Bibliographical review on the Poiseuille-Rayleigh-Bénard flows: The mixed convection flows in horizontal rectangular ducts heated from below

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Table of contents

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§1 - The Poiseuille-Rayleigh-Bénard (PRB) flows are laminar mixed convection flows in horizontal rectangular ducts uniformly heated from below and uniformly cooled from above. The three most common configurations in PRB flows are the Poiseuille flow and two thermoconvective flow configurations : the transversal rolls ($R_{\perp}$) and the longitudinal rolls ($R_{//}$). These flow patterns are presented on *figures 1, 2 and 3* respectively, with their parameter range. This paper aims at making a comprehensive review of the literature on the PRB flows and a synthesis of the qualitative and quantitative data which describe them. Notably, we complete Kelly's reviews [1, 2] by analysing more than ninety new references on this subject, among which fifty are later than 1993. The temporal linear stability analyses of the Poiseuille flow vis-à-vis the $R_{\perp}$ and the $R_{//}$ give the diagrams presented on *figures 4 and 5* for infinite and finite spanwise aspect ratio ducts, respectively. The results of these diagrams have been known since early eighties (cf. §4). They are qualitative results which only give a simplified idea of the stability of the PRB flows. The spatio-temporal linear stability analyses (dating from early nineties), which imply the concepts of convective and absolute instabilities, are more appropriate to describe the stability of these flows (cf. §5). Furthermore, recent experimental, theoretical and numerical studies, carried out in the non-linear domain, have shown the existence of three-dimensional unsteady thermoconvective structures whose shape is more complex than the $R_{\perp}$ and the $R_{//}$. A review of these complex structures and on their stability is proposed in §8. The industrial applications of the PRB flows are presented in §2 and the pioneer experiments are presented in §3. A review on the heat transfers and on the characteristics growth length of the $R_{\perp}$ and of the $R_{//}$ is proposed in §6, and a synthesis of the quantitative data on the spatial and temporal characteristics of the $R_{\perp}$ and of the $R_{//}$ is given in §7.

§2 - The two main industrial applications of the PRB flows concern, first, the development of the horizontal rectangular chemical vapor deposition (CVD) reactors which are used to produce thin and uniform films of inorganic matter (cf. §2.1) and, second, the cooling of the electronic equipments (cf. §2.2). A general presentation of the CVD process is done in §2.1.1. In §2.1.2, it is shown that the presence of $R_{\perp}$ flows, of $R_{//}$ flows or of a steady transversal roll located just above the entrance of the heated zone (cf. *figure 6* and *table 1* where this roll is noted ''$R_{\perp}$ fixe'') must be eliminated of the CVD reactors because these thermoconvective structures cause non-uniform deposits. Recent studies [15, 18-20] show that the aperiodic or chaotic flows observed at high Rayleigh and low Reynolds numbers should be used to get uniform deposits in CVD reactors. A synthesis of the CVD numerical studies for configurations close to the PRB flows is presented in §2.1.3 and in *table I*. The studies on the cooling of electronic equipments, close to the PRB flow configuration, are summed up in §2.2.

§3 - In this paragraph, we present a review of the pioneer experiments which took place from 1920 (with the first studies of Idrac [35, 36]) to 1938. They are mainly concerned with applications to meteorology. Their results were synthesized by Sir D. Brunt in 1951 [3].

§4 - From 1962 to 1984, the PRB flow studies were mainly based on linear stability analyses of the Poiseuille flow, completed by experiments to verify the results of the theoretical models. Some of these models (based on temporal linear stability analyses) and the associated experiments concentrated on the fully-established thermoconvective flows (cf. §4.2), while the others (based on spatial linear stability analyses) and the associated experiments concentrated on the thermal entrance zone (cf. §4.3). The whole theoretical analyses are summarized in *table II*, in which the parameter range, the form of the normal modes, the criteria of neutral stability and the type of the thermoconvective structures appearing above the neutral stability threshold are mentioned. In §4.2.1, the studies dealing with the stability of the Poiseuille flow vis-à-vis the first type and second type $R_{//}$ (cf. *figure 7*) are reviewed. In §4.2.2, a long discussion about the experimental verification of the increase with Re of the critical Rayleigh number for the $R_{\perp}$ ($Ra_{\perp}$*) is reported. Three analytical formulae, extracted from [67, 68], giving $Ra_{\perp}$* and the critical time pulsation and wave number for the $R_{\perp}$ as a function of Re and Pr, are also reported (cf. equations (1-3)). In §4.2.3, we briefly comment on the results of the 3D temporal linear stability analysis in finite lateral extension ducts [52, 53, 54]. Some of these results are presented in *figures 5, 8-10*. The mathematical formulation of the temporal linear stability analysis is given in annexe A. In §4.2.4, the weaknesses of this temporal linear stability analysis are brought out : first, it predicts either an odd or an even number of $R_{//}$ to appear depending on the aspect ratio B (cf. *figure 8*), whereas all the experiments and numerical simulations predict an even number of $R_{//}$ for fully-established flows at Re>0 and for B>1.1 (cf. [76] for the explanation) ; second, it rightly predicts the critical Rayleigh number $Ra_{//}$* for the appearance of the $R_{//}$, but fails to predict the critical Rayleigh number for the appearance of the $R_{\perp}$ and the critical Reynolds number $Re^*_{\perp-//}$ determined experimentally and numerically (cf. *figure 10* and *table III*). To remedy this, the space and time growth of the perturbations has to be analysed, i.e. a convective/absolute stability analysis is necessary (cf. §5). In §4.3, three studies on the PRB flow stability in the entrance zone of infinite lateral extension ducts are analysed (cf. *table II*). The used stability criteria are schematically represented in *figure 12*. *Figure 11* shows that the $R_{\perp}$ are more unstable than the $R_{//}$ just at the entrance of the heated zone for small Reynolds numbers, which is widely verified by several recent papers. As shown in *figure 13*, the stability criterion used in [87] permits to find the experimental results again with a good agreement.

§5 - In §5.1 and §5.2, we remind the reader of the general notions about the convective and absolute intabilities (cf. *figure 14* in the case of 2D PRB flows) and about the amplitude and Ginzburg-Landau equations. The criteria of stability and of absolute and convective instability for the Ginzburg-Landau equation (5) in its linearised version are given by equations (6-8) respectively. In §5.3, the two main contributions to the analyse of the convective and absolute instabilities in PRB flows are reviewed. Müller et al. [67, 68, 78, 91] analyse the transitions between Poiseuille flow and $R_\perp$ in 2D PRB flows by 2D numerical simulations and with the Ginzburg-Landau equation (9) (cf. *table IV* for the coefficient values of this equation and cf. *figure 10* for the convective/absolute transition curve $Ra_\perp^{conv}$ computed from this theory). Carrière and Monkewitz [69] analyse the convective/absolute instability of the 3D PRB flows between two infinite horizontal walls by directly computing the Green function and by analysing its long-time behaviour. The original result of [69], which contradicts the ones obtained from the models based on two amplitude equations [103-106], is that the $R_{//}$ are never absolutely unstable when Re>0. Furthermore, when $Ra<Ra_\perp^{conv}$, the most amplified mode, located at the center of the wave packet, corresponds to the appearence of $R_{//}$. It is shown that complementary theoretical or experimental studies would be necessary to understand the convective/absolute transitions for the $R_\perp$ and the $R_{//}$ in the case of finite cross section ducts. Two possible scenarios are proposed to try to explain why in the experiments and in the numerical simulations the $R_{//}$ rise not far from the inlet and for Ra close to $Ra_{//}$*, if we consider that the $R_{//}$ are convectively unstable like in the case of flows between two infinite plates. In §5.4 and *figure 15,* the results of Müller et al. [67, 68, 91] on the influence of a white noise on the convectively unstable $R_\perp$ flows are

presented and the references of studies comparing the influence of local and global noise on convectively unstable PRB flows are given [108-109].

§6 - In §6.1, the studies on the axial variation of the time and spanwise average Nusselt number in the thermal entrance zone of PRB flows are reviewed. This variation is shown for $R_{//}$ flows on *figure 16*, where three zones are revealed. Equations (11-12), equation (13) and equation (16) are correlation laws respectively for the coordinates x_1 of the beginning of zone (2), x_2 of Nu^{max} in zone (2) and x_3 at which Nu exceeds the forced convection value of zone (1) by 3%. Other correlation laws (equations (14-15)) for the appearance length and the establishment length of the $R_{//}$, determined from LDA experiments, are reported in §6.3.1. The results and the variation laws for the space and time average Nusselt number in fully-established PRB flows are reviewed in §6.2 and *table V*. This Nusselt number is shown to keep constant whatever Re in the case of $R_{//}$ flows, but it is shown to decrease when Re increases in the case of $R_{\perp}$ flows. Most of the heat transfer studies are carried out in ducts of very large or infinite (2D) cross section. Very few data on the spanwise confinement influence on heat transfers are available. §6.3.2 is dedicated to the determination of the characteristic growth length l_e of the $R_{\perp}$. As shown in *figure 17* and in the nomenclature, l_e is determined from the stationary envelope Wmax(x). The curve $L_e=f(V_g)$ drawn on *figure 18*, representing the variation of the reduced growth length L_e as a function of the reduced group velocity V_g, is shown to be a universal curve in the sense that it is independent of the Prandtl number, as long as Re and Ra are not too large [68, 78]. The divergence of the curve $L_e=f(V_g)$ at $V_g=2$ corresponds to the transition between the absolutely unstable $R_{\perp}$ ($V_g<2$) and the convectively unstable $R_{\perp}$ ($V_g>2$). The values of Ra and Re at which l_e diverge are shown to be a very precise criterion to define the transition curve $Ra_{\perp}^{conv}(Re)$ (compare *figures 10 and 17*).

§7 - In this section, the fine spatial and temporal structure of the $R_{\perp}$ is analysed. In §7.1, the influence on this structure of the open boundary conditions (OBC) that must be imposed at the outlet boundary of the computational domains for the numerical simulations is reviewed. Among all the tested OBC, the Orlanski type boundary conditions (equation (17)) is shown to be the one which perturbs the least the $R_{\perp}$ flows. The influence of the inlet boundary conditions (IBC) on the $R_{\perp}$ development in the numerical simulations and in the experiments is reviewed in §7.2. The IBC influence on the $R_{\perp}$ is shown to be more important than the one of the OBC. The growth length l_e, the wave length λ and the frequency f of the $R_{\perp}$ vary depending on whether the IBC is of Dirichlet or of Neumann type [143] (cf. *figure 21*). At very small but non-vanishing flows, the $R_{\perp}$ are shown to be pinned by the IBC (cf. *figure 20*). A new critical Reynolds number Re** indicating the transition between the stationary and the moving $R_{\perp}$ is introduced. An analytical expression for Re** is proposed by [68, 91] (cf. equation (19)). It is compared to 2D numerical results in *table VI*. While the origin of the $R_{\perp}$ pinning with the Neumann inlet conditions is clearly identified, it is not explained with the Dirichlet inlet conditions. For fully-established $R_{\perp}$ flows, the variations of f, λ and Vr/U° (the ratio of the $R_{\perp}$ velocity to the average velocity of the channel flow) as a function of the flow parameters are reviewed in §7.3, 7.4 and 7.5 respectively. While the variations of f, λ and Vr/U° as a function of Ra, Re and Pr are quite well known in the case of 2D problems ($B\rightarrow\infty$), in the case of finite cross section ducts, the variations of these quantity as a function of Pr and B are very badly known because of a lack of data (cf. *figures 20-23* and *tables VII and VIII*). The linear correlation laws giving Vr/U° as a function of Ra are summarized in *table VIII* and in *figures 22 and 23* for the infinite (2D) and finite cross section ducts, respectively. Note that these laws are independent of Re, except in one case [128].

§8 - The aim of this section is to synthetize the data obtained these last ten years regarding the description and the stability of the complex thermoconvective patterns of the PRB flows. In §8.1, the two most precise and complete experimental stability diagrams of the PRB flows (cf. *figures 24 and 26*) are presented (§8.1.1) and analysed (§8.1.2). They are compared with each other and with other results of literature. In §8.1.2, we particularly concentrate on patterns with splitting and merging of $R_{//}$, with hysteresis effects and with superposition of $R_{\perp}$ and $R_{//}$ (cf. *figures 27 and 28* for this third case). In §8.1.1, the seventeen very recent experimental and numerical papers of T. F. Lin et al. about PRB flows are summarized. In §8.2, the stability diagram and two flow patterns obtained from a model based on two coupled Ginzburg-Landau equations (equations (21)) by Müller et al. [104] are analysed (cf. *figures 29 and 30*), despite the criticism of Carrière and Monkewitz [69] about the validity of this model. Kato and Fujimura's (2000) [54] weakly non-linear stability analysis, based on the equations (22), is also reported. It analyses the stability of the three thermoconvective patterns that are presented on *figure 31* (the $R_{//}$, the 3D horse-shoe shapped $R_{\perp}$ and the mixed modes) in the neighborhood of the critical point (Ra, Re)=($Ra_{//}$*, Re*). In §8.3.1, the stability diagram of the fully-established stationary $R_{//}$ vis-à-vis the secondary wavy instability of *figure 33* (cf. *figure 32*), obtained by Clever and Busse [150] from a temporal linear stability analysis, is compared to the other experimental and numerical results of literature. It is shown that the in phase wavy $R_{//}$ of *figure 33* have been observed in horizontal ducts only three times [46, 48, 81] and also in sligthly inclined ducts [126]. On the other hand, $R_{//}$ waving in opposition of phase (cf. *figure 34*) have been observed in numerous experimental and numerical studies, particularly in small spanwise aspect ratio ducts (B≤6) [18, 75, 81, 129]. Two stability diagrams of these periodic wavy rolls are presented in *figure 35*. Complementary studies should be carried out to explain theoretically the presence of these $R_{//}$ oscillating in opposition of phase in the experiments and to confirm experimentally or numerically the existence of the $R_{//}$ oscillating in phase. Other complex flow patterns (inclined or V-shaped $R_{//}$), only observed by Yu et al. [73], are reproduced on *figure 36*. In §8.3.2, the successive transitions in PRB flows, from the symmetric and periodic $R_{//}$ flows to the asymmetric and periodic $R_{//}$ flows (cf. *figure 34(d)*), then to the aperiodic flows and to the chaotic flows, are reviewed. It is shown that numerical simulations with non-Boussinesq codes will have to be envisaged in order to better agree with the experiments at high Rayleigh numbers.

§9 - As a conclusion, the badly known or badly understood aspects of the PRB flows are summed up and subjects for future investigations are proposed.

Annexe A – This annexe presents the mathematical formulation of the temporal linear stability analyses of the purely conductive Poiseuille flow for PRB flows between two infinite plates (§A.1) and in finite lateral extension channels (§A.2).

Nomenclature

Latin letters

A	longitudinal aspect ratio of the channel, L/h
A(x, t)	temporal and spatial modulation of the amplitude of the perturbations whose variations are given by the amplitude equation theory
B	transversal aspect ratio of the channel, l/h
c_0, c_1, c_2	coefficients of the Ginzburg-Landau equation
c_x, c_y	streamwise and spanwise spatial amplification coefficients of the perturbations (c_x, $c_y \in$ IR)
D_h	hydraulic diameter, (m)
f	frequency of the $R_\perp$, (s^{-1})
f	scalar or vectorial variable replacing a set of variables verifying the same mathematical relation
g	gravitational acceleration, ($m.s^{-2}$)
h	channel height, (m)
i	horizontal unit vector pointing to the average flow direction
j	horizontal unit vector such as **j**=**k**∧**i**
k	upward vertical unit vector
k_x, k_y	streamwise and spanwise dimensionless wave numbers of the perturbations (k_x, $k_y \in$ IR)
l	channel width, (m)
l_e	dimensionless characteristic growth length of the $R_\perp$ defined by $Wmax(l_e)=W_s/2$
L	channel length, (m)
L_e	reduced characteristic growth length of the $R_\perp$, $\sqrt{\mu}\, l_e / \xi_0$

p	thermodynamic pressure
P	p+ρgz
q	heat flux, (W.m^{-2})
$R_{//}$	longitudinal rolls
$R_{\perp}$	transversal rolls
$R_{\perp-//}$	superposition of the $R_{\perp}$ and of the $R_{//}$
r_T	reduced temperature, $(T_c-T_f)/T_f$
t	dimensionless time
T	dimensionless fluid temperature
T_c, T_f	temperatures of the bottom hot wall and of the top cold wall, respectively, (K)
ΔT	temperature difference between the hot and cold walls, T_c-T_f, (K)
U, V, W	streamwise (x), spanwise (y) and vertical (z) velocity component
U°	average velocity of the flow, (m.s^{-1})
V	dimensional (m.s^{-1}) or dimensionless velocity vector, (U, V, W)
v_g	group velocity in the Ginzburg-Landau equation ; $v_g=(\partial\sigma_i/\partial k_x)^*$
V_g	reduced group velocity, $v_g\tau_0/(\mu\xi_0^2(1+c_1^2))^{1/2}$
Vr	velocity of the $R_{\perp}$
Wmax	stationary envelope of the maximum vertical velocity component along the channel axis (function of x), (m.s^{-1})
W_s	saturation amplitude of W : value of Wmax(x) for $x\in[x_e, x_s]$, (m.s^{-1})
x_e, x_s	axial coordinates indicating the end of the entrance zone and the beginning of the zone perturbed by the outlet boundary conditions, respectively
x, y, z	axial, spanwise and vertical coordinates (dimensionless in general)

Dimensionless parameters

Nu	according to the context, local or space and/or time average Nusselt number computed on the horizontal walls of the channel
Gr	Grashof number, $Ra/Pr=g\beta(T_c-T_f)h^3/\nu^2$
Pe	Peclet number, $RePr=U°h/\alpha$
Pr	Prandtl number, ν/α
Ra	Rayleigh number, $g\beta(T_c-T_f)h^3/(\nu\alpha)$
Ra'	modified Rayleigh number based on the heat flux q through the heated wall, $g\beta qh^4/(\kappa\nu\alpha)$
Ra*	$\min(Ra_{\perp}^*, Ra_{//}^*)$
Ra^*_0	critical Rayleigh number at Re=0 ; $Ra^*_0=1708$ when $A\rightarrow\infty$ and $B\rightarrow\infty$
$Ra_{\perp}^*$	critical Rayleigh number determined from the temporal linear stability analysis of the Poiseuille flow vis-à-vis the convectively unstable $R_{\perp}$ ($Ra_{\perp}^*$ increases with Re)
$Ra_{//}^*$	like $Ra_{\perp}^*$ but for the $R_{//}$ ($Ra_{//}^*$ is independent of Re)
$Ra_{//}^{*II}$	critical Rayleigh number determined from the temporal linear stability analysis of the stationary $R_{//}$ flows vis-à-vis the periodic $R_{//}$ flows
$Ra_{\perp}^{conv}$	critical Rayleigh number determined from the spatio-temporal linear stability analysis between the convectively and absolutely unstable zones for the $R_{\perp}$
$Ra_{//}^{conv}$	like $Ra_{\perp}^{conv}$ but for the $R_{//}$
Re	Reynolds number, $U°h/\nu$
Re'	Reynolds number, $U^{max}h/\nu$

Re*	critical Reynolds number, determined from the temporal linear stability analysis, corresponding to the transition between Poiseuille flow, $R_{\perp}$ and $R_{//}$ (for $Ra=Ra_{\perp}*=Ra_{//}*$)
$Re*_{\perp-//}$	critical Reynolds number, determined experimentaly or by numerical simulations, corresponding to the transition between the $R_{\perp}$ and the $R_{//}$ in the non-linear domain (for $Ra>Ra_{\perp}*$ and $Ra>Ra_{//}*$)
Re**	critical Reynolds number corresponding to the threshold of the $R_{\perp}$ pinning observed for very small flows : it indicates the transition between the stationary $R_{\perp}$ and the moving $R_{\perp}$

Greek letters

α	thermal diffusivity, ($m^2.s^{-1}$)
β	thermal expansion coefficient, (K^{-1})
γ	coefficient of the Ginzburg-Landau equation
ε	=(Ra-1708)/1708, relative distance to the critical Rayleigh number $Ra*_0=1708$
$\varepsilon_{\perp}*$	$=(Ra_{\perp}*-1708)/1708$
$\varepsilon_{//}*$	$=(Ra_{//}*-1708)/1708$
$\varepsilon_{\perp}^{conv}$	$=(Ra_{\perp}^{conv}-1708)/1708$
$\varepsilon_{//}^{conv}$	$=(Ra_{//}^{conv}-1708)/1708$
κ	thermal conductivity, ($W.m^{-1}.K^{-1}$)
λ	dimensionless wave length of the $R_{\perp}$ (h is the reference length)
μ	$=(Ra-Ra_{\perp}*)/Ra_{\perp}*$, relative distance to the critical Rayleigh number $Ra_{\perp}*(Re)$
ν	kinematic viscosity, ($m^2.s^{-1}$)
ξ_0^2	coefficient of the Ginzburg-Landau equation

ρ mass per unit volume, ($kg.m^{-3}$)

σ dimensionless complex time amplification coefficient of the perturbations, $\sigma_r+i\sigma_i$

σ_i dimensionless time pulsation of the perturbations

τ_0 coefficient of the Ginzburg-Landau equation

ψ stream function, given by $U=-\partial\psi/\partial z$ et $W=\partial\psi/\partial x$

Ω vorticity, $\partial U/\partial z-\partial W/\partial x=-\nabla^2\psi$

Abbreviations and symbols

CVD Chemical Vapor Deposition

IBC Inlet Boundary Condition

LDA Laser Doppler Anemometry (Anémométrie Laser Doppler)

OBC Open or Outlet Boundary Condition

PBC Periodic Boundary Condition

PRB Poiseuille-Rayleigh-Bénard

∇ nabla operator, vector ($\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$) in cartesian coordinates

Superscripts

° reference state or average value

conv indicates parameters or variables at the transition between the convectively and absolutely unstable zones

max, min indicate a maximum and a minimum, respectively

* indicates parameters or variables at the transition between the linearly stable and convectively unstable zones. In a general way, indicates a critical value

— basic state for the linear stability study

’ infinitesimal perturbation

^ perturbation amplitudes when they are written in the normal mode form

Subscripts

i, r imaginary part and real part of a complex number

$\perp$, $//$ indicate variables corresponding to the $R_{\perp}$ and to the $R_{//}$, respectively