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Experimental identification of a prior tensor-valued random field for the elasticity properties of cortical bones using in vivo ultrasonic measurements

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The biomechanical materials are among the most complex mechanical systems. Most often, their micro-structure are complex and random. This is the case for the human cortical bones which are considered in this paper. For such a system, the micro-structure can be altered near its interface with the marrow (osteoporosis). A gradient of porosity is then observed in the thickness direction but, in this case, none of the usual theories of porous materials can be applied. For this reason, we present a simplified model with gradient for the elasticity tensor. The elasticity tensor is modelled by a random field. In this paper, the parameters of this probabilistic model are identified with experimental observations in ultrasonic range.

1 Introduction

A cortical bone layer is a biomechanical system that is difficult to model in regard to the complexity level of its microstructure. The experimental identification of its effective mechanical properties at the macroscale is usually carried out using the axial transmission technique which is often modelled with a simplified mean mechanical model. In this paper, the simplified mean mechanical model is a fluid-solid semi-infinite multilayer system (skin and muscles/cortical layer/marrow). It is also assumed that the effective elasticity properties of the solid layer (cortical bone) have spatial variations in the thickness direction but, in this case, none of the usual theories of porous materials can be applied. For these reasons, these systems are often modelled using a simplified mechanical model which corresponds to a rough approximation of the real system. The uncertainties introduced in the construction of this simplified mean model are taken into account with an a priori probabilistic model in which the elasticity tensor is a non-homogeneous and non-Gaussian tensor-valued random field which has been constructed by C. Soize using the information theory and the maximum entropy principle. The parameters of this probabilistic model are (1) the mean value of the effective thickness and the mean value of the elasticity tensor of the cortical bone and (2) the parameters controlling the level of uncertainties which depends on the spatial coordinates. A method and an application are presented for the identification of these parameters using in vivo experimental measurements in ultrasonic range with the axial transmission technique.

2 Simplified model

The properties of the human cortical bone are studied by using in vivo measurements obtained with the axial transmission technique: an acoustic pulse is applied on the skin layer in the ultrasonic range and the velocity of the first arriving signal is measured. A simplified model of the human cortical bone with the skin, the coupling gel with a probe that signal is measured. A simplified model of the human cortex is presented for the identification of these parameters using in vivo experimental measurements in ultrasonic range with the axial transmission technique.

\[ Q_1 = \{x_1 \in \mathbb{R}, \quad x_2 \in \mathbb{R}, x_3 = z_1\} \]
\[ Q_2 = \{x_1 \in \mathbb{R}, \quad x_2 \in \mathbb{R}, x_3 = z_2\} \]

2 Simplified model for a porous medium with gradient

It is well-known that bone medium are made of porous material. However, for the human cortical bones, the pore sizes are not small with respect to the thickness of the cortical layer. In addition, the pore size increases along the transverse direction \( x_3 \). In case of osteoporosis, this gradient of porosity is such that, near interface \( \Sigma_2 \), the cortical material is mostly made up of a fluid. No usual theory on porous medium [1, 2, 3] is suitable for modelling such properties.
Hereafter, we then propose an approach that allows the modelling of the elasticity matrix \([ C(x_3)]\) to be still constructed within the usual framework of the continuum mechanics. For all \(x_3\) in \([a, b]\), the material in the cortical layer is assumed to be locally an homogeneous transverse isotropic medium and for all \(x_3\) in \([c, b]\) it is assumed to be a fluid. Consequently, (1) for all \(x_3\) in \([0, a]\), we have \([ C(x_3)] = [ C^S]\) and \(\rho(x) = \rho^S\); (2) for all \(x_3\) in \([c, b]\) we have \([ C^T]\) and \(\rho(x) = \rho_T\); where \([ C^S]\) is the elasticity matrix of a transverse isotropic medium, \([ C^T]\) is the elasticity matrix of a fluid medium, \(\rho^S\) is the mass density of the cortical layer without taking into account the porosity and \(\rho_T\) is the mass density of the second fluid (the marrow). All components of \([ C^S]\) are zero except the following

\[
[C^S]_{11} = \frac{e^S_2(1 - \nu_T)}{(e_L - e_L \nu_T - 2e_T \nu_T^2)},
\]

\[
[C^S]_{12} = \frac{e_T \nu_T}{(e_L - e_L \nu_T - 2e_T \nu_T^2)},
\]

\[
[C^S]_{22} = \frac{e^S_2(1 + \nu_T)}{(1 + \nu_T)(e_L - e_L \nu_T - 2e_T \nu_T^2)},
\]

\[
[C^S]_{23} = \frac{e_T \nu_T}{(1 + \nu_T)(e_L - e_L \nu_T - 2e_T \nu_T^2)},
\]

\[C^S_{44} = \rho_T, \quad [C^S]_{45} = g_{TL},\]

with \([ C^S]_{12} = [ C^S]_{13} = [ C^S]_{12} = [ C^S]_{11}, [ C^S]_{23} = [ C^S]_{12} \) and \([ C^S]_{45} = [ C^S]_{66}\) and where \(e_L\) and \(e_T\) are the longitudinal and transversal Young moduli, \(g_{TL}\) and \(g_{TT}\) are the longitudinal and transversal shear moduli and \(\nu_T\) and \(\nu_T\) are the longitudinal and transversal Poisson coefficients such that \(\nu_T = e_T / 2(1 + \nu_T)\). All components of \([ C^T]\) are zero except \([ C^T]_{11}, [ C^T]_{12}, [ C^T]_{13}, [ C^T]_{22}, [ C^T]_{23}, [ C^T]_{33}, [ C^T]_{12}, [ C^T]_{13}\) that are all equal to \(\rho_T c_T^2\). The model of \([ C(x_3)]\) and \(\rho(x)\) is the following

\[
[C(x_3)] = (1 - f(x_3))[ C^S] + f(x_3)[ C^T],
\]

\[
\rho(x_3) = (1 - f(x_3)) \rho^S + f(x_3) \rho_T,
\]

where \(f(x_3) = 1\) if \(x_3 < b\), \(f(x_3) = 1\) if \(x_3 > a\) and \(f(x_3) = \sum_{k=0}^4 \alpha_k x_3^k \) if \(b \leq x_3 \leq a\) in which coefficients \(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4\) are such that \(f(a) = 0, f(b) = 1\) and the derivative of \(f\) with respect to \(x_3\) is such that \(f'(a) = f'(b) = 0\).

4 Probabilistic model of the thickness and elasticity matrix of the cortical layer

At the mesoscale modeling, the cortical bone constituting the elastic solid layer is a heterogeneous anisotropic material for which the elasticity properties field is modeled by a matrix-valued random field \([ C] = \{ [ C(x_3)], x_3 \in [b, 0]\}\). The prior probabilistic model of \([ C]\) is chosen in the ensemble of tensor-valued random field adapted to elliptic operator, defined in [10, 11]. This probability model of the uncertain parameters are constructed by using the maximum entropy principle [9, 6, 7]. For all \(b \leq x_3 \leq 0\), \([ C(x_3)]\) is a positive-definite random matrix which is written as

\[\text{Probabilistic model: } [ C(x_3)] = [ L(x_3)] G(x_3) + [ C_0(x_3)],\]

in which the deterministic matrix \([ C_0(x_3)]\) is positive-definite and the matrix \([ G(x_3)]\) is a positive-definite random matrix; these two matrices are defined below. In the definition of \([ C(x_3)]\), an uppercase \(T\) denotes the transpose operator. By construction, one has

\[E([ C(x_3)]) = [ C(x_3)] , \quad \forall x_3 \in [b, 0],\]

in which \([ C] = \{ [ C(x_3)], x_3 \in [b, 0]\}\) is the mean value field defined in the previous section and the operator \(E\) denotes the mathematical expectation. Positive-definite matrix \([ C_0(x_3)]\) must be such that, for all \(x_3 \in [b, 0]\), \([ C(x_3)] - [ C_0(x_3)]\) is positive-definite. The \(n \times n\) matrix \([ L(x_3)]\) corresponds to the Cholesky decomposition of the positive-definite matrix \([ C(x_3)] - [ C_0(x_3)]\), that is to say \([ C(x_3)] - [ C_0(x_3)] = [ L(x_3)]^T [ L(x_3)]\). The matrix-valued random field \([ G] = \{ [ G(x_3)], x_3 \in [b, 0]\}\) is defined as a non-linear mapping of 21 independent second-order centered homogeneous Gaussian random fields \(U_{ jj}(x_3)\), \(x_3 \in [b, 0]\) with \(1 \leq j \leq j' \leq n\) for which the autocorrelation functions \(R_{ U_{ jj}}(\xi) = E[ U_{ jj}(x_3 + \xi, x_3)]\) are all equal chosen to a same unique function \((2 \pi^2 \xi^2 \sin^2(\pi \xi / 2 \ell)\) depending only on a spatial correlation length denoted \(\ell\). The explicit expression of this non-linear mapping can be found in [10, 11]. Let parameter \(\delta\) be the dispersion coefficient defined as \(\delta^2 = (E[ || [ G(x_3)] ||^2 ] - 1) / n\). The probability density function \(P_{ G_{(x_3)}}\) of random matrix \([ G(x_3)]\) with respect to the measure \(dA = 2^{(n-1)/4} \prod_{1 \leq j \leq n, [A]_{jj}}\) on the set \(\mathbb{M}^n\) of the symmetric positive \(n \times n\) real matrices is then written as

\[p_{ G_{(x_3)}}(A) = \frac{\delta}{\mathcal{M}^n}(|| A ||) \exp[-n \operatorname{tr}(A)] ,\]

in which \(\alpha_n = (n + 1) / (2e^S), b_n = a_n(1 - \delta)\) where \(\delta\) is a dispersion coefficient and \(\mathcal{M}^n|| A ||\) is equal to 1 if \(A\) belongs to \(\mathbb{M}^n\) and is equal to zero if \(A\) does not belong to \(\mathbb{M}^n\). It can be seen that \(P_{ G_{(x_3)}}\) does not correspond to the probability density function of a Gaussian random matrix. In addition, the probability density functions \(p_{ C_0(x_3)}\) of random matrix \([ C_0(x_3)]\) with respect to the measure \(dA\) on the set \(\mathbb{M}^n\) is written as

\[p_{ C_0(x_3)}(A) = \mathcal{N}_{ || A - [ C_0(x_3)] ||^2, \lambda} \exp[-n \frac{1}{2} \operatorname{tr}(A)] ,\]

where \(\lambda\) is the trace operator; \(\mathcal{N}\) is a normalizing constant; \(\lambda\) is a positive real parameters that depends on the statistical fluctuation of random matrices \([ C(x_3)]\). It can be seen that \(p_{ C_0(x_3)}\) does not correspond to the probability density function of a Gaussian random matrix.

For the application to the cortical bone, we do not have any information concerning matrix \([ C_0(x_3)]\) which is only introduced to preserve the ellipticity property of the stiffness operator. This matrix can be chosen, for \(x_3\) in \([b, 0]\), as \([ C_0(x_3)] = \eta_0 [ C_0(x_3)]\) in which \(0 < \eta_0 < 1\). In this case, \(\eta_0\) can be chosen very small if no information concerning \([ C_0(x_3)]\) is available.

With such a stochastic modeling, the displacement field of the elastic solid layer and the two acoustic pressure fields of the acoustic fluid layers are random fields denoted by \(U\), \(P_1\) and \(P_2\).

5 Application

In a previous paper [5], the components of matrix \([ C^S]\) has been identified with an experimental database using mea-
Figure 2: graph of $a \mapsto F(a, t)$ is shown for different values of $t$. Vertical axis: $F(a, t)$. Horizontal axis: $a$


