Let \( u \) be a label such that \( z \in \langle u \rangle \). and \( z \notin \langle u \rangle_{\mu_k} \) (if \( z \notin \langle u \rangle \), or \( z \in \langle u \rangle_{\mu_k} \), then the support of \( u \) in the image \( \mu_k + (z, \mu(z)) \) is equal to the support of \( u \) in the image \( \mu_k \). Observe that \( \mu_k(z) = \mu(z) \) and \( \mu_k(x) = \mu(y) \). Then, from \( z \notin \langle u \rangle_{\mu_k} \), we derive \( \perp \neq \nu(z) \lor u = (\mu(z) \lor \mu(y)) \lor u = (\mu_k(z) \lor \mu_k(x)) \lor u = (\mu_k(z) \land \mu_k(x)) \lor u = \mu_k(x) \lor u \) (the last equality follows from \( z \notin \langle u \rangle_{\mu_k} \), whence \( \mu_k(z) \land u = \perp \)). Thus, \( x \notin \langle u \rangle_{\mu_k} \). As \( z \notin \langle u \rangle_{\mu_k} \) (since \( z \notin \langle u \rangle_{\mu_k} \) and \( \mu \) is non-decreasing, no point in \( z^* \) is in the support of \( u \) in \( \mu_k \). Moreover, as \( \mu_k \) is in \( X \setminus x^* \), no point in \( z^* \setminus x^* \) is in the support of \( u \) in \( \mu_k \). Hence, \( z \lor \langle u \rangle_{\mu_k} \) has a minimal element, \( x \), and is contractible (Property 3). Therefore, \( z \) is \( \beta \)-simple for \( \langle u \rangle_{\mu_k} \cup \langle z \rangle \). This establishes that \( z \) is simple for the label \( \nu(z) \) in the image \( \mu_k \). Thus, the images \( \nu \) and \( \mu' \) are equivalent, strongly equivalent if \( X = \mathbb{P}^n \).

3. Let \( u = \bigvee_{z \in x^* \setminus \{x\}} \mu(z) \). We prove now that \( y \) is simple for the label \( u \) in the image \( \nu \). Remember that \( \nu(y) = \mu(y) \).
and $u \leq \mu(y)$ for $\mu$ is non-decreasing. Therefore $w \land u \leq w \land \nu(y)$ for all $w \in T$. Let $w$ be a label such that $w \land \nu(y) \neq \bot$ and $w \land u = \bot$. Obviously, for all $z \in y^t \setminus \{x\}$, we have $\mu(z) \leq u$ and, thereafter, $w \land \mu(z) \leq w \land u = \bot$. Hence, $y^t \setminus \{x\} \subseteq \{x\}$. Now, $x \in \langle w \rangle_{\nu}$ for $\nu(x) = \nu(y)$. Thus, $y^t \setminus \{w\} = \{x\}$ and $y$ is $\beta$-simple for the support of $w$ in the image $v$. We derive that $y$ is simple for the label $u$ in the image $v$ and that the images $v + (y, u)$ and $v$ are equivalent, strongly equivalent if $X = \mathbb{P}^n$.

4. Finally, let us prove that $x$ is simple for the label $u = \bigvee_{z \in y^t \setminus \{x\}} \mu(z)$ in the image $v' = v + (y, u)$ in which the label of $x$ is $\nu(x) = \mu(y)$. Remember that we have established that $w \land u \leq w \land \mu(y)$ for all $w \in T$. Let $w$ be a label such that $w \land \mu(y) \neq \bot$ and $w \land u = \bot$. Since $\nu$ is non-decreasing and $x \in \langle w \rangle_{\nu}$, one has $x^t \setminus \{w\} = \{y\}$ and therefore $x^t \setminus \{w\} = \{y\} \setminus \{y\}$. Now, by hypothesis, $X$ has the pierced sphere property. Then $x^t \setminus \{y\}$ is homotopically trivial and $x$ is a $\gamma$-simple point for $\langle w \rangle_{\nu}$. Furthermore $x^t \setminus \{w\} = \{y\}$ is clearly contractible so $x$ is a $\beta$-simple point (and thus a $\gamma$-simple point) for $\langle w \rangle_{\nu}$. Hence (Property (2)), for all labels $w, \langle w' \rangle_{\nu}$ and $\langle w \rangle_{\nu \vee (x, u)}$ are weakly homotopy equivalent and $\langle w \rangle_{\nu}$ and $\langle w \rangle_{\nu \vee (x, u)}$ are weakly homotopy equivalent (if $w$ is such that $w \land \mu(y) = \bot$ or $w \land u \neq \bot$, the above equivalences are equalities). It is plain that the image $v' + (x, u)$ is equal to the cut $\mu_{x, u}$. Thus $\mu_{x, u}$ and $v'$ are equivalent (strongly equivalent if $X = \mathbb{P}^n$).

By transitivity, $\mu_{x,u}$ and $\mu$ are equivalent (strongly equivalent if $X = \mathbb{P}^n$).

5 Regular label images

In this section, we are interested in labels images constructed from label digital images, that is, images defined on $\mathbb{Z}^n$. The particularity of these label images is that they are entirely determined by their values on the xels (the minimal points of $X$, which are also –by identification– the points of $\mathbb{Z}^n$).

As $X$ is locally finite, for any point $x \in X$ the set $x^- = \{y \in x^t \mid b(y) = 0\}$ is not empty and is finite. Thus, we can define the label of a point $x$ in $X$ depending only on the labels of the elements of $x^-$. 

5.1 Regular and regularised images

**Definition 48 (Regular label image)** A label image $\mu : X \to T$ is a regular (label) image if, for all $x \in X$,

$$\mu(x) = \bigvee_{y \in x^-} \mu(y)$$

**Proposition 49** Let $\mu : X \to T$ be a regular label image. Then, $\mu$ is a closed support image.

**Proof** It is plain that, for any point $x, y \in X$, $x \leq y \Rightarrow x^- \subseteq y^- \Rightarrow \mu(x) \leq \mu(y)$. Hence, a regular label image is non-decreasing and thereafter is a closed support image (Proposition 35).

The *regularisation* of a label image $\mu$ is the regular image $\mu'$ which coincides with $\mu$ on the xels of $X$.

If $\mu$ is a closed support image and $\mu'$ is its regularisation, then $\mu'(x) = \bigvee_{y \leq x} \mu'(y) = \bigvee_{y \leq x} \mu(y)$ (for $\mu$ is non-decreasing) for all $x \in X$. It can easily be seen that the regularisation of a closed support image is the smallest closed support image which coincides with $\mu$ on the xels of $X$.

Let us define the function $\zeta : L^Z \to T^X$ which maps, in a one-to-one manner, a label digital image on a regular image. Given a label digital image $\lambda : Z^n \to L, \zeta(\lambda) : X \to T$ is the only regular image such that, for any xel $x \in X$, $\zeta(\lambda)(x) = \lambda(x)$ (actually, $\zeta(\lambda)(i(j)) = j(\lambda(x))$ where $i$ and $j$ are respectively the embedding of $Z^n$ in $X$ and of $L$ in $T$).

In general, the binary images $\mu \land \ell$, where $\mu$ is a regular image and $\ell \in L$ is a proto-label, are not regular (see Counterexample 65 in Appendix D). Nevertheless, if we regularise these binary images, we find that any regular image is a supremum of regular binary images.

**Proposition 50** Let $\mu : X \to T$ be a regular label image. Let $(l)_{i=1}^\ell (\ell \geq 1)$ be the set of the atoms of $T$. Then $\mu = \bigvee_{i=1}^\ell \mu'_i$ where, for all $i \in [1, \ell)$, $\mu'_i$ denotes the regularisation of the binary image $\mu_i = \mu \land l_i$.

**Proof** We prove first that a supremum of regular images is regular:

$$\left( \bigvee_{i=1}^\ell \mu'_i \right)(x) = \bigvee_{i=1}^\ell \mu'_i(x) = \bigvee_{i=1}^\ell \bigvee_{y \in x^-} \mu'_i(y)$$

$$= \bigvee_{y \in x^-} \bigvee_{i=1}^\ell \mu'_i(y) = \bigvee_{y \in x^-} \left( \bigvee_{i=1}^\ell \mu'_i \right)(y).$$

Now, obviously, $\mu'_i(y) = \mu_i(y)$ for all xel $y$ and therefore, $\bigvee_{i=1}^\ell \mu'_i(y) = \bigvee_{i=1}^\ell \mu_i(y)$ for all xel $y$. From Proposition 30 we have $\mu = \bigvee_{i=1}^\ell \mu_i$. Thus, $\mu$ and $\bigvee_{i=1}^\ell \mu'_i$ are regular images which coincide on the xels of $X$. Hence, $\mu = \bigvee_{i=1}^\ell \mu'_i$.

5.2 Regular images onto a Boolean lattice

In this subsection, we assume the lattice $T$ to be Boolean. For all pair $(u, v)$ of labels, we set $u \setminus v = u \land u'$ where $u'$ is the complement of $u$ in $T$.

The next proposition establishes that the reduction of the number of labels, by identification of some labels with the background, preserves the regularity of the image.
Proposition 51 Let $\mu : X \to T$ be a regular image. Let $t \in T$ be a label. Then, the image $\mu \land t : X \to t'$ defined, for all $x \in X$, by $(\mu \land t)(x) = \mu(x) \land t$, is regular.

Proof For any point $x \in X$, we have $(\mu \land t)(x) = \mu(x) \land t = (\bigvee_{y \in x'} \mu(y)) \land t = \bigvee_{y \in x'} (\mu(y) \land t)$. Therefore, the image $\mu \land t$ is regular.

Applied to proto-labels $l$, Proposition 51 says that the binary images $\mu \land l$, whose support is $\delta$, are regular.

With the following proposition, we show that the function $\zeta$ permutes with the reduction of the lattice $T$ to $t$ for any label $t \in T$.

Proposition 52 Let $\lambda : \mathbb{Z}^n \to L$ be a label digital image. The $\zeta$ for all $t \in T$, $\zeta(\lambda \land t) = \zeta(\lambda \land t)$.

In other words, we have the following commutative diagram:

$\begin{array}{ccc}
L_{\mathbb{Z}^n} & \xrightarrow{\zeta} & T^X \\
\lambda \mapsto \lambda \land t & & \mu \mapsto \mu \land t \\
\zeta & \downarrow & \zeta \\
L_{\mathbb{Z}^n} & \xrightarrow{\zeta} & T^X
\end{array}$

Proof Since the images $\zeta(\lambda \land t)$ and $\zeta(\lambda \land t)$ are regular (from Proposition 51) and the very definition of $\zeta$, it suffices to show that they are equal on the xels of $X$. Let $x$ be a xel. On one side, one has $\zeta(\lambda \land t)(x) = (\lambda \land t)(x) = \lambda(x) \land t$ and on the other side, $(\zeta(\lambda \land t))(x) = (\zeta(\lambda \land t))(x) = \lambda(x) \land t$. Thus, $\zeta(\lambda \land t)(x) = (\zeta(\lambda \land t))(x)$.

After reducing the number of labels by taking the infimum with a particular label $t$, we can consider the remaining labels as a unique label. The result is a binary image whose support is $\delta$. Starting from a label digital image, the following proposition shows that this operation can be made before or after the use of the function $\zeta$. Combining this proposition with Proposition 52 and the results established in 5.2, it means that the connected components and the digital fundamental groups of any binary digital image obtained by just considering a particular union of labels in a label digital image are isomorphic to the ones obtained by the same operation in the label image.

In Proposition 53, the lattice $T$ need not be distributive.

Proposition 53 Let $\lambda : \mathbb{Z}^n \to L$ be a label digital image. Let $B : L \to \{\bot, \top\}$ be the binary image defined by $B(l)(z) = \bot$ if $l(z) = \bot$ and $B(l)(z) = \top$ otherwise. Let $B \zeta(\lambda) : X \to \{\bot, \top\}$ be the binary image defined by $B \zeta(\lambda)(z) = \bot$ if $\zeta(\lambda)(z) = \bot$ and $B \zeta(\lambda)(z) = \top$ otherwise. Then, $B \zeta(\lambda) = \zeta(B \lambda)$.

Proof The proof consists of showing that 1. $B \zeta (\lambda)$ is regular and 2. the functions $B \zeta (\lambda)$ and $\zeta(B \lambda)$ coincide on the xels of $X$.

1. Let $\mu : X \to T$ be a regular image, $B \mu : X \to \{\bot, \top\}$ be the binary image defined by $B \mu(z) = \bot$ if $\mu(z) = \bot$ and $B \mu(z) = \top$ otherwise and $x$ be a point of height greater than or equal to 1. We have: $B \mu(x) = \bot \iff \mu(x) = \bot \iff \bigvee_{y \in x'} \mu(y) = \bot \iff \forall y \in x', \mu(y) = \bot \iff \forall y \in x', B \mu(y) = \bot \iff \bigvee_{y \in x'} B \mu(y) = \bot$. We can straightforwardly conclude that $B \mu$ is regular.

2. Let $x$ be a xel. We have: $B \zeta(\lambda)(x) = \bot \iff \zeta(\lambda)(x) = \bot \iff \lambda(x) = \bot \iff B \lambda(x) = \bot \iff \zeta(B \lambda)(x) = \bot$. Hence, $B \zeta (\lambda) = \zeta(B \lambda)$ are equal on the xels of $X$ and, since they are regular, they are equal.

The following lemma gives a way to locally regularise some closed support images. We will use this lemma in Subsection 5.3 to regularise a label image after a cut.

Lemma 54 Let $\mu : X \to T$ be a closed support image and $\mu'$ be the regularisation of $\mu$. Let $(x, y)$ be a free pair for the label $t = \mu(x) \setminus \mu'(x)$ in the image $\mu$ such that $\mu(x) = \mu(y)$.

Then, the cut $\mu_{x,y}$ is equal to $\mu$ on $X \setminus \{x, y\}$ and to $\mu'$ on $\{x, y\}$.

Proof Since $\mu(x) = \mu(y)$ and $\mu$ is non-decreasing, $\mu(y) \leq \mu(z)$ for all $z \in x^t$. Now, for any point $z \in x^t \setminus \{y\}$, by Definition 5.2, $\mu_{x,z}(z) = \mu(z)(y)$ and thereafter, $\mu_{x,z}(z) = \mu(z)$. By Definition 5.2 again, $\mu_{x,z}(z) = \mu(z)$ for any point $z$ in $x^t \setminus \{y\}$. Thus, $\mu_{x,z}$ is equal to $\mu$ on $X \setminus \{x, y\}$.

As $(x, y)$ is a free pair for $t$, we derive that $t \neq \bot$. In particular, $x$ is not a xel (by definition, $\mu'$ coincides with $\mu$ on the xels of $X$). Then:

$\mu'(x) = \bigvee_{z \in x'} \mu(z) \leq \bigvee_{z \in x' \setminus \{y\}} \mu(z) = \mu_{x,z}(x)$;

$\bigvee_{z \in x' \setminus \{y\}} \mu(z) \leq \mu(y) = \mu(x)$ (for $\mu$ is non-decreasing);

since $(x, y)$ is a free pair for $t$, no point $z \in y^t \setminus \{x\}$, $z \neq x$, is in the support of the label $t$; thus, $\mu_{x,z}(x) = \mu_{x,z}(z) \land t = \bigvee_{z \in x' \setminus \{y\}} \mu(z) \land t = \bot$.

The lattice $T$ is distributive, so it is modular (see Appendix X).

Then, since $\mu'(x) \leq \mu_{x,y}(x) \leq \mu(x)$ and $\mu_{x,y}(x) \land t = \bot$, we get: $\mu_{x,y}(x) = \mu_{x,y}(x) \land \mu(x) = \mu_{x,y}(x) \land (t \lor \mu'(x)) = (\mu_{x,y}(x) \land t) \lor \mu'(x) = \mu'(x)$.

As regards the point $y$, we have $\mu_{x,y}(y) = \mu_{x,y}(x) = \mu'(x) \leq \mu'(y) = \bigvee_{z \in y^t} \mu(z) \leq \bigvee_{z \in y^t \setminus \{x\}} \mu(z) = \mu_{x,y}(y)$. Hence, $\mu_{x,y}(y) = \mu'(y)$.

5.3 Digitally simple xels

A cut in a regular image is seldom regular. For instance, the cut of Figure 19(b) is not regular since the 1-face under $x$ is
magenta instead of red. But, since, most of the time, the domain of the initial image is a subset of $\mathbb{Z}^n$, one may want the final image, after processing, to be also defined on $\mathbb{Z}^n$. Unfortunately, it is not correct (from a topological viewpoint) to extract a label digital image from a label image by just retaining the xels (for instance, in Figure 19(b), the support of the label $(g, b)$ is connected thanks to a 1-face in magenta but the support of this label is disconnected in the underlying digital image). To properly overcome this issue, it is necessary to use the inverse function of $\zeta$, the function we used to construct the label image. Since $\zeta$ is a bijection between label digital images and regular images (topologically sound as we have seen in Subsection 5.2), we need to improve cuts in order to have a means to locally modify a pure and regular image in such a way that the result is still a pure and regular image. Figure 23 exemplifies the following definition.

**Definition 55 (Digitally simple xel)** Let $\mu : X \rightarrow T$ be a regular image and $t \in T$ be a label. A xel $x \in X$ is digitally simple for $t$ if there exists a sequence of cuts $(\mu_i)_{i=0}^{r}$, $r \geq 0$, where $\mu_0 = \mu$, $\mu_i$ is a cut in $\mu_{i-1}$ for all $i \in [1, r]$, $\mu_i$ is regular, $x \in (t)_0$, and $\mu(y) = \mu_i(y)$ for any xel $y$ distinct from $x$.

In the sequel, so we do not impose the space $X$ to be $\mathbb{F}^n$, we borrow the notion of attachment to authors that have worked on image processing in the framework of cubical complexes [5, 23].

Let $\mu : X \rightarrow T$ be a regular image, $x$ a xel in $X$ and $t$ a label in $T$. We set $\text{Att}(x, t) = x^* \cap (t)_\mu$ where the image $\mu'$ is the regularisation of $\mu + (x, \bot)$. The points in $\text{Att}(x, t)$ are the points that "attach" the xel $x$ to the support of $t$ (see Figure 24).

![Figure 24](image)

(a) A regular image $\mu : \mathbb{F}^2 \rightarrow 2^{[0, b]}$. (b) The set $\text{Att}(x, [b])$. (c) The set $\text{Att}(x, [r])$.

We set also $\text{Card}((t) = \text{Card}(t \in L^* | \mu \leq t) = \text{Card}(t \cap L^*)$. The integer $\text{Card}((t)$ is the number of proto-labels under the label $t$.

The following proposition provides a sufficient condition for a xel $x$ to be digitally simple for a label $t \in T$ in a pure and regular image $\mu$. It is required the existence of a free pair $(x, y)$ for the label $\mu(x)$ with $y \in (t)$ (condition (i)), the possibility to shrink $x^+$ onto $\text{Att}(x, \mu(x))$ by withdrawal of (combinatorial) free pairs in such a way that the points whose label is less than or equal to the label of $y$ are removed first (condition (ii)) and that no point in $x^+ \setminus \text{Att}(x, \mu(x))$ has more than one proto-label distinct from those of $y$ (condition (iii)). The proof consists of regularising step by step (thanks to Lemma 54) the labels of the points of $x^+ \setminus \{x, y\}$ in the non-regular image $\mu_{x,y}(x)$, beginning by the points whose label is less than or equal to the one of $y$. Figure 25 illustrates some of these steps. In Appendix D, Counterexample 66 shows that in the following proposition, condition (iii) is not necessary. This condition is used in the second part of the proof to ensure that for any free pair considered, the two points share the same label. Thereby, our example is built in such a way that this last property is true, even if condition (iii) is not respected.

**Proposition 56** Let $\mu : X \rightarrow T$ be a pure and regular image whose codomain $T$ is distributive and whose domain $X$ is such that any point in $X$ that cover a xel covers at most one other xel and no other points. Let $t$ be a label of $T$ and $x$ a xel of $X$, not in $(t)$. If:

(i) there exists a point $y \in (t)$ such that $(x, y)$ is a free pair for the label $\mu(x)$,

(ii) $x^+ \setminus \{x, y\} \cup (x^+ \setminus \mu^{-1}(\mu(y)^*)) \cup A \setminus A$,

(iii) for any point $z \in x^* \setminus A$, $\text{Card}(z \setminus \mu(y)) \leq 1$,

where $A = \text{Att}(x, \mu(x))$, then the xel $x$ is digitally simple for the label $t$.

**Proof** We set $t_0 = \mu(x)$. Since $(x, y)$ is a free pair for the label $t_0$, it is also a combinatorial free pair for the set $\{x\}$. Let $((x_i, y_i))_{i=0}^{r}$ be a sequence of combinatorial free pairs from $x^+$ to $A = \text{Att}(x, t_0)$ such that $x_0 = x, y_0 = y$ and $\bigcup_{i=0}^{r} \{x_i, y_i\} = (x^+ \setminus \mu^{-1}(\mu(y)^*)) \cup A$ with $k \in [0, r]$. We set $t_i = \mu(y_i) \setminus t_0$. From the hypothesis on $X$ and $\mu$, we derive that $t_1$ is an atom and,
thereafter, that $t_1 \leq t$. Let $\mu_1$ be the cut $\mu_{y_{t_1}}$. By Definition 42, $\mu_1(h) = t_1$ if $h \in \{x, y\}$, $\mu_1(h) = \mu(h) \lor t_1$ if $h \in x^1 \backslash \{x, y\}$ and $\mu_1(h) = \mu(h)$ otherwise. In particular, $t_0 \lor t_1 \leq t_1(h)$ for any point $h \in \bigcup_{i=0}^{k-1}[x_i, y_i]$. $\mu_1(h)$ is the regularisation of $\mu$ and $V_t(h) = t_0$ if $h \in x^1 \backslash (\{x, y\} \cup A)$ and $V_t(h) = \perp$ otherwise ($V_1 = \mu \lor \mu'$). Now $x_1 \in \langle t_0 \rangle_{\mu_1}$ and $\langle t_0 \rangle_{\mu_1} \cap \{x^1 \backslash \{x, y\}\} = \langle t_0 \rangle_{\mu_1} \cap \{x^1 \backslash \{x, y\}\}$; $x^1 \backslash \{x, y\}$ is empty for $y_1 \notin A$. Thus, $(x_1, y_1)$, which is a combinatorial free pair in $x^1 \backslash \{x, y\}$, is also a combinatorial free pair in $\langle t_0 \rangle_{\mu_1}$. Then, from Proposition 48, $(x_1, y_1)$ is a free pair for the label $t_0 = V_1(x_1)$.

The cut $\mu_2 = (\mu_1)_{t_0}$ verifies $\mu_2(h) = \mu'(h)$ if $h \in \{x_1, y_1\}$ and $\mu_2(h)$ is $\mu(h)$ otherwise (Lemma 44). Thereby, gradually, we can show that the pairs $(x_0, y_0)$, $1 \leq i \leq k$, are free for $t_0$ in the image $\mu = \mu' \lor V_i$ where $V_{t_0}(h) = t_0$ for all $h \in x^1 \backslash (A \cup \bigcup_{j=0}^{i} \bigcup_{j=0}^{k-1}[x_j, y_j])$ and $V_{t_0}(h) = \perp$ otherwise.

The pair $(x_{k+1}, y_{k+1})$ is in $x^1 \backslash \mu^{-1}(\{t_0, t_0 \lor t_1\})$ thus we have $t_0 \lor t_1 < \mu_{(x_{k+1})} = \mu_{(x_{k+1})} \lor t_0$ and $t_0 \lor t_1 < \mu_{(y_{k+1})} = \mu_{(y_{k+1})} \lor t_1$. Now, $\Card(\mu_{(x_{k+1})} \backslash \{t_0 \lor t_1\}) = \Card(\mu_{(y_{k+1})} \backslash \{t_0 \lor t_1\}) \leq 1$ (Hypothesis 47). Hence, necessarily, we have $\Card(\mu_{(x_{k+1})} \backslash \{t_0 \lor t_1\}) = \Card(\mu_{(y_{k+1})} \backslash \{t_0 \lor t_1\}) = 1$.

Since $\mu_{k}(x_{k+1}) \leq \mu_{k}(y_{k+1})$, for $\mu$ is a closed support image and cuts of closed support images are closed support images (Proposition 45), we have $\mu_{k_{k+1}} = \mu_{k(y_{k+1})}$. Thereafter we deduce as above that $(x_{k+1}, y_{k+1})$ is a free pair in $\mu_{k}$ for $t_0$ and the cut $\mu_{k+1}$ is equal to $\mu' \lor V_{t_0}$ with $V_{t_0}(h) = t_0$ for all $h \in x^1 \backslash (A \cup \bigcup_{j=0}^{k} \bigcup_{j=0}^{k-1}[x_j, y_j])$ and $V_{t_0}(h) = \perp$ otherwise. We continue the same reasoning on each pair $(x_i, y_i)$ for $k + 2 \leq i \leq r$. The last cut is $\mu_r$ with $\mu_r = \mu' \lor V_r$ where $V_r(h) = t_0$ for all $h \in x^1 \backslash (A \cup \bigcup_{j=0}^{k} \bigcup_{j=0}^{k-1}[x_j, y_j])$ and $V_r(h) = \perp$ otherwise, i.e., $V_r = \perp$ and $\mu_r = \mu'$. So, we are done. □

In [55], Couprie and Bertrand have established a “confluence” property for collapses inside a cubical cell of dimension 2, 3 or 4: if $x^1 \searrow \Att(x, t)$ and $X$ is a complex such that $\Att(x, t) \subset X \subset x^1$, then $x^1 \searrow X$ iff $X \searrow \Att(x, t)$. Thanks to this property, we can apply Proposition 56 to test whether a xel $x \in \mathbb{Z}^n (n \leq 4)$ is digitally simple for a label $t$ by the mean of the following greedy algorithm. Of course if the following algorithm returns “false”, it just means that the hypothesis of Proposition 56 are not all satisfied and, since this proposition only provides sufficient conditions, the tested xel can nevertheless be digitally simple. Figure 26 provides examples of images obtained from the same label digital image by applying the following algorithm to perform thinning or growing on the support of a label.

**Algorithm 1**

**Require:** $(x, y)$: a free pair for the label $t$

**Ensure:** Boolean

1: $Y \leftarrow x^1 \setminus \Att(x, t)$
2: $T \leftarrow \{z \in Y \mid \Card(\mu(z) \setminus \mu(y)) > 1\}$
3: if $T \neq \emptyset$ then
4: return false
5: end if
6: $Z \leftarrow \{z \in Y \mid \mu(z) \leq \mu(y)\}$
7: while $\exists (h, h') \in Z \times Z, (h, h')$ free pair in $Y$ do
8: $Z \leftarrow Z \setminus \{h', h\}$, $Y = Y \setminus \{h, h'\}$
9: end while
10: if $Z \neq \emptyset$ then
11: return false
12: end if
13: while $\exists (h, h') \in Y \times Y, (h, h')$ free pair in $Y$ do
14: $Y = Y \setminus \{h, h'\}$
15: end while
16: if $Y \neq \emptyset$ then
17: return false
18: end if
19: return true

**6 Conclusion**

In this article we have proposed some tools to locally modify a label image with respect not only to the topologies of the labels but also to the topology of the partition, in the sense that the topologies of any union of labels can also be preserved (depending on the choice for the lattice of labels). Here, topology preservation is understood as the existence of weak homotopy equivalence: when a point $x$ is removed from a set $X$, the inclusion $i: X \setminus \{x\} \rightarrow X$ puts in one-to-one correspondence the connected components of $X \setminus \{x\}$ and $X$ and induces isomorphisms between the homotopy groups of both spaces.

Let us now have a look at some of the more relevant models for label images evoked in the introduction. Assum-
Fig. 26 (a) A label digital image \( A_0 : \mathbb{Z}^2 \to L \) (the background is not depicted). (b) The regular image \( \mu = \zeta(L_0) : \mathbb{Z}^2 \to 2^L \). (c) The pre-image \( A_1 = \zeta^{-1}(\mu_1) \) where \( \mu_1 \) is obtained from \( \mu \) by applying Algorithm 1 to shrink the green label. (d) The regular image \( \mu_1 \). (e) The pre-image \( A_2 = \zeta^{-1}(\mu_2) \) where \( \mu_2 \) is obtained from \( \mu \) by applying Algorithm 1 to expand the green label. (f) The regular image \( \mu_2 \). (g–i) The same detail in the images \( A_0, A_1, A_2 \). (j) A part of the above detail in the image \( \mu_1 \). Observe that the isolated green square is not digitally simple for the brown label: the change of label will fill a hole in the brown label.

Furthermore, some questions remain. Can Theorem 27 be extended to a wider family of spaces? This would ensure strong equivalences between label images in other spaces than \( \mathbb{R}^n \). Is it possible to enrich the model in order to be able to work with other types of regular images as those defined in \( [22] \)? This could be interesting for the modelling of the \( (18, 6) \)-adjacency pair in \( \mathbb{Z}^3 \). We hope to be able to give some answers to these issues in further works.

### A Lattices

In this appendix, we recall some vocabulary and properties used in the article. More information on lattices can be found in, e.g., [56] or [57].

**Lattice.** A lattice is a poset in which every pair \( (a, b) \) of elements have a supremum, denoted \( a \lor b \), and a infimum, denoted \( a \land b \). Thereafter in a finite lattice, there exists a least and a greatest element.

**Atom/Atomistic.** In a lattice, an element is an atom if it covers the minimal element. An atomistic lattice is a lattice in which each element that is not the least element is a supremum of a set of atoms.

**Modular.** A lattice is modular if \( x \leq z \) implies \( x \lor (y \land z) = (x \lor y) \land z \).

**Distributive.** A lattice is distributive if \( x \lor (y \land z) = (x \lor y) \land (x \lor z) \), or, equivalently, if \( x \land (y \lor z) = (x \land y) \lor (x \land z) \). A distributive lattice is modular.

**Boolean.** In a lattice, \( y \) is a complement of \( x \) if the infimum of \( x \) and \( y \) is the minimum element and the supremum of \( x \) and \( y \) is the maximum element. A lattice is Boolean if it is distributive and each element has a complement. Moreover, in this case, this complement is unique. A finite distributive lattice is Boolean iff it is atomistic.

**Opening.** Let \( L \) be a lattice. A function \( \varphi : L \to L \) is an opening if \( \varphi \) is anti-extensive (\( \varphi(x) \leq x \) for all \( x \in L \)) and \( \varphi(x) \leq y \Rightarrow \varphi(x) \leq \varphi(y) \) for all \( x, y \in L \). An opening is order-preserving (\( x \leq y \Rightarrow \varphi(x) \leq \varphi(y) \)) and idempotent (\( \varphi \circ \varphi = \varphi \)). Let \( A \) be a subset of \( L \). The
function \( \varphi_A : L \to L \) defined by \( \varphi_A(x) = \bigvee \{a \in A \mid a \leq x\} \) is an opening.

### B Proof of Proposition 26

The proof of Proposition 26 (Subsection 26) relies on some combinatorial properties of cubical and simplicial complexes that we establish hereafter.

**Lemma 57** Let \( X \) be a cubical or a simplicial complex equipped with the order \( \preceq \). Let \( x \in X \) be an m-face (0 ≤ m ≤ \( \dim(X) \)).

(i) Let \( y \in x^1 \) be a k-face (0 ≤ k ≤ m). There exist exactly \( m-k \) faces in \( x^1 \) of dimension \( (k+1) \) which include \( y \).

(ii) Let \( x_1, x_2 \) be two faces in \( x^1 \) such that \( \dim(x_1) = m-1 \), \( x = x_1 \cup x_2 \). Let \( Z \) be the set of faces in \( x^1 \) that intersect both \( x_1 \) and \( x_2 \). The function \( \theta : Z \to x^1 \) defined by \( \theta(z) = z \cap x_1 \) is a bijection and \( \dim(\theta(z)) = \dim(z) - 1 \) for all \( z \in Z \).

**Proof** (i) If \( k = m \), Lemma 57 is trivial. We suppose now that \( k < m \).

If \( X \) is a cubical complex, there are \( m+1 \) vertices in \( x \) and \( k+1 \) vertices in \( y \). Hence, there exist exactly \( (m+1)-(k+1) = m-k \) faces of \( x \) of dimension \( k+1 \) including \( y \) (thus containing the \( k+1 \) vertices of \( y \) plus one). If \( X \) is a cubical n-complex, we can assume without loss of generality that \( x = \bigcup_{i=1}^{m} I_i \) where \( I_i \subseteq P^1 \) if \( i \leq m \), \( I_i \subseteq P^0 \) otherwise (see Subsection 26) and \( y = \bigcup_{i=1}^{k} J_i \) where \( \emptyset \subseteq J_i \subseteq I_i \) if \( i \leq k \) and \( J_i = I_i \) otherwise. It is plain that the only \( (k+1) \)-faces included in \( x \) and including \( y \) are the \( m-k \) faces \( z_j \), \( 1 \leq j \leq m-k \) defined by \( z_j = \bigcap_{i=1}^{m} K_i \) with \( K_i = J_i \) if \( i \leq j \) and \( K_i = I_i \) otherwise.

(ii) If \( X \) is a simplicial complex, because \( \dim(x_1) = \dim(x) - 1 \) and \( x = x_1 \cup x_2 \), \( x_2 \) is a singleton. Then, for all \( z \in Z \), \( \theta(z) = z \cap x_1 \) is a \( k \)-face. So, it is plain that \( \theta \) is a bijection whose inverse \( \theta^{-1} \) is defined by \( \theta^{-1}(z) = z \cup x_2 \). Furthermore, for all \( z \in Z \), \( \dim(z) > 0 \) and the simplex \( z \cap x_1 \subset z \) has dimension \( k+1 \).

If \( X \) is a cubical complex, because \( x = x_1 \cup x_2 \), we have \( \dim(x_2) = \dim(x_1) + 1 \). As above, we can assume that \( x = \bigcup_{i=1}^{m} I_i \) where \( I_i \subseteq P^1 \) if \( i \leq m \), \( I_i \subseteq P^0 \) otherwise, \( x_1 = \bigcup_{i=1}^{m} J_i \) and \( x_2 = \bigcup_{i=m+1}^{k} J_i \) with \( J_i = I_i \) if \( i \leq m \), \( \emptyset \subseteq J_i \subseteq I_i \) and \( J_i = \emptyset \) otherwise. These conditions, it can easily be seen that \( Z = \bigcup_{i=1}^{m} K_i \) with \( K_i = I_i \) if \( i \leq m \), \( K_i = I_i \) if \( i \leq k \), \( \emptyset \subseteq K_i \subseteq I_i \). Hence, \( \theta(z) = \bigcap_{i=1}^{m} K_i \) and \( \theta^{-1}(z) = \bigcup_{i=1}^{m} K_i \) with \( K_i = I_i \) and \( K_i = I_i \) otherwise. Hence, \( \theta \) is injective. Moreover, obviously, \( Card(\{z \mid K_i \subseteq P^1\}) = Card(\{z \mid K_i \subseteq P^0\}) = 1 \).

We establish below a proposition which straightforwardly provides Proposition 26 as a corollary. This proposition will be used in the proof of Theorem 27 (see Appendix K). Some steps of the proof are depicted in Figure 27.

**Proposition 58** Let \( X \) be a cubical or a simplicial complex equipped with the order \( \preceq \). Let \( x, y \in X \) be two faces with \( \dim(y) = \dim(x) - 1 \). Let \( Y \) be a subset of \( x^1 \), containing \( y \). Then, \( x^1 \setminus Y \) is contractible.

**Proof** We set \( m = \dim(x) \) and \( X_0 = x^1 \setminus Y \). If \( m = 1 \), Proposition 58 is trivial (\( X_0 \) is a singleton). Suppose now that \( m \geq 2 \). We denote by \( y' \) the face opposite to \( y \) in \( x^1 \). Let \( y' \cup y \). Observe that \( \dim(y') = 0 \) if \( x \) is a simplicial complex and \( \dim(y') = m-1 \) if \( x \) is a cubical complex. We will shrink \( X_0 \) to \( y' \) by removing unipolar points from \( X_0 \). First, we remove the faces of \( X_0 \) that are in \( x^1 \), in decreasing order relatively to their dimension. For any \( (m-2) \)-face \( z \in y' \) \( y \) we derive from Lemma 57 that there are two \((m-1)\)-faces in \( x^1 \) including \( z \), one of which is \( y \). Hence, \( z \) is down unipolar in \( X_0 \) and, thanks to Properties 6 and 11.

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10 We write \( \cup \) for the disjoint union.
\[
\begin{align*}
\text{(ii)} & \quad \text{Proposition 58 could be stated in terms of } y \text{ such that } x \subseteq \text{ be a set that has a maximum or a minimum, then, the point } x \text{ is } y \text{-simple in } X \cup Y. \\
\text{Proof} & \quad \text{Proving that } x \text{ is } y \text{-simple in } X \cup Y \text{ comes down to establish that } x^* \cap Y \text{ is homotopically trivial. First case: } Y \text{ has a minimum noted } y. \text{ The proof is made by induction on } m, \text{ the dimension of } x. \text{ If } m = 1, \text{ the result is obvious. We assume now that } m \geq 2. \text{ If } \dim(y) = m - 1, \text{ we apply Proposition 58.} \\
& \quad \text{If } \dim(y) \leq m - 2, \text{ let } z \text{ be an } (m - 1)\text{-face of } x^* \text{ including } y. \text{ From the induction hypothesis, } z^* \cap Y \text{ is homotopically trivial. Hence, } z \text{ is a } y \text{-point in } x^* \cap Y. \text{ So, } x^* \setminus (Y \cup \{z\}) \text{ is weak-homotopy equivalent to } x^* \setminus Y \text{ (Property 22). Now, from Proposition 58 we deduce that } x^* \setminus (Y \cup \{z\}) \text{ is contractible and we are done: by transitivity, } x^* \cap Y \text{ is homotopically trivial.} \\
& \quad \text{Second case: } Y \text{ has a maximum noted } y. \text{ The proof is made by induction on } \text{Card}(Y). \text{ If } \text{Card}(Y) = 1, \text{ i.e., } y = \{y\}, \text{ we use the first part of the proof to conclude. Suppose now that } \text{Card}(Y) \geq 2. \text{ Let } z, z \neq y, \text{ be a face in } Y \text{ such that } \dim(z) = \min\{\dim(t) \mid t \in Y \setminus \{y\}\}. \text{ We observe that } z^* \cap Y \neq \{y\}. \text{ Now, we set } Z = (x^* \cap Y) \cup \{z\} = x^* \setminus (Y \cup \{z\}). \text{ As } \text{Card}(Z) < \text{Card}(Y), \text{ we deduce from the induction hypothesis that } Z \text{ is homotopically trivial. Let us prove that } z \text{ is a } y \text{-point for } Z. \text{ We have } z^* \cap Z = z^* \setminus \{y\} \text{ which, from the first part of the proof, is homotopically trivial. Hence, } z \text{ is a } y \text{-simple point for } Z. \text{ Thereafter, the injection } i : x^* \setminus Y \rightarrow Z \text{ is a weak homotopy equivalence and we conclude straightforwardly.} \\
\begin{proof} (Theorem 27) \\
\text{Let } y \neq \emptyset \text{ be a } y \text{-simple point in } Y, \text{ then } y^* \cap Y \text{ or } y^* \cap Y \text{ is contractible. We suppose first that } y^* \cap Y \text{ is contractible. From Corollary 9 we know that there exists a sequence } (x_j)_{j \in [0, r]} \text{ such that } y^* \cap Y = (x_j)_{j \in [0, r]} \text{ and } x_j \text{ is unipolar in } x_{j-1} \text{ for all } j \in [1, r]. \text{ The proof consists to establish that } x_j \text{ is a } y \text{-simple point in } y^* \setminus (x_j)_{j \in [0, r]} \text{ for all } j \in [1, r]. \text{ This will imply (by transitivity) that the injection of } y^* \cap Y \text{ in } y^* \setminus (x_j)_{j \in [0, r]} \text{ is a weak homotopy equivalence. Then Proposition 60 will permit us to conclude easily. So, let us suppose first that } x_j \text{ is unipolar in } x_{j-1} \text{ for some } j \in [1, r]. \text{ We set } Y_j = x_j^* \cap (x_{j-1})_0. \\
& \quad \text{From Proposition 60 we derive that } x_j^* \setminus Y_j \text{ is homotopically trivial (since } Y_j \text{ has a minimum). A } x_j^* \cap (x_j^* \cap x_{j-1}) = x_j^* \setminus Y_j, x_j \text{ is a } y \text{-simple point in } y^* \setminus (x_j)_{j \in [0, r]} \text{ and we set } Y_j = x_j^* \cap (x_{j-1})_0. \text{ We observe that } Y_j \text{ has a maximum. Thanks to Lemma 59 we can consider an homeomorphism } z : y^* \cap Y \rightarrow Z \text{ where } Z \text{ is a simplicial cell. From Property 2 (any continuous function between posets is non-decreasing), we derive that } w(y^* \cap x_j^*) = w(y^* \cap y) \text{ and that } w(Y_j) \text{ has a maximum for } (Y_j, Y_j). \text{ Then we invoke Proposition 61 to assert that } w(y^* \cap x_j^*) \setminus Y_j = w(y^* \cap Y_j) \text{ is homotopically trivial. Hence, } y^* \cap Y_j \text{ is homotopically trivial and } x_j \text{ is a } y \text{-point in } Y_j. \\
& \quad \text{We suppose now that } y^* \cap Y \text{ is contractible. Taking the reverse order on } (x = \mathbb{P}^x, (\leq_x) \text{ is homeomorphic to } (\leq_x)), \text{ we derive from Proposition 13 that } y \text{ is a } y \text{-simple point for } Y. \text{ From Corollary 10 that } y^* \cap Y \text{ is contractible. Then it follows from the first part of the proof that } y \text{ is a } y \text{-simple point for } (X \cup Y) \cup \{y\}\text{ equipped with the inclusion and we conclude, invoking Proposition 20 that } y \text{ is a } y \text{-simple point for } (X \cup Y) \cup \{y\} \text{ with the initial order.} \\
\end{proof}
\end{align*}
\]

D Counterexamples

**Counterexample 61 (Theorem 27)** Figure 28 illustrates the fact that Theorem 27 is generally false when the poset \((X, \leq)\) is a cubical complex, but \((X, \geq)\) is not a cubical complex.

**Counterexample 62 (Proposition 32)** Figure 29 illustrates the fact that Proposition 32 is generally false in a non-distributive lattice.

**Counterexample 63 (Proposition 44)** Figure 27 illustrates the fact that Proposition 44 is generally false in a non-distributive lattice. Furthermore, this figure shows that the number of connected components of the supports is not preserved by a cut in such a lattice. Therefore, this counterexample is also a counterexample for Theorem 47 when the lattice is not distributive.

**Counterexample 64 (Theorem 47)** Figure 37 illustrates the fact that Theorem 47 is generally false if the poset \(X\) has not the pierced sphere property.

**Counterexample 65 (Proposition 50)** Figure 32 shows that if the lattice \(T\) is not distributive, the binary image \(\mu \wedge a\) where \(\mu\) is a regular image and \(a\) is an atom of \(T\) can be non-regular.

**Counterexample 66 (Proposition 50)** Figure 37 shows that in Proposition 50 Condition (iii) is not necessary.

E Comparison between ML-simple points and digitally simple points

In Figure 44 we borrow the images used in 15 to illustrate the notion of ML-simple point in label digital images in order to compare this notion with our own notion of digitally simple point in regular label images (we have omitted the first image of 15 which is very similar to the second one).

References

Fig. 28 (a) A set $X$ which is a cubical complex but whose dual is not a complex (because of the boundary). In black, a subset $Y$ of $X$. The point $y$ is a 1-face of $Y$. In light grey, the complement of $Y$ in $X$, $X \setminus Y$. (b) In black, the set $Y \cup \{y\}$. In light grey, the set $(X \setminus Y) \cup \{y\}$. Clearly, $y$ is a $β$-simple for $Y$ (is up-unipolar in $Y$) but $y$ is not $γ$-simple for $(X \setminus Y) \cup \{y\}$ since this later set has not the same number of connected components as $X \setminus Y$.

Fig. 29 (a) A label image $μ : X \rightarrow T$. (b) The Hasse diagram of $T$ (which is not distributive). The labels $\bot, R, G, B, Y, \top$ are depicted respectively in white, red, green, blue, yellow and black. The yellow 2-face $x$ is not simple for the label $\top$ since the label $G$ is such that $G \land Y = \bot$ and $G \land \top = \top$ but $x$ is not $β$-simple for $(G) \cup \{x\}$. (c) The label image $ϕ \circ μ : X \rightarrow ϕ(T)$ (for the definition of $ϕ$, see Proposition 32). In this image, the point $x$ is simple for the label $\top$.

Fig. 30 (a) A closed supports label image $μ : X \rightarrow T$ with $T = \{\emptyset, \{r\}, \{g\}, \{b\}, \{r, g\}, \{g, b\}, \{r, g, b, t\}\}$, equipped with the inclusion. (b) The Hasse diagram of $T$. The labels $\{r\}, \{g\}, \{b\}, \{r, g\}, \{g, b\}, \{r, g, b, t\}$ are depicted respectively in red, green, blue, yellow, cyan and black. (c) The cut $μ_{\{g\}}, t$. In the image $μ$, the support of $t$ is empty. But, in the cut $μ_{\{g\}}, t$ the support of $t$ is no longer empty (it contains the three points in black).

Fig. 31 (a) An image $μ : X \rightarrow 2^{[0,1]}$. In the poset $X$, the points $z$ and $z'$ are identified. Thus, $X$ has not the pierced sphere property ($x^* \setminus \{y\}$ is a ring). The support of $\{g\}$ is a ball. (b) The cut $μ_{\{g\}}, t$. The support of $\{g\}$ is a ring.

Fig. 32 (a) A regular image $μ : X \rightarrow T$ with $T = \{\emptyset, \{r\}, \{g\}, \{b\}, \{r, g, b\}\}$ equipped with the inclusion. (b) The binary image $μ \land \{b\}$ which is not regular.

Fig. 33 (a) A regular image $μ : X \rightarrow 2^{[r,g,b,e]}$ where the four proto-labels $r, g, b, e$ are depicted respectively in red, green, blue and grey. We take the notations of the proof of Proposition 55. The xel $x$ is at the center of the image. Its label is $ι_b = \{e\}$. (b) The label digital image associated to $μ$ (in $2^{[1]}$). (c) The cut $μ_1 = μ_{ι_b}$. (d) The cut $μ_2 = (μ_{ι_b})_{ι_b}$. (e) The cut $μ_3 = (μ_{ι_b})_{ι_b}$. (f) The cut $μ_4 = (μ_{ι_b})_{ι_b}$, which is regular. Hence, the xel $x$ is digitally simple. Nevertheless we have $μ(τ_y) = \lor\{r, g, b, e\}$, so Condition (iii) of Proposition 55 is not satisfied.
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