Advanced computational dissipative structural acoustics and fluid-structure interaction in low- and medium-frequency domains. Reduced-order models and uncertainty quantification

Roger Ohayon, Christian Soize

Advanced Computational Dissipative Structural Acoustics and Fluid-Structure Interaction in Low- and Medium-Frequency Domains. Reduced-Order Models and Uncertainty Quantification

R. Ohayon[a], C. Soize[b]

[a]*Structural Mechanics and Coupled Systems Laboratory, Conservatoire National des Arts et Metiers (CNAM), 2 rue Conte, 75003 Paris, France*
[b]*Laboratoire Modélisation et Simulation Multi-Echelle (MSME UMR 8208 CNRS), Université Paris-Est, 5 bd Descartes, 77454 Marne-la-Vallée, France*

Abstract

This paper presents an advanced computational method for the prediction of the responses in the frequency domain of general linear dissipative structural-acoustic and fluid-structure systems, in the low- and medium-frequency domains, including uncertainty quantification. The system under consideration is constituted of a deformable dissipative structure, coupled with an internal dissipative acoustic fluid including wall acoustic impedances and surrounded by an infinite acoustic fluid. The system is submitted to given internal and external acoustic sources and to prescribed mechanical forces. An efficient reduced-order computational model is constructed using a finite element discretization for the structure and the internal acoustic fluid. The external acoustic fluid is treated using an appropriate boundary element method in the frequency domain. All the required modeling aspects for the analysis of the medium-frequency domain have been introduced namely, a viscoelastic behavior for the structure, an appropriate dissipative model for the internal acoustic fluid including wall acoustic impedance and a model of uncertainty in particular for modeling errors. This advanced computational formulation, corresponding to new extensions and complements with respect to the state-of-the-art, is well adapted for developing new generation of software, in particular for parallel computers.

Keywords: Computational mechanics, Structural acoustics, Vibroacoustic, Fluid-structure interaction, Uncertainty quantification, Reduced-order model, Medium frequency, Low frequency, Dissipative system, Viscoelasticity, Wall acoustic impedance, Finite element discretization, Boundary element method

Nomenclature

a_{ijkh} = elastic coefficients of the structure
b_{ijkh} = damping coefficients of the structure
c_0 = speed of sound in the internal acoustic fluid
c_E = speed of sound in the external acoustic fluid
f = vector of the generalized forces for the internal acoustic fluid
$\mathbf{f}^S$ = vector of the generalized forces for the structure
g = mechanical body force field in the structure
i = imaginary complex number i
k = wave number in the external acoustic fluid
n = number of internal acoustic DOF
n_S = number of structural DOF
n_j = component of vector **n**
n = outward unit normal to $\partial\Omega$
n_j^S = component of vector $\mathbf{n}^S$
$\mathbf{n}^S$ = outward unit normal to $\partial\Omega_S$

Email addresses: roger.ohayon@cnam.fr (R. Ohayon), christian.soize@univ-paris-est.fr (C. Soize)

p = internal acoustic pressure field
p_E = external acoustic pressure field
$p_E|_{\Gamma_E}$ = value of the external acoustic pressure field on Γ_E
p_{given} = given external acoustic pressure field
$p_{\text{given}}|_{\Gamma_E}$ = value of the given external acoustic pressure field on Γ_E
$\mathbf{q}$ = vector of the generalized coordinates for the internal acoustic fluid
$\mathbf{q}^S$ = vector of the generalized coordinates for the structure
s_{ij}^{damp} = component of the damping stress tensor in the structure
t = time
$\mathbf{u}$ = structural displacement field
$\mathbf{v}$ = internal acoustic velocity field
x_j = coordinate of point $\mathbf{x}$
$\mathbf{x}$ = generic point of $\mathbb{R}^3$
$[A]$ = reduced dynamical matrix for the internal acoustic fluid
$[\mathbf{A}]$ = random reduced dynamical matrix for the internal acoustic fluid
$[\mathbb{A}]$ = dynamical matrix for the internal acoustic fluid
$[A_{\text{BEM}}]$ = reduced matrix of the impedance boundary operator for the external acoustic fluid
$[\mathbb{A}_{\text{BEM}}]$ = matrix of the impedance boundary operator for the external acoustic fluid
$[A_{\text{FSI}}]$ = reduced dynamical matrix for the fluid-structure coupled system
$[\mathbf{A}_{\text{FSI}}]$ = random reduced dynamical matrix for the fluid-structure coupled system
$[\mathbb{A}_{\text{FSI}}]$ = dynamical matrix for the fluid-structure coupled system
$[A^S]$ = reduced dynamical matrix for the structure
$[\mathbf{A}^S]$ = random reduced dynamical matrix for the structure
$[\mathbb{A}^S]$ = dynamical matrix for the structure
$[A^Z]$ = reduced dynamical matrix associated with the wall acoustic impedance
$[\mathbb{A}^Z]$ = dynamical matrix associated with the wall acoustic impedance
$[C]$ = reduced coupling matrix between the internal acoustic fluid and the structure
$[\mathbf{C}]$ = random reduced coupling matrix between the internal acoustic fluid and the structure
$[\mathbb{C}]$ = coupling matrix between the internal acoustic fluid and the structure
$[D]$ = reduced damping matrix for the internal acoustic fluid
$[\mathbf{D}]$ = random reduced damping matrix for the internal acoustic fluid
$[\mathbb{D}]$ = damping matrix for the internal acoustic fluid
$[D^S]$ = reduced damping matrix for the structure
$[\mathbf{D}^S]$ = random reduced damping matrix for the structure
$[\mathbb{D}^S]$ = damping matrix for the structure
DOF = degrees of freedom
$\mathbb{F}$ = vector of discretized acoustic forces
$\mathbb{F}^S$ = vector of discretized structural forces
$G_{ijkh}(0)$ = initial elasticity tensor for viscoelastic material
$G_{ijkh}(t)$ = relaxation functions for viscoelastic material
$\mathbf{G}$ = mechanical surface force field on $\partial\Omega_S$
$[\mathbf{G}]$ = random matrix
$[\mathbf{G}_0]$ = random matrix
$[K]$ = reduced "stiffness" matrix for the internal acoustic fluid
$[\mathbf{K}]$ = random reduced "stiffness" matrix for the internal acoustic fluid
$[\mathbb{K}]$ = "stiffness" matrix for the internal acoustic fluid
$[K^S]$ = reduced stiffness matrix for the structure
$[\mathbf{K}^S]$ = random reduced stiffness matrix for the structure
$[\mathbb{K}^S]$ = stiffness matrix for the structure
$[M]$ = reduced "mass" matrix for the internal acoustic fluid
$[\mathbf{M}]$ = random reduced "mass" matrix for the internal acoustic fluid
$[\mathbb{M}]$ = "mass" matrix for the internal acoustic fluid
$[M^S]$ = reduced mass matrix for the structure
$[\mathbf{M}^S]$ = random reduced mass matrix for the structure
$[\mathbb{M}^S]$ = mass matrix for the structure
$\mathbb{P}_\alpha$ = internal acoustic mode
$[\mathcal{P}]$ = matrix of internal acoustic modes
Q = internal acoustic source density
Q_E = external acoustic source density
$\mathbf{Q}$ = random vector of the generalized coordinates for the internal acoustic fluid

$\mathbf{Q}^S$	= random vector of the generalized coordinates for the structure
$\mathbf{P}$	= random vector of internal acoustic pressure DOF
$\mathbb{P}$	= vector of internal acoustic pressure DOF
$\mathbf{U}$	= random vector of structural displacement DOF
$\mathbb{U}$	= vector of structural displacement DOF
$\mathbb{U}_\alpha$	= elastic structural mode α
$[\mathcal{U}]$	= matrix of elastic structural modes
Z	= wall acoustic impedance
$\mathbf{Z}_{\Gamma_E}$	= impedance boundary operator for external acoustic fluid
δ	= dispersion parameter
ε_{kh}	= component of the strain tensor in the structure
ω	= circular frequency in rad/s
ρ_0	= mass density of the internal acoustic fluid
ρ_E	= mass density of the external acoustic fluid
ρ_S	= mass density of the structure
σ	= stress tensor in the structure
σ_{ij}	= component of the stress tensor in the structure
$\sigma_{ij}^{\text{elas}}$	= component of the elastic stress tensor in the structure
τ	= damping coefficient for the internal acoustic fluid
$\partial\Omega$	= boundary of Ω
$\partial\Omega_E$	= boundary of Ω_E equal to Γ_E
$\partial\Omega_S$	= boundary of Ω_S
Γ	= coupling interface between the structure and the internal acoustic fluid
Γ_E	= coupling interface between the structure and the external acoustic fluid
Γ_Z	= coupling interface between the structure and the internal acoustic fluid with acoustical properties
Ω	= internal acoustic fluid domain
Ω_i	= $\mathbb{R}^3\backslash(\Omega_E \cup \Gamma_E)$
Ω_E	= external acoustic domain
Ω_S	= structural domain

1. Introduction

The fundamental objective of this paper is to present an advanced computational method for the prediction of the responses in the frequency domain of general linear dissipative structural-acoustic and fluid-structure systems in the low- and medium-frequency domains. The system under consideration is constituted of a deformable dissipative structure, coupled with an internal dissipative acoustic fluid including wall acoustic impedances. The system is surrounded by an infinite acoustic fluid and is submitted to given internal and external acoustic sources, and to prescribed mechanical forces.

Instead of presenting an exhaustive review of such a problem in this introductory section, we have preferred to move the review discussions in each relevant sections.

Concerning the appropriate formulations for computing the elastic, acoustic and elastoacoustic modes of the associated conservative fluid-structure system, including substructuring techniques, and for constructing reduced-order computational models in fluid-structure interaction and for structural-acoustic systems, we refer the reader to Morand and Ohayon (1995); Ohayon et al. (1997); Ohayon and Soize (1998); Ohayon (2004b,a). For dissipative complex systems, the reader can find the details of the basic formulations in Ohayon and Soize (1998).

In this paper, the proposed formulation, which corresponds to new extensions and complements with respect to the state-of-the-art, can be used for the development of a new generation of computational software in particular, in the context of parallel computers. We present here an advanced computational formulation which is based on an efficient reduced-order model in the frequency domain and for which all the required modeling aspects for the analysis of the medium-frequency domain have been taken into account. More precisely, we have introduced a viscoelastic modeling for the structure, an appropriate dissipative model for the internal acoustic fluid including wall acoustic impedance and finally, a global model of uncertainty. It should be noted that model uncertainties must absolutely be taken into account in the computational models of complex vibroacoustic systems in order to improve the prediction of the responses in the medium-frequency range. The reduced-order computational model is constructed using the finite element discretization for the structure and for the internal acoustic fluid.

The external acoustic fluid is treated using an appropriate boundary element method in the frequency domain.

The sections of the paper are:

1. *Introduction*
2. *Statement of the problem in the frequency domain*
3. *External inviscid acoustic fluid equations*
4. *Internal dissipative acoustic fluid equations*
5. *Structure equations*
6. *Boundary value problem in terms of* $\{\boldsymbol{u}, p\}$
7. *Computational model*
8. *Reduced-order computational model*
9. *Uncertainty quantification*
10. *Symmetric boundary element method without spurious frequencies for the external acoustic fluid*
11. *Conclusion*

The *References* are given at the end of the paper.

2. Statement of the Problem in the Frequency Domain

We consider a mechanical system made up of a damped linear elastic free-free structure Ω_S containing a dissipative acoustic fluid (gas or liquid) which occupies a domain Ω. This system is surrounded by an infinite external inviscid acoustic fluid domain Ω_E (gas or liquid) (see Fig. 2). A part Γ_Z of the internal fluid-structure interface is assumed to be dissipative and is modeled by a wall acoustic local impedance Z. This system is submitted to a given internal acoustic source in the acoustic cavity and to given mechanical forces applied to the structure. In the infinite external acoustic fluid domain, external acoustic sources are given. It is assumed that the external forces are in equilibrium.

We are interested in the responses in the *low*- and *medium-frequency* domains for the displacement field in the structure, the pressure field in the acoustic cavity and the pressure fields on the external fluid-structure interface and in the external acoustic fluid (near and far fields). It is now well established that the predictions in the medium-frequency domain must be improved by taking into account both the system-parameter uncertainties and the model uncertainties induced by modeling errors. Such aspects will be considered in the last section of the paper devoted to *Uncertainty Quantification* (UQ) in structural acoustics and in fluid-structure interaction.

2.1. Main notations

The physical space $\mathbb{R}^3$ is referred to a cartesian reference system and we denote the generic point of $\mathbb{R}^3$ by $\mathbf{x} = (x_1, x_2, x_3)$. For any function $f(\mathbf{x})$, the notation $f_{,j}$ means the partial derivative with respect to x_j. We also use the classical convention for summations over repeated Latin indices, but not over Greek indices. As explained above, we are interested in vibration problems formulated in the frequency domain for structural-acoustic and fluid-structure interaction systems. Therefore, we introduce the Fourier transform for various quantities involved. For instance, for the displacement field $\mathbf{u}$, the stress tensor σ_{ij} and the strain tensor ε_{ij} of the structure, we will use the following simplified notation consisting in using the same symbol for a quantity and its Fourier transform. We then have,

$$\mathbf{u}(\mathbf{x}, \omega) = \int_{-\infty}^{+\infty} e^{-i\omega t}\, \mathbf{u}(\mathbf{x}, t)\, dt\,, \tag{1}$$

$$\sigma_{ij}(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega t}\, \sigma_{ij}(t)\, dt\,, \tag{2}$$

$$\varepsilon_{ij}(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega t}\, \varepsilon_{ij}(t)\, dt\,, \tag{3}$$

in which the circular frequency ω is real. Nevertheless, for other quantities, some exceptions to this rule are done and in such a case, the Fourier transform of a function f will be noted $\widehat{f}$,

$$\widehat{f}(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega t}\, f(t)\, dt\,. \tag{4}$$

2.2. Geometry - Mechanical and acoustical hypotheses - Given loadings

The coupled system is assumed to be in linear vibrations around a static equilibrium state taken as a natural state at rest.

Structure Ω_S. In general, a complex structure is composed of a main part called the *master structure*, defined as the "primary" structure accessible to conventional modeling including *uncertainties* modeling, and a secondary part called the *fuzzy substructure* related to the structural complexity and including for example many equipment units attached to the master structure. In the present paper, we will not consider fuzzy substructures and concerning the fuzzy structure theory, we refer the reader to Soize (1986, 1993), to Chapter 15 of Ohayon and Soize (1998) for a synthesis, and to Fernandez et al. (2009) for extension of the theory to uncertain complex vibroacoustic system with fuzzy interface modeling. Consequently, the so-called ”master structure” will be simply called here ”structure”.

The structure at equilibrium occupies the three-dimensional bounded domain Ω_S with a boundary $\partial\Omega_S$ which is made up of a part Γ_E which is the coupling interface between the structure and the external acoustic fluid, a part Γ which is a coupling interface between the structure and the internal acoustic fluid and finally, the part Γ_Z which is another part of the coupling interface between the structure and the internal acoustic fluid with acoustical properties. The structure is assumed to be free (free-free structure), *i.e.* not fixed on any part of boundary $\partial\Omega_S$. The outward unit normal to $\partial\Omega_S$ is denoted as $\mathbf{n}^S = (n_1^S, n_2^S, n_3^S)$ (see Fig. 2). The displacement field in Ω_S is denoted by $\mathbf{u}(\mathbf{x},\omega) = (u_1(\mathbf{x},\omega), u_2(\mathbf{x},\omega), u_3(\mathbf{x},\omega))$. A surface force field $\mathbf{G}(\mathbf{x},\omega) = (G_1(\mathbf{x},\omega), G_2(\mathbf{x},\omega), G_3(\mathbf{x},\omega))$ is given on $\partial\Omega_S$ and a body force field $\mathbf{g}(\mathbf{x},\omega) = (g_1(\mathbf{x},\omega), g_2(\mathbf{x},\omega), g_3(\mathbf{x},\omega))$ is given in Ω_S. The structure is a dissipative medium whose viscoelastic constitutive equation is defined in Section 5.2.

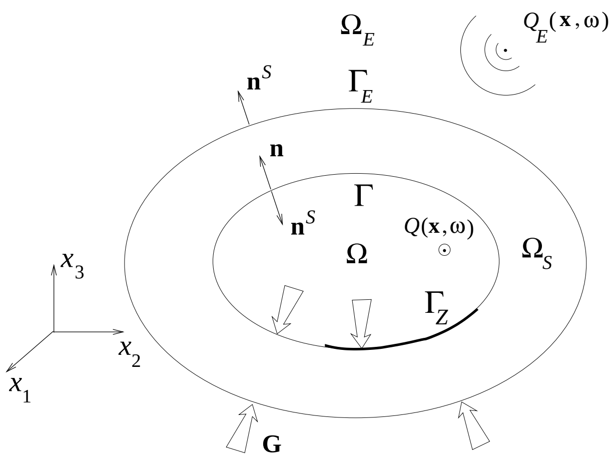

Figure 1: Configuration of the system.

Internal dissipative acoustic fluid Ω. Let Ω be the internal bounded domain filled with a dissipative acoustic fluid (gas or liquid) as described in Section 4. The boundary $\partial\Omega$ of Ω is $\Gamma \cup \Gamma_Z$. The outward unit normal to $\partial\Omega$ is denoted as $\mathbf{n} = (n_1, n_2, n_3)$ and we have $\mathbf{n} = -\mathbf{n}^S$ on $\partial\Omega$ (see Fig. 2). Part Γ_Z of the boundary has acoustical properties modeled by a wall acoustic impedance $Z(\mathbf{x},\omega)$ satisfying the hypotheses defined in Section 4.2. We denote the pressure field in Ω as $p(\mathbf{x},\omega)$ and the velocity field as $\mathbf{v}(\mathbf{x},\omega)$. We assume that there is no Dirichlet boundary condition on any part of $\partial\Omega$. An acoustic source density $Q(\mathbf{x},\omega)$ is given inside Ω.

External inviscid acoustic fluid Ω_E. The structure is surrounded by an external inviscid acoustic fluid (gas or liquid) as described in Section 10. The fluid occupies the infinite three-dimensional domain Ω_E whose boundary $\partial\Omega_E$ is Γ_E. We introduce the bounded open domain Ω_i defined by $\Omega_i = \mathbb{R}^3\backslash(\Omega_E \cup \Gamma_E)$. Note that, in general, Ω_i does not coincide with the internal acoustic cavity Ω. The boundary $\partial\Omega_i$ of Ω_i is then Γ_E. The outward unit normal to $\partial\Omega_i$ is $\mathbf{n}^S$ defined above (see Fig. 2). We denote the pressure field in Ω_E as $p_E(\mathbf{x},\omega)$. We assume that there is no Dirichlet boundary condition on any part of Γ_E. An acoustic source density $Q_E(\mathbf{x},\omega)$ is given in Ω_E. This acoustic source density induces a pressure field $p_{\text{given}}(\omega)$ on Γ_E defined in Section 10. For the sake of brevity, we do not consider here the case of an incident plane wave and we refer the reader to Ohayon and Soize (1998) for this case.

3. External Inviscid Acoustic Fluid Equations

An inviscid acoustic fluid occupies the infinite domain Ω_E and is described by the acoustic pressure field $p_E(\mathbf{x},\omega)$ at point $\mathbf{x}$ of Ω_E and at circular frequency ω. Let ρ_{E} be the constant mass density of the external acoustic fluid at equilibrium. Let c_{E} be the constant speed of sound in the external acoustic fluid at equilibrium and let $k = \omega/c_{\text{E}}$ be the wave number at frequency ω. The pressure is then solution of the classical exterior Neumann problem related to the Helmholtz equation with a source term,

$$\nabla^2 p_E + k^2\, p_E = -i\omega\, Q_E \quad \text{in} \quad \Omega_E\,, \tag{5}$$

$$\frac{\partial p_E}{\partial \mathbf{n}^S} = \omega^2\, \rho_E\, \mathbf{u}\cdot\mathbf{n}^S \quad \text{on} \quad \Gamma_E\,, \tag{6}$$

$$|\, p_E \,| = O(\frac{1}{R}) \quad , \quad \left| \frac{\partial p_E}{\partial R} + i\, k\, p_E \right| = O(\frac{1}{R^2})\,, \tag{7}$$

with $R = \|\mathbf{x}\| \rightarrow +\infty$, where $\partial/\partial R$ is the derivative in the radial direction and where $\mathbf{u}\cdot\mathbf{n}^S$ is the normal displacement field on Γ_E induced by the deformation of the structure. Equation (7) corresponds to the outward Sommerfeld radiation condition at infinity. In Section 10, it is proven that the value $p_E|_{\Gamma_E}$ of the pressure field p_E on the external fluid-structure interface Γ_E is related to $p_{\text{given}}|_{\Gamma_E}$ and to $\mathbf{u}$ by Eq. (141),

$$p_E|_{\Gamma_E}(\omega) = p_{\text{given}}|_{\Gamma_E}(\omega) + i\omega\, \mathbf{Z}_{\Gamma_E}(\omega)\{\mathbf{u}(\omega)\cdot\mathbf{n}^S\}\,, \tag{8}$$

in which the different quantities are defined in Section 10 which is a self-contained section describing the computational modeling of the external inviscid acoustic fluid by an appropriate boundary element method. It should be noted that, in Eq. (8), the pressure field $p_E|_{\Gamma_E}(\omega)$ is related to the value of the normal displacement field $\mathbf{u}(\omega)\cdot\mathbf{n}^S$ on the external fluid-structure interface Γ_E through the operator $\mathbf{Z}_{\Gamma_E}(\omega)$.

4. Internal Dissipative Acoustic Fluid Equations

4.1. *Internal dissipative acoustic fluid equations in the frequency domain*

The fluid is assumed to be homogeneous, compressible and dissipative. In the reference configuration, the fluid is at rest. The fluid is either a gas or a liquid and gravity effects are neglected (see Andrianarison and Ohayon (2006) to take into account both gravity and compressibility effects for an inviscid internal fluid). Such a fluid is called a *dissipative acoustic fluid*. Generally, there are two main physical dissipations. The first one is an internal acoustic dissipation inside the cavity due to the viscosity and the thermal conduction of the fluid. These dissipation mechanisms are assumed to be small. In the model proposed, we consider only the dissipation due to the viscosity. This correction introduces an additional dissipative term in the Helmholtz equation without modifying the conservative part. The second one is the dissipation generated inside the "wall viscothermal boundary layer" of the cavity and is neglected here. We then consider only the acoustic mode (irrotational motion) predominant in the volume. The vorticity and entropy modes which mainly play a role in the "wall viscothermal boundary layer" are not modeled. For additional details concerning dissipation in acoustic fluids, we refer the reader to Lighthill (1978); Pierce (1989); Landau and Lifchitz (1992); Bruneau (2006).

The dissipation due to thermal conduction is neglected and the motions are assumed to be irrotational. Let ρ_0 be the mass density and c_0 be the constant speed of sound in the fluid at equilibrium in the reference configuration Ω. We have (see the details in Ohayon and Soize (1998)),

$$i\omega\, p = -\rho_0\, c_0^2\; \nabla\cdot\mathbf{v} + c_0^2\, Q\,, \tag{9}$$

$$i\omega\, \rho_0\, \mathbf{v} + \nabla p = \tau c_0^2 \nabla Q - i\omega\, \tau \nabla p\,, \tag{10}$$

in which τ is given by

$$\tau = \frac{1}{\rho_0 c_0^2}\Big(\frac{4}{3}\eta + \zeta\Big) > 0\,. \tag{11}$$

The constant η is the dynamic viscosity, $\nu = \eta/\rho_0$ is the kinematic viscosity and ζ is the second viscosity which can depend on ω. Therefore, τ can depend on frequency ω. To simplify the notation, we write τ instead of $\tau(\omega)$. Eliminating $\mathbf{v}$ between Eqs. (9) and (10), then dividing by ρ_0, yield the Helmholtz equation with a dissipative term and a source term,

$$\begin{aligned} -\frac{\omega^2}{\rho_0 c_0^2}\, p - i\omega \frac{\tau}{\rho_0} \nabla^2 p - \frac{1}{\rho_0}\, \nabla^2 p \\ = \frac{1}{\rho_0}(i\omega Q - \tau c_0^2 \nabla^2 Q) \quad \text{in} \quad \Omega\,. \end{aligned} \tag{12}$$

Taking $\tau = 0$ and $Q = 0$ in Eq. (12) yields the usual Helmholtz equation for wave propagation in inviscid acoustic fluid.

4.2. *Boundary conditions in the frequency domain*

(i) *Neumann boundary condition on* Γ. Using Eq. (10) and $\mathbf{v}\cdot\mathbf{n} = i\omega\,\mathbf{u}\cdot\mathbf{n}$ on Γ, yields the following Neumann boundary condition ,

$$(1 + i\omega\,\tau)\, \frac{\partial p}{\partial \mathbf{n}} = \omega^2\, \rho_0\, \mathbf{u}\cdot\mathbf{n} + \tau\, c_0^2\, \frac{\partial Q}{\partial \mathbf{n}} \quad \text{on} \quad \Gamma\,. \tag{13}$$

(ii) *Neumann boundary condition on* Γ_Z *with wall acoustic impedance*. The part Γ_Z of the boundary $\partial\Omega$ has acoustical properties modeled by a *wall acoustic impedance* $Z(\mathbf{x},\omega)$ defined for $\mathbf{x} \in \Gamma_Z$, with complex values. The wall impedance boundary condition on Γ_Z is written as

$$p(\mathbf{x},\omega) = Z(\mathbf{x},\omega)\,\{\mathbf{v}(\mathbf{x},\omega)\cdot\mathbf{n} - i\omega\,\mathbf{u}(\mathbf{x},\omega)\cdot\mathbf{n}\}\,. \quad (14)$$

Wall acoustic impedance $Z(\mathbf{x},\omega)$ must satisfy appropriate conditions in order to ensure that the problem is correctly stated (see Ohayon and Soize (1998) for a general formulation and see Deü et al. (2008) for a simplified model of the Voigt type with an internal inviscid fluid). Using Eq. (10), $\mathbf{v}\cdot\mathbf{n} = i\omega\,\mathbf{u}\cdot\mathbf{n}$ and Eq. (14) on Γ, yields the following Neumann boundary condition with wall acoustic impedance,

$$\begin{aligned}(1+i\omega\,\tau)\,\frac{\partial p}{\partial \mathbf{n}} &= \omega^2\,\rho_0\,\mathbf{u}\cdot\mathbf{n} \\ &-i\omega\rho_0\frac{p}{Z} + \tau\,c_0^2\,\frac{\partial Q}{\partial \mathbf{n}} \quad \text{on} \quad \Gamma_Z\,, \end{aligned} \quad (15)$$

4.3. *Case of a free surface for a liquid*

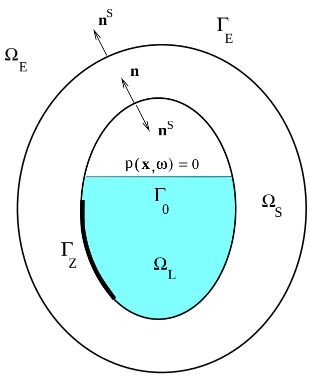

Figure 2: Configuration of the structural-acoustic system for a liquid with free surface.

Cavity Ω is partially filled with a liquid (dissipative acoustic fluid) occupying the domain Ω_L. It is assumed that the complementary part $\Omega\backslash\Omega_L$ is a vacuum domain. The boundary $\partial\Omega_L$ of Ω_L is constituted of three boundaries, Γ_Z, Γ_0 corresponding to the free surface of the liquid and a part Γ_L of Γ. The Neumann boundary condition on Γ_L is given by Eq. (13), on Γ_Z is given by Eq. (15) and, neglecting gravity effects, the following Dirichlet condition is written on the free surface,

$$p = 0 \quad \text{on} \quad \Gamma_0\,. \quad (16)$$

5. Structure Equations

5.1. *Structure equations in the frequency domain*

The equation of the structure occupying domain Ω_S is written as

$$-\omega^2\,\rho_{\mathrm{S}}\,u_i - \sigma_{ij,j}(\mathbf{u}) = g_i \quad \text{in} \quad \Omega_S\,, \quad (17)$$

in which $\rho_{\mathrm{S}}(\mathbf{x})$ is the mass density of the structure. The constitutive equation (linear viscoelastic model, see Section 5.2, Eq. (31)) is such that the symmetric stress tensor σ_{ij} is written as

$$\sigma_{ij}(\mathbf{u}) = (a_{ijkh}(\omega) + i\omega\,b_{ijkh}(\omega))\,\varepsilon_{kh}(\mathbf{u})\,, \quad (18)$$

in which the symmetric strain tensor $\varepsilon_{kh}(\mathbf{u})$ is such that

$$\varepsilon_{kh}(\mathbf{u}) = \frac{1}{2}(u_{k,h}(\mathbf{x},\omega) + u_{h,k}(\mathbf{x},\omega))\,, \quad (19)$$

and where the tensors $a_{ijkh}(\omega)$ and $b_{ijkh}(\omega)$ depend on ω (see Section 5.2). The boundary condition on the fluid-structure external interface Γ_E is such that

$$\sigma_{ij}(\mathbf{u})n_j^S = G_i - p_E|_{\Gamma_E}\,n_i^S \quad \text{on} \quad \Gamma_E\,, \quad (20)$$

in which $p_E|_{\Gamma_E}$ is given by Eq. (8) and yields

$$\begin{aligned}\sigma_{ij}(\mathbf{u})\,n_j^S &= G_i - p_{\text{given}}|_{\Gamma_E}\,n_i^S \\ &-i\omega\,\mathbf{Z}_{\Gamma_E}(\omega)\{\mathbf{u}\cdot\mathbf{n}^S\}\,n_i^S \quad \text{on} \quad \Gamma_E\,. \end{aligned} \quad (21)$$

Since $\mathbf{n}^S = -\mathbf{n}$, the boundary condition on $\Gamma\cup\Gamma_Z$ is written as

$$\sigma_{ij}(\mathbf{u})n_j^S = G_i + p\,n_i \quad \text{on} \quad \Gamma\cup\Gamma_Z\,. \quad (22)$$

in which p is the internal acoustic pressure field defined in Section 4.

5.2. Viscoelastic constitutive equation

In dynamics, the structure must always be modeled as a dissipative continuum. For the conservative part of the structure, we use the linear elasticity theory which allows the structural modes to be introduced. This was justified by the fact that, in the low-frequency range, the conservative part of the structure can be modeled as an elastic continuum. In this section, we introduce damping models for the structure based on the general linear theory of viscoelasticity presented in Truesdell (1973) (see also Bland (1960); Fung (1968)). Complementary developments are presented with respect to the viscoelastic constitutive equation detailed in Ohayon and Soize (1998).

In this section, $\mathbf{x}$ is fixed in Ω, and we rewrite the stress tensor $\sigma_{ij}(\mathbf{x},t)$ as $\sigma_{ij}(t)$, the strain tensor $\varepsilon_{ij}(\mathbf{x},t)$ as $\varepsilon_{ij}(t)$ and its time derivative $\dot\varepsilon_{ij}(\mathbf{x},t)$ as $\dot\varepsilon_{ij}(t)$.

Constitutive equation in the time domain. The stress tensor $\sigma_{ij}(t)$ is written as

$$\sigma_{ij}(t) = G_{ijkh}(0)\,\varepsilon_{kh}(t) + \int_0^{+\infty} \dot G_{ijkh}(\tau)\varepsilon_{kh}(t-\tau)d\tau\,, \tag{23}$$

where $\sigma_{ij}(t) = 0$ and $\varepsilon(t) = 0$ for $t \leq 0$. The real functions $G_{ijkh}(\mathbf{x},t)$, denoted as $G_{ijkh}(t)$, are called the *relaxation functions*. The tensor $G_{ijkh}(t)$ (and thus $\dot G_{ijkh}(t)$) has the usual property of symmetry and $G_{ijkh}(0)$, which is called the *initial elasticity tensor*, is positive definite. The relaxation functions are defined on $[0\,,+\infty[$, are differentiable with respect to t on $]0\,,+\infty[$, their derivatives are denoted as $\dot G_{ijkh}(t)$ and are assumed to be integrable on $[0\,,+\infty[$. Functions $G_{ijkh}(t)$ can be written as

$$G_{ijkh}(t) = G_{ijkh}(0) + \int_0^t \dot G_{ijkh}(\tau)\,d\tau\,. \tag{24}$$

Therefore, the limit of $G_{ijkh}(t)$, denoted as $G_{ijkh}(\infty)$, is finite as t tends to $+\infty$,

$$G_{ijkh}(\infty) = G_{ijkh}(0) + \int_0^{+\infty} \dot G_{ijkh}(\tau)\,d\tau\,. \tag{25}$$

The tensor $G_{ijkh}(\infty)$, called the *equilibrium modulus* at $\mathbf{x}$, is symmetric, positive definite and corresponds to the usual elasticity coefficients of the elastic material for a static deformation. In effect, the static equilibrium state is obtained for t tends to infinity.

For all $\mathbf{x}$ fixed in Ω, we introduce the real functions $t \mapsto g_{ijkh}(\mathbf{x},t)$, denoted as $g_{ijkh}(t)$, such that

$$g_{ijkh}(t) = 0 \;\text{ if}\, t < 0\,,$$
$$g_{ijkh}(t) = \dot G_{ijkh}(t) \;\text{ if}\, t \geq 0\,. \tag{26}$$

Since $g_{ijkh}(t) = 0$ for $t < 0$, we deduce that $g_{ijkh}(t)$ is a causal function.

Using Eq. (26), Eq. (23) can be rewritten as

$$\sigma_{ij}(t) = G_{ijkh}(0)\,\varepsilon_{kh}(t) + \int_{-\infty}^{+\infty} g_{ijkh}(\tau)\varepsilon_{kh}(t-\tau)d\tau\,, \tag{27}$$

It should be noted that Eq. (27) corresponds to the most general formulation in the time domain within the framework of the linear theory of viscoelasticity. The usual approach which consists in modeling the constitutive equation in time domain by a linear differential equation in $\sigma(t)$ and $\varepsilon(t)$ (see for instance Truesdell (1973); Dautray and Lions (1992)), corresponds to a particular case which is an approximation of the general Eq. (27). An alternative approximation of Eq. (27) consists in representing the integral operator by a differential operator acting on additional hidden variables. This type of approximation can efficiently be described using fractional derivative operators (see for instance Deü and Matignon (2010); Bagley and Torvik (1983)).

Constitutive equation in the frequency domain. The general constitutive equation in the frequency domain is written as

$$\sigma_{ij}(\omega) = \sigma_{ij}^{\text{elas}}(\omega) + i\omega\, s_{ij}^{\text{damp}}(\omega)\,, \tag{28}$$

in which

$$\sigma_{ij}^{\text{elas}}(\omega) = a_{ijkh}(\omega)\;\varepsilon_{kh}(\omega)\,, \tag{29}$$
$$s_{ij}^{\text{damp}}(\omega) = b_{ijkh}(\omega)\;\varepsilon_{kh}(\omega)\,. \tag{30}$$

Equation (28) can then be rewritten as

$$\sigma_{ij}(\omega) = (a_{ijkh}(\omega) + i\omega\, b_{ijkh}(\omega))\ \varepsilon_{kh}(\omega)\,. \tag{31}$$

Tensors $a_{ijkh}(\omega)$ and $b_{ijkh}(\omega)$ must satisfy the symmetry properties

$$a_{ijkh}(\omega) = a_{jikh}(\omega) = a_{ijhk}(\omega) = a_{khij}(\omega)\,, \tag{32}$$
$$b_{ijkh}(\omega) = b_{jikh}(\omega) = b_{ijhk}(\omega) = b_{khij}(\omega)\,, \tag{33}$$

and the positive-definiteness properties, *i.e.*, for all second-order real symmetric tensors X_{ij},

$$a_{ijkh}(\omega)\, X_{kh}\, X_{ij}\ \geq\ c_a(\omega)\, X_{ij}\, X_{ij}\,, \tag{34}$$
$$b_{ijkh}(\omega)\, X_{kh}\, X_{ij}\ \geq\ c_b(\omega)\, X_{ij}\, X_{ij}\,, \tag{35}$$

in which the positive constants $c_a(\omega)$ and $c_b(\omega)$ are such that $c_a(\omega) \geq c_0 > 0$ and $c_b(\omega) \geq c_0 > 0$ where c_0 is a positive real constant independent of ω.

Since $g_{ijkh}(t)$ is an integrable function on $]-\infty\,, +\infty[$, its Fourier transform $\widehat{g}_{ijkh}(\omega)$, defined by

$$\begin{aligned} \widehat{g}_{ijkh}(\omega) &= \int_{-\infty}^{+\infty} e^{-i\omega t}\, g_{ijkh}(t)\, dt \\ &= \int_{0}^{+\infty} e^{-i\omega t}\, \dot{G}_{ijkh}(t)\, dt\,, \end{aligned} \tag{36}$$

is a complex function which is continuous on $]-\infty\,, +\infty[$ and such that

$$\lim_{|\omega|\to+\infty} |\widehat{g}_{ijkh}(\omega)| = 0\,. \tag{37}$$

The real part $\widehat{g}^R_{ijkh}(\omega) = \Re e\{\widehat{g}_{ijkh}(\omega)\}$ and the imaginary part $\widehat{g}^I_{ijkh}(\omega) = \Im m\{\widehat{g}_{ijkh}(\omega)\}$ of $\widehat{g}_{ijkh}(\omega)$ are even and odd functions, that is to say $\widehat{g}^R_{ijkh}(-\omega) = \widehat{g}^R_{ijkh}(\omega)$ and $\widehat{g}^I_{ijkh}(-\omega) = -\widehat{g}^I_{ijkh}(\omega)$. We can then deduce that

$$\widehat{g}^I_{ijkh}(0) = 0\,. \tag{38}$$

We can now take the Fourier transform of Eq. (27) and using Eq. (31) yield the relations,

$$a_{ijkh}(\omega) = G_{ijkh}(0) + \widehat{g}^R_{ijkh}(\omega)\,, \tag{39}$$
$$\omega\, b_{ijkh}(\omega) = \widehat{g}^I_{ijkh}(\omega)\,. \tag{40}$$

From Eqs. (37), (39) and (40) yields

$$\lim_{|\omega|\to+\infty} a_{ijkh}(\omega) = G_{ijkh}(0)\,, \tag{41}$$

$$\lim_{|\omega|\to+\infty} \omega\ b_{ijkh}(\omega) = 0\,. \tag{42}$$

From Eqs. (31), (41) and (42), we deduce that

$$\sigma_{ij}(\infty) = G_{ijkh}(0)\, \varepsilon_{kh}(\infty)\,. \tag{43}$$

Eq. (43) shows that viscoelastic materials behave elastically at high frequencies with elasticity coefficients defined by the initial elasticity tensor $G_{ijkh}(0)$ which differs from the equilibrium modulus tensor $G_{ijkh}(\infty)$ written, using Eqs. (25) and (38), as

$$G_{ijkh}(\infty) = G_{ijkh}(0) + \widehat{g}^R_{ijkh}(0)\,. \tag{44}$$

As pointed out before, the positive-definite tensor $G_{ijkh}(\infty)$ corresponds to the usual elasticity coefficients of a linear elastic material for a static deformation process. More specifically, for $\omega = 0$, using Eqs. (38) to (40) and Eq. (31) yield

$$\sigma_{ijkh}(0) = a_{ijkh}(0)\, \varepsilon_{ijkh}(0)\,. \tag{45}$$

in which $\sigma_{ijkh}(0) = \{\sigma_{ijkh}(\omega)\}_{\omega=0}$ and $\varepsilon_{ijkh}(0) = \{\varepsilon_{ijkh}(\omega)\}_{\omega=0}$, and where

$$a_{ijkh}(0) = G_{ijkh}(0) + \widehat{g}^R_{ijkh}(0) = G_{ijkh}(\infty)\,. \tag{46}$$

The reader should be aware of the fact that the constitutive equation of an elastic material in a static deformation process is defined by $G_{ijkh}(\infty)$ and not by the initial elasticity tensor $G_{ijkh}(0)$. Referring to Coleman (1964); Truesdell (1973), it has been proven that $G_{ijkh}(0) - G_{ijkh}(\infty)$ is a positive-definite tensor and consequently, $\widehat{g}^R_{ijkh}(0) = G_{ijkh}(\infty) - G_{ijkh}(0)$ is a negative-definite tensor.
Since $g_{ijkh}(t)$ is a causal function, the real part $\widehat{g}^R_{ijkh}(\omega)$ and the imaginary part $\widehat{g}^I_{ijkh}(\omega)$ of its Fourier transform $\widehat{g}_{ijkh}(\omega)$ are related by the following relations involving the Hilbert transform (see Papoulis (1977); Hahn (1996)),

$$\widehat{g}^R_{ijkh}(\omega) = \frac{1}{\pi}\text{p.v}\int_{-\infty}^{+\infty} \frac{\widehat{g}^I_{ijkh}(\omega')}{\omega - \omega'}\, d\omega'\,, \tag{47}$$

$$\widehat{g}^I_{ijkh}(\omega) = -\frac{1}{\pi}\text{p.v}\int_{-\infty}^{+\infty} \frac{\widehat{g}^R_{ijkh}(\omega')}{\omega - \omega'}\, d\omega'\,, \tag{48}$$

in which p.v denotes the Cauchy principal value defined as

$$\text{p.v}\int_{-\infty}^{+\infty} = \lim_{\ell\to+\infty,\eta\to 0^+} \{\int_{-\ell}^{-\eta} + \int_{\eta}^{\ell}\}\,. \tag{49}$$

The relations defined by Eqs. (47) and (48) are also called the Kramers and Kronig relations for function $g_{ijkh}(t)$ (see Kronig (1926); Kramers (1927)).

LF-range constitutive equation approximation. In the low-frequency range and in most cases, the coefficients $a_{ijkh}(\omega)$ given by the linear viscoelastic model defined by Eq. (39) are almost frequency independent. In such a case, they can be approximated by $a_{ijkh}(\omega) \simeq a_{ijkh}(0)$ which is independent of ω (but which depends on **x**). It should be noted that this approximation can only be made on a finite interval corresponding to the low-frequency range and cannot be used in all the frequency domain because Eqs. (47) and (48) are not satisfied and integrability property is lost.

MF range constitutive equation. In the medium-frequency range, the previous LF-range constitutive equation approximation is generally not valid and the full linear viscoelastic theory defined by Eq. (31) must be used.

Bibliographical comments concerning expressions of frequency-dependent coefficients. Some algebraic representations of functions $a_{ijkh}(\omega)$ and $b_{ijkh}(\omega)$ have been proposed in the literature (see for instance Bland (1960); Truesdell (1973); Bagley and Torvik (1983); Golla and Hughes (1985); Lesieutre and Mingori (1990); Dautray and Lions (1992); Mc Tavish and Hughes (1993); Dovstam (1995); Ohayon and Soize (1998); Lesieutre (2010)). Concerning linear hysteretic damping correctly written in the present context, we refer the reader to Inaudi and Kelly (1995); Makris (1997).

6. Boundary Value Problem in Terms of $\{\mathbf{u}, p\}$

The boundary value problem in terms of $\{\mathbf{u}, p\}$ is written as follows. For all real ω and for given $\mathbf{G}(\omega)$, $\mathbf{g}(\omega)$, $p_{\text{given}}|_{\Gamma_E}(\omega)$ and $Q(\omega)$, find $\mathbf{u}(\omega)$ and $p(\omega)$, such that

$$-\omega^2\,\rho_{\text{S}}\,\mathbf{u} - \text{div}\,\sigma(\mathbf{u}) = \mathbf{g} \quad \text{in} \quad \Omega_S\,, \tag{50}$$

$$\begin{aligned}\sigma(\mathbf{u})\,\mathbf{n}^S &= \mathbf{G} - p_{\text{given}}|_{\Gamma_E}\,\mathbf{n}^S \\ &\quad -i\omega\,\mathbf{Z}_{\Gamma_E}(\omega)\{\mathbf{u}\cdot\mathbf{n}^S\}\,\mathbf{n}^S \quad \text{on} \quad \Gamma_E\,,\end{aligned} \tag{51}$$

$$\sigma(\mathbf{u})\,\mathbf{n}^S = \mathbf{G} + p\,\mathbf{n} \quad \text{on} \quad \Gamma\cup\Gamma_Z\,. \tag{52}$$

$$\begin{aligned}&-\frac{\omega^2}{\rho_0 c_0^2}\,p - i\omega\frac{\tau}{\rho_0}\boldsymbol{\nabla}^2 p - \frac{1}{\rho_0}\,\boldsymbol{\nabla}^2 p \\ &\quad = \frac{1}{\rho_0}(i\omega Q - \tau c_0^2\boldsymbol{\nabla}^2 Q) \quad \text{in} \quad \Omega\,.\end{aligned} \tag{53}$$

$$(1+i\omega\,\tau)\,\frac{\partial p}{\partial\mathbf{n}} = \omega^2\,\rho_0\,\mathbf{u}\cdot\mathbf{n} + \tau\,c_0^2\,\frac{\partial Q}{\partial\mathbf{n}} \quad \text{on} \quad \Gamma\,. \tag{54}$$

$$\begin{aligned}(1+i\omega\,\tau)\,\frac{\partial p}{\partial\mathbf{n}} &= \omega^2\,\rho_0\,\mathbf{u}\cdot\mathbf{n} \\ &\quad -i\omega\rho_0\frac{p}{Z} + \tau\,c_0^2\,\frac{\partial Q}{\partial\mathbf{n}} \quad \text{on} \quad \Gamma_Z\,.\end{aligned} \tag{55}$$

In case of a free surface in the internal acoustic cavity (see Section 4.3, we must add the following boundary condition

$$p = 0 \quad \text{on} \quad \Gamma_0\,. \tag{56}$$

Comments.

- We are interested in studying the linear vibrations of the coupled system around a static equilibrium which is consider as a natural state at rest (then the external solid and acoustic forces are assumed to be in equilibrium).
- Eq. (50) corresponds to the structure equation (see Eqs. (17) and (28)), in which $\{\text{div}\,\sigma(\mathbf{u})\}_i = \sigma_{ij,j}(\mathbf{u})$.
- Eqs. (51) and (52) are the boundary conditions for the structure (see Eqs. (21) and (22)).
- Eq. (53) corresponds to the internal dissipative acoustic fluid equation (see Eq. (12)).
- Finally, Eqs. (54) and (55) are the boundary conditions for the acoustic cavity (see Eqs. (13) and (15)).

- It is important to note that the external acoustic pressure field p_E has been eliminated as a function of **u** using the acoustic impedance boundary operator $\mathbf{Z}_{\Gamma_E}(\omega)$ while the internal acoustic pressure field p is kept.

7. Computational Model

The computational model is constructed using the finite element discretization of the boundary value problem. We consider a finite element mesh of structure Ω_S and a finite element mesh of internal acoustic fluid Ω. We assume that the two finite element meshes are compatible on interface $\Gamma \cup \Gamma_Z$. The finite element mesh of surface Γ_E is the trace of the mesh of Ω_S (see Fig. 3). We classically use

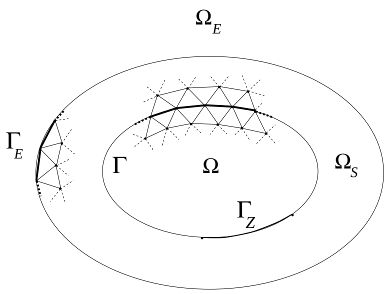

Figure 3: Example of structure and internal fluid finite element meshes.

the finite element method to construct the discretization of the variational formulation of the boundary value problem defined by Eqs. (50) to (55), with additional boundary condition defined by Eq. (56) in case of a free surface for an internal liquid. For the details concerning the practical construction of the finite element matrices, we refer the reader to Ohayon and Soize (1998). Let $\mathbb{U}(\omega)$ be the complex vector of the n_S degrees-of-freedom (DOFs) which are the values of $\mathbf{u}(\omega)$ at the nodes of the finite element mesh of domain Ω_S. For the internal acoustic fluid, let $\mathbb{P}(\omega)$ be the complex vectors of the n DOFs which are the values of $p(\omega)$ at the nodes of the finite element mesh of domain Ω. The finite element method yields the following complex matrix equation,

$$[\mathbb{A}_{\text{FSI}}(\omega)] \begin{bmatrix} \mathbb{U}(\omega) \\ \mathbb{P}(\omega) \end{bmatrix} = \begin{bmatrix} \mathbb{F}^S(\omega) \\ \mathbb{F}(\omega) \end{bmatrix} \quad , \tag{57}$$

in which the complex matrix $[\mathbb{A}_{\text{FSI}}(\omega)]$ is defined by

$$\begin{bmatrix} [\mathbb{A}^S(\omega)] - \omega^2 [\mathbb{A}_{\text{BEM}}(\omega/c_{\text{E}})] & [\mathbb{C}] \\ \omega^2\, [\mathbb{C}]^T & [\mathbb{A}(\omega)] + [\mathbb{A}^Z(\omega)] \end{bmatrix} . \tag{58}$$

In Eq. (58), the symmetric $(n_S \times n_S)$ complex matrix $[\mathbb{A}^S(\omega)]$ is defined by

$$[\mathbb{A}^S(\omega)] = -\omega^2 [\mathbb{M}^S] + i\omega\, [\mathbb{D}^S(\omega)] + [\mathbb{K}^S(\omega)] \, , \tag{59}$$

in which $[\mathbb{M}^S]$, $[\mathbb{D}^S(\omega)]$ and $[\mathbb{K}^S(\omega)]$ are symmetric $(n_S \times n_S)$ real matrices which represent the mass matrix, the damping matrix and the stiffness matrix of the structure. Matrix $[\mathbb{M}^S]$ is positive and invertible (positive definite) and matrices $[\mathbb{D}^S(\omega)]$ and $[\mathbb{K}^S(\omega)]$ are positive and not invertible (positive semidefinite) due to the presence of six rigid body motions since the structure has been considered as a free-free structure. The symmetric $(n \times n)$ complex matrix $[\mathbb{A}(\omega)]$ is defined by

$$[\mathbb{A}(\omega)] = -\omega^2 [\mathbb{M}] + i\omega\, [\mathbb{D}(\omega)] + [\mathbb{K}] \, , \tag{60}$$

in which $[\mathbb{M}]$, $[\mathbb{D}(\omega)]$ and $[\mathbb{K}]$ are symmetric $(n \times n)$ real matrices. Matrix $[\mathbb{M}]$ is positive and invertible and matrices $[\mathbb{D}(\omega)]$ and $[\mathbb{K}]$ are positive and not invertible with rank $n-1$. From Eq. (53), it can easily be deduced that $[\mathbb{D}(\omega)] = \tau(\omega)\, [\mathbb{K}]$ in which $\tau(\omega)$ is defined by Eq. (11). The internal fluid-structure coupling matrix $[\mathbb{C}]$, related to the coupling between the structure and the internal fluid on the internal fluid-structure interface, is a $(n_S \times n)$ real matrix which is only related to the values of $\mathbb{U}$ and $\mathbb{P}$ on the internal fluid-structure interface. The wall acoustic impedance matrix $[\mathbb{A}^Z(\omega)]$ is a symmetric $(n \times n)$ complex matrix depending on the wall acoustic impedance $Z(\mathbf{x}, \omega)$ on Γ_Z and which is only related to the values of $\mathbb{P}$ on boundary Γ_Z. The boundary element matrix $[\mathbb{A}_{\text{BEM}}(\omega/c_{\text{E}})]$, which depends on ω/c_{E}, is a symmetric $(n_S \times n_S)$ complex matrix which is only related to the values of $\mathbb{U}$ on the external fluid-structure interface Γ_E. This matrix is written as

$$[\mathbb{A}_{\text{BEM}}(\omega/c_{\text{E}})] = -\rho_{\text{E}}\, [\mathbb{N}]^T\, [B_{\Gamma_E}(\omega/c_{\text{E}})]\, [\mathbb{N}] \, , \tag{61}$$

in which $[B_{\Gamma_E}(\omega/c_{\text{E}})]$ is the full symmetric $(n_E \times n_E)$ complex matrix defined in Section 10.7 and where $[\mathbb{N}]$ is a sparse $(n_E \times n_S)$ real matrix related to the finite element discretization.

8. Reduced-Order Computational Model

The strategy used for constructing the reduced-order computational model consists in using the projection basis constituted of:

- the undamped elastic structural modes of the structure in vacuo for which the constitutive equation corresponds to elastic materials (see Eq. (45)), and consequently, the stiffness matrix has to be taken for $\omega = 0$.

- the undamped acoustic modes of the acoustic cavity with fixed boundary and without wall acoustic impedance. Two cases must be considered: one for which the internal pressure varies with a variation of the volume of the cavity (a cavity with a sealed wall called a closed cavity) and the other one for which the internal pressure does not vary with the variation of the volume of the cavity (a cavity with a non sealed wall called an almost closed cavity).

8.1. Computation of the elastic structural modes

This step concerns the finite element calculation of the undamped elastic structural modes of structure Ω_S in vacuo for which the constitutive equation corresponds to elastic materials. Setting $\lambda^S = \omega^2$, we then have the following classical $(n_S \times n_S)$ generalized symmetric real eigenvalue problem

$$[\mathbb{K}^S(0)]\,\mathbb{U} = \lambda^S\,[\mathbb{M}^S]\,\mathbb{U}\,. \tag{62}$$

It can be shown that there is a zero eigenvalue with multiplicity 6 (corresponding to the six rigid body motions) and that there is an increasing sequence of $n_S - 6$ strictly positive eigenvalues (corresponding to the elastic structural modes), each positive eigenvalue can be multiple (case of a structure with symmetries),

$$0 < \lambda_1^S \leq \ldots \leq \lambda_\alpha^S \leq \ldots\,. \tag{63}$$

Let $\mathbb{U}_1, \ldots, \mathbb{U}_\alpha, \ldots$ be the eigenvectors (the elastic structural modes) associated with $\lambda_1^S, \ldots, \lambda_\alpha^S, \ldots$. Let $0 < N_S \leq n_S - 6$. We introduce the $(n_S \times N_S)$ real matrix of the N_S elastic structural modes $\mathbb{U}_\alpha$ associated with the first N_S strictly positive eigenvalues,

$$[\,\mathcal{U}\,] = [\,\mathbb{U}_1 \ldots \mathbb{U}_\alpha \ldots \mathbb{U}_{N_S}\,]\,. \tag{64}$$

One has the classical orthogonality properties,

$$[\,\mathcal{U}\,]^T\,[\mathbb{M}^S]\,[\,\mathcal{U}\,] = [M^S]\,, \tag{65}$$

$$[\,\mathcal{U}\,]^T\,[\mathbb{K}^S(0)]\,[\,\mathcal{U}\,] = [K^S(0)]\,, \tag{66}$$

in which $[M^S]$ is a diagonal matrix of positive real numbers and where $[K^S(0)]$ is the diagonal matrix of the eigenvalues such that $[K^S(0)]_{\alpha\beta} = \lambda_\alpha^S\,\delta_{\alpha\beta}$ (the eigenfrequencies are $\omega_\alpha^S = \sqrt{\lambda_\alpha^S}$).

8.2. Computation of the acoustic modes

This step concerns the finite element calculation of the undamped acoustic modes of a closed (sealed wall) or an almost closed (non sealed wall) acoustic cavity Ω. Setting $\lambda = \omega^2$, we then have the following classical $(n \times n)$ generalized symmetric real eigenvalue problem

$$[\mathbb{K}]\,\mathbb{P} = \lambda\,[\mathbb{M}]\,\mathbb{P}\,. \tag{67}$$

It can be shown that there is a zero eigenvalue with multiplicity 1, denoted as λ_0 (corresponding to constant eigenvector denoted as $\mathbb{P}_0$) and that there is an increasing sequence of $n-1$ strictly positive eigenvalues (corresponding to the acoustic modes), each positive eigenvalue can be multiple (case of an acoustic cavity with symmetries),

$$0 < \lambda_1 \leq \ldots \leq \lambda_\alpha \leq \ldots\,. \tag{68}$$

Let $\mathbb{P}_1, \ldots, \mathbb{P}_\alpha, \ldots$ be the eigenvectors (the acoustic modes) associated with $\lambda_1, \ldots, \lambda_\alpha, \ldots$.

- *Closed (sealed wall) acoustic cavity*. Let be $0 < N \leq n$. We introduce the $(n \times N)$ real matrix of the constant eigenvector $\mathbb{P}_0$ and of the $N-1$ acoustic modes $\mathbb{P}_\alpha$ associated with the first $N-1$ strictly positive eigenvalues,

$$[\,\mathcal{P}\,] = [\,\mathbb{P}_0, \mathbb{P}_1 \ldots \mathbb{P}_\alpha \ldots \mathbb{P}_{N-1}\,]\,. \tag{69}$$

- *Almost closed (non sealed wall) acoustic cavity*. Let be $0 < N \leq n-1$. We introduce the $(n \times N)$ real matrix of the N acoustic modes $\mathbb{P}_\alpha$ associated with the first N strictly positive eigenvalues,

$$[\,\mathcal{P}\,] = [\,\mathbb{P}_1 \ldots \mathbb{P}_\alpha \ldots \mathbb{P}_N\,]\,. \tag{70}$$

One has the classical orthogonality properties,

$$[\mathcal{P}]^T\,[\mathbb{M}]\,[\mathcal{P}] = [M]\,, \tag{71}$$

$$[\mathcal{P}]^T\,[\mathbb{K}]\,[\mathcal{P}] = [K]\,, \tag{72}$$

in which $[M]$ is a diagonal matrix of positive real numbers and where $[K]$ is the diagonal matrix of the eigenvalues such that $[K]_{\alpha\beta} = \lambda_\alpha\,\delta_{\alpha\beta}$ (for non zero eigenvalue, the eigenfrequencies are $\omega_\alpha = \sqrt{\lambda_\alpha}$).

8.3. Construction of the reduced-order computational model

The reduced-order computational model, of order $N_S \ll n_S$ and $N \ll n$, is obtained by projecting Eq. (57) as follows,

$$\mathbb{U}(\omega) = [\mathcal{U}]\,\mathbf{q}^S(\omega)\,, \tag{73}$$

$$\mathbb{P}(\omega) = [\mathcal{P}]\,\mathbf{q}(\omega)\,. \tag{74}$$

The complex vectors $\mathbf{q}^S(\omega)$ and $\mathbf{q}(\omega)$ of dimension N_S and N are the solution of the following equation

$$[A_{\mathrm{FSI}}(\omega)]\,\begin{bmatrix}\mathbf{q}^S(\omega)\\ \mathbf{q}(\omega)\end{bmatrix} = \begin{bmatrix}\mathbf{f}^S(\omega)\\ \mathbf{f}(\omega)\end{bmatrix}\quad , \tag{75}$$

in which the complex matrix $[A_{\mathrm{FSI}}(\omega)]$ is defined by

$$\begin{bmatrix}[A^S(\omega)] - \omega^2[A_{\mathrm{BEM}}(\omega/c_{\mathrm{E}})] & [C]\\ \omega^2\,[C]^T & [A(\omega)] + [A^Z(\omega)]\end{bmatrix}. \tag{76}$$

In Eq. (76), the symmetric $(N_S \times N_S)$ complex matrix $[A^S(\omega)]$ is defined by

$$[A^S(\omega)] = -\omega^2[M^S] + i\omega\,[D^S(\omega)] + [K^S(\omega)]\,, \tag{77}$$

in which $[M^S]$, $[D^S(\omega)]$ and $[K^S(\omega)]$ are positive-definite symmetric $(N_S \times N_S)$ real matrices such that $[D^S(\omega)] = [\mathcal{U}]^T\,[\mathbb{D}^S(\omega)]\,[\mathcal{U}]$ and $[K^S(\omega)] = [\mathcal{U}]^T\,[\mathbb{K}^S(\omega)]\,[\mathcal{U}]$. The symmetric $(N \times N)$ complex matrix $[A(\omega)]$ is defined by

$$[A(\omega)] = -\omega^2[M] + i\omega\,[D(\omega)] + [K]\,, \tag{78}$$

in which $[M]$, $[D(\omega)]$ and $[K]$ are symmetric $(N \times N)$ real matrices. Matrix $[M]$ is positive and invertible. The diagonal $(N \times N)$ real matrix $[D(\omega)]$ is written as $[D(\omega)] = \tau(\omega)\,[K]$ in which $\tau(\omega)$ is defined by Eq. (11). For a closed (sealed wall) acoustic cavity, matrix $[K]$ is positive and not invertible with rank $N - 1$, while for an almost closed (non sealed wall) acoustic cavity, matrix $[K]$ is positive and invertible. The $(N_S \times N)$ real matrix $[C]$ is written as $[C] = [\mathcal{U}]^T\,[\mathbb{C}]\,[\mathcal{P}]$. The symmetric $(N \times N)$ complex matrix $[A^Z(\omega)]$ is such that $[A^Z(\omega)] = [\mathcal{P}]^T\,[\mathbb{A}^Z(\omega)]\,[\mathcal{P}]$ and finally, the symmetric $(N_S \times N_S)$ complex matrix $[A_{\mathrm{BEM}}(\omega/c_{\mathrm{E}})]$ is given by $[A_{\mathrm{BEM}}(\omega/c_{\mathrm{E}})] = [\mathcal{U}]^T\,[\mathbb{A}_{\mathrm{BEM}}(\omega/c_{\mathrm{E}})]\,[\mathcal{U}]$. The given forces are written as $\mathbf{f}^S(\omega) = [\mathcal{U}]^T\,\mathbb{F}^S(\omega)$ and $\mathbf{f}(\omega) = [\mathcal{P}]^T\,\mathbb{F}(\omega)$.

9. Uncertainty Quantification

9.1. Short overview on uncertainty quantification

In this section, we summarize the fundamental concepts related to uncertainties and their stochastic modeling in computational structural-acoustic models (extracted from Soize (2012a,b)).

9.1.1. Uncertainty and variability

The *designed structural-acoustic system* is used to manufacture the *real system* and to construct the nominal computational model (also called the *mean computational model* or sometimes, the mean model) using a mathematical-mechanical modeling process for which the main objective is the prediction of the responses of the real system. The real system can exhibit a variability in its responses due to fluctuations in the manufacturing process and due to small variations of the configuration around a nominal configuration associated with the designed structural-acoustic system. The mean computational model which results from a mathematical-mechanical modeling process of the designed structural-acoustic system, has parameters (such as geometry, mechanical properties, boundary conditions) which can be uncertain (for example, parameters related to the structure, the internal acoustic fluid, the wall acoustic impedance). In this case, there are *uncertainties on the computational model parameters*. In the other hand, the modeling process induces some modeling errors defined as the *model uncertainties*. Fig 4 summarizes the two types of uncertainties in a computational model and the variabilities of a real system. It is important to take into account both the uncertainties on the computational model parameters and the model uncertainties to improve the predictions in order to use such a

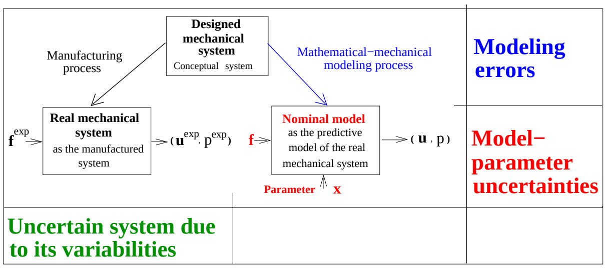

Figure 4: Variabilities and types of uncertainties in computational structural acoustics and fluid-structure interaction

computational model to carry out robust optimization, robust design and robust updating with respect to uncertainties. Today, it is well understood that, as soon as the probability theory can be used, then the stochastic approach of uncertainties is the most powerful, efficient and effective tool for modeling and for solving direct problem and inverse problem related to the identification. The developments presented below are carried out within the framework of the probability theory.

9.1.2. *Types of approach for stochastic modeling of uncertainties*

The *parametric probabilistic approach* consists in modeling the *uncertain parameters of the computational model* by random variables and then, in constructing the stochastic model of these random variables using the available information. Such an approach is very well adapted and very efficient to take into account the uncertainties in the computational model parameters. Many works have been published and a state-of-the-art can be found, for instance, in Ghanem and Spanos (1991, 2003); Mace et al. (2005); Schueller (2005, 2007); Deodatis and Spanos (2008).

Concerning *model uncertainties* induced by *modeling errors*, it is well understood that the prior and posterior probability models of the uncertain parameters of the computational model are not sufficient and do not have the capability to take into account model uncertainties in the context of computational mechanics as explained, for instance, in Beck and Katafygiotis (1998) and in Soize (2000, 2001, 2005b). Two main methods can be used to take into account model uncertainties (modeling errors).

(i) *Output-prediction-error method*. It consists in introducing a stochastic model of the system output which is the difference between the real system output and the computational model output. If there are no experimental data, then this method cannot really be used because there is generally no information concerning the probability model of the noise which is added to the computational model output. If experiments are available, the observed prediction error is then the difference between the measured real system output and the computational model output. A posterior probability model can then be constructed (Beck and Katafygiotis, 1998; Beck and Au, 2002) using the Bayesian method (Spall, 2003; Kaipio and Somersalo, 2005). Such an approach is efficient but requires experimental data. In this case, the posterior probability model of the uncertain parameters of the computational model strongly depends on the probability model of the noise which is added to the model output and which is often unknown. In addition, for many problems, it can be necessary to take into account the modeling errors at the operators level of the mean computational model. For instance, such an approach seems to be necessary to take into account the modeling errors on the mass and the stiffness operators of a computational dynamical model in order to analyze the generalized eigenvalue problem. It is also the case for the robust design optimization performed with an uncertain computational model for which the design parameters of the computational model are not fixed but vary inside an admissible set of values.

(ii) *Nonparametric probabilistic approach* of model uncertainties induced by modeling errors. This approach, proposed in Soize (2000) as an alternative method to the previous output-prediction-error method, allows modeling errors to be taken into account at the operators level by introducing random operators and not at the model output level by introducing an additive noise. It should be noted that this second approach allows a prior probability model of model uncertainties to be constructed even if no experimental data are available. This nonparametric probabilistic approach is based on the use of a reduced-order model and the random matrix theory. It consists in directly constructing the stochastic modeling of the operators of the mean computational model. The random matrix theory (Mehta, 1991) and its developments in the context of dynamics, vibration and acoustics (Soize, 2000, 2001, 2005b, 2010b; Wright and Weaver, 2010) is used to construct the prior probability distribution of the random matrices modeling the uncertain operators of the mean computational model. This prior probability distribution is constructed by using the maximum entropy principle (Jaynes, 1957), in the context of Information Theory (Shannon, 1948), for which the constraints are defined by the available information (Soize, 2000, 2001, 2003a, 2005a,b, 2010b). Since the basic paper Soize (2000), many works have been published in order:

- to validate, using experimental results, the nonparametric probabilistic approach of both the computational model-parameter uncertainties and the model uncertainties induced by modeling errors (Chebli and Soize, 2004; Soize, 2005b; Chen et al., 2006; Duchereau and Soize, 2006; Soize et al., 2008a; Durand et al., 2008; Fernandez et al., 2009, 2010),

- to extend the applicability of the theory to other areas (Soize, 2003b; Soize and Chebli, 2003; Capiez-Lernout and Soize, 2004; Desceliers et al., 2004; Capiez-Lernout et al., 2005; Cottereau et al., 2007; Soize, 2008; Das and Ghanem, 2009; Kassem et al., 2009),

- to extend the theory to new ensembles of positive-definite random matrices yielding a more flexible description of the dispersion levels (Mignolet and Soize, 2008a),

- to apply the theory for the analysis of complex dynamical systems in the medium-frequency range, including structural-acoustic systems, (Ghanem and Sarkar, 2003; Soize, 2003b; Chebli and Soize, 2004; Capiez-Lernout et al., 2006; Duchereau and Soize, 2006; Arnst et al., 2006; Durand et al., 2008; Pellissetti et al., 2008; Desceliers et al., 2009; Fernandez et al., 2009, 2010; Kassem et al., 2011; Soize, 2012a),

- to analyze nonlinear dynamical systems (i) with local nonlinear elements (Desceliers et al., 2004; Sampaio and Soize, 2007a,b; Batou and Soize, 2009b,a; Ritto et al., 2009, 2010; Wang et al., 2011) and (ii) with nonlinear geometrical effects (Mignolet and Soize, 2008b; Capiez-Lernout et al., 2012).

Concerning the coupling of the parametric probabilistic approach of uncertain computational model parameters, with the nonparametric probabilistic approach of model uncertainties induced by modeling errors, a methodology has recently been proposed (Soize, 2010a; Batou et al., 2011). This *generalized probabilistic approach* of uncertainties in computational dynamics uses the random matrix theory. The proposed approach allows the prior probability model of each type of uncertainties (uncertainties on the computational model parameters and model uncertainties) to be separately constructed and identified.
Concerning robust updating or robust design optimization which consists in updating a computational model or in optimizing the design of a mechanical system with a computational model, in taking into account the uncertainties in the computational model parameters and the modeling uncertainties. An overview of the computational methods in optimization considering uncertainties can be found in Schueller and Jensen (2008). Robust updating and robust design developments with uncertainties in the computational model parameters are developed in Papadimitriou et al. (2001); Taflanidis and Beck (2008); Goller et al. (2009) while robust updating and robust design optimization with modeling uncertainties can be found in Capiez-Lernout and Soize (2008b,a,c); Soize et al. (2008b); Ritto et al. (2010).

9.2. *Uncertainties and stochastic reduced-order computational structural-acoustic model*

This section is devoted to the construction of the stochastic model of both computational model-parameters uncertainties and modeling errors using the nonparametric probabilistic approach and random matrix theory (for the details, see Durand et al. (2008); Soize (2010b, 2012a,b)). We apply this methodology to the reduced-order computational structural acoustic model defined by Eqs. (73) to (78). It is assumed that there is no uncertainties in the boundary element matrix $[A_{\rm BEM}(\omega/c_{\rm E})]$ and in the wall acoustic impedance matrix $[A^Z(\omega)]$. Consequently, for fixed values N_S and N, the stochastic reduced-order computational structural-acoustic model of order N_S and N is written as

$$\mathbf{U}(\omega) = [\mathcal{U}]\,\mathbf{Q}^S(\omega)\,, \qquad (79)$$

$$\mathbf{P}(\omega) = [\mathcal{P}]\,\mathbf{Q}(\omega)\,, \tag{80}$$

in which, for all fixed ω, the complex random vectors $\mathbf{Q}^S(\omega)$ and $\mathbf{Q}(\omega)$ of dimension N_S and N are the solution of the following equation

$$[\mathbf{A}_{\mathrm{FSI}}(\omega)] \begin{bmatrix} \mathbf{Q}^S(\omega) \\ \mathbf{Q}(\omega) \end{bmatrix} = \begin{bmatrix} \mathbf{f}^S(\omega) \\ \mathbf{f}(\omega) \end{bmatrix} \quad , \tag{81}$$

and where the complex random matrix $[\mathbf{A}_{\mathrm{FSI}}(\omega)]$ is written as

$$\begin{bmatrix} [\mathbf{A}^S(\omega)] - \omega^2 [A_{\mathrm{BEM}}(\omega/c_{\mathrm{E}})] & [\,\mathbf{C}\,] \\ \omega^2\,[\,\mathbf{C}\,]^T & [\mathbf{A}(\omega)] + [A^Z(\omega)] \end{bmatrix} . \tag{82}$$

The symmetric $(N_S \times N_S)$ complex random matrix $[\mathbf{A}^S(\omega)]$ is defined by

$$[\mathbf{A}^S(\omega)] = -\omega^2 [\mathbf{M}^S] + i\omega\,[\mathbf{D}^S(\omega)] + [\mathbf{K}^S(\omega)]\,, \tag{83}$$

in which the positive-definite symmetric $(N_S \times N_S)$ real matrices $[\mathbf{M}^S]$, $[\mathbf{D}^S(\omega)]$ and $[\mathbf{K}^S(\omega)]$ are random matrices whose probability distributions are constructed in Sections 9.4 and 9.5. The symmetric $(N \times N)$ complex random matrix $[\mathbf{A}(\omega)]$ is written as

$$[\mathbf{A}(\omega)] = -\omega^2 [\mathbf{M}] + i\omega\,[\mathbf{D}(\omega)] + [\mathbf{K}]\,, \tag{84}$$

in which $[\mathbf{M}]$, $[\mathbf{D}(\omega)]$ and $[\mathbf{K}]$ are symmetric $(N \times N)$ real random matrices. Random matrix $[\mathbf{M}]$ is positive definite. The diagonal $(N \times N)$ real random matrix $[\mathbf{D}(\omega)]$ is written as

$$[\mathbf{D}(\omega)] = \tau(\omega)\,[\mathbf{K}]\,, \tag{85}$$

in which $\tau(\omega)$ is deterministic and defined by Eq. (11). For a closed (sealed wall) acoustic cavity, random matrix $[\mathbf{K}]$ is positive and not invertible with rank $N-1$, while for an almost closed (non sealed wall) acoustic cavity, random matrix $[\mathbf{K}]$ is positive definite. The probability distributions of random matrices $[\mathbf{M}]$, $[\mathbf{K}]$ and of the $(N_S \times N)$ real random matrix $[\mathbf{C}]$ are constructed in Sections 9.6 to 9.8.

9.3. Preliminary results for the stochastic modeling of the random matrices for the stochastic reduced-order computational structural-acoustic model

In the framework of the nonparametric probabilistic approach of uncertainties, the probability distributions and the generators of independent realizations of such random matrices are constructed using random matrix theory (Mehta, 1991) and the maximum entropy principle (Jaynes, 1957; Soize, 2008) from Information Theory (Shannon, 1948), in which Shannon introduced the notion of entropy as a measure of the level of uncertainties for a probability distribution. For instance, if $p_X(x)$ is a probability density function on a real random variable X, the entropy $\mathcal{E}(p_X)$ of p_X is defined by $\mathcal{E}(p_X) = -\int_{-\infty}^{+\infty} p_X(x)\,\log(p_X(x))\,dx$. The maximum entropy principle consists in maximizing the entropy, that is to say, maximizing the uncertainties, under the constraints defined by the available information. Consequently, it is important to define the algebraic properties of the random matrices for which the probability distributions have to be constructed. Let E be the mathematical expectation. For instance, $E\{X\} = \int_{-\infty}^{+\infty} x\,p_X(x)\,dx$. Consequently, we have $\mathcal{E}(p_X) = -E\{\log(p_X(X))\}$. In order to construct the probability distributions of the random matrices introduced in Section 9.2, we need to define a basic ensemble of random matrices.

It is well known that a real Gaussian random variable can take negative values. Consequently, the Gaussian orthogonal ensemble (GOE) of random matrices (Mehta, 1991), which is the generalization for the matrix case of the Gaussian random variable, cannot be used when positiveness property of the random matrix is required. Therefore, new ensembles of random matrices are required to implement the nonparametric probabilistic approach of uncertainties. Below, we summarize the construction (Soize, 2000, 2001) of an ensemble of positive-definite symmetric $(m \times m)$ real random matrices.

9.3.1. Definition of the available information

For the probabilistic construction using the maximum entropy principle, the available information corresponds to two constraints. The first one is the mean value which is given and equal to the identity matrix. The second one is an integrability condition which has to be imposed in order to ensure the decreasing of the probability density function around the origin. These two constraints are written as

$$E\{[\mathbf{G}_0]\} = [I_m] \quad , \quad E\{\log(\det[\mathbf{G}_0])\} = \chi\,, \tag{86}$$

in which $|\chi|$ is finite and where $[I_m]$ is the $(m \times m)$ identity matrix.

9.3.2. *Probability density function*

The value of the probability density function of the random matrix $[\mathbf{G}_0]$ for the matrix $[G]$ is noted $p_{[\mathbf{G}_0]}([G])$ and satisfies the usual normalization condition,

$$\int p_{[\mathbf{G}_0]}([G])\ \widetilde{d}G = 1\,, \tag{87}$$

in which the integration is carried out on the set of all the positive-definite symmetric $(m \times m)$ real matrices and where it can be shown that the volume element $\widetilde{d}G$ is written as $\widetilde{d}G = 2^{m(m-1)/4}\,\Pi_{1\leq j\leq k\leq m}\, dG_{jk}$.

Let δ be the positive real number defined by

$$\delta = \left\{\frac{1}{m}E\{\|\,[\mathbf{G}_0] - [I_m]\,\|_F^2\}\right\}^{1/2}\,, \tag{88}$$

which will allow the dispersion of the probability model of random matrix $[\mathbf{G}_0]$ to be controlled and where $\|\,\mathcal{M}\,\|_F$ is the Frobenius matrix norm of the matrix $[\mathcal{M}]$ such that $\|\,\mathcal{M}\,\|_F^2 = \mathrm{tr}\{[\mathcal{M}]^T[\mathcal{M}]\}$. For δ such that $0 < \delta < (m+1)^{1/2}(m+5)^{-1/2}$, the use of the maximum entropy principle under the two constraints defined by Eq. (86) and the normalization condition defined by Eq. (87), yields, for all positive-definite symmetric $(m \times m)$ real matrix $[G]$,

$$p_{[\mathbf{G}_0]}([G]) = c_0\big(\det[G]\big)^{c_1}\exp\{-c_2\,\mathrm{tr}[G]\}\,, \tag{89}$$

in which the positive constant of normalization c_0, the constant $c_1 = (m+1)(1-\delta^2)/(2\delta^2)$ and the constant $c_2 = (m+1)/(2\delta^2)$ depend on m and δ.

9.3.3. *Generator of independent realizations*

The generator of independent realizations (which is required to solve the random equations with the Monte Carlo method) is constructed using the following algebraic representation. Using the Cholesky decomposition, random matrix $[\mathbf{G}_0]$ is written as $[\mathbf{G}_0] = [\mathbf{L}]^T\,[\mathbf{L}]$ in which $[\mathbf{L}]$ is an upper triangular $(m \times m)$ random matrix such that:

- random variables $\{[\mathbf{L}]_{jj'}, j \leq j'\}$ are independent;
- for $j < j'$, the real-valued random variable $[\mathbf{L}]_{jj'}$ is written as $[\mathbf{L}]_{jj'} = \sigma_m U_{jj'}$ in which $\sigma_m = \delta(m+1)^{-1/2}$ and where $U_{jj'}$ is a real-valued Gaussian random variable with zero mean and variance equal to 1;
- for $j = j'$, the positive-valued random variable $[\mathbf{L}]_{jj}$ is written as $[\mathbf{L}]_{jj} = \sigma_m\sqrt{2V_j}$ in which V_j is a positive-valued Gamma random variable with probability density function $\Gamma(a_j, 1)$ in which $a_j = \frac{m+1}{2\delta^2} + \frac{1-j}{2}$.

9.3.4. *Ensemble* SG_ε^+ *of random matrices*

Let $0 \leq \varepsilon \ll 1$ be a positive number (for instance, ε can be chosen as 10^{-6}). We then define the ensemble $\mathrm{SG}_\varepsilon^+$ of all the random matrices such that

$$[\mathbf{G}] = \frac{1}{1+\varepsilon}\{[\mathbf{G}_0] + \varepsilon\,[I_m]\}\,, \tag{90}$$

in which $[\mathbf{G}_0]$ is a random matrix whose probability density function is defined in Section 9.3.2 and whose generator of independent realizations is defined in Section 9.3.3.

9.3.5. *Cases of several random matrices*

It can be proven (Soize, 2005b) that, if there are several random matrices for which there is no available information concerning their statistical dependencies, then the use of the maximum entropy principle yields that the best model which maximizes the entropy (the uncertainties) is a stochastic model for which all these random matrices are independent.

9.4. *Stochastic modeling of random matrix* $[\boldsymbol{M}^S]$

Since there is no available information concerning the statistical dependency of $[\mathbf{M}^S]$ with the other random matrices of the problem, then random matrix $[\mathbf{M}^S]$ is independent of all the other random matrices. The deterministic matrix $[M^S]$ is positive definite and consequently, can be written as $[M^S] = [L_{M^S}]^T\,[L_{M^S}]$ in which $[L_{M^S}]$ is an upper triangular real matrix. Using the nonparametric probabilistic approach of uncertainties, the stochastic model of the positive-definite symmetric random matrix $[\mathbf{M}^S]$ is then defined by

$$[\mathbf{M}^S] = [L_{M^S}]^T\,[\mathbf{G}_{M^S}]\,[L_{M^S}]\,, \tag{91}$$

where $[\mathbf{G}_{M^S}]$ is a $(N_S \times N_S)$ random matrix belonging to ensemble $\mathrm{SG}_\varepsilon^+$ defined in Section 9.3.4 and whose probability distribution and generator of independent realizations depend only on dimension N_S and on the dispersion parameter δ_{M^S}.

9.5. *Stochastic modeling of the family of random matrices* $[\boldsymbol{D}^S(\omega)]$ *and* $[\boldsymbol{K}^S(\omega)]$

Since there is no available information concerning the statistical dependency of the random matrices $\{[\mathbf{D}^S(\omega)], [\mathbf{K}^S(\omega)]\}$ with the other random matrices of the problem, then $\{[\mathbf{D}^S(\omega)], [\mathbf{K}^S(\omega)]\}$ are independent of all the other random matrices. But we will see below that $[\mathbf{D}^S(\omega)]$ and $[\mathbf{K}^S(\omega)]$ are statistically dependent random matrices. For stochastic modeling of $[\mathbf{D}^S(\omega)]$ and $[\mathbf{K}^S(\omega)]$ related to the linear viscoelastic structure, we propose to use the new extension presented in Soize and Poloskov (2012) which is based on the Hilbert transform Papoulis (1977) in the frequency domain to express the causality properties (similarly to the transforms used in Section 5.2). The nonparametric probabilistic approach of uncertainties then consists in modeling the positive-definite symmetric $(N_S \times N_S)$ real matrices $[D^S(\omega)]$ and $[K^S(\omega)]$ by random matrices $[\mathbf{D}^S(\omega)]$ and $[\mathbf{K}^S(\omega)]$ such that,

$$E\{[\mathbf{D}^S(\omega)]\} = [D^S(\omega)]\, ,\ E\{[\mathbf{K}^S(\omega)]\} = [K^S(\omega)], \quad (92)$$

$$[\mathbf{D}^S(-\omega)] = [\mathbf{D}(^S\omega)]\, , \quad [\mathbf{K}^S(-\omega)] = [\mathbf{K}^S(\omega)]\, . \quad (93)$$

For $\omega \geq 0$, the construction of the stochastic model of the family of random matrices $[\mathbf{D}^S(\omega)]$ and $[\mathbf{K}^S(\omega)]$ is carried out as follows.

- Constructing the family $[\mathbf{D}^S(\omega)]$ of random matrices such that $[\mathbf{D}^S(\omega)] = [L_{D^S}(\omega)]^T\,[\mathbf{G}_{D^S}]\,[L_{D^S}(\omega)]$, where $[L_{D^S}(\omega)]$ is such that $[D^S(\omega)] = [L_{D^S}(\omega)]^T\,[L_{D^S}(\omega)]$ and where $[\mathbf{G}_{D^S}]$ is a $(N_S \times N_S)$ random matrix belonging to ensemble $\mathrm{SG}_\varepsilon^+$, defined in Section 9.3.4. Its probability distribution and its generator of independent realizations depend only on dimension N_S and on the dispersion parameter δ_{D^S} which allows the level of uncertainties to be controlled.

- Defining the family $[\widehat{\mathbf{N}}^I(\omega)]$ of random matrices such that $[\widehat{\mathbf{N}}^I(\omega)] = \omega\,[\mathbf{D}^S(\omega)]$.

- Constructing the family $[\widehat{\mathbf{N}}^R(\omega)]$ of random matrices using the equation

$$[\widehat{\mathbf{N}}^R(\omega)] = \frac{2}{\pi}\mathrm{p.v}\int_0^{+\infty} \frac{\omega'^{\,2}}{\omega^2 - \omega'^{\,2}}\,[\mathbf{D}^S(\omega')]\,d\omega'\, , \quad (94)$$

or equivalently, using the two following equations which are useful for computation:

$$[\widehat{\mathbf{N}}^R(0)] = -\frac{2}{\pi}\int_0^{+\infty}[\mathbf{D}^S(\omega)]\,d\omega\, , \quad (95)$$

and, for $\omega > 0$,

$$\begin{aligned} [\widehat{\mathbf{N}}^R(\omega)] &= \frac{2}{\pi}\,\mathrm{p.v}\int_0^{+\infty}\frac{u^2}{1-u^2}\,\omega\,[\mathbf{D}^S(\omega u)]\,du\, , \\ &= \frac{2}{\pi}\lim_{\eta\to 0}\{\int_0^{1-\eta} + \int_{1+\eta}^{+\infty}\}\, . \end{aligned} \quad (96)$$

- Defining the family $[\widehat{\mathbf{N}}(\omega)]$ of random matrices such that $[\widehat{\mathbf{N}}(\omega)] = [\widehat{\mathbf{N}}^R(\omega)] + i\,[\widehat{\mathbf{N}}^I(\omega)]$.

- Constructing the random matrix $[\mathbf{K}^S(0)] = [L_{K^S(0)}]^T\,[\mathbf{G}_{K^S(0)}]\,[L_{K^S(0)}]$ where $[L_{K^S(0)}]$ is such that $[K^S(0)] = [L_{K^S(0)}]^T\,[L_{K^S(0)}]$ and where $[\mathbf{G}_{K^S(0)}]$ is a $(N_S \times N_S)$ random matrix belonging to ensemble $\mathrm{SG}_\varepsilon^+$ defined in Section 9.3.4 and whose probability distribution and generator of independent realizations depend only on dimension N_S and on the dispersion parameter $\delta_{K^S(0)}$ which allows the level of uncertainties to be controlled. It should be noted that random matrix $[\mathbf{G}_{K^S(0)}]$ is independent of random matrix $[\mathbf{G}_{D^S}]$.

- Computing the random matrix $[\mathbf{D}^+] = -[\widehat{\mathbf{N}}^R(0)] = \frac{2}{\pi}\int_0^{+\infty}[\mathbf{D}^S(\omega)]\,d\omega$.

- Defining the random matrix $[\mathbf{K}_0^S] = [\mathbf{K}^S(0)] + [\mathbf{D}^+]$.

- Constructing the random matrix $[\mathbf{K}^S(\omega)] = [\mathbf{K}_0^S] + [\widehat{\mathbf{N}}^R(\omega)]$ and verifying that $[\mathbf{K}^S(\omega)]$ is effectively an increasing function on $[0\,, +\infty[$.

9.6. Stochastic modeling of random matrix [$\mathbf{M}$]

Since there is no available information concerning the statistical dependency of [$\mathbf{M}$] with the other random matrices of the problem, then random matrix [$\mathbf{M}$] is independent of all the other random matrices. The deterministic matrix $[M]$, is positive definite and consequently, can be written as $[M] = [L_M]^T\,[L_M]$ in which $[L_M]$ is an upper triangular real matrix. Using the nonparametric probabilistic approach of uncertainties, the stochastic model of the positive-definite symmetric random matrix [$\mathbf{M}$] is then defined by

$$[\mathbf{M}] = [L_M]^T\,[\mathbf{G}_M]\,[L_M]\,, \tag{97}$$

where $[\mathbf{G}_M]$ is a $(N \times N)$ random matrix belonging to ensemble $\mathrm{SG}_\varepsilon^+$ defined in Section 9.3.4 and whose probability distribution and generator of independent realizations depend only on dimension N and on the dispersion parameter δ_M.

9.7. Stochastic modeling of random matrix [$\mathbf{K}$]

Since there is no available information concerning the statistical dependency of [$\mathbf{K}$] with the other random matrices of the problem, then random matrix [$\mathbf{K}$] is independent of all the other random matrices. For the stochastic modeling of [$\mathbf{K}$], two cases have to be considered.

- *Closed (sealed wall) acoustic cavity*. In such a case, the symmetric positive matrix $[K]$ is of rank $N-1$ and can then be written as $[K] = [L_K]^T\,[L_K]$ in which $[L_K]$ is a rectangular $(N, N-1)$ real matrix. Using the nonparametric probabilistic approach of uncertainties, the stochastic model of the positive symmetric random matrix [$\mathbf{K}$] of rank $N-1$ is then defined (Soize, 2005b) by

$$[\mathbf{K}] = [L_K]^T\,[\mathbf{G}_K]\,[L_K]\,, \tag{98}$$

  where $[\mathbf{G}_K]$ is a $((N-1)\times(N-1))$ random matrix belonging to ensemble $\mathrm{SG}_\varepsilon^+$ defined in Section 9.3.4 and whose probability distribution and generator of independent realizations depend only on dimension $N-1$ and on the dispersion parameter δ_K.

- *Almost closed (non sealed wall) acoustic cavity*. The matrix $[K]$ is positive definite and thus invertible. Consequently, it can be written as $[K] = [L_K]^T\,[L_K]$ in which $[L_K]$ is an upper triangular (N, N) real matrix. Using the nonparametric probabilistic approach of uncertainties, the stochastic model of this positive symmetric random matrix yields

$$[\mathbf{K}] = [L_K]^T\,[\mathbf{G}_K]\,[L_K]\,, \tag{99}$$

  where $[\mathbf{G}_K]$ is a $(N \times N)$ random matrix belonging to ensemble $\mathrm{SG}_\varepsilon^+$ defined in Section 9.3 and whose probability distribution and generator of independent realizations depend only on dimension N and on the dispersion parameter δ_K.

9.8. Stochastic modeling of random matrix [$\mathbf{C}$]

Since there is no available information concerning the statistical dependency of [$\mathbf{C}$] with the other random matrices of the problem, then random matrix [$\mathbf{C}$] is independent of all the other random matrices. We use the construction proposed in (Soize, 2005b) in the context of the nonparametric probabilistic approach. Let us assumed that $N_S \geq N$ and that the $(N_S \times N)$ real matrix $[C]$ is such that $[C]\,\mathbf{q} = 0$ implies $\mathbf{q} = 0$. If $N \geq N_S$, the following construction must be applied to $[C]^T$ instead of $[C]$. Using the singular value decomposition of rectangular matrix $[C]$, one can write $[C] = [R]\,[T]$ in which the $(N_S \times N)$ real matrix $[R]$ is such that $[R]^T\,[R] = [\,I_N]$ and where the symmetric square matrix $[T]$ is a positive-definite symmetric $(N \times N)$ real matrix. Using the Cholesky decomposition, we then have $[T] = [L_T]^T\,[L_T]$ in which $[L_T]$ is an upper triangular matrix. The $(N_S \times N)$ real random matrix [$\mathbf{C}$] is then written as

$$[\mathbf{C}] = [R]\,[\mathbf{T}] \quad , \quad [\mathbf{T}] = [L_T]^T\,[\mathbf{G}_C]\,[L_T] \quad , \tag{100}$$

where $[\mathbf{G}_C]$ is a $(N \times N)$ random matrix belonging to ensemble $\mathrm{SG}_\varepsilon^+$ defined in Section 9.3.4 and whose probability distribution and generator of independent realizations depend only on dimension N_S,N and on the dispersion parameter δ_C.

9.9. Comments about the stochastic model parameters of uncertainties and the stochastic solver

The dispersion parameter δ of each random matrix [$\mathbf{G}$] allows its level of dispersion (statistical fluctuations) to be controlled. The dispersion parameters of random matrices $[\mathbf{G}_{M^S}]$, $[\mathbf{G}_{D^S}]$, $[\mathbf{G}_{K^S(0)}]$, $[\mathbf{G}_M]$, $[\mathbf{G}_K]$ and $[\mathbf{G}_C]$ is represented by a vector $\boldsymbol{\delta}$ such that

$$\boldsymbol{\delta} = (\delta_{M^S}\,,\, \delta_{D^S}\,,\, \delta_{K^S(0)}\,,\, \delta_M\,,\, \delta_K\,,\, \delta_C) \quad , \qquad (101)$$

which belongs to an admissible set $\mathcal{C}_{\boldsymbol{\delta}}$ and which allows the level of uncertainties to be controlled for each type of operators introduced in the stochastic reduced-order computational structural-acoustic model. Consequently, if no experimental data are available, then $\boldsymbol{\delta}$ has to be used to analyze the robustness of the solution of the structural-acoustic problem with respect to uncertainties by varying $\boldsymbol{\delta}$ in $\mathcal{C}_{\boldsymbol{\delta}}$.

For a given value of $\boldsymbol{\delta}$, there are two major classes of methods for solving the stochastic reduced-order computational structural-acoustic model defined by Eqs. (79) to (85). The first one belongs to the category of the spectral stochastic methods (see Ghanem and Spanos (1991, 2003); LeMaitre and Knio (2010)). The second one belongs to the class of the stochastic sampling techniques for which the Monte Carlo method is the most popular. Such a method is often called non-intrusive since it offers the advantage of only requiring the availability of classical deterministic codes. It should be noted that the Monte Carlo numerical simulation method (see for instance (Fishman, 1996; Rubinstein and Kroese, 2008) is a very effective and efficient one because it as the four following advantages,

- it is a non-intrusive method,
- it is adapted to massively parallel computation without any software developments,
- it is such that its convergence can be controlled during the computation,
- the speed of convergence is independent of the dimension.

If experimental data are available, there are several possible methodologies (whose one is the maximum likelihood method) to identify the optimal values of $\boldsymbol{\delta}$ (for sake of brevity, these aspects are not considered in this paper and we refer the reader to Soize (2012a)).

10. Symmetric Boundary Element Method Without Spurious Frequencies for the External Acoustic Fluid

The inviscid acoustic fluid occupies the infinite three-dimensional domain Ω_E whose boundary $\partial\Omega_E$ is Γ_E. This section is devoted to the construction of the frequency-dependent impedance boundary operator $\mathbf{Z}_{\Gamma_E}(\omega)$, for the external acoustic problem. We recall that the operator $\mathbf{Z}_{\Gamma_E}(\omega)$ is such that $p_E|_{\Gamma_E}(\omega) = \mathbf{Z}_{\Gamma_E}(\omega)\, v(\omega)$ which relates the pressure field $p_E|_{\Gamma_E}(\omega)$ exerted by the external fluid on Γ_E to the normal velocity field $v(\omega)$ induced by the deformation of this boundary Γ_E.

Many methods can be found in literature for solving this problem: the boundary element methods, the artificial boundary conditions and the local/nonlocal non-reflecting boundary condition (NRBC) to take into account the Sommerfeld radiation condition at infinity, the Dirichlet-to-Neumann (DtN) boundary condition related to a nonlocal artificial boundary condition which match analytical and numerical solutions, the infinite element method, the doubly asymptotic approximation method, the finite element method in unbounded domain and related *a posteriori* error estimation and, finally, the wave based method for unbounded domain, see for instance Geers and Felippa (1983); Givoli (1992); Harari et al. (1996); Astley (2000); Farhat et al. (2003, 2004); Oden et al. (2005); Bergen et al. (2010). This section is devoted to the presentation on the boundary element methods.

The frequency-dependent impedance boundary operator $\mathbf{Z}_{\Gamma_E}(\omega)$ can be constructed, either in time domain and then, taking the Fourier transform, or directly constructed in the frequency domain. One technique for constructing $\mathbf{Z}_{\Gamma_E}(\omega)$ consists in using boundary integral formulations (Jones, 1974; Costabel and Stephan, 1985; Jones, 1986; Kress, 1989; Colton and Kress, 1992; Dautray and Lions, 1992; Bonnet, 1999; Nedelec, 2001; Hsiao and Wendland, 2008). In the time domain, it uses the so-called Kirchhoff retarded potential formula (see for instance Baker and Copson (1949); Lee et al. (2009)). It should be noted that the formulations in the frequency domain can easily be implemented in massively parallel computers.

The finite element discretization of the boundary integral equations yields the Boundary Element Method (Brebbia and Dominguez, 1992; Chen and Zhou, 1992; Hackbusch, 1995; Ohayon and Soize, 1998; Gaul et al., 2003). Furthermore, most of those formulations yield unsymmetric fully populated complex matrices. The computational cost can then be reduced using the fast multipole methods (Greengard and Rokhlin, 1987; Gumerov and Duraiswami, 2004; Schanz and Steinbach, 2007; Bonnet et al., 2009; Brunner et al., 2009).

A major drawback of the classical boundary integral formulations for the exterior Neumann problem related to the Helmholtz equation is related to the uniqueness problem although the boundary value problem has a unique solution for all real frequencies (Sanchez-Hubert and Sanchez-Palencia, 1989; Dautray and Lions, 1992). Precisely, there is not a unique solution of the physical problem for a sequence of real frequencies called *spurious* or *irregular frequencies,* also called *Jones eigenfrequencies* (Burton and Miller, 1971; Jones, 1983; Colton and Kress, 1992; Luke and Martin, 1995; Jentsch and Natroshvili, 1999). Various methods are proposed in the literature to overcome this mathematical difficulty arising in the boundary element method (Panich, 1965; Schenck, 1968; Burton and Miller, 1971; Angelini and Hutin, 1983; Mathews, 1986; Amini and Harris, 1990; Amini et al., 1992; Ohayon and Soize, 1998).

In this section, we present a method, initially developed in Angelini and Hutin (1983), yielding an appropriate symmetric boundary element method valid for all real values of the frequency which is numerically stable and very efficient. This method is detailed in Ohayon and Soize (1998) and does not require introducing additional degrees of freedom in the numerical discretization for treatment of irregular frequencies. This method has been extended to the Maxwell equations (Angelini et al., 1993). In the case of an external liquid domain with a zero-pressure free surface (which is not presented here for sake of brevity) the method presented below can be adapted using the image method (for the details, see Ohayon and Soize (1998)).

10.1. Exterior Neumann problem related to the Helmholtz equation

The geometry is defined in Fig. 5. The inviscid fluid

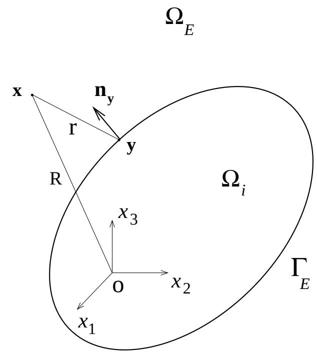

Figure 5: Geometry of the external infinite domain.

occupies the infinite domain Ω_E. For practical computational considerations, the exterior Neumann problem related to the Helmholtz equation (see Eqs. (5) to (7)) is rewritten in terms of a velocity potential $\psi(\mathbf{x},\omega)$. Let $\mathbf{v}(\mathbf{x},\omega) = \boldsymbol{\nabla}\psi(\mathbf{x},\omega)$ be the velocity field of the fluid. The acoustic pressure $p(\mathbf{x},\omega)$ is related to $\psi(\mathbf{x},\omega)$ by the following equation,

$$p(\mathbf{x},\omega) = -i\omega\,\rho_{\mathrm{E}}\,\psi(\mathbf{x},\omega) \quad \text{in} \quad \Omega_E\,, \tag{102}$$

where ρ_{E} is the constant mass density of the external fluid at equilibrium. Let c_{E} be the constant speed of sound in the external fluid at equilibrium and let $k = \omega/c_{\mathrm{E}}$ be the wave number at frequency ω. The exterior Neumann problem is written as

$$\nabla^2\psi(\mathbf{x},\omega) + k^2\,\psi(\mathbf{x},\omega) = 0 \quad \text{in} \quad \Omega_E\,, \tag{103}$$

$$\frac{\partial\psi(\mathbf{y},\omega)}{\partial\mathbf{n_y}} = v(\mathbf{y}) \quad \text{on} \quad \Gamma_E\,, \tag{104}$$

$$|\,\psi\,| = O(\frac{1}{R}) \quad , \quad \left|\frac{\partial\psi}{\partial R} + i\,k\,\psi\right| = O(\frac{1}{R^2})\,, \tag{105}$$

with $R = \|\mathbf{x}\| \to +\infty$, where $\partial/\partial R$ is the derivative in the radial direction and where $v(\mathbf{y})$ is the prescribed normal velocity field on Γ_E. Equation (103) is the Helmholtz equation in the external acoustic fluid, Eq. (104) is the Neumann condition on external fluid-structure interface Γ_E and Eq. (105) corresponds to the outward Sommerfeld radiation condition at infinity.

10.2. Pressure field in Ω_E and on Γ_E

For arbitrary real $\omega \neq 0$, it can be shown that the boundary value problem defined by Eqs. (103) to (105) admits a unique solution denoted $\psi^{\rm sol}$ which depends linearly of the normal velocity v (Sanchez-Hubert and Sanchez-Palencia, 1989; Dautray and Lions, 1992). Let $\psi^{\rm sol}_{\Gamma_E}$ be the value of $\psi^{\rm sol}$ on Γ_E. For all $\mathbf{x}$ in Ω_E, let us introduce the linear operator $\mathbf{R}(\mathbf{x}, \omega/c_{\rm E})$ such that

$$\psi^{\rm sol}(\mathbf{x}, \omega) = \mathbf{R}(\mathbf{x}, \omega/c_{\rm E})\, v\, . \tag{106}$$

We also introduce the linear boundary operator $\mathbf{B}_{\Gamma_E}(\omega/c_{\rm E})$ such that

$$\psi^{\rm sol}_{\Gamma_E} = \mathbf{B}_{\Gamma_E}(\omega/c_{\rm E})\, v\, . \tag{107}$$

Using Eq. (102), for all $\mathbf{x}$ in Ω_E, the pressure field $p(\mathbf{x}, \omega)$ is written as

$$p(\mathbf{x}, \omega) = \mathbf{Z}_{\rm rad}(\mathbf{x}, \omega)\, v\, , \tag{108}$$

in which $\mathbf{Z}_{\rm rad}(\mathbf{x}, \omega)$ is called the *radiation impedance operator* which can then be written as

$$\mathbf{Z}_{\rm rad}(\mathbf{x}, \omega) = -i\,\omega\,\rho_{\rm E}\,\mathbf{R}(\mathbf{x}, \omega/c_{\rm E})\, . \tag{109}$$

Similarly, the pressure field $p|_{\Gamma_E}(\omega)$ on Γ_E is written as

$$p|_{\Gamma_E}(\omega) = \mathbf{Z}_{\Gamma_E}(\omega)\, v\, , \tag{110}$$

in which $\mathbf{Z}_{\Gamma_E}(\omega)$ is called the *acoustic impedance boundary operator* and which can then be written as

$$\mathbf{Z}_{\Gamma_E}(\omega) = -i\,\omega\,\rho_{\rm E}\,\mathbf{B}_{\Gamma_E}(\omega/c_{\rm E})\, . \tag{111}$$

Note that $\mathbf{Z}_{\Gamma_E}(\omega)$ is nonlocal operator.

10.3. Symmetry property of the acoustic impedance boundary operator

The transpose of operator $\mathbf{B}_{\Gamma_E}(\omega/c_{\rm E})$ is denoted by ${}^t\mathbf{B}_{\Gamma_E}(\omega/c_{\rm E})$. It can then be proven (see Ohayon and Soize (1998)) the following symmetry property,

$${}^t\mathbf{B}_{\Gamma_E}(\omega/c_{\rm E}) = \mathbf{B}_{\Gamma_E}(\omega/c_{\rm E})\, , \tag{112}$$

and from Eq. (111), we deduce that

$${}^t\mathbf{Z}_{\Gamma_E}(\omega) = \mathbf{Z}_{\Gamma_E}(\omega)\, . \tag{113}$$

It should be noted that these complex operators are symmetric but not hermitian.

10.4. Positivity of the real part of the acoustic impedance boundary operator

Operator $i\omega\mathbf{Z}_{\Gamma_E}(\omega)$ can be written as

$$i\omega\mathbf{Z}_{\Gamma_E}(\omega) = -\omega^2\,\mathbf{M}_{\Gamma_E}(\omega/c_{\rm E}) + i\omega\mathbf{D}_{\Gamma_E}(\omega/c_{\rm E}), \tag{114}$$

in which $\mathbf{M}_{\Gamma_E}(\omega/c_{\rm E})$ and $\mathbf{D}_{\Gamma_E}(\omega/c_{\rm E})$ are two linear operators such that

$$\omega\,\mathbf{M}_{\Gamma_E}(\omega/c_{\rm E}) = \Im m\,\mathbf{Z}_{\Gamma_E}(\omega)\, , \tag{115}$$

$$\mathbf{D}_{\Gamma_E}(\omega/c_{\rm E}) = \Re e\,\mathbf{Z}_{\Gamma_E}(\omega)\, . \tag{116}$$

It can be shown (Ohayon and Soize, 1998) the following positivity property of the real part $\mathbf{D}_{\Gamma_E}(\omega/c_{\rm E})$ of the acoustic impedance boundary operator, which is due to the Sommerfeld radiation condition at infinity.

10.5. Construction of the acoustic impedance boundary operator for all real value of the frequency

We present here the appropriate symmetric boundary element method without spurious frequencies, for which details can be found in Ohayon and Soize (1998). This formulation simultaneously uses two boundary singular integral equations on Γ_E. The first one is based on the use of a single- and double-layer potentials on Γ_E. The second integral equation is obtained by a normal derivative on Γ_E of the first one. We then obtained the following system relating $\psi^{\rm sol}_{\Gamma_E}$ to v which then allows $\mathbf{B}_{\Gamma_E}(\omega/c_{\rm E})$ to be defined using Eq. (107),

$$\begin{bmatrix} 0 \\ \psi^{\rm sol}_{\Gamma_E} \end{bmatrix} = \begin{bmatrix} -\mathbf{S}_{\rm T}(\omega/c_{\rm E}) & \frac{1}{2}\,{}^t\mathbf{I} - {}^t\mathbf{S}_{\rm D}(\omega/c_{\rm E}) \\ \frac{1}{2}\,\mathbf{I} - \mathbf{S}_{\rm D}(\omega/c_{\rm E}) & \mathbf{S}_{\rm S}(\omega/c_{\rm E}) \end{bmatrix} \begin{bmatrix} \psi_{\Gamma_E} \\ v \end{bmatrix} . \tag{117}$$

The linear boundary integral operators $\mathbf{S}_{\rm S}(\omega/c_{\rm E})$, $\mathbf{S}_{\rm D}(\omega/c_{\rm E})$ and $\mathbf{S}_{\rm T}(\omega/c_{\rm E})$ are defined by

$$< \mathbf{S}_{\rm S}(\omega/c_{\rm E})\, v\, , \delta v > = \int_{\Gamma_E}\int_{\Gamma_E} G(\mathbf{x}-\mathbf{y})\, v(\mathbf{y})\, \delta v(\mathbf{x})\, ds_{\mathbf{y}}\, ds_{\mathbf{x}}\, , \tag{118}$$

$$< \mathbf{S}_{\rm D}(\omega/c_{\rm E})\, \psi_{\Gamma_E}\, , \delta v > = \int_{\Gamma_E}\int_{\Gamma_E} \frac{\partial G(\mathbf{x}-\mathbf{y})}{\partial \mathbf{n}_{\mathbf{y}}}\, \psi_{\Gamma_E}(\mathbf{y})\, \delta v(\mathbf{x})\, ds_{\mathbf{y}}\, ds_{\mathbf{x}}\, , \tag{119}$$

$$< \mathbf{S}_{\mathrm{T}}(\omega/c_{\mathrm{E}})\psi_{\Gamma_E}, \delta\psi_{\Gamma_E} > =$$

$$-k^2\int_{\Gamma_E}\int_{\Gamma_E} G(\mathbf{x}-\mathbf{y})\, \mathbf{n_x}\cdot \mathbf{n_y}\, \psi_{\Gamma_E}(\mathbf{y})\, \delta\psi_{\Gamma_E}(\mathbf{x})\, ds_{\mathbf{y}}\, ds_{\mathbf{x}}$$

$$+\int_{\Gamma_E}\int_{\Gamma_E} G(\mathbf{x}-\mathbf{y})$$

$$\{\mathbf{n_y}\times\nabla_{\mathbf{y}}\psi_{\Gamma_E}(\mathbf{y})\}\cdot\{\mathbf{n_x}\times\nabla_{\mathbf{x}}\delta\psi_{\Gamma_E}(\mathbf{x})\}\, ds_{\mathbf{y}}\, ds_{\mathbf{x}}\,. \quad (120)$$

where $G(\mathbf{x}-\mathbf{y})$ is the Green function which is written as

$$G(\mathbf{x}-\mathbf{y}) = g(\|\mathbf{x}-\mathbf{y}\|) = -(4\pi)^{-1}\, e^{-i\,k\,r}/r\,, \quad (121)$$

in which $r = \|\mathbf{x}-\mathbf{y}\|$. In Eqs. (118) to (120), the brackets correspond to bilinear forms which allow the operators to be defined and the functions δv and $\delta\psi_{\Gamma_E}$ are associated with functions v and ψ_{Γ_E}. Considering Eq. (117), let $\mathbf{H}(\omega/c_{\mathrm{E}})$ be the operator defined by

$$\mathbf{H}(\omega/c_{\mathrm{E}}) = \begin{bmatrix} -\mathbf{S}_{\mathrm{T}}(\omega/c_{\mathrm{E}}) & \frac{1}{2}\,{}^t\mathbf{I} - {}^t\mathbf{S}_{\mathrm{D}}(\omega/c_{\mathrm{E}}) \\ \frac{1}{2}\,\mathbf{I} - \mathbf{S}_{\mathrm{D}}(\omega/c_{\mathrm{E}}) & \mathbf{S}_{\mathrm{S}}(\omega/c_{\mathrm{E}}) \end{bmatrix}. \quad (122)$$

It can be proven that operator $\mathbf{H}(\omega/c_{\mathrm{E}})$ has the symmetric property, ${}^t\mathbf{H}(\omega/c_{\mathrm{E}}) = \mathbf{H}(\omega/c_{\mathrm{E}})$. In Eq. (117), the first equation can be rewritten as $\mathbf{S}_{\mathrm{T}}(\omega/c_{\mathrm{E}})\,\psi_{\Gamma_E} = (\frac{1}{2}\,{}^t\mathbf{I} - {}^t\mathbf{S}_{\mathrm{D}}(\omega/c_{\mathrm{E}}))\, v$. This classical boundary equation which allows the velocity potential to be calculated for a given normal velocity, has a unique solution for all real ω which does not belong to the set of frequencies for which $\mathbf{S}_{\mathrm{T}}(\omega/c_{\mathrm{E}})$ has a null space which is not reduced to $\{0\}$. This set of frequencies is called the set of the *spurious* or *irregular* frequencies. Consequently, as proven in Ohayon and Soize (1998), for a spurious frequency, ψ_{Γ_E} is the sum of solution $\psi^{\mathrm{sol}}_{\Gamma_E}$ with an arbitrary element belonging to the null space of operator $\mathbf{S}_{\mathrm{T}}(\omega/c_{\mathrm{E}})$. The originality of the proposed method (Angelini and Hutin, 1983; Ohayon and Soize, 1998) (extended to the Maxwell equations in Angelini et al. (1993)), then consists in using the second equation which is written as $\psi^{\mathrm{sol}}_{\Gamma_E} = (\frac{1}{2}\,\mathbf{I} - \mathbf{S}_{\mathrm{D}}(\omega/c_{\mathrm{E}}))\,\psi_{\Gamma_E} + \mathbf{S}_{\mathrm{S}}(\omega/c_{\mathrm{E}})\, v$, and which yields solution $\psi^{\mathrm{sol}}_{\Gamma_E}$ for all real ω, because the elements belonging to the null space are filtered when ω is a spurious frequency. Concerning the practical construction of $\psi^{\mathrm{sol}}_{\Gamma_E}$, for all real values of ω, using Eq. (117), a particular elimination procedure will be described in Section 10.7.

10.6. Construction of the radiation impedance operator

The solution $\{\psi^{\mathrm{sol}}(\mathbf{x},\omega), \mathbf{x}\in\Omega_E\}$ of Eqs. (103) to (105) can be calculated using the following integral equation

$$\psi^{\mathrm{sol}}(\mathbf{x},\omega) = \int_{\Gamma_E} \{G(\mathbf{x}-\mathbf{y})\, v(\mathbf{y}) - \psi^{\mathrm{sol}}_{\Gamma_E}(\mathbf{y},\omega)\,\frac{\partial G(\mathbf{x}-\mathbf{y})}{\partial \mathbf{n_y}}\}\, ds_{\mathbf{y}}\,. \quad (123)$$

For all $\mathbf{x}$ fixed in Ω_E, we define the linear integral operators $\mathbf{R}_{\mathrm{S}}(\mathbf{x},\omega/c_{\mathrm{E}})$ and $\mathbf{R}_{\mathrm{D}}(\mathbf{x},\omega/c_{\mathrm{E}})$ by

$$\mathbf{R}_{\mathrm{S}}(\mathbf{x},\omega/c_{\mathrm{E}})\, v = \int_{\Gamma_E} G(\mathbf{x}-\mathbf{y})\, v(\mathbf{y})\, ds_{\mathbf{y}}\,, \quad (124)$$

$$\mathbf{R}_{\mathrm{D}}(\mathbf{x},\omega/c_{\mathrm{E}})\,\psi_{\Gamma_E} = \int_{\Gamma_E} \psi_{\Gamma_E}(\mathbf{y})\,\frac{\partial G(\mathbf{x}-\mathbf{y})}{\partial \mathbf{n_y}}\, ds_{\mathbf{y}}\,. \quad (125)$$

Using Eq. (107), Eq. (123) can be rewritten as

$$\psi^{\mathrm{sol}}(\mathbf{x},\omega) = \{\mathbf{R}_{\mathrm{S}}(\mathbf{x},\omega/c_{\mathrm{E}}) - \mathbf{R}_{\mathrm{D}}(\mathbf{x},\omega/c_{\mathrm{E}})\,\mathbf{B}_{\Gamma_E}(\omega/c_{\mathrm{E}})\}\, v\,. \quad (126)$$

From Eq. (106), we deduce that, for all $\mathbf{x}$ fixed in Ω_E,

$$\mathbf{R}(\mathbf{x},\omega/c_{\mathrm{E}}) = \mathbf{R}_{\mathrm{S}}(\mathbf{x},\omega/c_{\mathrm{E}}) - \mathbf{R}_{\mathrm{D}}(\mathbf{x},\omega/c_{\mathrm{E}})\,\mathbf{B}_{\Gamma_E}(\omega/c_{\mathrm{E}})\,, \quad (127)$$

and the radiation impedance operator $\mathbf{Z}_{\mathrm{rad}}(\mathbf{x},\omega)$ is calculated using Eqs. (109) and (127),

$$\mathbf{Z}_{\mathrm{rad}}(\mathbf{x},\omega) = -i\,\omega\,\rho_{\mathrm{E}}\,\{\mathbf{R}_{\mathrm{S}}(\mathbf{x},\omega/c_{\mathrm{E}}) - \mathbf{R}_{\mathrm{D}}(\mathbf{x},\omega/c_{\mathrm{E}})\,\mathbf{B}_{\Gamma_E}(\omega/c_{\mathrm{E}})\}\,. \quad (128)$$

10.7. Symmetric boundary element method without spurious frequencies

We use the finite element method to discretize the boundary integral operators $\mathbf{S}_{\mathrm{S}}(\omega/c_{\mathrm{E}})$, $\mathbf{S}_{\mathrm{D}}(\omega/c_{\mathrm{E}})$ and $\mathbf{S}_{\mathrm{T}}(\omega/c_{\mathrm{E}})$ (corresponding to a boundary element method). Let us consider a finite element mesh of boundary Γ_E. Let $\mathbf{V} = (V_1,\ldots,V_{n_E})$ and $\mathbf{\Psi}_{\Gamma_E} = (\Psi_{\Gamma_E,1},\ldots,\Psi_{\Gamma_E,n_E})$ be the complex vectors of the n_E degrees-of-freedom constituted of the values of v and ψ_{Γ_E} at the nodes of the mesh. Let $[S_{\rm S}(\omega/c_{\rm E})]$, $[S_{\rm D}(\omega/c_{\rm E})]$ and $[S_{\rm T}(\omega/c_{\rm E})]$ be the full complex matrices corresponding to the discretization of the operators defined in Eqs. (118) to (120). The complex matrices $[S_{\rm S}(\omega/c_{\rm E})]$ and $[S_{\rm T}(\omega/c_{\rm E})]$ are symmetric. The finite element discretization of Eq. (117) yields

$$\begin{bmatrix} \mathbf{0} \\ \mathbf{\Psi}^{\rm sol}_{\Gamma_E} \end{bmatrix} = [H(\omega/c_{\rm E})] \begin{bmatrix} \mathbf{\Psi}_{\Gamma_E} \\ \mathbf{V} \end{bmatrix} , \qquad (129)$$

in which the symmetric complex matrix $[H(\omega/c_{\rm E})]$ is the matrix

$$\begin{bmatrix} -[S_{\rm T}(\omega/c_{\rm E})] & \frac{1}{2}\,[\,E\,]^T - [S_{\rm D}(\omega/c_{\rm E})]^T \\ \frac{1}{2}\,[\,E\,] - [S_{\rm D}(\omega/c_{\rm E})] & [S_{\rm S}(\omega/c_{\rm E})] \end{bmatrix} . \qquad (130)$$

In Eq. (129), $\mathbf{\Psi}^{\rm sol}_{\Gamma_E}$ is the complex vector of the nodal unknowns corresponding to the finite element discretization of $\psi^{\rm sol}_{\Gamma_E}$. The matrix $[\,E\,]$ is the non-diagonal $(n_E \times n_E)$ real matrix corresponding to the discretization of identity operator $\mathbf{I}$. The elimination of $\mathbf{\Psi}_{\Gamma_E}$ in Eq. (129) yields a linear equation between $\mathbf{\Psi}^{\rm sol}_{\Gamma_E}$ and $\mathbf{V}$ which defines the symmetric $(n_E \times n_E)$ complex matrix $[B_{\Gamma_E}(\omega/c_{\rm E})]$ which corresponds to the finite element discretization of boundary integral operator $\mathbf{B}_{\Gamma_E}(\omega/c_{\rm E})$. We then have

$$\mathbf{\Psi}^{\rm sol}_{\Gamma_E} = [B_{\Gamma_E}(\omega/c_{\rm E})]\,\mathbf{V} . \qquad (131)$$

The particular elimination procedure discussed in Section 10.5, which avoids the spurious frequencies, is defined below. Vector $\mathbf{\Psi}_{\Gamma_E}$ is eliminated using a Gauss elimination with a partial row pivoting algorithm (Golub and Van Loan, 1989). If ω does not belong to the set of the spurious frequencies, then $[S_{\rm T}(\omega/c_{\rm E})]$ is invertible and the elimination in Eq. (129) is performed up to row number n_E. If ω coincides with a spurious frequency ω_α that is to say $\omega = \omega_\alpha$, then $[S_{\rm T}(\omega_\alpha/c_{\rm E})]$ is not invertible and its null space is a real subspace of $\mathbb{C}^{n_E}$ of dimension $n_\alpha < n_E$. In this case, the elimination in Eq. (129) is performed up to row number $n_E - n_\alpha$. In practice, n_α is unknown. During the Gauss elimination with a partial row pivoting algorithm, the elimination process is stopped when a "zero" pivot is encountered. It should be noted that when the elimination is stopped, the equations corresponding to row numbers $n_E - n_\alpha + 1, \ldots , n_E$ are automatically satisfied. From Eq. (111), we deduce that the $(n_E \times n_E)$ complex symmetric matrix $[Z_{\Gamma_E}(\omega)]$ of operator $\mathbf{Z}_{\Gamma_E}(\omega)$ is such that

$$[Z_{\Gamma_E}(\omega)] = -i\,\omega\,\rho_{\rm E}\,[B_{\Gamma_E}(\omega/c_{\rm E})] . \qquad (132)$$

Finally, the finite element discretization of the acoustic radiation impedance operator $\mathbf{Z}_{\rm rad}(\mathbf{x},\omega)$ defined by Eq. (129) is written as

$$[Z_{\rm rad}(\mathbf{x},\omega)] = -i\,\omega\,\rho_{\rm E}\,\{[R_{\rm S}(\mathbf{x},\omega/c_{\rm E})] - [R_{\rm D}(\mathbf{x},\omega/c_{\rm E})]\,[B_{\Gamma_E}(\omega/c_{\rm E})]\} . \qquad (133)$$

10.8. Acoustic response to prescribed wall displacement field and acoustic source density

We now consider the acoustic response of the infinite external acoustic fluid submitted to a prescribed external acoustic excitation, namely an acoustic source $Q_E(\mathbf{x},\omega)$, and to a prescribed normal velocity field on Γ_E which is written as $v = i\omega\,\mathbf{u}(\omega)\cdot\mathbf{n}^S$ in which $\mathbf{n}^S$ is the unit normal to Γ_E, external to structure Ω_S, and where $\mathbf{u}$ is the displacement field of the external fluid-structure interface Γ_E. This response is formulated using the results related to the exterior Neumann problem for the Helmholtz equation which have been presented in Sections 10.1 to 10.7 and using the linearity of the problem.

Pressure in Ω_E. At any point $\mathbf{x}$ fixed in Ω_E, the resultant pressure $p_E(\mathbf{x},\omega)$ is written as

$$p_E(\mathbf{x},\omega) = p_{\rm rad}(\mathbf{x},\omega) + p_{\rm given}(\mathbf{x},\omega) , \qquad (134)$$

in which $p_{\rm rad}(\mathbf{x},\omega)$ is the field radiated by the boundary Γ_E submitted to the prescribed velocity field v and written (see Eq. (108)) as

$$p_{\rm rad}(\mathbf{x},\omega) = i\omega\,\mathbf{Z}_{\rm rad}(\mathbf{x},\omega)\{\mathbf{u}(\omega)\cdot\mathbf{n}^S\} . \qquad (135)$$

The pressure $p_{\rm given}(\mathbf{x},\omega)$ is such that

$$p_{\rm given}(\mathbf{x},\omega) = p_{\rm inc,Q}(\mathbf{x},\omega) - \mathbf{Z}_{\rm rad}(\mathbf{x},\omega)\{\frac{\partial\psi_{\rm inc,Q}}{\partial\mathbf{n}^S}\} , \qquad (136)$$

where $p_{\rm inc,Q}(\mathbf{x},\omega)$ is the pressure in the free space induced by the acoustic source Q_E and which is written as

$$p_{\rm inc,Q}(\mathbf{x},\omega) = -i\omega \int_{K_Q} G(\mathbf{x}-\mathbf{x}')\,Q(\mathbf{x}',\omega)\,d\mathbf{x}' , \qquad (137)$$

in which the Green function G is defined by Eq. (121) and where $\partial\psi_{\text{inc,Q}}/\partial\mathbf{n}^S$ is deduced from Eqs. (137) and (102). The second term in the right-hand side of Eq. (136) corresponds to the scattering of the incident wave (induced by the external acoustic source) by the boundary Γ_E considered as rigid and fixed.

Pressure on Γ_E. The resultant pressure on Γ_E is then written as

$$p_E|_{\Gamma_E}(\omega) = p_{\text{rad}}|_{\Gamma_E}(\omega) + p_{\text{given}}|_{\Gamma_E}(\omega)\,, \tag{138}$$

in which $p_{\text{rad}}|_{\Gamma_E}(\omega)$ is written as

$$p_{\text{rad}}|_{\Gamma_E}(\omega) = i\omega\,\mathbf{Z}_{\Gamma_E}(\omega)\{\mathbf{u}(\omega)\cdot\mathbf{n}^S\}\,, \tag{139}$$

and the pressure field $p_{\text{given}}|_{\Gamma_E}(\omega)$ on Γ_E is such that

$$p_{\text{given}}|_{\Gamma_E}(\omega) = p_{\text{inc,Q}}|_{\Gamma_E}(\omega) - \mathbf{Z}_{\Gamma_E}(\omega)\{\frac{\partial\psi_{\text{inc,Q}}}{\partial\mathbf{n}^S}\}\,. \tag{140}$$

Substituting Eq. (139) in (138) yields

$$p_E|_{\Gamma_E}(\omega) = p_{\text{given}}|_{\Gamma_E}(\omega) + i\omega\,\mathbf{Z}_{\Gamma_E}(\omega)\{\mathbf{u}(\omega)\cdot\mathbf{n}^S\}. \tag{141}$$

For details, we refer the reader to Chapter 12 of Ohayon and Soize (1998).

10.9. Asymptotic formula for the radiated pressure far field

At point $\mathbf{x}$ in the external domain Ω_E, the radiated pressure $p(\mathbf{x},\omega)$ is given (see Eq. (108)) by $p(\mathbf{x},\omega) = \mathbf{Z}_{\text{rad}}(\mathbf{x},\omega)\,v$. Let R and $\mathbf{e}$ be such that (see Fig. 6.)

$$\mathbf{x} = R\,\mathbf{e} \quad \text{with} \quad R = \|\mathbf{x}\|\,. \tag{142}$$

Definition of integral operators $\mathbf{R}_{\text{S}}^{\infty}(\mathbf{x},\omega/c_{\text{E}})$ *and* $\mathbf{R}_{\text{D}}^{\infty}(\mathbf{x},\omega/c_{\text{E}})$. For all $\mathbf{x} = R\,\mathbf{e}$ fixed in external domain Ω_E, we define the linear integral operators $\mathbf{R}_{\text{S}}^{\infty}(\mathbf{x},\omega/c_{\text{E}})$ and $\mathbf{R}_{\text{D}}^{\infty}(\mathbf{x},\omega/c_{\text{E}})$ by

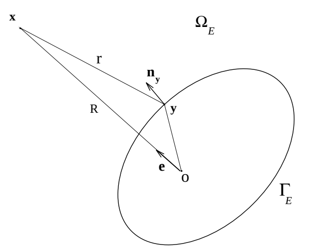

Figure 6: Geometrical configuration.

$$\mathbf{R}_{\text{S}}^{\infty}(\mathbf{x},\omega/c_{\text{E}})\,v = \frac{1}{R}e^{-i\omega R/c_{\text{E}}}\int_{\Gamma_E} N_{\mathbf{e}}(\mathbf{y})\,v(\mathbf{y})\,ds_{\mathbf{y}}\,, \tag{143}$$

$$\mathbf{R}_{\text{D}}^{\infty}(\mathbf{x},\omega/c_{\text{E}})\,\psi_{\Gamma_E} = \frac{i\omega}{c_{\text{E}}}\,\frac{1}{R}e^{-i\omega R/c_{\text{E}}}\int_{\Gamma_E}\mathbf{e}\cdot\mathbf{n}_{\mathbf{y}}\,N_{\mathbf{e}}(\mathbf{y})\psi_{\Gamma_E}(\mathbf{y})\;ds_{\mathbf{y}}\,,$$

in which $N_{\mathbf{e}}(\mathbf{y})$ is defined by

$$N_{\mathbf{e}}(\mathbf{y}) = -\frac{1}{4\pi}\,\exp(i\,\mathbf{e}\cdot\mathbf{y}\,\omega/c_{\text{E}})\,. \tag{144}$$

Asymptotic formula for radiation impedance operator $\mathbf{Z}_{\text{rad}}(\mathbf{x},\omega)$. We have the following asymptotic formulas

$$\lim_{R\to+\infty}\mathbf{R}_{\text{S}}(R\,\mathbf{e},\omega/c_{\text{E}}) = \mathbf{R}_{\text{S}}^{\infty}(R\,\mathbf{e},\omega/c_{\text{E}})\,, \tag{145}$$

$$\lim_{R\to+\infty}\mathbf{R}_{\text{D}}(R\,\mathbf{e},\omega/c_{\text{E}}) = \mathbf{R}_{\text{D}}^{\infty}(R\,\mathbf{e},\omega/c_{\text{E}})\,. \tag{146}$$

From Eq. (127), we deduce the asymptotic formula for the radiation impedance operator

$$\lim_{R\to+\infty}\mathbf{Z}_{\text{rad}}(R\mathbf{e},\omega) = -i\omega\rho_{\text{E}}\{\mathbf{R}_{\text{S}}^{\infty}(R\mathbf{e},\omega/c_{\text{E}}) - \mathbf{R}_{\text{D}}^{\infty}(R\mathbf{e},\omega/c_{\text{E}})\,\mathbf{B}_{\Gamma_E}(\omega\,c_{\text{E}})\}\,. \tag{147}$$

11. Conclusion

We have presented an advanced computational formulation for dissipative structural-acoustics systems and fluid-structure interaction which is adapted for developing new generation of software. An efficient stochastic reduced-order model in the frequency domain is proposed to analyze low- and medium-frequency ranges. All the required modeling aspects for the analysis of the medium-frequency domain have been introduced namely, a viscoelastic behavior for the structure, an appropriate dissipative model for the internal acoustic fluid including wall acoustic impedance and a model of uncertainty in particular for modeling errors.

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