An implicit finite volume scheme for a scalar hyperbolic problem with measure data related to piecewise deterministic Markov processes

Robert Eymard, Sophie Mercier, Alain Prignet

An implicit finite volume scheme for a scalar hyperbolic problem with measure data related to piecewise deterministic Markov processes

Robert Eymard, Sophie Mercier, Alain Prignet

Université Paris-Est, Laboratoire d'Analyse et de Mathématiques Appliquées, UMR CNRS 8050, 5 bd Descartes, 77454 Marne la Vallée Cedex 2, France

We are interested here in the numerical approximation of a family of probability measures, solution of the Chapman–Kolmogorov equation associated to some non-diffusion Markov process with uncountable state space. Such an equation contains a transport term and another term, which implies redistribution of the probability mass on the whole space. An implicit finite volume scheme is proposed, which is intermediate between an upstream weighting scheme and a modified Lax–Friedrichs one. Due to the seemingly unusual probability framework, a new weak bounded variation inequality had to be developed, in order to prove the convergence of the discretised transport term. Such an inequality may be used in other contexts, such as for the study of finite volume approximations of scalar linear or nonlinear hyperbolic equations with initial data in L^1. Also, due to the redistribution term, the tightness of the family of approximate probability measures had to be proven. Numerical examples are provided, showing the efficiency of the implicit finite volume scheme and its potentiality to be helpful in an industrial reliability context.

Keywords: Linear hyperbolic problems with measure solutions; Weak bounded variation inequalities; Chapman–Kolmogorov equations; Piecewise-deterministic Markov process; Growth–collapse Markov process

1. Introduction

Within a more and more competitive context, industrialists often have to assess as accurately as possible different quantities linked for example to economical and/or safety constraints. For instance, an industrial gas provider must be aware of the production availability of his gas plant, because of possible penalties to be paid in case of drop in its production rate. Lots of other quantities may of course be of interest to him, such as the mean functioning cost per unit time, the mean number of component failures up to some fixed horizon, and so on. All those quantities may be written as expectations of some functional of the underlying stochastic process which describes the time evolution of the studied system, or equivalently, as integrals with respect to the marginal distribution of the process. From an industrial point of view, such distributions then are essential to evaluate. Unfortunately, they often are unreachable in closed form, especially in the modern context of dynamic reliability, which is concerned with the study of so-called hybrid systems (details below). Such distributions (or their derivatives) are hence numerically evaluated, most of the time by Monte Carlo simulations which often entail very long computation times. New methods for their numerical assessment then are to be developed, from where most of the quantities with industrial interest may be derived, in the context of dynamic reliability.

The point of this paper hence is to develop some numerical scheme, to assess the marginal distribution of some process describing the time evolution of a hybrid system. Such a hybrid system is governed by two different types of dynamics: a discrete dynamic, which is related to the occurrence of events such as failures of components, some button switching, and so on, and a continuous dynamic, linked to the evolution of continuous characteristics, such as pressure, temperature, liquid level in a tank, and so on. The time evolution of such hybrid systems is modelled with so-called Piecewise-Deterministic Markov Processes (PDMPs), which are non-diffusion Markov processes with uncountable state space; see [10] or [17] for details. A PDMP hence is a Markov hybrid process $(I_t, X_t)_{t\geq 0}$, where the discrete part I_t takes range in a finite set E, and where the continuous part X_t takes range in $\mathbb{R}^N$, with $N \in \mathbb{N}^*$. Due to its Markovian characteristic, one may write its associated Chapman–Kolmogorov equation. A PDMP jumps at countable isolated times and such an equation comprises a transport term, which corresponds to a deterministic evolution of the process between jumps, and a redistribution term, which corresponds to jumps and entails redistribution of the probability mass on the whole space. In order to make the paper clearer, we now make the choice to specialise our exposure to so-called Markov Growth–Collapse processes (GCPs); see [5]. Such processes are PDMPs where E is reduced to a singleton, so that the discrete part $(I_t)_{t\geq 0}$ is constant and hence unnecessary. They typically describe the time evolution of a quantity, for instance the size of a queue, with successive phases of deterministic growth and random instantaneous collapse (or jump in the vocabulary of PDMPs). Their behaviour hence is typical of a PDMP and such a specialisation allows us to be much clearer in our exposure. Extension of our results for GCPs to PDMPs is straightforward and is exposed at the end of the paper, in Section 6.

Let us now be more specific and let $(X_t)_{t\geq 0}$ be a GCP: as already said, the process $(X_t)_{t\geq 0}$ jumps at isolated countable times. Between jumps, the deterministic evolution of $(X_t)_{t\geq 0}$ follows an ordinary differential equation:

$$\frac{\mathrm{d}X_t}{\mathrm{d}t} = \mathbf{v}(X_t), \quad \forall t \in [t_1, t_2) \tag{1}$$

where $\mathbf{v}$ is an application from $\mathbb{R}^N$ to $\mathbb{R}^N$, and t_1, t_2 are two successive jump times. At jump times, transitions from $X_{t^-} = x \in \mathbb{R}^N$ to $X_t = y \in \mathbb{R}^N$ are governed by a transition rate $\lambda(x)$ and by a probability measure $\mu(x)(\mathrm{d}y)$ which stands for the conditional distribution of X_t given that $X_{t^-} = x$.

We make the following assumptions on the data, which will be referred to as assumptions $\mathcal{H}_0$ in the following:

- the transition rate $\lambda : \mathbb{R}^N \to \mathbb{R}_+$ is continuous and bounded by $\Lambda = \|\lambda\|_\infty$,
- the velocity $\mathbf{v} : \mathbb{R}^N \to \mathbb{R}^N$ is Lipschitz continuous and bounded by $V = \|\mathbf{v}\|_\infty > 0$,
- let $\mathcal{P}(\mathbb{R}^N)$ be the set of probability measures on $\mathbb{R}^N$; the function $\mu : \mathbb{R}^N \to \mathcal{P}(\mathbb{R}^N)$ is such that for all $\psi \in C_b(\mathbb{R}^N)$ (continuous and bounded from $\mathbb{R}^N$ to $\mathbb{R}$), the function $x \mapsto \int_{\mathbb{R}^N} \psi(y)\mu(x)(\mathrm{d}y)$ is continuous (and bounded) from $\mathbb{R}^N$ to $\mathbb{R}$,
- the measure $\rho_{\mathrm{ini}}(\mathrm{d}x)$ is a probability measure on $\mathbb{R}^N$, and stands for the initial distribution of $(X_t)_{t\geq 0}$.

We may now write the Chapman–Kolmogorov (CK) equation associated to the Markov process $(X_t)_{t\geq 0}$ and we set $\rho(t)(\mathrm{d}x)$ to be the marginal distribution at time t of the process $(X_t)_{t\geq 0}$. Under $\mathcal{H}_0$, it is then known that $(\rho(t)(\mathrm{d}x))_{t\geq 0}$ is the single family of probability measures solution of the CK equation, which here is written as

$$\begin{aligned}
&\int_{\mathbb{R}^N} \varphi(x)\rho(t)(\mathrm{d}x) - \int_{\mathbb{R}^N} \varphi(x)\rho_{\mathrm{ini}}(\mathrm{d}x) - \int_0^t \int_{\mathbb{R}^N} (\mathbf{v}(x) \cdot \nabla\varphi(x))\rho(s)(\mathrm{d}x)\,\mathrm{d}s \\
&\quad = \int_0^t \int_{\mathbb{R}^N} \lambda(x) \left(\int_{\mathbb{R}^N} \varphi(y)\mu(x)(\mathrm{d}y) - \varphi(x) \right) \rho(t)(\mathrm{d}x)\,\mathrm{d}s \quad \forall t \in \mathbb{R}_+, \forall \varphi \in C_c^1(\mathbb{R}^N)
\end{aligned} \tag{2}$$

where $C_c^1(\mathbb{R}^N)$ stands for the set of continuously differentiable functions from $\mathbb{R}^N$ to $\mathbb{R}$ with a compact support; see [10] or [17] for the CK equation associated to a PDMP, and [7] for the uniqueness result. The third term in the left side of (2) is the transport term whereas the right side is the redistribution term.

This paper is dedicated to the study of the convergence of an implicit finite volume scheme towards the solution of the CK equation (2) and we now discuss the mathematical nature of some of the arising problems. To make it clearer and only for the purpose of this introduction, we consider the case where the data measures involved in the CK equation admit density with respect to Lebesgue measure on $\mathbb{R}^N$: we hence assume that $\mu(x)(\mathrm{d}y) = d_\mu(x, y)\, \mathrm{d}y$ and $\rho_{\mathrm{ini}}(\mathrm{d}x) = u_{\mathrm{ini}}(x)\ \mathrm{d}x$, where $u_{\mathrm{ini}} \in L^1(\mathbb{R}^N)$ because ρ_{ini} is a probability measure. It may then be proved that the solution $\rho(t)(\mathrm{d}x)$ of the CK equation admits a density and we set $\rho(t)(\mathrm{d}x) = u(x, t)\ \mathrm{d}x$. Writing again the CK equation, the function u then happens to be, at least formally, a weak solution of the equation

$$\frac{\mathrm{d}u}{\mathrm{d}t}(x,t) + \mathrm{div}(u(x,t)\mathbf{v}(x)) = \int_{\mathbb{R}^N} \lambda(y) d_\mu(y,x) u(y,t)\mathrm{d}y - \lambda(x)u(x,t) \tag{3}$$

for $(x, t) \in \mathbb{R}_+ \times \mathbb{R}^N$ with the initial condition

$$u(x, 0) = u_{\mathrm{ini}}(x), \quad \text{for } x \in \mathbb{R}^N. \tag{4}$$

Problem (3)–(4) can be seen as a linear hyperbolic problem with an integral form right-hand side, and an initial condition in $L^1(\mathbb{R}^N)$ instead of the standard framework $u_{\mathrm{ini}} \in L^\infty(\mathbb{R}^N)$. Only for the purpose of this introduction again, let us set $u_{h,k}$ to be an approximate solution, where h is a space step and k is a time step. Looking at the convergence study (see (62)–(63) in the proof of Lemma 9), one may see that the following condition is used for proving the convergence of $u_{h,k}$:

$$\forall R, T > 0, \quad \lim_{h\to 0} h \int_0^T |u_{h,k}(\cdot, t)|_{BV(\overline{B}(0,R))}\mathrm{d}t = 0, \tag{5}$$

where, for a function $v : \mathbb{R}^N \to \mathbb{R}$ and for any set $\Omega \subset \mathbb{R}^N$, we denote by

$$|v|_{BV(\Omega)} = \sup\left\{\left|\int_{\mathbb{R}^N} v(x)\mathrm{div}\varphi(x)\mathrm{d}x\right|, \varphi \in C^1(\mathbb{R}^N)_N, \|\varphi\|_\infty \le 1, \varphi(x) = 0 \text{ for all } x \in \mathbb{R}^N \setminus \Omega\right\} \tag{6}$$

the BV-semi-norm of v.

In order to prove (5), if u_{ini} were in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$, one might proceed as is done in [6] or in [14] for explicit or implicit finite volumes schemes on general meshes and $u_{\mathrm{ini}} \in L^\infty(\mathbb{R}^N)$, and prove some weak bounded variation (BV) inequality of the following shape:

$$\int_0^T |u_{h,k}(\cdot, t)|_{BV(\overline{B}(0,R))}\mathrm{d}t \le \frac{C\|u_{\mathrm{ini}}\|_{L^2(B(0,R'))}}{\sqrt{h}},$$

where C and R' depend on R and T. Unfortunately, such an inequality is no longer valid here for the present scheme and $u_{\mathrm{ini}} \in L^1(\mathbb{R}^N)$, which is imposed by our probabilistic context. We have hence been led to develop a new weak BV inequality:

$$\int_0^T |u_{h,k}(\cdot, t)|_{BV(\overline{B}(0,R))}\mathrm{d}t \le \frac{C}{h^{1/q}},$$

for some $1 < q < 2$ (see Lemma 4). This inequality, which is sufficient to get (5), is shown using the analogy between an upstream weighting scheme or a modified Lax–Friedrichs scheme, and a continuous problem including a vanishing viscosity term. We can hence use the tools developed in [11] or [16] for the convergence of finite volume approximations to the solution of elliptic problems with measure data, which themselves mimic the continuous framework provided in [4,3] for continuous parabolic problems. Nevertheless, the proofs given in such papers do not hold in the case of the classical upstream weighting scheme, since a non-vanishing viscosity term was required on the whole mesh, which has led us to introduce an intermediate scheme between an upstream weighting one and a modified Lax–Friedrichs scheme.

Also, in order to complete the convergence proof, a last technical problem is due to the redistribution term in the CK equation. Due to that term, we indeed have to control the probability mass which escapes outside compact sets for the family of approximate distributions. In other words, we have to prove tightness of this family. With that aim, we explicitly construct a Liapounov function, which classically allows us to conclude (see [12] e.g.).

Finally, note that we had already studied the convergence of an explicit finite volume scheme towards the solution of the CK equation (2) in a previous paper [8]. The convergence proof was established there only under an inverse CFL condition $h/k \to 0$, this condition being used in order to replace a weak bounded variation inequality which we did not have. The tools developed in the present paper would however not hold in the framework of [8], where the scheme is a purely upstream weighting one.

This paper is organised as follows. The numerical scheme is provided in Section 2 (in the case when E is a singleton). Some properties of the scheme are developed in Section 3, which are required to get some weak BV inequalities given in Section 4. Such weak BV inequalities are used for proving the convergence of the scheme, which is done in Section 5. We then present in Section 6 extensions of our results to the case of a general E, with possible jumps between discrete states of E. We finally conclude this paper in Section 7 with numerical experiments showing the efficiency and precision of the method, and its relevance in an industrial reliability context.

2. The finite volume scheme

Let us first give the definition of admissible meshes of $\mathbb{R}^N$.

Definition 1. An admissible mesh of $\mathbb{R}^N$ for problem (2) is a partition $\mathcal{M}$ of $\mathbb{R}^N$ such that

(1) for all $K \in \mathcal{M}$, K is bounded, the interior of K is an open convex subset of $\mathbb{R}^N$ and the N dimensional measure of K, denoted by $\mathrm{m}(K)$, is strictly positive,

(2) for all $K \in \mathcal{M}$, denoting by ∂K the boundary of K, and, for all $L \in \mathcal{M}$, denoting by $K|L = \partial K \cap \partial L$, there exists $\mathcal{N}_K \subset \mathcal{M}$ such that $K \notin \mathcal{N}_K$ and $\partial K = \bigcup_{L\in\mathcal{N}_K} K|L$, and, for all $L \in \mathcal{N}_K$, $K|L$, called an edge of K, is included in a hyperplane of $\mathbb{R}^N$, with a strictly positive $N-1$ dimensional measure equal to $\mathrm{m}(K|L)$; we then denote by $\mathbf{n}_{KL}$ the unit normal vector to $K|L$ oriented from K to L,

(3) the size of the mesh, defined by $h_{\mathcal{M}} = \sup_{K\in\mathcal{M}} \mathrm{diam}(K)$, is finite,

(4) there exists $C_1 \geq 1$ with

$$\frac{1}{C_1} h_{\mathcal{M}} \sum_{L\in\mathcal{N}_K} \mathrm{m}(K|L) \leq \mathrm{m}(K) \leq C_1\, h_{\mathcal{M}} \sum_{L\in\mathcal{N}_K} \mathrm{m}(K|L), \quad \forall K \in \mathcal{M}, \tag{7}$$

$$\frac{1}{C_1} h_{\mathcal{M}} \leq \mathrm{diam}(K) \leq h_{\mathcal{M}}, \quad \forall K \in \mathcal{M}. \tag{8}$$

and

$$\frac{1}{C_1} h_{\mathcal{M}}^N \leq \mathrm{m}(K) \leq C_1\, h_{\mathcal{M}}^N, \quad \forall K \in \mathcal{M}. \tag{9}$$

We then denote by $C_{\mathcal{M}}$ the infimum of all C_1 such that (7)–(9) hold.

Note that all classical regular meshes of $\mathbb{R}^N$ are admissible in the sense of the previous definition.

Now, let $\mathcal{M}$ be a fixed admissible mesh of $\mathbb{R}^N$. For such a mesh, we set

$$v_{K,L} = \frac{1}{\mathrm{m}(K|L)} \int_{K|L} \mathbf{v}(x) \cdot \mathbf{n}_{KL} \mathrm{d}s(x), \quad \forall K \in \mathcal{M}, \forall L \in \mathcal{N}_K, \tag{10}$$

where $\mathrm{d}s(x)$ stands for the $N-1$ dimensional measure on $K|L$ and

$$w_{K,L} = \max(|v_{K,L}|, \varepsilon), \quad \forall K \in \mathcal{M}, \forall L \in \mathcal{N}_K \tag{11}$$

for a given $\varepsilon \in [0, V]$. We also set

$$\lambda_K = \frac{1}{\mathrm{m}(K)} \int_K \lambda\,(x)\,\mathrm{d}x$$

$$a_{K,L} = \frac{1}{\mathrm{m}(K)} \int_K \lambda(x) \left(\int_L \mu(x)\,(\mathrm{d}y) \right) \mathrm{d}x$$

for $K, L \in \mathcal{M}$ with

$$\sum_{L\in\mathcal{M}} a_{K,L} = \lambda_K. \tag{12}$$

For a given time step $k > 0$, the scheme then is written as

$$u_K^{(0)} = \frac{1}{\mathrm{m}(K)} \int_K \mathrm{d}\rho_{\mathrm{ini}}(x), \quad \forall K \in \mathcal{M} \tag{13}$$

and

$$\begin{aligned} &\mathrm{m}(K)(u_K^{(n+1)} - u_K^{(n)}) + k \sum_{L\in\mathcal{N}_K} \mathrm{m}(K|L) \left(v_{K,L} \frac{u_K^{(n+1)} + u_L^{(n+1)}}{2} + \frac{w_{K,L}}{2} (u_K^{(n+1)} - u_L^{(n+1)}) \right) \\ &\quad = -k\mathrm{m}(K)\lambda_K u_K^{(n+1)} + k \sum_{L\in\mathcal{M}} \mathrm{m}(L) a_{L,K} u_L^{(n+1)}, \quad \forall K \in \mathcal{M}, \forall n \in \mathbb{N}. \end{aligned} \tag{14}$$

We first prove the existence and uniqueness of a solution to this numerical scheme in the following lemma.

Lemma 2. *Let us assume hypotheses $\mathcal{H}_0$ and let $\mathcal{M}$ be an admissible mesh of $\mathbb{R}^N$ in the sense of Definition* 1. *Let $k > 0$ and $\varepsilon \in [0, V]$ be given. Then there exists one and only one family of real numbers $(u_K^{(n)})_{K\in\mathcal{M}, n\in\mathbb{N}}$ such that* (13)–(14) *hold and $\sum_{K\in\mathcal{M}} \mathrm{m}(K) |u_K^{(n)}| < \infty$ for all $n \in \mathbb{N}$. Moreover, the following properties hold:*

$$u_K^{(n)} \geq 0, \quad \forall K \in \mathcal{M}, \forall n \in \mathbb{N}, \tag{15}$$

and

$$\sum_{K\in\mathcal{M}} \mathrm{m}(K)\, u_K^{(n)} = 1, \quad \forall n \in \mathbb{N}. \tag{16}$$

Proof. Let $\|\|_{\mathcal{L}_1}$ be the following norm on $\mathcal{L}_1 = \{u := (u_K)_{K\in\mathcal{M}} \text{ s.t. } \sum_{K\in\mathcal{M}} \mathrm{m}(K)\, |u_K| < +\infty\}$:

$$\|u\|_{\mathcal{L}_1} = \sum_{K\in\mathcal{M}} \mathrm{m}(K)\, |u_K|. \tag{17}$$

For $u \in \mathcal{L}_1$ fixed, let us consider ψ_u defined by $\psi_u(p) = r$ for $p \in \mathcal{L}_1$ with

$$\begin{aligned} C\mathrm{m}(K) r_K &= C\mathrm{m}(K) p_K - \frac{\mathrm{m}(K)}{k} (p_K - u_K) \\ &\quad - \sum_{L\in\mathcal{N}_K} \mathrm{m}(K|L) \left(v_{K,L} \frac{p_K + p_L}{2} + \frac{w_{K,L}}{2} (p_K - p_L) \right) - \mathrm{m}(K)\lambda_K p_K + \sum_{L\in\mathcal{M}} \mathrm{m}(L) a_{L,K} p_L \end{aligned} \tag{18}$$

$$\begin{aligned} &= \frac{\mathrm{m}(K)}{k} u_K + \mathrm{m}(K) \left(C - \frac{1}{k} - \frac{1}{2\mathrm{m}(K)} \sum_{L\in\mathcal{N}_K} \mathrm{m}(K|L) \left(v_{K,L} + w_{K,L} \right) - \lambda_K \right) p_K \\ &\quad + \frac{1}{2} \sum_{L\in\mathcal{N}_K} \mathrm{m}(K|L) \left(w_{K,L} - v_{K,L} \right) p_L + \sum_{L\in\mathcal{M}} \mathrm{m}(L) a_{L,K} p_L \end{aligned} \tag{19}$$

where $C > 0$ is a constant to be chosen such that the coefficient of p_K in (19) is non-negative.

Due to (7), $v_{K,L} + w_{K,L} \leq 2V$ and $\lambda_K \leq \Lambda$, we know

$$\frac{1}{k} + \frac{1}{2\mathrm{m}(K)} \sum_{L\in\mathcal{N}_K} \mathrm{m}(K|L) \left(v_{K,L} + w_{K,L} \right) + \lambda_K \leq \frac{1}{k} + V \frac{C_{\mathcal{M}}}{h_{\mathcal{M}}} + \Lambda.$$

We then take C such that

$$C \geq \frac{1}{k} + V \frac{C_{\mathcal{M}}}{h_{\mathcal{M}}} + \Lambda.$$

As the coefficients of p_L and p_K in (19) now are non-negative (remember $w_{K,L} - v_{K,L} > 0$), we derive that

$$
\begin{aligned}
&\|\psi_u(p)\|_{\mathcal{L}_1} \\
&\leq \frac{1}{C} \sum_{K\in\mathcal{M}} \left[\frac{\mathrm{m}(K)}{k} |u_K| + \mathrm{m}(K) \left(C - \frac{1}{k} - \frac{1}{2\mathrm{m}(K)} \sum_{L\in\mathcal{N}_K} \mathrm{m}(K|L) \left(v_{K,L} + w_{K,L}\right) - \lambda_K \right) |p_K| \right. \\
&\quad \left. + \frac{1}{2} \sum_{L\in\mathcal{N}_K} \mathrm{m}(K|L) \left(w_{K,L} - v_{K,L}\right) |p_L| + \sum_{L\in\mathcal{M}} \mathrm{m}(L) a_{L,K} |p_L| \right]
\end{aligned} \tag{20}
$$

$$
\begin{aligned}
&= \frac{1}{C} \left[\frac{1}{k} \|u\|_{\mathcal{L}_1} + \left(C - \frac{1}{k} \right) \|p\|_{\mathcal{L}_1} - \frac{1}{2} \sum_{K\in\mathcal{M}} \sum_{L\in\mathcal{N}_K} \mathrm{m}(K|L) \left(v_{K,L} + w_{K,L}\right) |p_K| - \sum_{K\in\mathcal{M}} \mathrm{m}(K) \lambda_K |p_K| \right. \\
&\quad \left. + \frac{1}{2} \sum_{K\in\mathcal{M}} \sum_{L\in\mathcal{N}_K} \mathrm{m}(K|L) \left(w_{K,L} - v_{K,L}\right) |p_L| + \sum_{L\in\mathcal{M}} \mathrm{m}(L) \lambda_L |p_L| \right]
\end{aligned} \tag{21}
$$

using (12).

Noting that $\sum_{K\in\mathcal{M}} \sum_{L\in\mathcal{N}_K} = \sum_{L\in\mathcal{M}} \sum_{K\in\mathcal{N}_L}$, $v_{L,K} = -v_{K,L}$, $\mathrm{m}(K|L) = \mathrm{m}(L|K)$ and that $w_{K,L} = w_{L,K}$, we easily get

$$
\sum_{K\in\mathcal{M}} \sum_{L\in\mathcal{N}_K} \mathrm{m}(K|L) \left(w_{K,L} - v_{K,L}\right) |p_L| = \sum_{K\in\mathcal{M}} \sum_{L\in\mathcal{N}_K} \mathrm{m}(K|L) \left(w_{K,L} + v_{K,L}\right) |p_K| \tag{22}
$$

and hence

$$
\|\psi_u(p)\|_{\mathcal{L}_1} \leq \frac{1}{C} \left(\frac{1}{k} \|u\|_{\mathcal{L}_1} + \left(C - \frac{1}{k} \right) \|p\|_{\mathcal{L}_1} \right) < +\infty. \tag{23}
$$

As a consequence, the function ψ_u maps $\mathcal{L}_1$ into $\mathcal{L}_1$ for any $u \in \mathcal{L}_1$. Besides, similar computations also give

$$
\left\| \psi_u(p) - \psi_u(p') \right\|_{\mathcal{L}_1} \leq \frac{C - \frac{1}{k}}{C} \left\| p - p' \right\|_{\mathcal{L}_1} \tag{24}
$$

for all $p, p' \in \mathcal{L}_1$. The function ψ_u then is a contraction on the Banach space $\mathcal{L}_1$ and, for all $u \in \mathcal{L}_1$, the function ψ_u admits a unique fixed point $p \in \mathcal{L}_1$.

Noting that $\psi_{u^{(n)}}\left(u^{(n+1)}\right) = u^{(n+1)}$ is equivalent to (14), the existence and uniqueness of $\left(u^{(n)}\right)_{n\in\mathbb{N}}$ in $\mathcal{L}_1$ such that (13)–(14) are true is then clear, recursively.

Let us now set

$$
\mathcal{C} = \left\{ u := (u_K)_{K\in\mathcal{M}} \in \mathcal{L}_1 \text{ s.t. } u_K \geq 0 \text{ for all } K \in \mathcal{M} \text{ and } \|u\|_{\mathcal{L}_1} = 1 \right\}.
$$

Taking $u \in \mathcal{C}$ and $p \in \mathcal{C}$, it is easy to check that $r := \psi_u(p) \in \mathcal{C}$, using the non-negativeness of the coefficients in (19), and noting that (20)–(21) and consequently the left inequality from (23) now are equalities.

Now, starting from $u \in \mathcal{C}$, the sequence $(p_n)_{n\in\mathbb{N}}$ recursively defined by $p_0 = u$ and $p_{n+1} = \psi_u(p_n)$ is such that $p_n \in \mathcal{C}$ for all $n \in \mathbb{N}$ and converges in $\mathcal{L}_1$ towards the single fixed point of ψ_u. We derive that for all $u \in \mathcal{C}$, the single fixed point of ψ_u in $\mathcal{L}_1$ actually is an element of $\mathcal{C}$.

Consequently, starting with $u^{(0)} \in \mathcal{C}$, the single sequence $\left(u^{(n)}\right)_{n\in\mathbb{N}}$ in $\mathcal{L}_1$ such that (13)–(14) are true has elements in $\mathcal{C}$, which achieves the proof. ■

3. Properties of the scheme

For all $s \in \mathbb{R}$, we denote by $\lfloor s \rfloor$ the greatest integer lower than or equal to s and for all $R \geq 0$, let $B(0, R) \subset \mathbb{R}^N$ be the open ball with centre 0 and radius R, with $B(0, 0) = \emptyset$.

Lemma 3. *Let us assume $\mathcal{H}_0$ and let $\mathcal{M}$ be an admissible mesh of $\mathbb{R}^N$ in the sense of Definition* 1. *Let $m \in (0, +\infty)$, $k > 0$ and $\varepsilon \in (0, V]$ be given and let $(u_K^{(n)})_{K\in\mathcal{M}, n\in\mathbb{N}}$ be the family of real numbers defined by* (10), (11), (13) *and*

(14) *such that* (15) *and* (16) *hold. Let* $R > 0$ *and* $T > 0$ *be given and let* $\theta^{(R)} : \mathbb{R}_+ \to [0,1]$ *be defined by* $\theta^{(R)}(s) = 1$ *for all* $s \in [0, R]$, $1 + R - s$ *for all* $s \in [R, R+1]$ *and 0 for all* $s \geq R+1$. *Let us denote*

$$\theta_K^{(R)} = \frac{1}{\mathrm{m}(K)} \int_K \theta^{(R)}(|x|)\mathrm{d}x,$$

and let us define $(\widehat{u}_K^{(n)})_{K\in\mathcal{M}, n\in\mathbb{N}}$ *by*

$$\widehat{u}_K^{(n)} = \theta_K^{(R)} u_K^{(n)}, \quad \forall K \in \mathcal{M}, \forall n \in \mathbb{N}. \tag{25}$$

Let C_1 *be such that* $C_{\mathcal{M}} \leq C_1$, *let* C_2 *be such that* $h_{\mathcal{M}} < C_2$ *and* $k < C_2$ *and let* T_1 *be the term defined by*

$$T_1 = \sum_{n=0}^{\lfloor T/k \rfloor} k \sum_{K\in\mathcal{M}} \sum_{L\in\mathcal{N}_K} \mathrm{m}(K|L) \frac{(\widehat{u}_K^{(n+1)} - \widehat{u}_L^{(n+1)})^2}{(1 + \max(\widehat{u}_K^{(n+1)}, \widehat{u}_L^{(n+1)}))^{m+1}}. \tag{26}$$

Then there exists C_3, *only depending on* N, R, T, m, $\mathbf{v}$, C_1, Λ, ε *and* C_2, *such that*

$$T_1 \leq C_3. \tag{27}$$

Proof. Let us introduce, as in [4], the function $\phi_m : \mathbb{R} \to \mathbb{R}$ such that $\phi_m(s) = (1 - 1/(1+|s|)^m) \times \mathrm{sign}(s)$ with $\phi_m(0) = 0$ and $\phi_m'(s) = m/(1+|s|)^{m+1}$, for all $s \in \mathbb{R}$. We define the real function Φ_m by $\Phi_m(s) = \int_0^s \phi_m(u)\mathrm{d}u$, for all $s \in \mathbb{R}$. We have $0 \leq \phi_m(s) \leq 1$ and $0 \leq \Phi_m(s) \leq s$ for $s \geq 0$. From (14), we get

$$\begin{aligned}
&\mathrm{m}(K)(u_K^{(n+1)} - u_K^{(n)}) + k\mathrm{m}(K)u_K^{(n+1)} D_K + \frac{k}{2} \sum_{L\in\mathcal{N}_K} \mathrm{m}(K|L) \left(w_{K,L} - v_{K,L}\right) (u_K^{(n+1)} - u_L^{(n+1)}) \\
&\quad = -k\mathrm{m}(K)\lambda_K u_K^{(n+1)} + k \sum_{L\in\mathcal{M}} \mathrm{m}(L) a_{L,K} u_L^{(n+1)}, \quad \forall K \in \mathcal{M}, \forall n \in \mathbb{N}
\end{aligned} \tag{28}$$

denoting by

$$D_K = \frac{1}{\mathrm{m}(K)} \int_K \mathrm{div}\,(\mathbf{v}(x))\,\mathrm{d}x = \frac{1}{\mathrm{m}(K)} \sum_{L\in\mathcal{N}_K} \mathrm{m}(K|L) v_{K,L}, \quad \forall K \in \mathcal{M}. \tag{29}$$

Thanks to $\mathcal{H}_0$, we get the existence of C_4 , only depending on $\mathbf{v}$, such that

$$|D_K| \leq C_4. \tag{30}$$

Let us multiply (28) by $\theta_K^{(R)} \phi_m(\theta_K^{(R)} u_K^{(n+1)})$, and sum the result on $K \in \mathcal{M}$ and $n = 0, \ldots, \lfloor T/k \rfloor$. We get $T_2 + T_3 + T_4 + T_5 = 0$, with

$$\begin{aligned}
T_2 &= \sum_{n=0}^{\lfloor T/k \rfloor} \sum_{K\in\mathcal{M}} \theta_K^{(R)} \phi_m(\theta_K^{(R)} u_K^{(n+1)}) \mathrm{m}(K)(u_K^{(n+1)} - u_K^{(n)}), \\
T_3 &= \sum_{n=0}^{\lfloor T/k \rfloor} k \sum_{K\in\mathcal{M}} \mathrm{m}(K)\theta_K^{(R)} \phi_m(\theta_K^{(R)} u_K^{(n+1)}) u_K^{(n+1)} D_K, \\
T_4 &= \frac{1}{2} \sum_{n=0}^{\lfloor T/k \rfloor} k \sum_{K\in\mathcal{M}} \theta_K^{(R)} \phi_m(\theta_K^{(R)} u_K^{(n+1)}) \sum_{L\in\mathcal{N}_K} \mathrm{m}(K|L) \left(w_{K,L} - v_{K,L}\right) (u_K^{(n+1)} - u_L^{(n+1)}),
\end{aligned}$$

and

$$T_5 = k \sum_{n=0}^{\lfloor T/k \rfloor} \sum_{K\in\mathcal{M}} \theta_K^{(R)} \phi_m(\theta_K^{(R)} u_K^{(n+1)}) \left(\mathrm{m}(K)\lambda_K u_K^{(n+1)} - \sum_{L\in\mathcal{M}} \mathrm{m}(L) a_{L,K} u_L^{(n+1)} \right).$$

We have $\widehat{u}_K^{(n+1)} \geq 0$ and $\sum_{K\in\mathcal{M}} \mathrm{m}(K)\widehat{u}_K^{(n+1)} \leq 1$ from Lemma 2, and $\widehat{u}_K^{(n+1)} = 0$ for all $K \in \mathcal{M}$ such that $K \not\subset B(0, R+1+C_2)$. The Taylor–Lagrange expansion formula yields that, for all $a, b \in \mathbb{R}$, there exists

$\Psi_{a,b} \in [\min(a,b), \max(a,b)]$ such that

$$\phi_m(a)(a-b) = \Phi_m(a) - \Phi_m(b) + \frac{1}{2}\phi_m'(\Psi_{a,b})(a-b)^2 \geq \Phi_m(a) - \Phi_m(b). \tag{31}$$

Applying (31) for $a = \widehat{u}_K^{(n+1)}$ and $b = \widehat{u}_K^{(n)}$, we get that

$$T_2 \geq \sum_{K\in\mathcal{M}} \mathrm{m}(K)\,\Phi_m(\widehat{u}_K^{(\lfloor T/k\rfloor+1)}) - \sum_{K\in\mathcal{M}} \mathrm{m}(K)\,\Phi_m(\widehat{u}_K^{(0)}), \tag{32}$$

with

$$0 \leq \sum_{K\in\mathcal{M}} \mathrm{m}(K)\,\Phi_m(\widehat{u}_K^{(0)}) \leq 1$$

due to $\Phi_m(s) \leq s$ for $s \geq 0$.

Thanks to (30) and $0 \leq \phi_m(s) \leq 1$ for all $s \geq 0$, we have

$$T_3 \geq -(T + C_2)C_4. \tag{33}$$

Let us write $T_4 = T_6 + T_7$, with

$$T_6 = \frac{1}{2}\sum_{n=0}^{\lfloor T/k\rfloor} k \sum_{K\in\mathcal{M}} \sum_{L\in\mathcal{N}_K} \mathrm{m}(K|L)\left(w_{K,L} - v_{K,L}\right)\phi_m(\widehat{u}_K^{(n+1)})(\widehat{u}_K^{(n+1)} - \widehat{u}_L^{(n+1)}),$$

and

$$T_7 = \frac{1}{2}\sum_{n=0}^{\lfloor T/k\rfloor} k \sum_{K\in\mathcal{M}} \phi_m(\widehat{u}_K^{(n+1)}) \sum_{L\in\mathcal{N}_K} \mathrm{m}(K|L)\left(w_{K,L} - v_{K,L}\right) u_L^{(n+1)}(\theta_L^{(R)} - \theta_K^{(R)}).$$

Changing the order of summation, we get

$$T_7 = \frac{1}{2}\sum_{n=0}^{\lfloor T/k\rfloor} k \sum_{L\in\mathcal{M}} u_L^{(n+1)} \sum_{K\in\mathcal{N}_L} \phi_m(\widehat{u}_K^{(n+1)})\mathrm{m}(K|L)\left(w_{K,L} - v_{K,L}\right)(\theta_L^{(R)} - \theta_K^{(R)}).$$

Using (8), we get $|\theta_L^{(R)} - \theta_K^{(R)}| \leq 2h_{\mathcal{M}}$. Hence, using (7), we get

$$0 \leq \sum_{K\in\mathcal{N}_L} \phi_m(\widehat{u}_K^{(n+1)})\mathrm{m}(K|L)\left(w_{K,L} - v_{K,L}\right)|\theta_L^{(R)} - \theta_K^{(R)}| \leq \mathrm{m}(L)4VC_1.$$

This leads to

$$T_7 \geq -2VC_1 \sum_{n=0}^{\lfloor T/k\rfloor} k \sum_{L\in\mathcal{M}} \mathrm{m}(L) u_L^{(n+1)} \geq -(T + C_2)2VC_1. \tag{34}$$

Let us apply (31) to T_6. We get $T_6 = T_8 + T_9$, with

$$T_8 = \frac{1}{2}\sum_{n=0}^{\lfloor T/k\rfloor} k \sum_{K\in\mathcal{M}} \sum_{L\in\mathcal{N}_K} \mathrm{m}(K|L)\left(w_{K,L} - v_{K,L}\right)(\Phi_m(\widehat{u}_K^{(n+1)}) - \Phi_m(\widehat{u}_L^{(n+1)})).$$

Thanks to the expression of ϕ_m', we have

$$T_9 = \frac{m}{4}\sum_{n=0}^{\lfloor T/k\rfloor} k \sum_{K\in\mathcal{M}} \sum_{L\in\mathcal{N}_K} \mathrm{m}(K|L)\left(w_{K,L} - v_{K,L}\right)\frac{(\widehat{u}_K^{(n+1)} - \widehat{u}_L^{(n+1)})^2}{(1 + \Psi_{\widehat{u}_K^{(n+1)},\widehat{u}_L^{(n+1)}})^{m+1}}.$$

We remark that $w_{K,L} - v_{K,L} \geq 0$ and

$$\frac{(\widehat{u}_K^{(n+1)} - \widehat{u}_L^{(n+1)})^2}{(1 + \Psi_{\widehat{u}_K^{(n+1)},\widehat{u}_L^{(n+1)}})^{m+1}} \geq \frac{(\widehat{u}_K^{(n+1)} - \widehat{u}_L^{(n+1)})^2}{(1 + \max(\widehat{u}_K^{(n+1)}, \widehat{u}_L^{(n+1)}))^{m+1}},$$

which implies that

$$T_9 \geq \frac{m}{4} \sum_{n=0}^{\lfloor T/k \rfloor} k \sum_{K\in\mathcal{M}} \sum_{L\in\mathcal{N}_K} \mathrm{m}(K|L) \left(w_{K,L} - v_{K,L}\right) \frac{(\widehat{u}_K^{(n+1)} - \widehat{u}_L^{(n+1)})^2}{(1+\max(\widehat{u}_K^{(n+1)}, \widehat{u}_L^{(n+1)}))^{m+1}}.$$

Gathering by edges the right-hand side of the above equation leads to

$$T_9 \geq \frac{m}{4} \sum_{n=0}^{\lfloor T/k \rfloor} k \sum_{K\in\mathcal{M}} \sum_{L\in\mathcal{N}_K} \mathrm{m}(K|L) w_{K,L} \frac{(\widehat{u}_K^{(n+1)} - \widehat{u}_L^{(n+1)})^2}{(1+\max(\widehat{u}_K^{(n+1)}, \widehat{u}_L^{(n+1)}))^{m+1}}.$$

Thanks to (11), we also know $w_{K,L} \geq \varepsilon$, and the term T_1 defined by (26) now satisfies

$$T_9 \geq \frac{m}{4} \varepsilon T_1. \tag{35}$$

We next write $T_8 = T_{10} + T_{11}$, with

$$T_{10} = \sum_{n=0}^{\lfloor T/k \rfloor} k \sum_{K\in\mathcal{M}} \sum_{L\in\mathcal{N}_K} \mathrm{m}(K|L) \left(v_{K,L} \frac{\Phi_m(\widehat{u}_K^{(n+1)}) + \Phi_m(\widehat{u}_L^{(n+1)})}{2} + \frac{w_{K,L}}{2} (\Phi_m(\widehat{u}_K^{(n+1)}) - \Phi_m(\widehat{u}_L^{(n+1)})) \right),$$

and

$$T_{11} = -\sum_{n=0}^{\lfloor T/k \rfloor} k \sum_{K\in\mathcal{M}} \mathrm{m}(K)\, \Phi_m(\widehat{u}_K^{(n+1)}) D_K.$$

Gathering by edges, we remark that

$$T_{10} = 0. \tag{36}$$

We also get that

$$T_{11} \geq -(T + C_2) C_4. \tag{37}$$

$$\begin{aligned} T_5 &= k \sum_{n=0}^{\lfloor T/k \rfloor} \sum_{K\in\mathcal{M}} \theta_K^{(R)} \phi_m(\theta_K^{(R)} u_K^{(n+1)}) \left(\mathrm{m}(K) \lambda_K u_K^{(n+1)} - \sum_{L\in\mathcal{M}} \mathrm{m}(L) a_{L,K} u_L^{(n+1)} \right) \\ &\geq -k \sum_{n=0}^{\lfloor T/k \rfloor} \sum_{K\in\mathcal{M}} \theta_K^{(R)} \phi_m(\theta_K^{(R)} u_K^{(n+1)}) \sum_{L\in\mathcal{M}} \mathrm{m}(L) a_{L,K} u_L^{(n+1)} \\ &\geq -\Lambda(T + C_2). \end{aligned}$$

Thanks to the relation $T_2 + T_3 + T_7 + T_9 + T_{10} + T_{11} + T_5 = 0$, and to (32), (33), (34), (36), (37), and (35), we deduce the existence of C_3, only depending on N, R, T, m, $\mathbf{v}$, C_1, ε, Λ and C_2, such that (27) holds. ■

4. A weak BV inequality

In this section, we apply many times Hölder's inequality

$$\sum_{i\in I} a_i b_i \leq \left(\sum_{i\in I} a_i^\alpha\right)^{\frac{1}{\alpha}} \left(\sum_{i\in I} b_i^\beta\right)^{\frac{1}{\beta}}, \quad \text{for } a_i, b_i \geq 0, \alpha, \beta > 1 \text{ with } \frac{1}{\alpha} + \frac{1}{\beta} = 1, \tag{38}$$

with various choices for I, a_i, b_i, α and β that we define each time.

A weak BV inequality is provided in the next lemma. Such an inequality is valid for general families $(\widehat{u}_K^{(n)})_{K\in\mathcal{M}, n\in\mathbb{N}}$ and does not depend on the scheme. Its proof requires some Sobolev inequalities which are given further in Lemmas 6 and 7. Such Sobolev inequalities were already given in [9], with alternate assumptions and proofs however (see the introduction of this paper).

Lemma 4 (*A Weak BV Inequality*). *Let $N \in \mathbb{N}^\star$ and let $\mathcal{M}$ be an admissible mesh of $\mathbb{R}^N$ in the sense of Definition* 1. *If $N \geq 2$, let $q \in (1, \frac{N+2}{N+1})$ be given and let $m := [(2-q)(N+1)/N] - 1 > 0$. If $N = 1$, let $q \in (1, \sqrt{2})$ and $m \in (0, \frac{2-q^2}{q})$ be given. Let $(\widehat{u}_K^{(n)})_{K\in\mathcal{M}, n\in\mathbb{N}}$ be a family of non-negative real numbers such that $\sum_{K\in\mathcal{M}} \mathrm{m}(K)\widehat{u}_K^{(n)} \leq 1$ for all $n \in \mathbb{N}$ and such that there exists $R > 0$ with $\widehat{u}_K^{(n)} = 0$, for all $n \in \mathbb{N}$, for all $K \in \mathcal{M}$ such that $K \not\subset B(0, R)$. Let $T > 0$ and $k > 0$ be given and let the terms T_1 and T_{12} be respectively defined by* (26) *and*

$$T_{12} = \sum_{n=0}^{\lfloor T/k \rfloor} k \sum_{K\in\mathcal{M}} \sum_{L\in\mathcal{N}_K} \mathrm{m}(K|L)|\widehat{u}_K^{(n+1)} - \widehat{u}_L^{(n+1)}|.$$

Let C_1 be such that $C_{\mathcal{M}} \leq C_1$, let C_3 be such that $T_1 \leq C_3$ and let C_2 be such that $h_{\mathcal{M}} < C_2$ and $k < C_2$. Then there exists C_5, only depending on N, R, T, q, m, C_1, C_2 and C_3 such that

$$T_{12} \leq C_5\, h_{\mathcal{M}}^{-1/q}. \tag{39}$$

Remark 5. For $\varepsilon \in (0, V]$, we know from (27) in Lemma 3 that the assumptions of the previous lemma are valid and hence that (39) is true for the family $(\widehat{u}_K^{(n)})_{K\in\mathcal{M}, n\in\mathbb{N}}$ associated to the scheme and constructed by (25).

Proof. In the course of this proof, we denote by C_i, for $i > 5$, various positive real numbers, only depending on N, R, T, q, C_1, C_2 and C_3. Let us first define T_{13} by

$$T_{13} = h_{\mathcal{M}}^{2-q} \sum_{n=0}^{\lfloor T/k \rfloor} k \sum_{K\in\mathcal{M}} \sum_{L\in\mathcal{N}_K} \mathrm{m}(K|L)|\widehat{u}_K^{(n+1)} - \widehat{u}_L^{(n+1)}|^q.$$

According to some ideas of [4,3], our aim is to prove that T_{13} is bounded independently of $h_{\mathcal{M}}$ (inequality (44)), which then provides the conclusion (39), applying Hölder's inequality, as we show at the end of this proof. Let $\theta^{(R)}$ be associated to R as in Lemma 3. We denote by $\mathcal{M}_{\theta^{(R)}}$ the set of all $K \in \mathcal{M}$ such that $\theta_K^{(R)} \neq 0$ or such that there exists $L \in \mathcal{N}_K$ with $\theta_L^{(R)} \neq 0$. We then apply Hölder's inequality (38) with $I = \{(n, K, L),\ n = 0, \ldots, \lfloor T/k \rfloor,\ K \in \mathcal{M}_{\theta^{(R)}}, L \in \mathcal{N}_K\}$, $\alpha = 2/q$, $\beta = 2/(2-q)$,

$$a_{n,K,L} = \left(k\, \mathrm{m}(K|L) \frac{(\widehat{u}_K^{(n+1)} - \widehat{u}_L^{(n+1)})^2}{(1 + \max(\widehat{u}_K^{(n+1)}, \widehat{u}_L^{(n+1)}))^{m+1}}\right)^{q/2},$$

and

$$b_{n,K,L} = (h_{\mathcal{M}}^2\, k\, \mathrm{m}(K|L))^{(2-q)/2} \left(1 + \max(\widehat{u}_K^{(n+1)}, \widehat{u}_L^{(n+1)})\right)^{(m+1)q/2}.$$

Hence we get

$$T_{13} \leq (T_1)^{q/2}(T_{14})^{1-q/2}, \tag{40}$$

defining T_{14} by

$$T_{14} = h_{\mathcal{M}}^2 \sum_{n=0}^{\lfloor T/k \rfloor} k \sum_{K\in\mathcal{M}_{\theta^{(R)}}} \sum_{L\in\mathcal{N}_K} \mathrm{m}(K|L) \left(1 + \max(\widehat{u}_K^{(n+1)}, \widehat{u}_L^{(n+1)})\right)^{(m+1)q/(2-q)}.$$

Note that the expression of T_{14} would be meaningless if we write the sum on $K \in \mathcal{M}$ instead of $K \in \mathcal{M}_{\theta^{(R)}}$. Thanks to $q \in (1, 2)$ and $m > 0$, which imply $r := (m+1)q/(2-q) > 1$, and thanks to the inequality $(a+b)^r \leq 2^{r-1}(a^r + b^r)$ for all $r \geq 1$, $a, b \geq 0$, we get the existence of C_6, only depending on N, m and q, such that

$$T_{14} \leq h_{\mathcal{M}} C_6\, (T_{15} + T_{16}), \tag{41}$$

defining T_{15} and T_{16} by

$$T_{15} = h_{\mathcal{M}} \sum_{n=0}^{\lfloor T/k \rfloor} k \sum_{K\in\mathcal{M}_{\theta^{(R)}}} \sum_{L\in\mathcal{N}_K} \mathrm{m}(K|L) \leq C_7, \tag{42}$$

with $C_7 = (T + C_2)C_1\,\mathrm{m}(B(0, R + 1C_2))$ and

$$T_{16} = h_{\mathcal{M}} \sum_{n=0}^{\lfloor T/k \rfloor} k \sum_{K \in \mathcal{M}_{\theta(R)}} \sum_{L \in \mathcal{N}_K} \mathrm{m}(K|L) \left(\max(\widehat{u}_K^{(n+1)}, \widehat{u}_L^{(n+1)}) \right)^{(m+1)q/(2-q)}.$$

We can then write

$$T_{16} \le h_{\mathcal{M}} \sum_{n=0}^{\lfloor T/k \rfloor} k \sum_{K \in \mathcal{M}_{\theta(R)}} \sum_{L \in \mathcal{N}_K} \mathrm{m}(K|L) \left(\left(\widehat{u}_K^{(n+1)}\right)^{(m+1)q/(2-q)} + \left(\widehat{u}_L^{(n+1)}\right)^{(m+1)q/(2-q)} \right),$$

which leads, thanks to (7), to

$$T_{16} \le 2C_1 \sum_{n=0}^{\lfloor T/k \rfloor} k \sum_{K \in \mathcal{M}_{\theta(R)}} \mathrm{m}(K) \left(\widehat{u}_K^{(n+1)}\right)^{(m+1)q/(2-q)}.$$

Let us now consider two cases:

(1) Case $N \ge 2$: thanks to the definition of m, we know $(m+1)\,q/\,(2-q) = q\,(N+1)\,/N$. Let us apply Hölder's inequality (38) with $I = \mathcal{M}_{\theta(R)}$, and

$$a_K = (\mathrm{m}(K)\widehat{u}_K^{(n+1)})^{q/N}, \qquad b_K = \mathrm{m}(K)^{1-q/N}(u_K^{(n+1)})^q,$$

$\alpha = N/q$, $\beta = N/(N-q)$ (this is possible since $N \ge 2$) and $q^\star = Nq/(N-q)$. We get

$$\sum_{K \in \mathcal{M}_{\theta(R)}} \mathrm{m}(K) \left(\widehat{u}_K^{(n+1)}\right)^{(m+1)q/(2-q)} \le \left(\sum_{K \in \mathcal{M}_{\theta(R)}} \mathrm{m}(K)\widehat{u}_K^{(n+1)} \right)^{\frac{q}{N}} \left(\sum_{K \in \mathcal{M}_{\theta(R)}} \mathrm{m}(K) \left(\widehat{u}_K^{(n+1)}\right)^{q^\star} \right)^{\frac{q}{q^\star}}.$$

(2) Case $N = 1$: let us define $q^\star$, α and β by $1/\alpha + 1/\beta = 1$, $1/\alpha + q^\star/\beta = (m+1)\,q/\,(2-q)$ and $\frac{1}{\beta} = \frac{q}{q^\star}$. This leads to $q^\star = q/(1 + q - (m+1)q/(2-q))$ with $q^\star \in (q, +\infty)$, since $q < (m+1)q/(2-q) < 1 + q$ due to $m \in (0, \frac{2-q^2}{q})$ with $q \in (1, \sqrt{2})$. Therefore, setting

$$a_K = (\mathrm{m}(K)\widehat{u}_K^{(n+1)})^{1/\alpha}, \qquad b_K = \mathrm{m}(K)^{1/\beta}(u_K^{(n+1)})^{q^\star/\beta},$$

we get

$$\sum_{K \in \mathcal{M}_{\theta(R)}} \mathrm{m}(K) \left(\widehat{u}_K^{(n+1)}\right)^{(m+1)q/(2-q)} \le \left(\sum_{K \in \mathcal{M}_{\theta(R)}} \mathrm{m}(K)\widehat{u}_K^{(n+1)} \right)^{1/\alpha} \left(\sum_{K \in \mathcal{M}_{\theta(R)}} \mathrm{m}(K) \left(\widehat{u}_K^{(n+1)}\right)^{q^\star} \right)^{\frac{q}{q^\star}}.$$

In both cases, thanks to the hypothesis $\sum_{K \in \mathcal{M}} \mathrm{m}(K)\widehat{u}_K^{(n+1)} \le 1$, we obtain

$$T_{16} \le 2C_1 \sum_{n=0}^{\lfloor T/k \rfloor} k \left(\sum_{K \in \mathcal{M}_{\theta(R)}} \mathrm{m}(K) \left(\widehat{u}_K^{(n+1)}\right)^{q^\star} \right)^{q/q^\star}.$$

We now apply Lemma 6 or 7 to the values $(\widehat{u}_K^{(n+1)})_{K \in \mathcal{M}}$: there exists C_8, only depending on N, q, m, R and C_1 such that

$$\left(\sum_{K \in \mathcal{M}_{\theta(R)}} \mathrm{m}(K) \left(\widehat{u}_K^{(n+1)}\right)^{q^\star} \right)^{q/q^\star} \le C_8\, h_{\mathcal{M}}^{1-q} \sum_{K \in \mathcal{M}} \sum_{L \in \mathcal{N}_K} \mathrm{m}(K|L) |\widehat{u}_K^{(n+1)} - \widehat{u}_L^{(n+1)}|^q,$$

which leads to

$$T_{16} \le h_{\mathcal{M}}^{1-q}\, 2C_1\, C_8 \sum_{n=0}^{\lfloor T/k \rfloor} k \sum_{K \in \mathcal{M}} \sum_{L \in \mathcal{N}_K} \mathrm{m}(K|L) |\widehat{u}_K^{(n+1)} - \widehat{u}_L^{(n+1)}|^q,$$

and therefore

$$h_{\mathcal{M}} T_{16} \leq 2C_1\, C_8\, T_{13}. \tag{43}$$

Hence we get from the hypothesis $T_1 \leq C_3$, (40)–(43)

$$T_{13} \leq (C_3)^{q/2} (C_6\, (h_{\mathcal{M}} C_7 + 2C_1\, C_8\, T_{13}))^{1-q/2}.$$

This leads, thanks to $h_{\mathcal{M}} \leq C_2$, to the existence of C_9, such that

$$T_{13} \leq C_9\,. \tag{44}$$

We now apply Hölder's inequality (38) to T_{12} with $I = \{(n, K, L),\ n = 0, \ldots, \lfloor T/k \rfloor, K \in \mathcal{M},\ L \in \mathcal{N}_K\}$, $\alpha = q$, $\beta = q/(q-1)$,

$$a_{n,K,L} = (h_{\mathcal{M}}^{2-q}\, k\, \mathrm{m}(K|L))^{1/q} |\widehat{u}_K^{(n+1)} - \widehat{u}_L^{(n+1)}|,$$

and

$$b_{n,K,L} = h_{\mathcal{M}}^{-1/q} (h_{\mathcal{M}}\, k\, \mathrm{m}(K|L))^{(q-1)/q}.$$

Hence we get

$$T_{12} \leq h_{\mathcal{M}}^{-1/q} T_{13}^{1/q} \left(h_{\mathcal{M}} \sum_{n=0}^{\lfloor T/k \rfloor} k \sum_{K \in \mathcal{M}} \sum_{L \in \mathcal{N}_K} \mathrm{m}(K|L) \right)^{1-1/q}.$$

Applying (42) and (44), we get

$$T_{12} \leq h_{\mathcal{M}}^{-1/q} C_9{}^{1/q}\, (C_7)^{1-1/q}\,,$$

which implies (39). ■

Lemma 6 (*General Discrete Sobolev Inequality, Case* $N = 1$). *Let* $N = 1$ *and let* $\mathcal{M}$ *be an admissible mesh in the sense of Definition* 1. *Let* $(\widehat{u}_K)_{K \in \mathcal{M}}$ *be a family of real numbers such that the number of* $K \in \mathcal{M}$ *such that* $\widehat{u}_K \neq 0$ *is finite, let* $A > 0$ *such that*

$$\sum_{K \in \mathcal{M}, \widehat{u}_K \neq 0} \mathrm{m}(K) \leq A,$$

and let $q \in (1, +\infty)$ *and* $q^\star \in (1, +\infty)$ *be given. Let* $C_1 > C_{\mathcal{M}}$ *be given. Then there exists* C_{10}, *only depending on* N, q, $q^\star$, A *and* C_1, *such that*

$$\left(\sum_{K \in \mathcal{M}} \mathrm{m}(K) |\widehat{u}_K|^{q^\star} \right)^{1/q^\star} \leq C_{10} \left(\sum_{K \in \mathcal{M}} \sum_{L \in \mathcal{N}_K} \mathrm{m}(K|L) h_{\mathcal{M}} \frac{|\widehat{u}_K - \widehat{u}_L|^q}{h_{\mathcal{M}}^q} \right)^{\frac{1}{q}}. \tag{45}$$

Proof. We have, for all $\bar{K} \in \mathcal{M}$,

$$|\widehat{u}_{\bar{K}}| \leq \sum_{K \in \mathcal{M}} \sum_{L \in \mathcal{N}_K} |\widehat{u}_K - \widehat{u}_L|.$$

Hence we get

$$\left(\sum_{\bar{K} \in \mathcal{M}} \mathrm{m}(\bar{K}) |\widehat{u}_{\bar{K}}|^{q^\star} \right)^{1/q^\star} \leq A^{1/q^\star} \sum_{K \in \mathcal{M}} \sum_{L \in \mathcal{N}_K} |\widehat{u}_K - \widehat{u}_L|.$$

Recalling that $\mathrm{m}(K|L) = 1$ for all $K \in \mathcal{M}$ and $L \in \mathcal{N}_K$ because $N = 1$, we have

$$\sum_{L \in \mathcal{N}_K} 1 \leq \frac{C_1}{h_{\mathcal{M}}} \mathrm{m}(K)$$

for all $K \in \mathcal{M}$ due to (7). Applying Hölder's inequality with $I = \{(K, L), K \in \mathcal{M},\ L \in \mathcal{N}_K \text{ s.t. } |\widehat{u}_K - \widehat{u}_L| \neq 0\}$, $\alpha = q$, $\beta = \frac{q}{q-1}$, $a_{K,L} = |\widehat{u}_K - \widehat{u}_L|$, $b_{K,L} = 1$ and using the fact that $|\widehat{u}_K - \widehat{u}_L| \neq 0$ implies $\widehat{u}_K \neq 0$ or $\widehat{u}_L \neq 0$ (hence $I \subset \{(K, L), K \in \mathcal{M}, L \in \mathcal{N}_K \text{ s.t. } \widehat{u}_K \neq 0\} \cup \{(K, L), L \in \mathcal{M},\ K \in \mathcal{N}_L \text{ s.t. } \widehat{u}_L \neq 0\}$), we now get

$$\left(\sum_{\bar{K} \in \mathcal{M}} \mathrm{m}(\bar{K}) |\widehat{u}_{\bar{K}}|^{q^\star} \right)^{1/q^\star} \leq A^{1/q^\star} \left(\sum_{K \in \mathcal{M}} \sum_{L \in \mathcal{N}_K} |\widehat{u}_K - \widehat{u}_L|^q \right)^{1/q} \left(\frac{2AC_1}{h_{\mathcal{M}}} \right)^{1-1/q},$$

which provides (45) using again $\mathrm{m}(K|L) = 1$. ■

Lemma 7 (*General Discrete Sobolev Inequality, Case* $N \geq 2$). *Let $N \in \mathbb{N}^\star$ with $N \geq 2$ and let $\mathcal{M}$ be an admissible mesh in the sense of Definition* 1. *Let $(\widehat{u}_K)_{K \in \mathcal{M}}$ be a family of real numbers such that the number of $K \in \mathcal{M}$ such that $\widehat{u}_K \neq 0$ is finite and let $q \in (1, N)$ be given. Let $C_1 > C_{\mathcal{M}}$ be given. Then there exists C_{11}, only depending on N, q and C_1, such that*

$$\left(\sum_{K \in \mathcal{M}} \mathrm{m}(K) |\widehat{u}_K|^{q^\star} \right)^{1/q^\star} \leq C_{11} \left(\sum_{K \in \mathcal{M}} \sum_{L \in \mathcal{N}_K} \mathrm{m}(K|L) h_{\mathcal{M}} \frac{|\widehat{u}_K - \widehat{u}_L|^q}{h_{\mathcal{M}}^q} \right)^{\frac{1}{q}}, \tag{46}$$

with $q^\star = \frac{Nq}{N-q}$.

Proof. Let $\gamma > 1$ be given (this value is chosen below as a function of N and q). We define the function $\widehat{v}$ on $\mathbb{R}^N$ by $\widehat{v}(x) = \widehat{v}_K$ for almost every $x \in K$, all $K \in \mathcal{M}$, where $\widehat{v}_K = |\widehat{u}_K|^\gamma$ for all $K \in \mathcal{M}$.

Due to Sobolev–Gagliardo–Nirenberg results, there is some $C_{12} > 0$ (which only depends on N) such that

$$\|\widehat{v}\|_{L^{N/(N-1)}(\mathbb{R}^N)} \leq C_{12} |\widehat{v}|_{BV(\mathbb{R}^N)} \tag{47}$$

where $|\widehat{v}|_{BV(\mathbb{R}^N)}$ is defined by (6). In the particular case of the piecewise constant function $\widehat{v}$, we can write

$$|\widehat{v}|_{BV(\mathbb{R}^N)} = \frac{1}{2} \sum_{K \in \mathcal{M}} \sum_{L \in \mathcal{N}_K} \mathrm{m}(K|L) |\widehat{v}_K - \widehat{v}_L|$$

(the factor $\frac{1}{2}$ resulting from the fact that each interface appears twice in the double sum), and

$$\|\widehat{v}\|_{L^{N/(N-1)}(\mathbb{R}^N)} = \left(\sum_{K \in \mathcal{M}} \mathrm{m}(K) (\widehat{v}_K)^{N/(N-1)} \right)^{(N-1)/N}.$$

Thanks to the inequality $||\widehat{u}_K|^\gamma - |\widehat{u}_L|^\gamma| \leq \gamma(|\widehat{u}_K|^{\gamma-1} + |\widehat{u}_L|^{\gamma-1})|\widehat{u}_K - \widehat{u}_L|$, we derive from (47)

$$\left(\sum_{K \in \mathcal{M}} \mathrm{m}(K) |\widehat{u}_K|^{\gamma N/(N-1)} \right)^{(N-1)/N} \leq \frac{C_{12}}{2} \sum_{K \in \mathcal{M}} \sum_{L \in \mathcal{N}_K} \mathrm{m}(K|L)\, \gamma(|\widehat{u}_K|^{\gamma-1} + |\widehat{u}_L|^{\gamma-1})|\widehat{u}_K - \widehat{u}_L|,$$

which provides, gathering terms by control volumes,

$$\left(\sum_{K \in \mathcal{M}} \mathrm{m}(K) |\widehat{u}_K|^{\gamma N/(N-1)} \right)^{(N-1)/N} \leq C_{12} \sum_{K \in \mathcal{M}} \sum_{L \in \mathcal{N}_K} \mathrm{m}(K|L)\, \gamma |\widehat{u}_K|^{\gamma-1} |\widehat{u}_K - \widehat{u}_L|.$$

Using Hölder's inequality with $I = \{(K, L), K \in \mathcal{M},\ L \in \mathcal{N}_K\}$, $\alpha = q$, $\beta = p \in \mathbb{R}_+$ such that $\frac{1}{p} + \frac{1}{q} = 1$, $a_{K,L} = (\mathrm{m}(K|L) h_{\mathcal{M}})^{1/p} |\widehat{u}_K|^{(\gamma-1)}$, $b_{K,L} = (\mathrm{m}(K|L) h_{\mathcal{M}})^{1/q} \frac{|\widehat{u}_K - \widehat{u}_L|}{h_{\mathcal{M}}}$ yields

$$\left(\sum_{K \in \mathcal{M}} \mathrm{m}(K) |\widehat{u}_K|^{\gamma N/(N-1)} \right)^{(N-1)/N} \leq C_{12} \gamma \left(\sum_{K \in \mathcal{M}} \sum_{L \in \mathcal{N}_K} \mathrm{m}(K|L)\, h_{\mathcal{M}} |\widehat{u}_K|^{p(\gamma-1)} \right)^{\frac{1}{p}}$$
$$\times \left(\sum_{K \in \mathcal{M}} \sum_{L \in \mathcal{N}_K} \mathrm{m}(K|L) h_{\mathcal{M}} \frac{|\widehat{u}_K - \widehat{u}_L|^q}{h_{\mathcal{M}}^q} \right)^{\frac{1}{q}}.$$

Since $\sum_{L\in\mathcal{N}_K} \mathrm{m}(K|L)h_{\mathcal{M}} \le C_1\,\mathrm{m}(K)$, this gives

$$\left(\sum_{K\in\mathcal{M}} \mathrm{m}(K)|\widehat{u}_K|^{\gamma N/(N-1)}\right)^{(N-1)/N} \le C_{12}\gamma C_1{}^{\frac{1}{p}} \left(\sum_{K\in\mathcal{M}} \mathrm{m}(K)|\widehat{u}_K|^{p(\gamma-1)}\right)^{\frac{1}{p}}$$
$$\times \left(\sum_{K\in\mathcal{M}}\sum_{L\in\mathcal{N}_K} \mathrm{m}(K|L)h_{\mathcal{M}}\frac{|\widehat{u}_K-\widehat{u}_L|^q}{h_{\mathcal{M}}^q}\right)^{\frac{1}{q}}.$$

We choose $\gamma = \frac{(N-1)q}{N-q}$. This leads to $p(\gamma-1) = \frac{Nq}{N-q} = q^\star = \gamma\frac{N}{N-1}$. Hence we get, since $\frac{N-1}{N}-\frac{1}{p} = \frac{N-q}{Nq} = 1/q^\star$,

$$\left(\sum_{K\in\mathcal{M}} \mathrm{m}(K)|\widehat{u}_K|^{q^\star}\right)^{1/q^\star} \le C_{12}\frac{(N-1)q}{N-q}C_1{}^{(q-1)/q}\left(\sum_{K\in\mathcal{M}}\sum_{L\in\mathcal{N}_K} \mathrm{m}(K|L)h_{\mathcal{M}}\frac{|\widehat{u}_K-\widehat{u}_L|^q}{h_{\mathcal{M}}^q}\right)^{\frac{1}{q}}, \tag{48}$$

which shows that (46) holds. ■

5. Convergence proof

5.1. Introduction

Under assumptions $\mathcal{H}_0$, for each admissible mesh $\mathcal{M}$ of $\mathbb{R}^N$ in the sense of Definition 1, each $k > 0$ and each $\varepsilon \in [0, V]$, we set

$$u_{\mathcal{M},k}(x,t) = u_K^{(n)}, \quad \text{for a.e. } x\in K, \forall K\in\mathcal{M}, \forall t\in[nk,(n+1)k), \forall n\in\mathbb{N}, \tag{49}$$

where $\left(u_K^{(n)}\right)_{n\in\mathbb{N},K\in\mathcal{M}}$ is the unique solution of the scheme, namely the single family of real numbers defined by (10), (11), (13) and (14) such that (15) and (16) hold. The marginal distribution $\rho(t)(\mathrm{d}x)$ of the process $(X_t)_{t\ge 0}$ is then approximated by the probability measure $u_{\mathcal{M},k}(x,t)\,\mathrm{d}x$. (Note the density with respect of Lebesgue measure for the approximate distribution.)

For $\varphi \in C_c^1(\mathbb{R}^N)$, we want to prove that $\int_{\mathbb{R}^N}\varphi(x)\;u_{\mathcal{M},k}(x,t)\,\mathrm{d}x$ converges to $\int_{\mathbb{R}^N}\varphi(x)\rho(t)(\mathrm{d}x)$ when $\max(h_{\mathcal{M}},k)$ goes to 0, where $(\rho(t)(\mathrm{d}x))_{t\in\mathbb{R}_+}$ is known to be the unique solution of the CK equation (2) (see the Introduction). With that aim, we have to prove that, if $\max(h_{\mathcal{M}},k)$ is small, then

$$\int_{\mathbb{R}^N}\varphi(x)\,u_{\mathcal{M},k}(x,t)\,\mathrm{d}x - \int_{\mathbb{R}^N}\varphi(x)\,u_{\mathcal{M},k}(x,0)\,\mathrm{d}x - \int_0^t\int_{\mathbb{R}^N}\mathbf{v}(x)\cdot\nabla\varphi(x)\,u_{\mathcal{M},k}(x,s)\,\mathrm{d}x\,\mathrm{d}s$$
$$\simeq \int_0^t\int_{\mathbb{R}^N}\lambda(x)\left(\int_{\mathbb{R}^N}\varphi(y)\mu(x)(\mathrm{d}y)-\varphi(x)\right)u_{\mathcal{M},k}(x,s)\,\mathrm{d}x\,\mathrm{d}s \tag{50}$$

for all $t > 0$. Also, for $t_1, t_2 \in \mathbb{R}_+$ such that $0 \le t_1 \le t_2$, we have to control the quantity

$$\left|\int_{\mathbb{R}^N}\varphi(x)\,u_{\mathcal{M},k}(x,t_2)\,\mathrm{d}x - \int_{\mathbb{R}^N}\varphi(x)\,u_{\mathcal{M},k}(x,t_1)\,\mathrm{d}x\right|,$$

which provides some kind of continuity with respect of t for $\int_{\mathbb{R}^N}\varphi(x)\;u_{\mathcal{M},k}(x,t)\,\mathrm{d}x$ when $\max(h_{\mathcal{M}},k)$ is small. Finally, due to the fact that the function $x \longmapsto \int_{\mathbb{R}^N}\varphi(y)\mu(x)(\mathrm{d}y)$ belongs to $C_b\left(\mathbb{R}^N\right)$ but usually not to $C_c(\mathbb{R}^N)$ even for φ with a compact support (think of $\mu(x) = \delta_0$ for instance), we also have to prove the tightness of $\left(u_{\mathcal{M},k}(x,t)\,\mathrm{d}x\right)_{\mathcal{M},k}$. All these technical results are provided in different lemmas in the next subsection. In Lemmas 8–10, we consider $t_1, t_2 \in \mathbb{R}_+$ fixed such that $0 \le t_1 \le t_2$. We set $n_1 := \left\lfloor\frac{t_1}{k}\right\rfloor$, $n_2 := \left\lfloor\frac{t_2}{k}\right\rfloor$ and, if $n_1 = n_2$, all the sums $\sum_{n=n_1}^{n_2-1}(\cdot)$ are set equal to zero. For $K\in\mathcal{M}$ and $\varphi\in C_c^1(\mathbb{R}^N)$, we also set

$$\varphi_K = \frac{1}{\mathrm{m}(K)}\int_K\varphi(x)\,\mathrm{d}x. \tag{51}$$

We start from (28), which we multiply by φ_K and sum on K and on $n_1 \le n \le n_2 - 1$. We get $T_{17}+T_{18}+T_{19}+T_{20}=0$, with

$$T_{17} = \sum_{n=n_1}^{n_2-1} \sum_{K\in\mathcal{M}} \mathrm{m}(K)(u_K^{(n+1)} - u_K^{(n)})\varphi_K, \tag{52}$$

$$T_{18} = k \sum_{n=n_1}^{n_2-1} \sum_{K\in\mathcal{M}} \mathrm{m}(K) u_K^{(n+1)} D_K \varphi_K, \tag{53}$$

$$T_{19} = \frac{k}{2} \sum_{n=n_1}^{n_2-1} \sum_{K\in\mathcal{M}} \sum_{L\in\mathcal{N}_K} \mathrm{m}(K|L)\left(w_{K,L} - v_{K,L}\right)(u_K^{(n+1)} - u_L^{(n+1)})\varphi_K, \tag{54}$$

$$T_{20} = k \sum_{n=n_1}^{n_2-1} \sum_{K\in\mathcal{M}} \left(\mathrm{m}(K)\lambda_K u_K^{(n+1)} - \sum_{L\in\mathcal{M}} \mathrm{m}(L) a_{L,K} u_L^{(n+1)}\right)\varphi_K. \tag{55}$$

We also introduce T_{21}, T_{22}, T_{23} and T_{24} with

$$T_{21} = \int_{\mathbb{R}^N} \varphi(x)\left(u_{\mathcal{M},k}(x,t_2) - u_{\mathcal{M},k}(x,t_1)\right)\mathrm{d}x, \tag{56}$$

$$T_{22} = \int_{t_1}^{t_2}\int_{\mathbb{R}^N} \varphi(x)\,\mathrm{div}\,(\mathbf{v}(x))\, u_{\mathcal{M},k}(x,t)\,\mathrm{d}x\ \mathrm{d}t, \tag{57}$$

$$T_{23} = -\int_{t_1}^{t_2}\int_{\mathbb{R}^N} \mathrm{div}\,(\varphi(x)\,\mathbf{v}(x))\, u_{\mathcal{M},k}(x,t)\,\mathrm{d}x\ \mathrm{d}t, \tag{58}$$

$$T_{24} = \int_{t_1}^{t_2}\int_{\mathbb{R}^N} \lambda(x)\left(\varphi(x) - \int_{\mathbb{R}^N} \varphi(y)\mu(x)(\mathrm{d}y)\right) u_{\mathcal{M},k}(x,t)\,\mathrm{d}x\ \mathrm{d}t \tag{59}$$

and

$$\begin{aligned} &T_{21}+T_{22}+T_{23}+T_{24} \\ &\quad = \int_{\mathbb{R}^N} \varphi(x)\left(u_{\mathcal{M},k}(x,t_2) - u_{\mathcal{M},k}(x,t_1)\right)\mathrm{d}x - \int_{t_1}^{t_2}\int_{\mathbb{R}^N} (\mathbf{v}(x)\cdot\nabla\varphi(x))\, u_{\mathcal{M},k}(x,t)\,\mathrm{d}x\ \mathrm{d}t \\ &\qquad + \int_{t_1}^{t_2}\int_{\mathbb{R}^N} \lambda(x)\left(\varphi(x) - \int_{\mathbb{R}^N}\varphi(y)\mu(x)(\mathrm{d}y)\right) u_{\mathcal{M},k}(x,t)\,\mathrm{d}x\ \mathrm{d}t. \end{aligned} \tag{60}$$

We first check that

$$T_{17} = T_{21} \tag{61}$$

and we now compare terms T_{18} and T_{22}, terms T_{19} and T_{23}, and terms T_{20} and T_{24}.

5.2. Convergence lemmas

Lemma 8. *Let us assume hypotheses $\mathcal{H}_0$ and let $\mathcal{M}$ be an admissible mesh of $\mathbb{R}^N$ in the sense of Definition* 1. *Let $\varepsilon \in [0, V]$ be given and for $k > 0$, $\varphi \in C_c^1(\mathbb{R}^N)$ and $0 \le t_1 < t_2$, let T_{18} and T_{22} respectively defined by* (53) *and* (57). *Then, for all $T > 0$, there is some C_{13} which depends only on φ, $\mathbf{v}$, T and C_2 such that*

$$|T_{18} - T_{22}| \le C_{13}\,(k + h_{\mathcal{M}})$$

for all $0 \le t_1 \le t_2 \le T$ and all $k \le C_2$.

Proof. Noting that

$$|T_{18} - T_{22}| = \left|\sum_{n=n_1}^{n_2-1}\sum_{K\in\mathcal{M}}\int_{nk}^{(n+1)k}\int_K \mathrm{div}\,(\mathbf{v}(x))\left(\varphi_K u_{\mathcal{M},k}(x,t+k) - \varphi(x) u_{\mathcal{M},k}(x,t)\right)\mathrm{d}x\ \mathrm{d}t\right|,$$

we first write $|T_{18} - T_{22}| \le T_{25} + T_{26}$ with

$$T_{25} = \left| \sum_{n=n_1}^{n_2-1} \sum_{K \in \mathcal{M}} \int_{nk}^{(n+1)k} \int_K \operatorname{div}(\mathbf{v}(x))\, \varphi_K \left(u_{\mathcal{M},k}(x, t+k) - u_{\mathcal{M},k}(x,t)\right) \mathrm{d}x\, \mathrm{d}t \right|,$$

$$T_{26} = \left| \sum_{n=n_1}^{n_2-1} \sum_{K \in \mathcal{M}} \int_{nk}^{(n+1)k} \int_K \operatorname{div}(\mathbf{v}(x))\, (\varphi_K - \varphi(x))\, u_{\mathcal{M},k}(x,t)\, \mathrm{d}x\, \mathrm{d}t \right|.$$

Besides, setting $C_{\mathbf{v}}$ such that $|\operatorname{div}(\mathbf{v}(x))| \le C_{\mathbf{v}}$ a.s.

$$\begin{aligned} T_{25} &= \left| \sum_{K \in \mathcal{M}} \int_K \operatorname{div}(\mathbf{v}(x))\, \varphi_K \left(\int_{n_2 k}^{(n_2+1)k} u_{\mathcal{M},k}(x,t)\, \mathrm{d}t - \int_{n_1 k}^{(n_1+1)k} u_{\mathcal{M},k}(x,t)\, \mathrm{d}t \right) \mathrm{d}x \right| \\ &\le C_{\mathbf{v}} \|\varphi\|_\infty k \sum_{K \in \mathcal{M}} m_K \left| u_K^{(n_2)} - u_K^{(n_1)} \right| \\ &\le 2 C_{\mathbf{v}} \|\varphi\|_\infty k \end{aligned}$$

and, using the existence of C_φ such that

$$|\varphi_K - \varphi(x)| \le C_\varphi h_{\mathcal{M}} \quad \text{for } x \in K,$$

we get

$$\begin{aligned} T_{26} &= \left| \sum_{n=n_1}^{n_2-1} \sum_{K \in \mathcal{M}} \int_{nk}^{(n+1)k} \int_K \operatorname{div}(\mathbf{v}(x))\, (\varphi_K - \varphi(x))\, u_{\mathcal{M},k}(x,t)\, \mathrm{d}x\, \mathrm{d}t \right| \\ &\le C_\varphi h_{\mathcal{M}} C_{\mathbf{v}} \left| \sum_{n=n_1}^{n_2-1} \sum_{K \in \mathcal{M}} \int_{nk}^{(n+1)k} \int_K u_{\mathcal{M},k}(x,t)\, \mathrm{d}x\, \mathrm{d}t \right| \\ &\le C_\varphi h_{\mathcal{M}} C_{\mathbf{v}} (t_2 - t_1 + k). \end{aligned}$$

Whence the result. ■

Lemma 9. *Let us assume hypotheses $\mathcal{H}_0$. Let $N \in \mathbb{N}^\star$ and let $\mathcal{M}$ be an admissible mesh of $\mathbb{R}^N$ in the sense of Definition* 1. *Let (q, m) as in Lemma* 4*; namely, if $N \ge 2$, let $q \in (1, \frac{N+2}{N+1})$ be given and let $m := (2-q)(N+1)/N - 1$; if $N = 1$, let $q \in (1, \sqrt{2})$ and $m \in (0, \frac{2-q^2}{q})$ be given. Let $\varepsilon \in (0, V]$ be given and for $k > 0$, $\varphi \in C_c^1(\mathbb{R}^N)$ and $0 \le t_1 < t_2$, let T_{19} and T_{23} respectively defined by* (54) *and* (58)*. Then, for all $T > 0$, there is some C_{14} which depends only on φ, $\mathbf{v}$, ε, N, T, q, m, Λ, C_1 and C_2 such that*

$$|T_{19} - T_{23}| \le C_{14} \left(k + h_{\mathcal{M}}^{1-1/q} \right)$$

for all $0 \le t_1 \le t_2 \le T$, all $k \le C_2$, all admissible mesh $\mathcal{M}$ such that $h_{\mathcal{M}} \le C_2$.

Proof. We first write $T_{19} = T_{27} + T_{28}$ where

$$T_{27} = \frac{k}{2} \sum_{n=n_1}^{n_2-1} \sum_{K \in \mathcal{M}} \sum_{L \in \mathcal{N}_K} \mathrm{m}(K|L) w_{K,L} (u_K^{(n+1)} - u_L^{(n+1)}) \varphi_K,$$

$$T_{28} = -\frac{k}{2} \sum_{n=n_1}^{n_2-1} \sum_{K \in \mathcal{M}} \sum_{L \in \mathcal{N}_K} \mathrm{m}(K|L) v_{K,L} (u_K^{(n+1)} - u_L^{(n+1)}) \varphi_K.$$

It is easy to check that

$$T_{27} = \frac{k}{4} \sum_{n=n_1}^{n_2-1} \sum_{K \in \mathcal{M}} \sum_{L \in \mathcal{N}_K} \mathrm{m}(K|L) w_{K,L} (u_K^{(n+1)} - u_L^{(n+1)}) (\varphi_K - \varphi_L).$$

Choosing $R > 0$ such that the support of φ is included in $B(0, R - 3C_2)$, we may restrict the summation on K in T_{27} to $K \in \mathcal{M}_{R-C_2}$, where $\mathcal{M}_{R'} = \left\{ K \in \mathcal{M} \text{ s.t. } K \subset B\left(0, R'\right) \right\}$ for all $R' > 0$. Indeed, it is easy to check that $\varphi_K \ne 0$

implies that $K \in \mathcal{M}_{R-3C_2+h_{\mathcal{M}}} \subset \mathcal{M}_{R-2C_2}$ and that $\varphi_L \neq 0$ implies that $L \in \mathcal{M}_{R-2C_2}$ and hence $K \in \mathcal{M}_{R-C_2}$ because $L \in \mathcal{N}_K$. For such an $R > 0$, we now consider $\hat{u}$ defined as in Lemma 3, and we set

$$T_{29} = h_{\mathcal{M}} k \sum_{n=0}^{\lfloor T/k \rfloor} \sum_{K \in \mathcal{M}_{R-C_2}} \sum_{L \in \mathcal{N}_K} \mathrm{m}(K|L) \left| u_K^{(n+1)} - u_L^{(n+1)} \right|.$$

Due to $\hat{u}_L^{(n+1)} = u_L^{(n+1)}$ for all $L \subset B(0, R)$ (the same for K), we have

$$T_{29} = h_{\mathcal{M}} k \sum_{n=0}^{\lfloor T/k \rfloor} \sum_{K \in \mathcal{M}_{R-C_2}} \sum_{L \in \mathcal{N}_K} \mathrm{m}(K|L) \left| \hat{u}_K^{(n+1)} - \hat{u}_L^{(n+1)} \right|.$$

Using $|\varphi_K - \varphi_L| \leq 2C_\varphi h_{\mathcal{M}}$ and $w_{K,L} \leq V$, we get

$$|T_{27}| \leq \frac{1}{2} C_\varphi V T_{29} \leq \frac{1}{2} C_\varphi V h_{\mathcal{M}} T_{12} \leq \frac{1}{2} C_\varphi V C_5 \, h_{\mathcal{M}}^{1-1/q} \tag{62}$$

using (39) (and (27); see Remark 5).

Now, let us set

$$T_{30} = -\sum_{n=n_1+1}^{n_2} \int_{nk}^{(n+1)k} \int_{\mathbb{R}^N} \operatorname{div}(\varphi(x)\,\mathbf{v}(x))\, u_{\mathcal{M},k}(x,t)\,\mathrm{d}x\,\mathrm{d}t.$$

We first note that

$$\begin{aligned} |T_{23} - T_{30}| &= \left| -\int_{t_1}^{(n_1+1)k} \int_{\mathbb{R}^N} \operatorname{div}(\varphi(x)\,\mathbf{v}(x))\, u_{\mathcal{M},k}(x,t)\,\mathrm{d}x\,\mathrm{d}t \right. \\ &\quad \left. + \int_{t_2}^{(n_2+1)k} \int_{\mathbb{R}^N} \operatorname{div}(\varphi(x)\,\mathbf{v}(x))\, u_{\mathcal{M},k}(x,t)\,\mathrm{d}x\,\mathrm{d}t \right| \\ &\leq kC \end{aligned}$$

with C some non-negative constant which depends only on φ and $\mathbf{v}$.

Besides,

$$\begin{aligned} T_{30} &= -k \sum_{n=n_1+1}^{n_2} \sum_{K \in \mathcal{M}_{R-C_2}} u_K^{(n)} \int_K \operatorname{div}(\varphi(x)\,\mathbf{v}(x))\,\mathrm{d}x \\ &= -k \sum_{n=n_1}^{n_2-1} \sum_{K \in \mathcal{M}_{R-C_2}} u_K^{(n+1)} \sum_{L \in \mathcal{N}_K} \int_{K|L} \varphi(x)\,(\mathbf{v}(x) \cdot \mathbf{n}_{KL})\,\mathrm{d}s(x) \end{aligned}$$

and

$$T_{28} = -k \sum_{n=n_1}^{n_2-1} \sum_{K \in \mathcal{M}_{R-C_2}} u_K^{(n+1)} \sum_{L \in \mathcal{N}_K} \mathrm{m}(K|L) v_{K,L} \frac{\varphi_K + \varphi_L}{2}.$$

We get

$$T_{28} - T_{30} = k \sum_{n=n_1}^{n_2-1} \sum_{K \in \mathcal{M}_{R-C_2}} u_K^{(n+1)} \sum_{L \in \mathcal{N}_K} \mathrm{m}(K|L) \hat{\varphi}_{K,L}$$

with

$$\begin{aligned} \hat{\varphi}_{K,L} &:= \frac{1}{\mathrm{m}(K|L)} \int_{K|L} \varphi(x)\,(\mathbf{v}(x) \cdot \mathbf{n}_{KL})\,\mathrm{d}s(x) - v_{K,L} \frac{\varphi_K + \varphi_L}{2} \\ &= -\hat{\varphi}_{L,K}. \end{aligned}$$

We derive

$$T_{28} - T_{30} = \frac{k}{2} \sum_{n=n_1}^{n_2-1} \sum_{K \in \mathcal{M}_{R-C_2}} \sum_{L \in \mathcal{N}_K} \left(u_K^{(n+1)} - u_L^{(n+1)} \right) \mathrm{m}(K|L) \hat{\varphi}_{K,L}$$

with

$$\begin{aligned}
\left| \hat{\varphi}_{K,L} \right| &\leq \frac{1}{\mathrm{m}(K|L)} \int_{K|L} \left| \varphi(x) - \frac{\varphi_K + \varphi_L}{2} \right| \times |\mathbf{v}(x) \cdot \mathbf{n}_{KL}| \,\mathrm{d}s(x) \\
&\leq C_\varphi h_{\mathcal{M}} \frac{1}{\mathrm{m}(K|L)} \int_{K|L} |\mathbf{v}(x) \cdot \mathbf{n}_{KL}| \,\mathrm{d}s(x) \\
&\leq C_\varphi h_{\mathcal{M}} V.
\end{aligned}$$

Hence

$$\begin{aligned}
|T_{28} - T_{30}| &\leq C_\varphi h_{\mathcal{M}} V \frac{k}{2} \sum_{n=n_1}^{n_2-1} \sum_{K \in \mathcal{M}_{R-C_1}} \sum_{L \in \mathcal{N}_K} \left| u_K^{(n+1)} - u_L^{(n+1)} \right| \mathrm{m}(K|L) \\
&\leq \frac{C_\varphi}{2} V T_{29} \\
&\leq \frac{C_\varphi}{2} V C_5 \, h_{\mathcal{M}}^{1-1/q} \qquad (63)
\end{aligned}$$

and the result, using

$$|T_{19} - T_{23}| \leq |T_{27}| + |T_{28} - T_{30}| + |T_{30} - T_{23}|. \quad \blacksquare$$

Lemma 10. *Let us assume hypotheses $\mathcal{H}_0$ and let $\mathcal{M}$ be an admissible mesh of $\mathbb{R}^N$ in the sense of Definition* 1. *Let $\varepsilon \in [0, V]$ be given and for $k > 0$, $\varphi \in C_c^1(\mathbb{R}^N)$ and $0 \leq t_1 < t_2$, let T_{20} and T_{24} respectively defined by* (55) *and* (59). *Then, for all $T > 0$, there is some C_{15} which depends only on φ, T and Λ such that*

$$|T_{20} - T_{24}| \leq C_{15} \, (k + h_{\mathcal{M}})$$

for all $0 \leq t_1 \leq t_2 \leq T$.

Proof. We first set $T_{20} = T_{31} + T_{32}$ and $T_{24} = T_{33} + T_{34}$ with

$$\begin{aligned}
T_{31} &= k \sum_{n=n_1}^{n_2-1} \sum_{K \in \mathcal{M}} \mathrm{m}(K) \lambda_K u_K^{(n+1)} \varphi_K, \\
T_{32} &= -k \sum_{n=n_1}^{n_2-1} \sum_{K \in \mathcal{M}} \sum_{L \in \mathcal{M}} \mathrm{m}(L) a_{L,K} u_L^{(n+1)} \varphi_K, \\
T_{33} &= \int_{t_1}^{t_2} \int_{\mathbb{R}^N} \lambda(x) \varphi(x) u_{\mathcal{M},k}(x,t) \,\mathrm{d}x \,\mathrm{d}t, \\
T_{34} &= -\int_{t_1}^{t_2} \int_{\mathbb{R}^N} \lambda(x) \int_{\mathbb{R}^N} \varphi(y) \mu(x)(\mathrm{d}y) u_{\mathcal{M},k}(x,t) \,\mathrm{d}x \,\mathrm{d}t.
\end{aligned}$$

We have

$$\begin{aligned}
T_{31} &= \sum_{n=n_1}^{n_2-1} \sum_{K \in \mathcal{M}} \int_K \int_{nk}^{(n+1)k} \lambda(x) \varphi_K u_{\mathcal{M},k}(x, t+k) \,\mathrm{d}x \\
&= \int_{(n_1+1)k}^{(n_2+1)k} \int_{\mathbb{R}^N} \lambda(x) \bar{\varphi}(x) \, u_{\mathcal{M},k}(x,t) \,\mathrm{d}x \,\mathrm{d}t
\end{aligned}$$

with

$$\bar{\varphi}(x) = \varphi_K \quad \text{for } x \in K.$$

Setting $T_{35} = \int_{t_1}^{t_2} \int_{\mathbb{R}^N} \lambda(x)\bar{\varphi}(x)\, u_{\mathcal{M},k}(x,t)\,\mathrm{d}x\,\mathrm{d}t$, we may write

$$|T_{31} - T_{33}| \le |T_{31} - T_{35}| + |T_{35} - T_{33}|$$

with

$$\begin{aligned} |T_{31} - T_{35}| &\le \left| \int_{t_2}^{(n_2+1)k} \int_{\mathbb{R}^N} \lambda(x)\varphi_K u_{\mathcal{M},k}(x,t)\,\mathrm{d}x\,\mathrm{d}t - \int_{t_1}^{(n_1+1)k} \int_{\mathbb{R}^N} \lambda(x)\varphi_K u_{\mathcal{M},k}(x,t)\,\mathrm{d}x\,\mathrm{d}t \right| \\ &\le 2k\Lambda \|\varphi\|_\infty \end{aligned}$$

and, due to $|\bar{\varphi}(x) - \varphi(x)| \le C_\varphi h_{\mathcal{M}}$ (on each $K \in \mathcal{M}$)

$$\begin{aligned} |T_{35} - T_{33}| &= \left| \int_{t_1}^{t_2} \int_{\mathbb{R}^N} \lambda(x)\left(\bar{\varphi}(x) - \varphi(x)\right) u_{\mathcal{M},k}(x,t)\,\mathrm{d}x\,\mathrm{d}t \right| \\ &\le C_\varphi h_{\mathcal{M}} \Lambda (t_2 - t_1). \end{aligned}$$

Similarly as for T_{31}, we now set

$$T_{36} = -\int_{t_1}^{t_2} \int_{\mathbb{R}^N} \lambda(x) \left(\int_{\mathbb{R}^N} \bar{\varphi}(y)\ \mu(x)(\mathrm{d}y) \right) u_{\mathcal{M},k}(x,t)\ \mathrm{d}x\ \mathrm{d}t.$$

We may write

$$|T_{32} - T_{34}| \le |T_{32} - T_{36}| + |T_{36} - T_{34}|.$$

Noting that

$$T_{32} = -\int_{(n_1+1)k}^{(n_2+1)k} \int_{\mathbb{R}^N} \lambda(x) \left(\int_{\mathbb{R}^N} \bar{\varphi}(y)\ \mu(x)(\mathrm{d}y) \right) u_{\mathcal{M},k}(x,t)\ \mathrm{d}x\ \mathrm{d}t,$$

we easily derive

$$|T_{32} - T_{36}| \le 2k\Lambda \|\varphi\|_\infty.$$

Besides,

$$\begin{aligned} |T_{36} - T_{34}| &= \left| \int_{t_1}^{t_2} \int_{\mathbb{R}^N} \lambda(x) \int_{\mathbb{R}^N} \left(\varphi(y) - \bar{\varphi}(y)\right) \mu(x)(\mathrm{d}y) u_{\mathcal{M},k}(x,t)\,\mathrm{d}x\,\mathrm{d}t \right| \\ &\le C_\varphi h_{\mathcal{M}} \Lambda (t_2 - t_1). \end{aligned}$$

We finally get

$$|T_{20} - T_{24}| \le 4k\Lambda \|\varphi\|_\infty + 2C_\varphi h_{\mathcal{M}} \Lambda (t_2 - t_1).$$

Hence the result. ■

We now state some continuity result of the approximate solution with respect to the time variable.

Lemma 11. *Let us assume hypotheses $\mathcal{H}_0$. Let $N \in \mathbb{N}^\star$ and let $\mathcal{M}$ be an admissible mesh of $\mathbb{R}^N$ in the sense of Definition 1. Let (q, m) as in Lemma 4, namely, if $N \ge 2$, let $q \in (1, \frac{N+2}{N+1})$ be given and let $m := (2-q)(N+1)/N - 1$; if $N = 1$, let $q \in (1, \sqrt{2})$ and $m \in (0, \frac{2-q^2}{q})$ be given. Let $\varepsilon \in (0, V]$, $k > 0$, $\varphi \in C_c^1(\mathbb{R}^N)$ and $T > 0$ be given. Then, there exists C_{16}, which only depends on φ, $\mathbf{v}$, ε, N, T, q, m, Λ, C_1 and C_2 such that*

$$\left| S_\varphi(\mathcal{M}, k, t_1, t_2) \right| \le C_{16} \left(|t_2 - t_1| + k + h_{\mathcal{M}}^{1-1/q} \right) \tag{64}$$

for all $0 \le t_1 \le t_2 \le T$, all $k \le C_2$, and all admissible mesh $\mathcal{M}$ such that $h_{\mathcal{M}} \le C_2 \le 1$, with

$$S_\varphi(\mathcal{M}, k, t_1, t_2) := \int_{\mathbb{R}^N} \varphi(x)\ u_{\mathcal{M},k}(x, t_2)\,\mathrm{d}x - \int_{\mathbb{R}^N} \varphi(x)\ u_{\mathcal{M},k}(x, t_1)\,\mathrm{d}x.$$

Proof. Starting from (60), we have

$$\begin{aligned}
\left|S_\varphi(\mathcal{M},k,t_1,t_2)\right| &= \Bigg|T_{21}+T_{22}+T_{23}+T_{24}+\int_{t_1}^{t_2}\int_{\mathbb{R}^N}(\mathbf{v}(x)\cdot\nabla\varphi(x))\,u_{\mathcal{M},k}(x,t)\,\mathrm{d}x\ \mathrm{d}t \\
&\quad -\int_{t_1}^{t_2}\int_{\mathbb{R}^N}\lambda(x)\left(\varphi(x)-\int_{\mathbb{R}^N}\varphi(y)\mu(x)(\mathrm{d}y)\right)u_{\mathcal{M},k}(x,t)\,\mathrm{d}x\ \mathrm{d}t\Bigg| \\
&\le \Bigg|T_{17}+T_{18}+T_{19}+T_{20}+\int_{t_1}^{t_2}\int_{\mathbb{R}^N}(\mathbf{v}(x)\cdot\nabla\varphi(x))\,u_{\mathcal{M},k}(x,t)\,\mathrm{d}x\ \mathrm{d}t \\
&\quad -\int_{t_1}^{t_2}\int_{\mathbb{R}^N}\lambda(x)\left(\varphi(x)-\int_{\mathbb{R}^N}\varphi(y)\mu(x)(\mathrm{d}y)\right)u_{\mathcal{M},k}(x,t)\,\mathrm{d}x\ \mathrm{d}t\Bigg| \\
&\quad +|T_{17}-T_{21}|+|T_{18}-T_{22}|+|T_{19}-T_{23}|+|T_{20}-T_{24}| \\
&\le \left|\int_{t_1}^{t_2}\int_{\mathbb{R}^N}\mathbf{v}(x)\,\nabla\varphi(x)u_{\mathcal{M},k}(x,t)\,\mathrm{d}x\ \mathrm{d}t\right| \\
&\quad +\left|\int_{t_1}^{t_2}\int_{\mathbb{R}^N}\lambda(x)\left(\varphi(x)-\int_{\mathbb{R}^N}\varphi(y)\mu(x)(\mathrm{d}y)\right)u_{\mathcal{M},k}(x,t)\,\mathrm{d}x\ \mathrm{d}t\right| \\
&\quad +C_{13}(k+h_{\mathcal{M}})+C_{14}\left(k+h_{\mathcal{M}}^{1-1/q}\right)+C_{15}(k+h_{\mathcal{M}})
\end{aligned}$$

due to $T_{17}+T_{18}+T_{19}+T_{19}=0$, equality (61) and Lemmas 8–10. We easily derive the result, using $h_{\mathcal{M}}^{1-1/q}\ge h_{\mathcal{M}}$ since $h_{\mathcal{M}}\le 1$. ■

We now come to the tightness of the family of approximate distributions and we begin with the construction of our Liapounov function (see the Introduction).

Lemma 12. *Let $N\in\mathbb{N}^*$ and let μ as in $\mathcal{H}_0$, namely $\mu:\mathbb{R}^N\to\mathcal{P}(\mathbb{R}^N)$ such that for all $\psi\in C_b(\mathbb{R}^N,\mathbb{R})$, the function $x\to\int\psi(y)\mu(x)(\mathrm{d}y)$ is continuous (and bounded) from $\mathbb{R}^N$ to $\mathbb{R}$. Let $\rho_{\mathrm{ini}}\in\mathcal{P}(\mathbb{R}^N)$ be given. Then there exists a Liapounov function $\mathcal{V}:\mathbb{R}^N\to\mathbb{R}_+$, and a real value $C_{17}>0$ independent of the data such that*

$$\mathcal{V}\ \textit{is Lipschitz continuous},\ \textit{with a constant lower or equal to } 1, \tag{65}$$

$$\lim_{|x|\to+\infty}\mathcal{V}(x)=+\infty, \tag{66}$$

$$\int_{\mathbb{R}^N}\mathcal{V}\,\mathrm{d}\rho_{\mathrm{ini}}<+\infty, \tag{67}$$

$$\int_{\mathbb{R}^N}\mathcal{V}(y)\ \mu(x)(\mathrm{d}y)\le\mathcal{V}(x)+C_{17}\quad \textit{for all } x\in\mathbb{R}^N. \tag{68}$$

Proof. We first remark that, thanks to the hypothesis on μ, the function $F_\mu:x\longmapsto\mu(x)(\mathrm{d}y)$ is continuous from $\mathbb{R}^N$ to $\mathcal{P}(\mathbb{R}^N)$ endowed with the weak topology. This implies that for all $R>0$, the family $\{\mu(x)(\mathrm{d}y):|x|\le R\}=F_\mu\left(\overline{B(0,R)}\right)$ is weakly compact. According to Prohorov's Theorem, the family $\{\mu(x)(\mathrm{d}y)\}_{x\in B(0,R)}$ is then tight (see, e.g., [2]), which allows us to introduce the function $f_\mu:\mathbb{R}_+^\star\times\mathbb{R}_+^\star\to\mathbb{R}_+^\star$, given by

$$\forall R\in\mathbb{R}_+^\star,\quad \forall\alpha\in\mathbb{R}_+^\star,\ f_\mu(R,\alpha)=\inf\left\{r\ge R,\ \sup_{x\in B(0,R)}\int_{\mathbb{R}^N\setminus B(0,r)}\mu(x)(\mathrm{d}y)\le\alpha\right\}. \tag{69}$$

Tightness of the probability measure ρ_{ini} allows to also introduce the function $f_{\mathrm{ini}}:\mathbb{R}_+^\star\times\mathbb{R}_+^\star\to\mathbb{R}_+^\star$ such that

$$\forall R\in\mathbb{R}_+^\star,\quad \forall\alpha\in\mathbb{R}_+^\star,\ f_{\mathrm{ini}}(R,\alpha)=\inf\left\{r\ge R,\ \int_{\mathbb{R}^N\setminus B(0,r)}\rho_{\mathrm{ini}}(\mathrm{d}y)\le\alpha\right\}. \tag{70}$$

We then recursively define the sequence $(R_n)_{n\in\mathbb{N}}$ by $R_0=0$ and

$$R_{n+1}=1+\max(f_\mu(R_n+1,1/(n+2)^3),\ f_{\mathrm{ini}}(R_n+1,1/(n+2)^3))$$

where 1 is added to the max to ensure $R_{n+1} > \max(f_\mu(R_n, 1/(n+2)^3), f_{\mathrm{ini}}(R_n, 1/(n+2)^3))$ and also added to R_n inside f_μ and f_{ini} because both functions are only defined for $R > 0$ whereas $R_0 = 0$.

We get the following properties:

$$\forall x \in B(0, R_n), \quad \int_{\mathbb{R}^N \setminus B(0, R_{n+1})} \mu(x)(\mathrm{d}y) \le \frac{1}{(n+2)^3} \tag{71}$$

and

$$\int_{\mathbb{R}^N \setminus B(0, R_{n+1})} \rho_{\mathrm{ini}}(\mathrm{d}y) \le \frac{1}{(n+2)^3}.$$

We now introduce the function $\mathcal{V} : \mathbb{R}^N \to \mathbb{R}$, given by

$$\forall x \in B(0, R_{n+1}) \setminus B(0, R_n), \quad \mathcal{V}(x) = n + \frac{|x| - R_n}{R_{n+1} - R_n} \tag{72}$$

with

$$n \le \mathcal{V}(x) < n + 1$$

for all $B(0, R_{n+1}) \setminus B(0, R_n)$.

Let $x \in \mathbb{R}^N$ be fixed and let $n \in \mathbb{N}^*$ be such that $x \in B(0, R_n) \setminus B(0, R_{n-1})$. (Note that the integer n exists and is unique because $B(0, R_0) = \emptyset$ and $R_m \to +\infty$ when $m \to +\infty$.)

Now, for all $m \ge n + 1$, we have

$$\begin{aligned}\int_{B(0, R_{m+1}) \setminus B(0, R_m)} \mathcal{V}(y)\mu(x)(\mathrm{d}y) &\le (m+1) \int_{\mathbb{R}^N \setminus B(0, R_m)} \mu(x)(\mathrm{d}y) \\ &\le \frac{m+1}{(m+1)^3} \\ &= \frac{1}{(m+1)^2}\end{aligned}$$

using $\mathcal{V}(y) < m + 1$, (71) and $x \in B(0, R_{m-1})$.

We derive

$$\begin{aligned}\int_{\mathbb{R}^N \setminus B(0, R_{n+1})} \mathcal{V}(y)\mu(x)(\mathrm{d}y) &= \sum_{m=n+1}^{\infty} \int_{B(0, R_{m+1}) \setminus B(0, R_m)} \mathcal{V}(y)\mu(x)(\mathrm{d}y) \\ &\le \sum_{m=n+2}^{\infty} \frac{1}{m^2}.\end{aligned} \tag{73}$$

Besides,

$$\int_{B(0, R_{n+1})} \mathcal{V}(y)\mu(x)(\mathrm{d}y) \le (n+1) \int_{B(0, R_{n+1})} \mu(x)(\mathrm{d}y) \le n + 1$$

and as $x \in B(0, R_n) \setminus B(0, R_{n-1})$, we also have $n + 1 \le \mathcal{V}(x) + 2$ and consequently

$$\int_{B(0, R_{n+1})} \mathcal{V}(y)\mu(x)(\mathrm{d}y) \le \mathcal{V}(x) + 2. \tag{74}$$

Summing (73) and (74) now gives

$$\int_{\mathbb{R}^N} \mathcal{V}(y)\mu(x)(\mathrm{d}y) \le \mathcal{V}(x) + C_{17}$$

for all $x \in \mathbb{R}^N$ with

$$C_{17} = 2 + \sum_{m=2}^{\infty} \frac{1}{m^2}.$$

In the same way, one easily gets that

$$\int_{\mathbb{R}^N} \mathcal{V}(y)\rho_{\text{ini}}(\mathrm{d}y) = \sum_{m=1}^{\infty} \int_{B(0,R_{m+1})\setminus B(0,R_m)} \mathcal{V}(y)\rho_{\text{ini}}(\mathrm{d}y) + \int_{B(0,R_1)} \mathcal{V}(y)\rho_{\text{ini}}(\mathrm{d}y)$$
$$\leq \sum_{m=1}^{\infty} \frac{1}{m^2} < +\infty,$$

which concludes the proof of the lemma. ∎

Lemma 13. *Let $t > 0$ be fixed. Then, under assumptions $\mathcal{H}_0$, the family $\left(u_{\mathcal{M},k}(x,s)\,\mathrm{d}x\right)_{(\varepsilon,\mathcal{M},k,s)\in\mathcal{F}_t}$ is tight where $\mathcal{F}_t = \{(\varepsilon,\mathcal{M},k,s) : \varepsilon \in [0,V]$, $\mathcal{M}$ is an admissible mesh in the sense of Definition 1 such that $h_{\mathcal{M}} \leq C_2$, $k \leq C_2$ and $s \in [0,t]\}$, namely: for all $\alpha > 0$, there is some $R > 0$ only depending on μ, $\mathbf{v}$, Λ, ρ_{ini}, C_1, C_2 and α such that*

$$\int_{\mathbb{R}^N\setminus B(0,R)} u_{\mathcal{M},k}(x,s)\,\mathrm{d}x < \alpha$$

for all $s \leq t$, all $k \leq C_2$, and all $\mathcal{M}$ such that $h_{\mathcal{M}} \leq C_2$.

Proof. Let $\mathcal{V}$ be the Liapounov function constructed in Lemma 12. For $n \in \mathbb{N}$, we set

$$s^{(n)}(\mathcal{V}) = \sum_{K\in\mathcal{M}} \mathrm{m}(K) u_K^{(n)} \mathcal{V}_K \quad \text{with} \quad \mathcal{V}_K = \frac{1}{\mathrm{m}(K)} \int_K \mathcal{V}(x)\ \mathrm{d}x$$

for all $K \in \mathcal{M}$. Our aim is to prove that $\sup_{\substack{h_{\mathcal{M}},\,k\leq C_2\\ n \text{ s.t. } nk\leq t}} s^{(n)}(\mathcal{V}) < +\infty$, which allows to conclude (see the end of this proof).

Starting from the scheme (14) multiplied by $\mathcal{V}_K$ and summing in K, we get

$$\begin{aligned} s^{(n+1)}(\mathcal{V}) &= s^{(n)}(\mathcal{V}) - k \sum_{K\in\mathcal{M}} \sum_{L\in\mathcal{N}_K} \mathrm{m}(K|L) \left(v_{K,L} \frac{u_K^{(n+1)} + u_L^{(n+1)}}{2} + \frac{w_{K,L}}{2}(u_K^{(n+1)} - u_L^{(n+1)}) \right) \mathcal{V}_K \\ &\quad - k \sum_{K\in\mathcal{M}} \mathrm{m}(K)\lambda_K u_K^{(n+1)} \mathcal{V}_K + k \sum_{K\in\mathcal{M}} \sum_{L\in\mathcal{M}} \mathrm{m}(L) a_{L,K} u_L^{(n+1)} \mathcal{V}_K \\ &= s^{(n)}(\mathcal{V}) + \frac{k}{2} \sum_{K\in\mathcal{M}} \sum_{L\in\mathcal{N}_K} \mathrm{m}(K|L) \left(w_{K,L} + v_{K,L}\right) u_K^{(n+1)} (\mathcal{V}_L - \mathcal{V}_K) \\ &\quad + k \sum_{K\in\mathcal{M}} \mathrm{m}(K) u_K^{(n+1)} \left(\sum_{L\in\mathcal{M}} a_{K,L}\mathcal{V}_L - \lambda_K \mathcal{V}_K \right). \end{aligned} \tag{75}$$

Using Lipschitz continuity of $\mathcal{V}$, we have $\mathcal{V}_L \leq |\mathcal{V}_L - \mathcal{V}(y)| + \mathcal{V}(y) \leq h_{\mathcal{M}} + \mathcal{V}(y)$ for all $y \in L$ and consequently

$$\begin{aligned} \sum_{L\in\mathcal{M}} a_{K,L}\mathcal{V}_L &= \frac{1}{\mathrm{m}(K)} \sum_{L\in\mathcal{M}} \int_K \lambda(x) \left(\int_L \mathcal{V}_L\, \mu(x)(\mathrm{d}y) \right) \mathrm{d}x \\ &\leq \frac{1}{\mathrm{m}(K)} \left(h_{\mathcal{M}} \int_K \lambda(x)\mathrm{d}x + \int_K \lambda(x) \left(\int_{\mathbb{R}^N} \mathcal{V}(y)\ \mu(x)(\mathrm{d}y) \right) \mathrm{d}x \right). \end{aligned} \tag{76}$$

Due to (68), we also have

$$\begin{aligned} \int_K \lambda(x) \left(\int_{\mathbb{R}^N} \mathcal{V}(y)\ \mu(x)(\mathrm{d}y) \right) \mathrm{d}x &\leq \int_K \lambda(x)(\mathcal{V}(x) + C_{17})\ \mathrm{d}x \\ &\leq \int_K \lambda(x)(|\mathcal{V}(x) - \mathcal{V}_K| + C_{17})\ \mathrm{d}x + \int_K \lambda(x)\mathcal{V}_K\ \mathrm{d}x \\ &\leq (h_{\mathcal{M}} + C_{17})\mathrm{m}(K)\lambda_K + \mathrm{m}(K)\lambda_K\mathcal{V}_K. \end{aligned}$$

Inequality (76) now gives

$$\sum_{L\in\mathcal{M}} a_{K,L}\mathcal{V}_L - \lambda_K\mathcal{V}_K \leq (2h_{\mathcal{M}} + C_{17})\,\lambda_K.$$

Using this last inequality, $|\mathcal{V}_L - \mathcal{V}_K| \le 2h_{\mathcal{M}}$ for $L \in \mathcal{N}_K$, $w_{K,L} + v_{K,L} \le 2V$ and (7), we derive from (75)

$$\begin{aligned} s^{(n+1)}(\mathcal{V}) &\le s^{(n)}(\mathcal{V}) + 2kh_{\mathcal{M}}V \sum_{K\in\mathcal{M}} \frac{C_1}{h_{\mathcal{M}}} \mathrm{m}(K) u_K^{(n+1)} + k\left(2h_{\mathcal{M}} + C_{17}\right)\Lambda \sum_{K\in\mathcal{M}} \mathrm{m}(K) u_K^{(n+1)} \\ &= s^{(n)}(\mathcal{V}) + 2kVC_1 + k\left(2h_{\mathcal{M}} + C_{17}\right)\Lambda. \end{aligned}$$

Setting $C_{18} := 2VC_1 + (2C_2 + C_{17})\Lambda$, we get

$$s^{(n+1)}(\mathcal{V}) \le s^{(n)}(\mathcal{V}) + kC_{18}$$

and hence

$$s^{(n)}(\mathcal{V}) \le s^{(0)}(\mathcal{V}) + C_{18}t$$

for all n such that $nk \le t$.

Besides,

$$\left|s^{(0)}(\mathcal{V}) - \int_{\mathbb{R}^N} \mathcal{V}\, \mathrm{d}\rho_{\mathrm{ini}}\right| \le \sum_{K\in\mathcal{M}} \left|\int_K \left(\mathcal{V}_K - \mathcal{V}(x)\right)\rho_{\mathrm{ini}}(\mathrm{d}x)\right| \le h_{\mathcal{M}} \sum_{K\in\mathcal{M}} \int_K \mathrm{d}\rho_{\mathrm{ini}} \le C_2$$

so that setting $C_{19} := \int_{\mathbb{R}^N} \mathcal{V}\, \mathrm{d}\rho_{\mathrm{ini}} + C_2 + C_{18}t$, we get

$$\sup_{\substack{h_{\mathcal{M}},k\le C_2 \\ n \text{ s.t. } nk\le t}} s^{(n)}(\mathcal{V}) \le C_{19} < +\infty$$

using (67).

We conclude using a classical argument, see e.g. [12] Proposition 1.II.5: due to (66), we know that, for each $\alpha > 0$, there is some R such that $\mathcal{V}(x) \ge \frac{C_{19}}{\alpha}$ for all $x \in B(0,R)^c$. We derive that for all $s \le t$

$$\frac{C_{19}}{\alpha} \int_{B(0,R)^c} u_{\mathcal{M},k}(x,s)\, \mathrm{d}x \le \int_{B(0,R)^c} u_{\mathcal{M},k}(x,s)\, \mathcal{V}(x)\, \mathrm{d}x \le \sup_{\substack{h_{\mathcal{M}},k\le C_2 \\ n \text{ s.t. } nk\le t}} s^{(n)}(\mathcal{V}) \le C_{19}$$

and

$$\int_{B(0,R)^c} u_{\mathcal{M},k}(x,s)\, \mathrm{d}x \le \alpha$$

which completes the proof. ■

5.3. *The convergence theorem*

We can now conclude with the convergence theorem:

Theorem 14. *Let $(\mathcal{M}_l, k_l)_{l\in\mathbb{N}}$ be a sequence such that, for all $l \in \mathbb{N}$, $\mathcal{M}_l$ is an admissible mesh in the sense of Definition* 1*, and such that* $\max\left(h_{\mathcal{M}_l}, k_l\right) \to 0$ *as $l \to \infty$. Let $\varepsilon \in (0, V]$ be fixed. Then, for all $t \in \mathbb{R}_+$, the sequence of probability measures $\left(u_{\mathcal{M}_l,k_l}(x,t)\,\mathrm{d}x\right)_{l\in\mathbb{N}}$ on $\mathbb{R}^N$ weakly converges to $\rho(t)(\cdot, \mathrm{d}x)$, where $(\rho(t)(\mathrm{d}x))_{t\ge 0}$ is the unique solution of the Chapman–Kolmogorov equation (see the Introduction).*

Proof. Let $\left(t_p\right)_{p\in\mathbb{N}}$ be a sequence of real numbers, dense in $\mathbb{R}_+$. Due to the tightness of $\left(u_{\mathcal{M}_l,k_l}\left(x,t_p\right)\mathrm{d}x\right)_{l\in\mathbb{N}}$ (Lemma 13) and Prohorov's theorem, for each $t_p \ge 0$, we may extract from $(\mathcal{M}_l, k_l)_{l\in\mathbb{N}}$ a sub-sequence $\left(\mathcal{M}_{\sigma_p(l)}, k_{\sigma_p(l)}\right)_{l\in\mathbb{N}}$ such that $\left(u_{\mathcal{M}_{\sigma_p(l)},k_{\sigma_p(l)}}\left(x,t_p\right)\mathrm{d}x\right)_{l\in\mathbb{N}}$ is weakly convergent (in the dual of $C_b\left(\mathbb{R}^N, \mathbb{R}\right)$) to some probability measure $\bar{\rho}(t_p)(\mathrm{d}x)$. Using a diagonal method, we can choose a sub-sequence $\left(\mathcal{M}_{\sigma(l)}, k_{\sigma(l)}\right)_{l\in\mathbb{N}}$ of $(\mathcal{M}_l, k_l)_{l\in\mathbb{N}}$, such that $\left(u_{\mathcal{M}_{\sigma(l)},k_{\sigma(l)}}\left(x,t_p\right)\mathrm{d}x\right)_{l\in\mathbb{N}}$ is weakly convergent for all $p \in \mathbb{N}$. Such a sub-sequence is again denoted by $(\mathcal{M}_l, k_l)_{l\in\mathbb{N}}$ in the following and for all $p \in \mathbb{N}$, $\left(u_{\mathcal{M}_l,k_l}\left(x,t_p\right)\mathrm{d}x\right)_{l\in\mathbb{N}}$ is now weakly convergent to the probability measure $\bar{\rho}(t_p)(\mathrm{d}x)$.

Let us prove that, for all $\varphi \in C_c^1(\mathbb{R}^N)$, the sequence $\left(\int_{\mathbb{R}^N} \varphi(x)\, u_{\mathcal{M}_l,k_l}(x,t)\,\mathrm{d}x\right)_{l\in\mathbb{N}}$ is a Cauchy sequence for all $t>0$. Let $t>0$ be fixed and let t_p be such that $|t-t_p| \le \alpha$ with $\alpha>0$. Since the sequence $\left(u_{\mathcal{M}_l,k_l}(x,t_p)\,\mathrm{d}x\right)_{l\in\mathbb{N}}$ is weakly convergent, there exists some l_0 such that, for all $l,n \ge l_0$,

$$\left|\int_{\mathbb{R}^N} \varphi(x)\, \left(u_{\mathcal{M}_l,k_l}(x,t_p) - u_{\mathcal{M}_n,k_n}(x,t_p)\right)\mathrm{d}x\right| \le \alpha.$$

We may now write, using (64),

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N} \varphi(x)\, u_{\mathcal{M}_l,k_l}(x,t)\,\mathrm{d}x - \int_{\mathbb{R}^N} \varphi(x)\, u_{\mathcal{M}_n,k_n}(x,t)\,\mathrm{d}x\right| \\
&\quad\le \left|S_\varphi\left(\mathcal{M}_l,k_l,\varepsilon,t,t_p\right)\right| + \left|S_\varphi\left(\mathcal{M}_n,k_n,\varepsilon,t,t_p\right)\right| + \left|\int_{\mathbb{R}^N} \varphi(x)\, \left(u_{\mathcal{M}_l,k_l}(x,t_p) - u_{\mathcal{M}_n,k_n}(x,t_p)\right)\mathrm{d}x\right| \\
&\quad\le C_{16}\left(2|t-t_p| + k_l + h_{\mathcal{M}_l}^{1-1/q} + k_n + h_{\mathcal{M}_n}^{1-1/q}\right) + \alpha \\
&\quad\le C_{16}\left(2\alpha + k_l + h_{\mathcal{M}_l}^{1-1/q} + k_n + h_{\mathcal{M}_n}^{1-1/q}\right) + \alpha
\end{aligned}$$

which shows that $\left(\int_{\mathbb{R}^N} \varphi(x)\, u_{\mathcal{M}_l,k_l}(x,t)\,\mathrm{d}x\right)_{l\in\mathbb{N}}$ is a Cauchy sequence and hence convergent.

We easily obtain the same property for all $\varphi \in C_c(\mathbb{R}^N)$ by the density of $C_c^1(\mathbb{R}^N)$. The function

$$\varphi \longmapsto \lim_{l\to+\infty} \int_{\mathbb{R}^N} \varphi(x)\, u_{\mathcal{M}_l,k_l}(x,t)\,\mathrm{d}x$$

now is a positive linear functional on $C_c(\mathbb{R}^N)$ and is then issued from a Radon measure $\bar{\rho}(t)(\mathrm{d}x)$. Due to tightness, the sequence $\left(u_{\mathcal{M}_l,k_l}(x,t)\,\mathrm{d}x\right)_{l\in\mathbb{N}}$ actually converges to $\bar{\rho}(t)(\mathrm{d}x)$ in the dual of $C_b(\mathbb{R}^N)$ and $\bar{\rho}(t)(\mathrm{d}x)$ is a probability measure.

Besides, for all $\varphi \in C_c^1(\mathbb{R}^N)$ and $0 \le t_1 \le t_2$,

$$\begin{aligned}
\lim_{l\to+\infty}(T_{21}+T_{22}+T_{23}+T_{24}) &= \lim_{l\to+\infty}(T_{17}+T_{18}+T_{19}+T_{20}) \\
&= 0
\end{aligned}$$

thanks to the relation $T_{17}+T_{18}+T_{19}+T_{20}=0$, to (61) and to Lemmas 8–10. Taking the limit in (60) with $t_1=0$ and $t_2=t$, we derive that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \varphi(x)\, \left(\bar{\rho}(t)(\mathrm{d}x) - \bar{\rho}(0)(\mathrm{d}x)\right) - \int_0^t \int_{\mathbb{R}^N} \left(\mathbf{v}(x)\cdot\nabla\varphi(x)\right)\bar{\rho}(u)(\mathrm{d}x)\,\mathrm{d}u \\
&\quad= \int_0^t \int_{\mathbb{R}^N} \lambda(x)\left(\int_{\mathbb{R}^N}\varphi(y)\mu(x)(\mathrm{d}y) - \varphi(x)\right) u_{\mathcal{M},k}(x,t)\,\mathrm{d}x\;\mathrm{d}t
\end{aligned}$$

for all $\varphi \in C_c^1(\mathbb{R}^N)$ and all $t \ge 0$.

$\bar{\rho}$ is then a function from $\mathbb{R}_+$ to $\mathcal{P}(\mathbb{R}^N)$ such that (2) is true for all $\varphi \in C_c^1(\mathbb{R}^N)$ and all $T>0$. Since $\bar{\rho}(0)(\mathrm{d}x) = \rho_{\mathrm{ini}}(\mathrm{d}x)$, we derive that $(\bar{\rho}(t)(\mathrm{d}x))_{t\in\mathbb{R}_+}$ is a solution of our initial problem. By the uniqueness result from [7], we now know that $\left(u_{\mathcal{M}_l,k_l}(x,t)\,\mathrm{d}x\right)_{l\in\mathbb{N}}$ weakly converges to $\rho(t)(\mathrm{d}x)$. We deduce in a classical way that the whole initial sequence converges in the same sense. ■

6. Extension to the case of a general PDMP

We are now interested in extending the results provided up to here for a GCP to the case of a general PDMP $(I_t, X_t)_{t\ge0}$ with values in $E\times\mathbb{R}^N$ or eventually $E\times F$, where E is a finite space and F is some Borel set of $\mathbb{R}^N$. Note that our appellation of a general PDMP is a little abusive because we do not envisage here possible jumps when the process reaches the state-space frontier as is usually considered for very general PDMPs; see [10].

Let us first specify our settings for a PDMP: just as a GCP, a PDMP $(I_t, X_t)_{t\ge0}$ randomly jumps at countable isolated times and between jumps, I_t is constant while X_t is deterministic: given that $I_t = i$ for $t\in[t_1,t_2)$, the environmental variable X_t follows the ordinary differential equation

$$\frac{\mathrm{d}X_t}{\mathrm{d}t} = \mathbf{v}(i, X_t), \quad \forall t\in\mathbb{R}_+ \text{ s.t. } I_t = i, \forall i\in E \tag{77}$$

where $\mathbf{v}$ is an application from $E \times \mathbb{R}^N$ to $\mathbb{R}^N$. Note that the deterministic evolution of $(X_t)_{t\geq 0}$ is now dependent on the discrete state $I_t = i$. At jump times, transitions from $(I_{t^-}, X_{t^-}) = (i, x) \in E \times \mathbb{R}^N$ to $(I_t, X_t) = (j, y) \in E \times \mathbb{R}^N$ are governed by a transition rate $a(i, j, x)$ from discrete state i to discrete state j which depends on $X_{t^-} = x$, and by a probability measure $\mu(i, j, x)(\mathrm{d}y)$ which stands for the conditional distribution of X_t given that $(I_{t^-}, I_t, X_{t^-}) = (i, j, x)$. All the previous results have been established in the case where E is reduced to a single element.

The assumptions $\mathcal{H}_0$ must now be extended to the following ones, denoted by $\mathcal{H}_{0,E}$ in what follows:

- the transition rate $a(i, j, \cdot) : \mathbb{R}^N \to \mathbb{R}_+$ is continuous and bounded for all $i, j \in E$; we set $\lambda(i, x) := \sum_{j\in E} a(i, j, x)$ and $\Lambda := \|\lambda\|_\infty$ with $\|\lambda\|_\infty := \sup_{x\in\mathbb{R}^N, i\in E} \lambda(i, x)$,
- the velocity $\mathbf{v} : E \times \mathbb{R}^N \to \mathbb{R}^N$ is such that $\mathbf{v}(i, \cdot) : x \longmapsto \mathbf{v}(i, x)$ is Lipschitz continuous and bounded by $V = \|\mathbf{v}\|_\infty > 0$ for all $i \in E$,
- the function $\mu : E^2 \times \mathbb{R}^N \to \mathcal{P}(\mathbb{R}^N)$ is such that for all $\psi \in C_b(\mathbb{R}^N)^E$ (continuous and bounded from $E \times \mathbb{R}^N \to \mathbb{R}$), the function $x \to \int_{\mathbb{R}^N} \psi(j, y)\mu(i, j, x)(\mathrm{d}y)$ is continuous from $\mathbb{R}^N$ to $\mathbb{R}$,
- the measure $\rho_{\mathrm{ini}}(\cdot, \mathrm{d}x)$ is a probability measure on $E \times \mathbb{R}^N$.

Under $\mathcal{H}_{0,E}$, it has been proven in [7] that the marginal distribution $\rho(t)(\cdot, \mathrm{d}x)$ of $(I_t, X_t)_{t\geq 0}$ is the single solution to the Chapman–Kolmogorov equation associated to the Markov process $(I_t, X_t)_{t\geq 0}$, which here is written as

$$\begin{aligned}
&\sum_{i\in E} \int_{\mathbb{R}^N} \varphi(i, x)\rho(t)(i, \mathrm{d}x) - \sum_{i\in E} \int_{\mathbb{R}^N} \varphi(i, x)\rho_{\mathrm{ini}}(i, \mathrm{d}x) - \sum_{i\in E} \int_0^t \int_{\mathbb{R}^N} \mathbf{v}(i, x) \cdot \nabla\varphi(i, x)\rho(s)(i, \mathrm{d}x)\, \mathrm{d}s \\
&\quad = \int_0^t \sum_{i\in E} \int_{\mathbb{R}^N} \sum_{j\in E} a(i, j, x) \left(\int_{\mathbb{R}^N} \varphi(j, y)\mu(i, j, x)(\mathrm{d}y) - \varphi(i, x) \right) \rho(s)(i, \mathrm{d}x)\, \mathrm{d}s, \\
&\quad \forall t \in \mathbb{R}_+, \forall \varphi \in C_c^1(\mathbb{R}^N)^E, \qquad (78)
\end{aligned}$$

where $C_c^1(\mathbb{R}^N)^E$ stands for the set of continuously differentiable functions from $E \times \mathbb{R}^N$ to $\mathbb{R}$ with a compact support.

Let us now adapt the scheme given in Section 2 to this more general framework.

Note that though a single mesh of $\mathbb{R}^N$ is used here, one could easily extend the results to meshes depending on $i \in E$ and even to a state space for $(I_t, X_t)_{t\geq 0}$ of the shape $\prod_{i\in E} \{i\} \times F_i$ with F_i a Borel set of some $\mathbb{R}^{N_i}$ with associated mesh $\mathcal{M}_i$ of F_i (all $i \in E$). Here again, the choice of simplicity has been made and a single mesh is used on a single space $F = \mathbb{R}^N$. This mesh $\mathcal{M}$ is assumed to be admissible in the sense of Definition 1.

For such a mesh, we set

$$v_{K,L}^{(i)} = \frac{1}{\mathrm{m}(K|L)} \int_{K|L} \mathbf{v}^{(i)}(x) \cdot \mathbf{n}_{KL} \mathrm{d}s(x), \quad \forall K \in \mathcal{M}, \forall L \in \mathcal{N}_K, \qquad (79)$$

and

$$w_{K,L}^{(i)} = \max(|v_{K,L}^{(i)}|, \varepsilon), \quad \forall K \in \mathcal{M}, \forall L \in \mathcal{N}_K, \forall i \in E \qquad (80)$$

for a given $\varepsilon \in [0, V]$.

For $K, L \in \mathcal{M}$, $i, j \in E$, we also set

$$\begin{aligned}
\lambda_K^{(i)} &= \frac{1}{\mathrm{m}(K)} \int_K \lambda^{(i)}(x)\, \mathrm{d}x \\
a_{K,L}^{(i,j)} &= \frac{1}{\mathrm{m}(K)} \int_K a^{(i,j)}(x) \left(\int_L \mu(i, j, x)(\mathrm{d}y) \right) \mathrm{d}x
\end{aligned}$$

with

$$\sum_{j\in E} \sum_{L\in\mathcal{M}} a_{K,L}^{(i,j)} = \lambda_K^{(i)} \qquad (81)$$

for all $i \in E$, all $K \in \mathcal{M}$.

For a given time step $k > 0$, the scheme then is written as

$$u_{i,K}^{(0)} = \frac{1}{\mathrm{m}(K)} \int_K \mathrm{d}\rho_{\mathrm{ini}}(i,x), \quad \forall K \in \mathcal{M}, \forall i \in E \tag{82}$$

with $u_{i,K}^{(0)} \geq 0$ for all $K \in \mathcal{M}$, all $i \in E$ and $\sum_{i\in E}\sum_{K\in\mathcal{M}} \mathrm{m}(K)u_{i,K}^{(0)} = 1$, because ρ_{ini} is a probability measure, and

$$\begin{aligned}
&\mathrm{m}(K)(u_{i,K}^{(n+1)} - u_{i,K}^{(n)}) + k \sum_{L\in\mathcal{N}_K} \mathrm{m}(K|L) \left(v_{K,L}^{(i)} \frac{u_{i,K}^{(n+1)} + u_{i,L}^{(n+1)}}{2} + \frac{w_{K,L}^{(i)}}{2} (u_{i,K}^{(n+1)} - u_{i,L}^{(n+1)}) \right) \\
&\quad = -k\mathrm{m}(K)\lambda_K^{(i)} u_{i,K}^{(n+1)} + k \sum_{j\in E} \sum_{L\in\mathcal{M}} \mathrm{m}(L) a_{L,K}^{(j,i)} u_{j,L}^{(n+1)}, \quad \forall K \in \mathcal{M}, \forall i \in E, \forall n \in \mathbb{N}.
\end{aligned} \tag{83}$$

Looking at the differences between (81)–(83) and (12)–(14), one can see that the only difference between the case of E reduced to a singleton and the general case is that each time a summation on $L \in \mathcal{M}$ (but not on $L \in \mathcal{N}_K$) appears in the case of E reduced to a singleton, one should sum on $(j, L) \in E \times \mathcal{M}$ in the case of a general E. This is the key to extending all results and proofs of the present paper to the case of a general E. In this way, under assumptions $\mathcal{H}_{0,E}$, for each admissible mesh $\mathcal{M}$ of $\mathbb{R}^N$ in the sense of Definition 1, each $k > 0$ and each $\varepsilon \in [0, V]$, we easily show the existence and uniqueness of a family $(u_{i,K}^{(n)})_{i\in E, K\in\mathcal{M}, n\in\mathbb{N}}$ such that (82)–(83) hold and $\sum_{i\in E}\sum_{K\in\mathcal{M}} \mathrm{m}(K) |u_{i,K}^{(n)}| < \infty$ for all $n \in \mathbb{N}$. Moreover, this family is such that $u_{i,K}^{(n)} \geq 0, \forall K \in \mathcal{M}, \forall n \in \mathbb{N}, \forall i \in E$ and $\sum_{i\in E}\sum_{K\in\mathcal{M}} \mathrm{m}(K)\ u_{i,K}^{(n)} = 1, \forall n \in \mathbb{N}$.

This allows us to associate to such a family an approximate solution $u_{\mathcal{M},k}(\cdot, x, t)\ \mathrm{d}x = \left(u_{\mathcal{M}_l,k_l}(i, x, t)\,\mathrm{d}x\right)_{i\in E}$ of the CK equation (78), setting

$$u_{\mathcal{M},k}(i, x, t) = u_{i,K}^{(n)}, \quad \text{for a.e. } x \in K, \forall K \in \mathcal{M}, \forall t \in [nk, (n+1)k), \forall n \in \mathbb{N}, \forall i \in E. \tag{84}$$

This leads to the following convergence theorem for a general E, the proof of which is a straightforward adaptation from that of Theorem 14.

Theorem 15. *Let $(\mathcal{M}_l, k_l)_{l\in\mathbb{N}}$ be a sequence such that, for all $l \in \mathbb{N}$, $\mathcal{M}_l$ is an admissible mesh in the sense of Definition* 1, *and such that* $\max\left(h_{\mathcal{M}_l}, k_l\right) \to 0$ *as $l \to \infty$. Let $\varepsilon \in (0, V]$ be fixed. Then, for all $t \in \mathbb{R}_+$, the sequence of probability measures $\left(u_{\mathcal{M}_l,k_l}(\cdot, x, t)\,\mathrm{d}x\right)_{l\in\mathbb{N}}$ on $E\times\mathbb{R}^N$ weakly converges to $\rho(t)(\cdot, \mathrm{d}x) = (\rho(t)(i, \mathrm{d}x))_{i\in E}$, where $(\rho(t)(\cdot, \mathrm{d}x))_{t\geq 0}$ is the unique solution of the Chapman–Kolmogorov equation* (78).

7. Numerical results

We here provide two numerical examples. The first one is very simple and is devoted to observe the diffusion due to the non-vanishing viscosity term in the case when $\varepsilon > 0$. The second one is a simplified case of an industrial problem.

7.1. First example

We here consider a GCP $(X_t)_{t\geq 0}$ (no I_t) with no jump ($\lambda(x) = 0$) and with X_0 uniformly distributed on $[-1, 0)$:

$$\rho_{\mathrm{ini}}(\mathrm{d}x) = \mathbf{1}_{[-1,0)}(x)\,\mathrm{d}x.$$

The velocity $\mathbf{v}$ is given by

$$\mathbf{v}(x) = \begin{cases} 1 & \text{if } x < -1 \\ -x & \text{if } -1 \leq x < +1 \\ -1 & \text{if } x \geq +1. \end{cases}$$

As there is no jump, X_t simply is the single solution of $\frac{\mathrm{d}y}{\mathrm{d}t} = \mathbf{v}(y)$ such that $y(0) = X_0$, with $X_0 \in [-1, 0)$. We derive $X_t = X_0 e^{-t}$ and X_t is uniformly distributed on $[-e^{-t}, 0)$:

$$\rho(t)(\mathrm{d}x) = \rho_t(x)\ \mathrm{d}x$$

with $\rho_t(x) = \mathbf{1}_{[-e^{-t},0)}(x)\,e^t$.

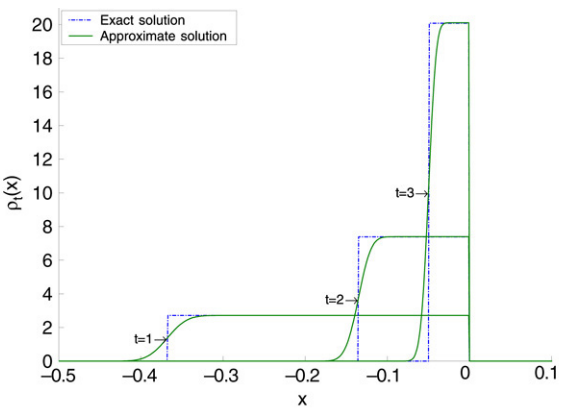

Fig. 1. Exact and approximate solutions for $\rho_t(x)$, for the case $\varepsilon = 0$.

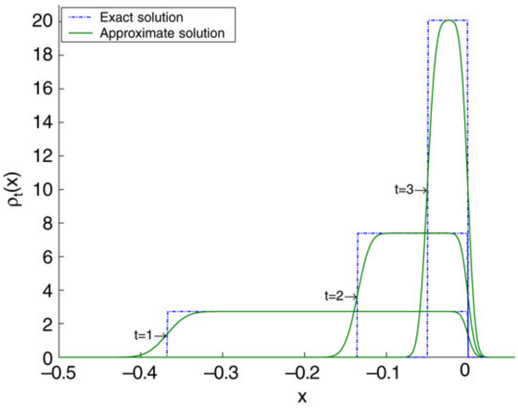

Fig. 2. Exact and approximate solutions for $\rho_t(x)$, for the case $\varepsilon = 10^{-1}$.

Taking $\mathcal{M} = \{[kh, (k+1)h), k \in \mathbb{Z}\}$ with $h = 10^{-3}$ as space step and $k = 10^{-3}$ as time step, the exact and approximate solutions for the probability density function (p.d.f.) $\rho_t(x)$ are plotted in Figs. 1 and 2 for $t = 1, 2, 3$, with respectively $\varepsilon = 0$ and $\varepsilon = 0.1$.

Fig. 1 corresponds to a classical upstream weighting scheme and all the mass of probability of the approximate solution is concentrated in $[-1, 0)$. In contrast, Fig. 2 corresponds to a scheme with a non-vanishing viscosity term and, as expected, we can observe diffusion around 0. Such a diffusion is however not very important: it is concentrated on the band where $|\mathbf{v}(x)| \leq \varepsilon$, namely on $[-\varepsilon, \varepsilon]$, and taking smaller ε easily leads to indistinguishable figures. Due to that, the results for the next example are only provided for $\varepsilon = 0$.

7.2. *Second example*

This second example is mainly drawn from a benchmark proposed by the French company Air Liquide [1], already studied in [13,15,18]. In order not to spend too long in describing the test-case, we here consider a simplified version: a gas production device is considered. It is composed of one production unit, which can be up or down, under repair. When up, the production rate of the unit varies between nominal and maximal rates, with nominal rate $\phi_{\text{nom}} = 7\,500$ m^3/h and maximal rate $\phi_{\max} = 10\,000$ m^3/h. When down, the production rate of the unit is zero. The device is required to produce gas at the nominal rate ϕ_{nom}. In order to prevent the device production to be stopped due to failures of the unit, a reservoir $\mathcal{R}$ is used, with maximal capacity $R = 2 \times 10^6$ m^3: when the unit is down, the device production is achieved by taking in $\mathcal{R}$ the required production, at least as long as the level in $\mathcal{R}$ is not too low. When the unit is up, its production rate is nominal as long as $\mathcal{R}$ is full. When the level in $\mathcal{R}$ is lower, the unit produces at a higher rate (maximal rate as long as the level in $\mathcal{R}$ is not too high) and the complementary production is used to refill $\mathcal{R}$. The device production rate then is a function of the unit state and of the level in $\mathcal{R}$.

We assume the following:

- the repair time of the unit is log-normally distributed, with p.d.f. $f_{(\bar{t},\sigma)}(t)$:

$$f_{(\bar{t},\sigma)}(t) = \frac{1}{t\,\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\ln(t/\bar{t})}{\sigma}\right)^2\right) \quad \text{for } t > 0,$$

$\bar{t} = \exp(0.23) \simeq 1.26\ h$, $\sigma = 2.25\ h$, with mean (Mean Down Time) $MDT = 15.8\ h$ and with a standard deviation equal to 198 h. The associated hazard rate function then is

$$h_{(\bar{t},\sigma)}(t) = f_{(\bar{t},\sigma)}(t) \Big/ \int_t^{+\infty} f_{(\bar{t},\sigma)}(s)\,\mathrm{d}s \quad \text{for } t > 0 \tag{85}$$

- the time to failure of the unit is Weibull distributed with associated hazard rate function

$$h_{(\alpha,\beta)} = \alpha\beta t^{\beta-1} \quad \text{for } t > 0,$$

$\alpha = \frac{1}{10^3}\ h$, $\beta = 1.01$, with mean (Mean Up Time) $MUT = 930\ h$ and with a standard deviation equal to 921 h.

We set $E = \{0, 1\}$, where 0, 1 are the down and up states for the unit, respectively. We then describe the time-evolution of the system by a PDMP $(I_t, X_t)_{t\geq 0}$ with values in $E \times \mathbb{R}^2$ where $X_t = (X_{1,t}, X_{2,t})$: component $X_{1,t}$ stands for the time elapsed in the current discrete state; for instance, if the unit is down at time t, component $X_{1,t}$ then stands for the time elapsed since the beginning of the on-going repair at time t. Component $X_{2,t}$ stands for the level in the reservoir.

The transition rate functions $a(i, j, x)$ and the probability measures $\mu(i, j, x)(\mathrm{d}y)$ are then given by

$$a(1, 0, x) = \begin{cases} h_{(\alpha,\beta)}(x_1) & \text{if } x_1 \geq 0 \\ h_{(\alpha,\beta)}(0) = 0 & \text{if } x_1 < 0 \end{cases}$$

$$a(0, 1, x) = \begin{cases} h_{(\bar{t},\sigma)}(x_1) & \text{if } x_1 \geq 0 \\ h_{(\bar{t},\sigma)}(0) = 0 & \text{if } x_1 < 0 \end{cases}$$

$$\mu(1, 0, x)(\mathrm{d}y) = \mu(0, 1, x)(\mathrm{d}y) = \delta_{(0,x_2)}(\mathrm{d}y)$$

where $x = (x_1, x_2) \in \mathbb{R}^2$.

We assume the speed of filling–emptying of the reservoir to be as follows: we set $r = \frac{R}{10}$ and, for $x = (x_1, x_2) \in \mathbb{R}^2$, we take

$$\begin{aligned} \mathbf{v}(1, x) &= \begin{cases} (1, \phi_{\max} - \phi_{\mathrm{nom}}) & \text{if } x_2 < R - r \\ \left(1, (\phi_{\max} - \phi_{\mathrm{nom}})\dfrac{(R - x_2)^+}{r}\right) & \text{if } R - r \leq x_2 \end{cases} \\ &= \left(1, (\phi_{\max} - \phi_{\mathrm{nom}}) \min\left(\frac{(R - x_2)^+}{r}, 1\right)\right) \end{aligned}$$

and

$$\begin{aligned} \mathbf{v}(0, x) &= \begin{cases} \left(1, -\phi_{\mathrm{nom}}\dfrac{x_2^+}{r}\right) & \text{if } x_2 < r \\ (1, -\phi_{\mathrm{nom}}) & \text{if } r \leq x_2 \end{cases} \\ &= \left(1, -\phi_{\mathrm{nom}} \min\left(\frac{x_2^+}{r}, 1\right)\right). \end{aligned}$$

The initial state is $(1, (0, R))$. Due to the shape of $\mathbf{v}(i, x)$ for $i = 0, 1$, we can see that $(I_t, X_t)_{t\geq 0}$ actually takes range in $E \times \mathbb{R}_+ \times (0, R]$ and the reservoir may never be emptied entirely. Similarly, once the reservoir has been emptied a little, the level cannot reach R any more and we hence know that $X_t \in \mathbb{R}_+ \times (0, R)$ after a while.

We assume the device production rate function to be given by

$$\phi(1, x_2) = \phi_{\mathrm{nom}}$$

$$\phi(0, x_2) = \phi_{\mathrm{nom}} \min\left(\frac{x_2}{r}, 1\right).$$

Quantities of interest for this benchmark are (e.g.)

- The availability at time t, namely the probability that the system is up at time t:

$$A_t = \mathbb{P}(I_t = 1) = \int_{\mathbb{R}^2} \rho(t)(1, \mathrm{d}x).$$

- The production availability at time t, namely the mean ratio at time t of effective production by nominal production:

$$\begin{aligned} PA_t &= \frac{\mathbb{E}\left(\phi\left(I_t, X_{2,t}\right)\right)}{\phi_{\text{nom}}} = A_t + \mathbb{E}\left(\mathbf{1}_{\{I_t=0\}} \times \min\left(\frac{X_{2,t}}{r}, 1\right)\right) \\ &= A_t + \int_{\mathbb{R}^2} \min\left(\frac{x_2}{r}, 1\right) \rho(t)(0, (\mathrm{d}x_1, \mathrm{d}x_2)). \end{aligned}$$

As already mentioned in the Introduction, note that most quantities of industrial interest in the reliability field may similarly be written in the form

$$\mathbb{E}(f(I_t, X_t)) = \sum_{i \in E} \int_{\mathbb{R}^N} f(i, x) \rho(t)(i, \mathrm{d}x)$$

or

$$\mathbb{E}\left(\int_0^t f(I_u, X_u)\, \mathrm{d}u\right) = \sum_{i \in E} \int_0^t \int_{\mathbb{R}^N} f(i, x) \rho(s)(i, \mathrm{d}x)\ \mathrm{d}s$$

with f continuous and bounded.

In our example, to get approximate A_t and PA_t as well as their asymptotic values, we have computed the approximate marginal distribution provided by the numerical scheme up to t large enough so that the results are stabilised ($t \simeq 10^6$). Since the repair and producing times are not bounded, we have approximated the domain for $X_{1,t}$ by some $[0, T]$ with T large (2.10^6 hours), and we have used a logarithmic step in order to be as accurate as possible for short times. As for the level in the reservoir $X_{2,t}$, its domain has been divided into 400 parts. This has led to 4×10^6 cells for the whole domain of the environmental condition $X_t = \left(X_{1,t}, X_{2,t}\right)$. Using easiness of the implicit scheme, the time step has been taken as variable, and adjusted in order to observe an average variation lower that 10^{-4} for the unit availability. Thanks to this method, the number of time steps needed for the computation over a time period equal to 10^6 hours is 174, the first time step being equal to 0.1 hour, and the last one being greater than 10^5 hours. The linear systems involved in the computations have been solved by BiCGStab. The computations take about 5 minutes on a standard PC.

In order to control our results, 95% confidence bands have also been computed for A_t and PA_t by Monte Carlo (MC) simulation with sample size equal to 10^6. Due to the lengths of the MC simulation, such bands have only been computed on [0, 5000]. Another control is given by the asymptotic availability A_∞, which is here reachable by another method: as the evolution of the unit is independent of the level in the reservoir, the unit actually evolves according to an alternate renewal process. We derive

$$A_\infty = \frac{MUT}{MUT + MDT} \simeq 0.983\,25$$

where MUT and MDT have been defined earlier with the data. (Note that the asymptotic production availability cannot be similarly computed.)

The asymptotic availability and production availability obtained by the numerical scheme (for $t \simeq 10^6$) respectively are

$$\hat{A}_\infty \simeq 0.983\,2$$
$$\widehat{PA}_\infty \simeq 0.994\,85.$$

The approximate solutions for A_t and PA_t are plotted in Figs. 3 and 4 respectively, as well as the 95% confidence bands for both figures, with moreover A_∞ in Fig. 3. We can see in such figures that our approximate solutions are entirely compatible with the MC results and with the asymptotic availability. Also, comparing both figures, we can observe that the availability immediately starts diminishing for small t whereas the production availability first remains

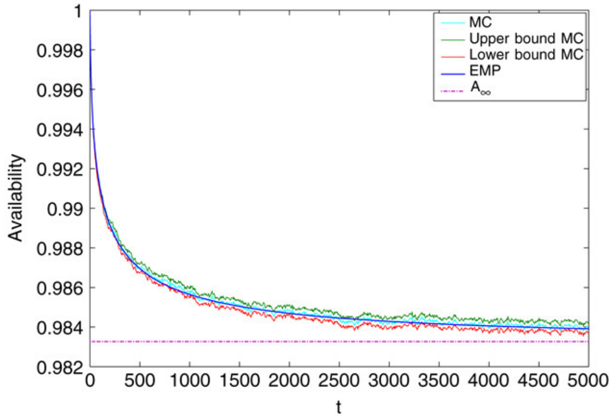

Fig. 3. Approximate availability by Monte Carlo simulation (MC) with 95% band (Upper and Lower bounds) and by the finite volume scheme (EMP), and asymptotic availability (A_∞).

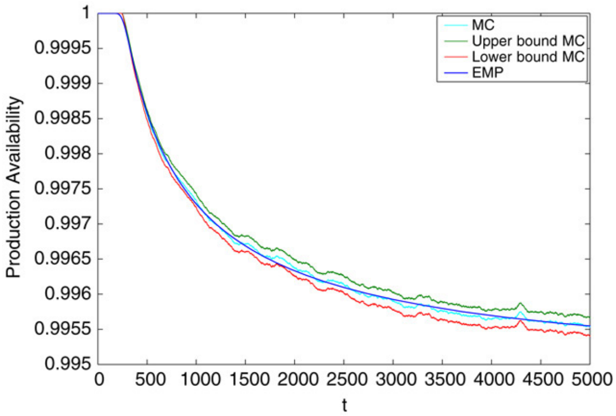

Fig. 4. Approximate production availability by Monte Carlo simulation (MC) with 95% band (Upper and Lower bounds) and by the finite volume scheme (EMP).

constant equal to 1. This is easily explained by the fact that the initially full reservoir prevents the production rate of the device from dropping below ϕ_{nom} for small t.

We next present in Figs. 5 and 6 the approximation of the marginal distributions in both failure and production states for $t \simeq 10^6$, namely roughly in the asymptotic case. Such plots represent the probability density function $u_{\mathcal{M},k}\left(i, (x_1, x_2), t\right)$ provided by the finite volume scheme (see (84)) for $i = 0, 1$ and $t \simeq 10^6$, with respect to the time elapsed in the current discrete state x_1 and to the level in the reservoir x_2. More precisely, we have actually plotted $\log\left(u_{\mathcal{M},k}\left(i, (x_1, x_2), t\right) + 10^{-12}\right)$ with respect to $\left(\log(x_1), x_2\right)$. Such a choice has been simply made to better bring to light both of the surfaces. Similarly, black lines have been added on sides of the surfaces with the same aim. We can observe in Fig. 5 that at the beginning of the repair, there is some peak on states with high level in the reservoir and that the p.d.f. smoothly decreases with the level. Such a peak and smooth decrease may also be observed at the end of the up period in Fig. 6. This means that the failure mostly occurs with a nearly full reservoir. However, a significant probability mass is located in the not yet filled-in reservoir state.

In the same way, we can see in Fig. 5 that there are two peaks at the end of the repair which correspond to a nearly full/empty reservoir, with a higher peak for the full case. Such peaks may also be observed in Fig. 6 at the beginning of the up period. This means that the repair is mostly completed before the reservoir significantly starts to empty, with a significant peak corresponding to a nearly empty reservoir however, and a consequently very low production rate for the device. This clearly lowers the asymptotic production availability.

As a conclusion, this example demonstrates the capability of the scheme to provide transient results as well as asymptotic ones. This is entirely due to the fact that it has been taken as implicit. Indeed, though the proof has not

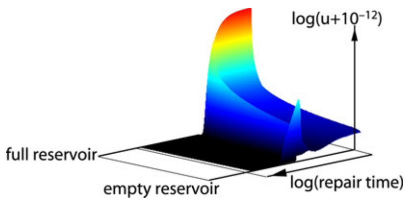

Fig. 5. Approximate probability density function in the failure state.

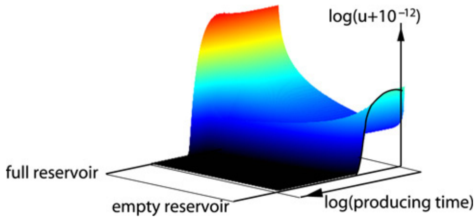

Fig. 6. Approximate probability density function in the producing state.

been written in such a setting, an implicit scheme allows us to take variable time steps, which may easily be adjusted to get accurate results, both for transient and asymptotic cases.

References

[1] Air Liquide, Exercise submitted to ESRA Technical Committee on dependability modelling, 2003.

[2] P. Billingsley, Convergence of Probability Measures, in: Wiley Series in Probability and Mathematical Statistics, John Wiley & sons, New York, 1968.

[3] L. Boccardo, T. Gallouët, Nonlinear elliptic and parabolic equations involving measure data, J. Funct. Anal. 87 (1989) 241–273.

[4] L. Boccardo, T. Gallouët, J.L. Vazquez, Nonlinear elliptic equations in Rn without growth restrictions on the data, J. Differential Equations 105 (2) (1993) 334–363.

[5] O. Boxma, D. Perry, W. Stadje, S. Zacks, A Markovian growth–collapse model, Adv. Appl. Probab. 38 (1) (2006) 221–243.

[6] S. Champier, T. Gallouët, R. Herbin, Convergence of an upstream finite volume scheme on a triangular mesh for a nonlinear hyperbolic equation, Numer. Math. 66 (1993) 139–157.

[7] Christiane Cocozza-Thivent, Robert Eymard, Sophie Mercier, Michel Roussignol, Characterization of the marginal distributions of Markov processes used in dynamic reliability, J. Appl. Math. Stoch. Anal. (2006) 1–18 (Article ID 92156).

[8] C. Cocozza-Thivent, R. Eymard, S. Mercier, A finite volume scheme for dynamic reliability models, IMA J. Numer. Anal. 26 (3) (2006) 446–471 (Advance Access published on 6 March 2006).

[9] Y. Coudière, T. Gallouët, R. Herbin, Discrete Sobolev inequalities and Lp error estimates for approximate finite volume solutions of convection diffusion equations, ESAIM: Math. Model. Numer. Anal. 35 (4) (1998) 767–778.

[10] M.H.A. Davis, Markov Models and Optimization, Chapman & Hall, London, 1993.

[11] J. Droniou, T. Gallouët, R. Herbin, A finite volume scheme for a noncoercive elliptic equation with measure data, SIAM J. Numer. Anal. 41 (6) (2003) 1997–2031.

[12] M. Duflo, Algorithmes Stochastiques, in: Collection: Mathématiques et Applications, vol. 23, Springer-Verlag, Berlin, Heidelberg, 1996.

[13] F. Dufour, Y. Dutuit, H. Zhang, F. Innal, Fiabilité dynamique et processus déterministes par morceaux, in: Proc. of $\lambda\mu$ 15, Lille, France, 2006 (in French).

[14] R. Eymard, T. Gallouët, R. Herbin, Existence and uniqueness of the entropy solution to a nonlinear hyperbolic equation, Chinese Ann. Math., Series B 16B (1) (1995) 1–14.

[15] Robert Eymard, Sophie Mercier, Comparison of numerical methods for the assessment of production availability of a hybrid system, Reliab. Engrg. Syst. Safety 93 (1) (2008) 169–178 (available online 29 December 2006).

[16] T. Gallouët, R. Herbin, Finite volume approximation of elliptic problems with irregular data, in: Finite Volume for Complex Analysis, Duisburg, Hermes, 1999.

[17] M. Jacobsen, Point Process Theory and Applications: Marked Point and Piecewise Deterministic Processes, in: Series: Probability and Its Applications, Birkhäuser, 2006.

[18] P.E. Labeau, Y. Dutuit, Fiabilité dynamique et disponibilité de production: un cas illustratif, in: Proc. of $\lambda\mu$ 14, Bourges, France, vol. 2, 2004, pp. 431–436 (in French).