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Existence of a solution for two phase flow in porous media: The case that the porosity depends on the pressure

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In this paper we prove the existence of a solution of a coupled system involving a two phase incompressible flow in the ground and the mechanical deformation of the porous medium where the porosity is a function of the global pressure. The model is strongly coupled and involves a nonlinear degenerate parabolic equation. In order to show the existence of a weak solution, we consider a sequence of related uniformly parabolic problems and apply the Schauder fixed point theorem to show that they possess a classical solution. We then prove the relative compactness of sequences of solutions by means of the Fréchet–Kolmogorov theorem; this yields the convergence of a subsequence to a weak solution of the parabolic system.

Keywords: Porous medium; Subsidence model; Nonlinear parabolic degenerate equations; Schauder fixed point theorem; Fréchet–Kolmogorov theorem

1. Introduction

In this article we prove the existence of a solution for a system involving a two phase incompressible flow in the ground and the mechanical deformation of a porous medium which is subject to the weakening water phenomena: this means that the medium absorbs the water which is already present or artificially injected to develop a draining process. The most well-known geophysical example is the Ekofisk deposit in the North Sea, where the rock is essentially made of chalk. The mathematical model which we present in Section 2 is a system of two coupled equations where the unknown functions are the saturation S and the pressure p_w of the water phase. In general, the coupling between the two phase flow and the mechanical deformations is taken into account by means of Biot's law; however Biot's law is not sufficient to describe the coupling in a sensitive porous medium in weakening water. Instead the model which we study involves the dependence of the porosity on the global pressure.

We rewrite the model in Section 2, where the saturation S and the global pressure p are the two unknown functions. We present two equivalent forms which both involve a parabolic equation which degenerates for $S = 1$: the first one allows to apply the maximum principle to the saturation S whereas the second one permits to obtain uniform estimates on some quantities depending on the saturation S. We indicate the precise hypotheses and present a definition of a weak solution of the problem. We consider a sequence $(\mathcal{P}_\epsilon)$ of regularized problems. In Section 3 prove that they possess a classical solution (S_ϵ, p_ϵ) by means of the Schauder fixed point theorem. We present a priori estimates uniform in the parameter ϵ for the pair of functions (p_ϵ, S_ϵ). In Section 4, we apply the Fréchet–Kolmogorov theorem to prove the relative compactness of the sequences and show the convergence of subsequences to a weak solution of the problem.

As related work let us mention the book of Gagneux and Madaune-Tort [5] about the mathematical analysis of petrol problems. Further Chen [2] considered a coupled system of equations modelling two phase flow in porous medium where the porosity only depends on space. He proved the existence of a weak solution which he obtained as a limit of solutions of associated discrete time problems. In [2], Chen proved the uniqueness of the solution under the hypothesis that the global pressure is a continuous Lipschitz function. Finally let us cite [4] where Eymard, Herbin and Michel deduced the existence of solutions of a general class of problems modelling two phase incompressible flow from the convergence proof of a finite volume method.

2. The mathematical model

We consider a system of nonlinear equations coupling two phase flow in the ground and the mechanical deformation of the porous media; the porosity $\varphi = \varphi(p)$ depends on the global pressure p introduced in [1]. The unknowns are the saturation S and the pressure p_w of the water phase. Problem $(\mathcal{P})$ is given by the coupled system

$$\left\{\begin{aligned} &\frac{\partial}{\partial t}\big(\varphi(p)S\big) = k\,\mathrm{div}\Big(\frac{\rho_w kr_{wo}(S)}{\mu_w}\nabla p_w\Big)\\ &\qquad\qquad + \alpha\big(f_w(S^*)(\bar{p}-p)^+ - f_w(S)(\bar{p}-p)^-\big),\\ &\frac{\partial}{\partial t}\big(\varphi(p)(1-S)\big) = k\,\mathrm{div}\Big(\frac{\rho_o kr_{ow}(S)}{\mu_o}\nabla(p_w+p_c)\Big)\\ &\qquad\qquad + \alpha\big(f_o(S^*)(\bar{p}-p)^+ - f_o(S)(\bar{p}-p)^-\big),\end{aligned}\right. \tag{2.1}$$

in $Q_T = \Omega \times (0, T]$, together with the boundary and initial conditions

$$
\begin{cases}
\nabla p_w \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \times (0, T], \\
\nabla (p_w + p_c) \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \times (0, T], \\
\nabla S.\mathbf{n} = 0 & \text{on } \partial\Omega \times (0, T], \\
S(\mathbf{x}, 0) = S_0(\mathbf{x}) & \text{in } \Omega, \\
p(\mathbf{x}, 0) = p_0(\mathbf{x}) & \text{in } \Omega,
\end{cases}
\tag{2.2}
$$

where

- $a^+ = \max(0, a)$ and $a^- = \max(0, -a)$,
- $p_\mathrm{c} = p_\mathrm{c}(S)$ is the capillary pressure, which is decreasing function,
- $kr_{ij} = kr_{ij}(S)$ is the mobility of the phase i in the presence of the phase j, where $i, j \in \{w, o\}$ and $i \neq j$. The functions kr_{wo} and kr_{ow} are respectively increasing and decreasing functions. Typical expressions in the case of an incompressible nonmiscible flow are given by

$$
kr_{wo}(S) = S^a, \quad kr_{ow}(S) = (1 - S)^a, \quad \text{with } a \in \{1, 2, 3\}, \tag{2.3}
$$

- ρ_i and μ_i are respectively the density and the viscosity of the phase i,
- S^* is a constant such that $S^* \in [0, 1]$,
- $p = p(\mathbf{x}, t)$ is the global pressure given by

$$
p = p_w + \int_0^S \frac{k_o(u)}{M(u)} p_\mathrm{c}'(u)\, du, \tag{2.4}
$$

where

$$
\begin{aligned}
&k_i(S) = \frac{\rho_i kr_{ij}(S)}{\mu_i} \quad \text{with } i, j \in \{w, o\}, \\
&M(S) = k_w(S) + k_o(S).
\end{aligned}
$$

For $i \in \{w, o\}$, we define the function f_i by

$$
f_i(S) = \frac{k_i(S)}{M(S)}. \tag{2.5}
$$

We remark that $0 \leqslant f_i(S) \leqslant 1$ and $f_w + f_o = 1$.

2.1. Equivalent form of Problem ($\mathcal{P}$)

We formulate below a problem equivalent to Problem ($\mathcal{P}$) with the saturation S and the global pressure p as unknown functions. We add up the two equations of system (2.1) to obtain:

$$
\frac{\partial}{\partial t}\varphi(p) = k \operatorname{div}\big(M(S)\nabla p\big) + \alpha(\overline{p} - p). \tag{2.6}
$$

We rewrite the first equation of system (2.1) as

$$
\begin{aligned}
\frac{\partial}{\partial t}\big(\varphi(p)S\big) = {} & k\Delta\psi(S) + k \operatorname{div}\big(f_w(S)M(S)\nabla p\big) \\
& + \alpha\big(f_w(S^*)(\overline{p} - p)^+ - f_w(S)(\overline{p} - p)^-\big),
\end{aligned}
\tag{2.7}
$$

where

$$\psi(S) = \int_0^S -\frac{k_o(u)k_w(u)}{M(u)} p_c'(u)\,du \tag{2.8}$$

is a continuously differentiable function on [0, 1]. We denote by κ its Lipschitz constant. This yields the first problem equivalent to Problem $(\mathcal{P})$ with the unknown functions (S, p) namely,

$$\begin{cases} \dfrac{\partial}{\partial t}\varphi(p) = k\,\mathrm{div}\big(M(S)\nabla p\big) + \alpha(\overline{p} - p), \\ \dfrac{\partial}{\partial t}\big(\varphi(p)S\big) = k\Delta\psi(S) + k\,\mathrm{div}\big(f_w(S)M(S)\nabla p\big) \\ \qquad\qquad\qquad + \alpha\big(f_w(S^*)(\overline{p} - p)^+ - f_w(S)(\overline{p} - p)^-\big), \end{cases} \tag{2.9}$$

in Q_T, with the boundary and initial conditions

$$\begin{cases} \nabla p \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \times (0, T], \\ \nabla\psi(S) \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \times (0, T], \\ S(\mathbf{x}, 0) = S_0(\mathbf{x}) & \text{in } \Omega, \\ p(\mathbf{x}, 0) = p_0(\mathbf{x}) & \text{in } \Omega. \end{cases} \tag{2.10}$$

Since it is difficult to apply the maximum principle to the second equation of (2.9) in order to show that $0 \leqslant S \leqslant 1$ we write a second formulation which allows us to obtain these estimates for S. We multiply the first equation of (2.9) by $f_w(S)$ and subtract it from the second equation of (2.9). We deduce from a formal computation that:

$$\begin{aligned} \big(S - f_w(S)\big)\frac{\partial}{\partial t}\varphi(p) + \varphi(p)\frac{\partial}{\partial t}S = k\Delta\psi(S) + kM(S)\nabla p.\nabla f_w(S) \\ + \alpha\big(f_w(S^*) - f_w(S)\big)(\overline{p} - p)^+, \end{aligned} \tag{2.11}$$

where the function ψ is given by (2.8). Consequently, a second form of Problem $(\mathcal{P})$ with the unknown functions (S, p) can be written as

$$\begin{cases} \dfrac{\partial}{\partial t}\varphi(p) = k\,\mathrm{div}\big(M(S)\nabla p\big) + \alpha(\overline{p} - p), \\ \dfrac{\partial}{\partial t}S = \dfrac{k}{\varphi(p)}\Delta\psi(S) + \dfrac{kM(S)}{\varphi(p)}\nabla p.\nabla f_w(S) + \dfrac{\alpha}{\varphi(p)}\big(f_w(S^*) - f_w(S)\big)(\overline{p} - p)^+ \\ \qquad - \dfrac{(S - f_w(S))}{\varphi(p)}\dfrac{\partial}{\partial t}\varphi(p), \end{cases} \tag{2.12}$$

with again the boundary and initial conditions (2.10).

2.2. *Hypothesis*

We suppose that the following hypotheses are satisfied:

$(\mathcal{H}_0)$ Ω is a bounded domain of $\mathbb{R}^n$, $\partial\Omega \in C^{5+\beta}$, with β a given constant in (0, 1);
$(\mathcal{H}_1)$ M is a differentiable function in [0, 1] given by

$$M(S) = \frac{\rho_w}{\mu_o}S^a + \frac{\rho_o}{\mu_o}(1 - S)^a, \tag{2.13}$$

so that there exist positive constants M_* and M^* such that

$$0 < M_* \leqslant M(S) \leqslant M^* \quad \text{in } [0,1]; \tag{2.14}$$

($\mathcal{H}_2$) $\alpha = \alpha(\mathbf{x})$ is a continuous function on $\overline{\Omega}$ and we define $\alpha^* = \sup_{\overline{\Omega}} \alpha$;

($\mathcal{H}_3$) $\overline{p} = \overline{p}(\mathbf{x}) \geqslant 0$ is a continuous function on $\overline{\Omega}$ and we also define $p_* = \inf_{\overline{\Omega}} \overline{p}$ and $p^* = \sup_{\overline{\Omega}} \overline{p}$, so that $0 \leqslant p_* \leqslant p^*$;

($\mathcal{H}_4$) φ is a continuously differentiable, strictly increasing function on $[p_*, p^*]$, such that $\varphi(p_*) > 0$;

($\mathcal{H}_5$) $\psi \in C^1([0,1])$ satisfies

$$C_* S^{m-1} \leqslant \psi'(S) \leqslant C^* S^{m-1} \quad \text{for } 0 \leqslant S \leqslant 1, \text{ with } m \in \mathbb{N}^*,$$

and the constant a in (2.13) satisfies

$$a \geqslant \max\left(1, \frac{m-1}{2}\right); \tag{2.15}$$

($\mathcal{H}_6$) f_w is a continuous differentiable and increasing function on [0, 1] such that

$$0 \leqslant f_w \leqslant 1, \quad f_w(0) = 0 \quad \text{and} \quad f_w(1) = 1; \tag{2.16}$$

($\mathcal{H}_7$) $(p_0, S_0) \in (L^2(\Omega))^2$ satisfies

$$p_* \leqslant p_0 \leqslant p^* \quad \text{a.e. } \Omega, \tag{2.17}$$

and

$$0 \leqslant S_0 \leqslant 1 \quad \text{a.e. } \Omega. \tag{2.18}$$

2.3. Notations

Further we recall the standard definitions of Hölder spaces (cf., [6]). Let $m \in \mathbb{N}$ and $0 < \beta < 1$. Let f be a continuous function defined in $\overline{Q}_T$ such that all its derivatives of the form $\frac{\partial^{r+s} f}{\partial t^r \partial x^s}$, with $2r + s \leqslant m$, are continuous. We define the norm $|f|_{Q_T}^{m+\beta, \frac{m+\beta}{2}}$ by

$$|f|_{Q_T}^{m+\beta, \frac{m+\beta}{2}} = \sum_{j=0}^{m} \langle f \rangle^{(j)} + \langle f \rangle_{Q_T}^{(m+\beta)}, \tag{2.19}$$

such that

$$\langle f \rangle^{(j)} = \sum_{2r+s=j} \max_{Q_T} \left| \frac{\partial^{r+s} f}{\partial t^r \partial x^s} \right|,$$

$$\langle f \rangle_{Q_T}^{(m+\beta)} = \sum_{2r+s=m} \left\langle \frac{\partial^{r+s} f}{\partial t^r \partial x^s} \right\rangle_{x,Q_T}^{(\beta)} + \sum_{2r+s=m} \left\langle \frac{\partial^{r+s} f}{\partial t^r \partial x^s} \right\rangle_{t,Q_T}^{(\frac{\beta}{2})},$$

$$\langle f \rangle_{x,Q_T}^{(\beta)} = \sup_{(\mathbf{x}_1,t),(\mathbf{x}_2,t) \in \overline{Q}_T} \frac{|f(\mathbf{x}_1,t) - f(\mathbf{x}_2,t)|}{|\mathbf{x}_1 - \mathbf{x}_2|^\beta},$$

$$\langle f \rangle_{t,Q_T}^{(\frac{\beta}{2})} = \sup_{(\mathbf{x},t_1),(\mathbf{x},t_2) \in \overline{Q}_T} \frac{|f(\mathbf{x},t_1) - f(\mathbf{x},t_2)|}{|t_1 - t_2|^{\frac{\beta}{2}}}.$$

We denote by $C^{m+\beta,\frac{m+\beta}{2}}(\overline{Q}_T)$ the functional space of functions f such that $|f|_{Q_T}^{m+\beta,\frac{m+\beta}{2}} < +\infty$: it is a Banach space with the norm $|.|_{Q_T}^{m+\beta,\frac{m+\beta}{2}}$.

2.4. Weak solutions of Problem $(\mathcal{P})$

In general, a solution of the degenerate parabolic Problem $(\mathcal{P})$ is not smooth. Hence, we have to define weak solutions.

Definition 2.4.1. We say that $(p, S) \in (L^\infty(Q_T))^2$ is a weak solution of Problem $(\mathcal{P})$, if it satisfies:

(i) $p \in L^2(0, T; H^1(\Omega))$ and $\psi(S) \in L^2(0, T; H^1(\Omega))$;
(ii) the integral equations

$$\begin{cases} \displaystyle\iint_{Q_T} \varphi(p)\partial_t v \, d\mathbf{x}\, dt = \iint_{Q_T} \big(kM(S)\nabla p.\nabla v - \alpha(\overline{p} - p)v\big)\, d\mathbf{x}\, dt \\ \displaystyle\qquad\qquad - \int_\Omega \varphi(p_0)v(0)\, d\mathbf{x}, \\ \displaystyle\iint_{Q_T} \varphi(p)S\partial_t v \, d\mathbf{x}\, dt = \iint_{Q_T} \Big(-k\psi(S)\Delta v + kf_w(S)M(S)\nabla p.\nabla v \\ \displaystyle\qquad\qquad - \alpha\big(f_w(S^*)(\overline{p} - p)^+ - f_w(S)(\overline{p} - p)^-\big)v\Big)\, d\mathbf{x}\, dt \\ \displaystyle\qquad\qquad - \int_\Omega \varphi(p_0)S_0 v(0)\, d\mathbf{x}, \end{cases} \tag{2.20}$$

for all v in

$$\mathcal{V} = \Big\{ w \in C^{2,1}(\overline{Q}_T),\ w(., T) = 0 \text{ in } \Omega,\ \frac{\partial w}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega \times [0, T] \Big\}.$$

3. A sequence of approximate problems $(\mathcal{P}_\epsilon)$

In order to prove the existence of a solution of Problem $(\mathcal{P})$, we introduce a sequence of regularized problems $(\mathcal{P}_\epsilon)$. Let ϵ be a real constant small enough, we define the sequences:

- $\{\varphi_\epsilon\}_{\epsilon>0} \in C^\infty(\mathbb{R})$ such that

$$\begin{aligned} &\varphi_\epsilon \xrightarrow[\epsilon\to 0]{} \varphi && \text{uniformly in the interval } [p_*, p^*], \\ &\varphi(p_*) \leqslant \varphi_\epsilon(p) \leqslant \varphi(p^*) + 1, && \\ &0 < c_\epsilon \leqslant \varphi'_\epsilon(p) && \text{for all } p \in \mathbb{R}, \\ &\varphi'_\epsilon(p) \leqslant \sup_{[p_*, p^*]} \varphi' && \text{for all } p \in [p_*, p^*], \end{aligned} \tag{3.1}$$

where c_ϵ is a constant depending on ϵ;

- $\alpha_\epsilon \in C_0^\infty(\overline{\Omega})$ such that
$$\alpha_\epsilon \xrightarrow[\epsilon\to 0]{} \alpha \quad \text{in } L^2(\Omega) \text{ and a.e. in } \Omega,\ \alpha_\epsilon \leqslant \alpha^*; \tag{3.2}$$
- $\overline{p}_\epsilon \in C_0^\infty(\overline{\Omega})$ such that
$$\overline{p}_\epsilon \xrightarrow[\epsilon\to 0]{} \overline{p} \quad \text{in } L^2(\Omega) \text{ and a.e. in } \Omega,\ p_* \leqslant \overline{p}_\epsilon \leqslant p^*; \tag{3.3}$$
- $S_{0,\epsilon} \in C_0^\infty(\overline{\Omega})$ such that
$$S_{0,\epsilon} \xrightarrow[\epsilon\to 0]{} S_0 \quad \text{in } L^2(\Omega) \text{ and a.e. in } \Omega,\ 0 \leqslant S_{0,\epsilon} \leqslant 1; \tag{3.4}$$
- $p_{0,\epsilon} \in C_0^\infty(\overline{\Omega})$ such that
$$p_{0,\epsilon} \xrightarrow[\epsilon\to 0]{} p_0 \quad \text{in } L^2(\Omega) \text{ and a.e. in } \Omega,\ p_* \leqslant p_{0,\epsilon} \leqslant p^*; \tag{3.5}$$
- $\psi_\epsilon \in C^\infty(\mathbb{R})$ such that
$$\begin{aligned} &\psi_\epsilon \xrightarrow[\epsilon\to 0]{} \psi \quad \text{in } C^1\big([0,1]\big), \\ &C_* S^{m-1} + \lambda_* \epsilon \leqslant \psi_\epsilon'(S) \leqslant C^* S^{m-1} + \lambda^* \epsilon \quad \text{in } [0,1], \end{aligned} \tag{3.6}$$
where λ_* and λ^* are strictly positive constants depending on ψ.

For all $r \in \mathbb{R}$, we define the functions
$$a^+(r) = r^+ \quad \text{and} \quad a^-(r) = r^-$$
and sequences of functions $\{a_\epsilon^+\}_{\epsilon>0}$ and $\{a_\epsilon^-\}_{\epsilon>0}$ in $C^\infty(\mathbb{R})$ such that
$$a_\epsilon^-(r) = a_\epsilon^+(r) - r, \tag{3.7}$$
and

- $a_\epsilon^+ \to a^+$ and $a_\epsilon^- \to a^-$ uniformly on each compact set of $\mathbb{R}$;
- $0 \leqslant a_\epsilon^{+\prime} \leqslant 1$, $-1 \leqslant a_\epsilon^{-\prime} \leqslant 0$, $a_\epsilon^+ \geqslant 0$ and $a_\epsilon^- \geqslant 0$.

Next we consider the sequence of regularized problems $(\mathcal{P}_\epsilon)$:
$$\begin{cases} \dfrac{\partial}{\partial t} S = \dfrac{k}{\varphi_\epsilon(p)} \Delta \psi_\epsilon(S) + \dfrac{kM(S)}{\varphi_\epsilon(p)} \nabla p . \nabla f_w(S) \\ \qquad\quad + \dfrac{\alpha_\epsilon}{\varphi_\epsilon(p)} \big(f_w(S^*) - f_w(S)\big) a_\epsilon^+(\overline{p}_\epsilon - p) - \dfrac{(S - f_w(S))}{\varphi_\epsilon(p)} \dfrac{\partial}{\partial t} \varphi_\epsilon(p), & \text{in } Q_T, \\ \dfrac{\partial}{\partial t} \varphi_\epsilon(p) = k \operatorname{div}\big(M(S)\nabla p\big) + \alpha_\epsilon(\overline{p}_\epsilon - p), & \text{in } Q_T, \end{cases} \tag{3.8}$$
with the following boundary and initial conditions:
$$\begin{cases} \nabla S \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \times (0,T], \\ \nabla p \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \times (0,T], \\ S(\mathbf{x},0) = S_{0,\epsilon}(\mathbf{x}) & \mathbf{x} \in \Omega, \\ p(\mathbf{x},0) = p_{0,\epsilon}(\mathbf{x}) & \mathbf{x} \in \Omega. \end{cases} \tag{3.9}$$

Theorem 3.1 *(Classical solution of the approximate problem $(\mathcal{P}_\epsilon)$). For all $\epsilon > 0$, there exists a classical solution $(p_\epsilon, S_\epsilon) \in (C^{5+\beta,\frac{5+\beta}{2}}(\overline{Q}_T))^2$ of the approximate problem $(\mathcal{P}_\epsilon)$. This solution satisfies*

$$p_* \leqslant p_\epsilon \leqslant p^* \quad \textit{and} \quad 0 \leqslant S_\epsilon \leqslant 1.$$

Proof. We define the closed convex set

$$K_\epsilon := \big\{ w \in C^{5+\beta,\frac{5+\beta}{2}}(\overline{Q}_T),\ p_* \leqslant w \leqslant p^* \text{ in } Q_T,\ w(\mathbf{x},0) = p_{0,\epsilon}(\mathbf{x}) \text{ in } \Omega, \\ \nabla w \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \times (0,T] \big\}. \tag{3.10}$$

For an arbitrary element p_ϵ of K_ϵ, we consider the problem $(\mathcal{P}_\epsilon^S)$

$$\begin{cases} \dfrac{\partial}{\partial t} S = \dfrac{k}{\varphi_\epsilon(p_\epsilon)} \Delta\psi_\epsilon(S) + \dfrac{kM(S)}{\varphi_\epsilon(p_\epsilon)} \nabla p_\epsilon . \nabla f_w(S) & \\ \qquad + \dfrac{\alpha_\epsilon}{\varphi_\epsilon(p_\epsilon)} \big(f_w(S^*) - f_w(S)\big) a_\epsilon^+(\overline{p}_\epsilon - p_\epsilon) & \\ \qquad - \dfrac{(S - f_w(S))}{\varphi_\epsilon(p_\epsilon)} \dfrac{\partial}{\partial t} \varphi_\epsilon(p_\epsilon) & \text{in } \Omega \times (0,T], \\ \nabla S \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \times (0,T], \\ S(\mathbf{x},0) = S_{0,\epsilon}(\mathbf{x}) & \text{in } \Omega. \end{cases} \tag{3.11}$$

The problem $(\mathcal{P}_\epsilon^S)$ is uniform parabolic so that it possesses a unique classical solution S_ϵ in $C^{2+\beta,1+\frac{\beta}{2}}(\overline{Q}_T)$ [6, Theorem 7.4, Chapter 5, p. 491]. Applying the comparison theorem [7, Lemma 3.6, Chapter 2, p. 61], we obtain

Lemma 3.1. *The function S_ϵ is such that*

$$0 \leqslant S_\epsilon \leqslant 1 \quad \textit{in } Q_T. \tag{3.12}$$

Since the coefficients of the differential operator in problem $(\mathcal{P}_\epsilon^S)$ are smooth, one can show that $S_\epsilon \in C^{5+\beta,\frac{5+\beta}{2}}(\overline{Q}_T)$ and that

$$|S_\epsilon|_{Q_T}^{5+\beta,\frac{5+\beta}{2}} \leqslant B_\epsilon^1 |p_\epsilon|_{Q_T}^{5+\beta,\frac{5+\beta}{2}} + B_\epsilon^2, \tag{3.13}$$

where the positive constants B_ϵ^1 and B_ϵ^2 only depend on ϵ and $S_{0,\epsilon}$. Indeed the idea is to apply four times [6, Theorem 5.3, Chapter 4, p. 320] to the problem (3.11) which we consider as linear. We successively show that $S_\epsilon \in C^{i+2+\beta,\frac{i+2+\beta}{2}}(\overline{Q}_T)$ for $i = 0, 1, 2, 3$. We remark that to obtain the desired regularity, we have to check compatibility conditions of order $i = 0, 1, 2$. Here, the condition of order i is given by

$$\nabla \frac{\partial^i S_\epsilon}{\partial t^i}\bigg|_{t=0} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega. \tag{3.14}$$

This condition is satisfied for all $i \in \{0,1,2\}$ thanks to the properties of the sequences $\{S_{0,\epsilon}\}, \{p_{0,\epsilon}\}, \{\overline{p}_\epsilon\}, \{\alpha_\epsilon\}$ and to the fact that the functions $\{p_\epsilon\}$ belong to the set K_ϵ. Hence, we obtain (3.13). Let S_ϵ be the solution of problem (3.11); we consider the problem $(\mathcal{P}_\epsilon^p)$ given by

$$
\begin{cases}
\dfrac{\partial}{\partial t}\varphi_\epsilon(p) = k\,\mathrm{div}\big(M(S_\epsilon)\nabla p\big) + \alpha_\epsilon(\overline{p}_\epsilon - p) & \text{in } \Omega \times (0,T],\\
\nabla p \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \times (0,T],\\
p(\mathbf{x},0) = p_{0,\epsilon}(\mathbf{x}) & \text{in } \Omega.
\end{cases}
\tag{3.15}
$$

The parabolic equation in problem $(\mathcal{P}_\epsilon^p)$ is uniform parabolic and we deduce from [6, Theorem 7.4, Chapter 5, p. 491] that it possesses a unique classical solution $\hat{p}_\epsilon \in C^{2+\beta,1+\frac{\beta}{2}}(\overline{Q}_T)$. Applying the comparison theorem [7, Lemma 3.6, Chapter 2, p. 61], we obtain the following bounds

Lemma 3.2. *The function $\hat{p}_\epsilon$ is such that*

$$
p_* \leqslant \hat{p}_\epsilon(\mathbf{x},t) \leqslant p^* \quad \text{in } Q_T. \tag{3.16}
$$

Since the coefficients of the partial differential equation in Problem $(\mathcal{P}_\epsilon^p)$ are smooth, we apply five times [6, Theorem 5.3, Chapter 4, p. 320] to problem (3.15) to deduce that $\hat{p}_\epsilon \in C^{6+\beta,\frac{6+\beta}{2}}(\overline{Q}_T)$ and that

$$
|\hat{p}_\epsilon|_{Q_T}^{6+\beta,\frac{6+\beta}{2}} \leqslant M_\epsilon^1 |S_\epsilon|_{Q_T}^{5+\beta,\frac{5+\beta}{2}} + M_\epsilon^2, \tag{3.17}
$$

where the positive constants M_ϵ^1 and M_ϵ^2 only depend on ϵ, $p_{0,\epsilon}$. To do that, we successively show that $\hat{p}_\epsilon \in C^{i+2+\beta,\frac{i+2+\beta}{2}}(\overline{Q}_T)$ for $i = 0,\dots,4$. We have to verify the compatibility conditions of order $i = 0,\dots,4$, which reduce to

$$
\nabla \frac{\partial^i \hat{p}_\epsilon}{\partial t^i}\bigg|_{t=0} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega. \tag{3.18}
$$

These conditions are satisfied thanks to the properties of sequences $\{S_{0,\epsilon}\}$, $\{p_{0,\epsilon}\}$, $\{\overline{p}_\epsilon\}$, $\{\alpha_\epsilon\}$ and $\{S_\epsilon\}$. Thus, we have defined a map $\mathcal{F}: p_\epsilon \to \hat{p}_\epsilon$ from K into itself. In order to apply the Schauder fixed point theorem, we have to show that the map $\mathcal{F}$ is compact and continuous.

Compactness: We define the closed bounded set of K_ϵ,

$$
\mathcal{B}_C = \Big\{ w \in K_\epsilon,\ |w|_{Q_T}^{5+\beta,\frac{5+\beta}{2}} \leqslant C \Big\}.
$$

We show below that $\mathcal{F}(\mathcal{B}_C)$ is compact in K_ϵ. Let p be arbitrary in $\mathcal{B}_C$ and let S_ϵ the solution of (3.11). Since $|p|_{Q_T}^{5+\beta,\frac{5+\beta}{2}} \leqslant C$ and in view of (3.13), we obtain

$$
|S_\epsilon|_{Q_T}^{5+\beta,\frac{5+\beta}{2}} \leqslant \widetilde{M}(C), \tag{3.19}
$$

where $\widetilde{M}(C)$ is a positive constant depending on C. Let $\hat{p}_\epsilon = \mathcal{F}(p)$ be the solution of problem (3.15). From (3.17), we get

$$
|\hat{p}_\epsilon|_{Q_T}^{6+\beta,\frac{6+\beta}{2}} \leqslant \widetilde{C}(C), \tag{3.20}
$$

where $\widetilde{C}(C)$ is a positive constant depending on C. Hence, we have proved that

$$
\mathcal{F}(\mathcal{B}_C) \subset \big\{ w \in C^{6+\beta,\frac{6+\beta}{2}}(\overline{Q}_T),\ |w|^{6+\beta,\frac{6+\beta}{2}}(\overline{Q}_T) \leqslant \widetilde{C}(C) \big\},
$$

so that $\mathcal{F}(\mathcal{B}_C)$ is bounded in $C^{6+\beta,\frac{6+\beta}{2}}(\overline{Q}_T)$ and since the embedding from $C^{6+\beta,\frac{6+\beta}{2}}(\overline{Q}_T)$ into $C^{5+\beta,\frac{5+\beta}{2}}(\overline{Q}_T)$ is compact, we deduce that $\overline{\mathcal{F}(\mathcal{B}_C)}$ is compact in K_ϵ and finally the map $\mathcal{F}$ is compact.

Continuity: Let $\{p^\delta\}_{\delta>0} \subset K_\epsilon$ be such that

$$p^\delta \xrightarrow[\delta\to 0]{} p \in K_\epsilon \quad \text{in } C^{5+\beta,\frac{5+\beta}{2}}(\overline{Q}_T). \tag{3.21}$$

There exist a sequence of functions $\{S_\epsilon^\delta\}_{\delta>0}$ and a function S_ϵ in $C^{5+\beta,\frac{5+\beta}{2}}(\overline{Q}_T)$ satisfying

$$\begin{cases} \dfrac{\partial}{\partial t} S_\epsilon^\delta = \dfrac{k}{\varphi_\epsilon(p^\delta)} \Delta\psi_\epsilon\big(S_\epsilon^\delta\big) + \dfrac{kM(S_\epsilon^\delta)}{\varphi_\epsilon(p^\delta)} \nabla p^\delta . \nabla f_w\big(S_\epsilon^\delta\big) \\ \qquad\qquad + \dfrac{\alpha_\epsilon}{\varphi_\epsilon(p^\delta)}\big(f_w(S^*) - f_w\big(S_\epsilon^\delta\big)\big) a_\epsilon^+\big(\overline{p}_\epsilon - p^\delta\big) \\ \qquad\qquad - \dfrac{(S_\epsilon^\delta - f_w(S_\epsilon^\delta))}{\varphi_\epsilon(p^\delta)} \dfrac{\partial}{\partial t}\varphi_\epsilon\big(p^\delta\big) & \text{in } \Omega\times[0,T], \\ \nabla S_\epsilon^\delta\cdot\mathbf{n} = 0 & \text{on } \partial\Omega\times[0,T], \\ S_\epsilon^\delta(\mathbf{x},0) = S_{0,\epsilon}(\mathbf{x}) & \text{in } \Omega, \end{cases} \tag{3.22}$$

and

$$\begin{cases} \dfrac{\partial}{\partial t} S_\epsilon = \dfrac{k}{\varphi_\epsilon(p)} \Delta\psi_\epsilon(S_\epsilon) + \dfrac{kM(S_\epsilon)}{\varphi_\epsilon(p)} \nabla p . \nabla f_w(S_\epsilon) \\ \qquad\qquad + \dfrac{\alpha_\epsilon}{\varphi_\epsilon(p)}\big(f_w(S^*) - f_w(S_\epsilon)\big) a_\epsilon^+(\overline{p}_\epsilon - p) \\ \qquad\qquad - \dfrac{(S_\epsilon - f_w(S_\epsilon))}{\varphi_\epsilon(p)} \dfrac{\partial}{\partial t}\varphi_\epsilon(p) & \text{in } \Omega\times[0,T], \\ \nabla S_\epsilon\cdot\mathbf{n} = 0 & \text{on } \partial\Omega\times[0,T], \\ S_\epsilon(\mathbf{x},0) = S_{0,\epsilon}(\mathbf{x}) & \text{in } \Omega. \end{cases} \tag{3.23}$$

Since the sequence $\{p^\delta\}_{\delta>0}$ is bounded, in view of (3.13) and of the compactness of the embedding from $C^{5+\beta,\frac{5+\beta}{2}}(\overline{Q}_T)$ into $C^{4+\beta,\frac{4+\beta}{2}}(\overline{Q}_T)$, there exist a function $\widehat{S}_\epsilon \in C^{5+\beta,\frac{5+\beta}{2}}(\overline{Q}_T)$ and a subsequence $\{S_\epsilon^{\delta_n}\}_{n\geqslant 0}$ such that $S_\epsilon^{\delta_n} \xrightarrow[\delta_n\to 0]{} \widehat{S}_\epsilon$ in $C^{4+\beta,\frac{4+\beta}{2}}(\overline{Q}_T)$. Let δ_n tend to 0 in problem (3.22); by the uniqueness of the solution of problem (3.23), we deduce that $\widehat{S}_\epsilon$ coincides with S_ϵ. Let $(\hat{p}_\epsilon^{\delta_n})_{n>0}$ be the sequence of solutions of the problems

$$\begin{cases} \dfrac{\partial}{\partial t}\varphi_\epsilon\big(\hat{p}_\epsilon^{\delta_n}\big) = k\,\mathrm{div}\big(M\big(S_\epsilon^{\delta_n}\big)\nabla\hat{p}_\epsilon^{\delta_n}\big) + \alpha_\epsilon\big(\overline{p}_\epsilon - \hat{p}_\epsilon^{\delta_n}\big) & \text{in } \Omega\times[0,T], \\ \nabla p_\epsilon^{\delta_n}\cdot\mathbf{n} = 0 & \text{in } \partial\Omega\times[0,T], \\ p_\epsilon^{\delta_n}(\mathbf{x},0) = p_{0,\epsilon}(\mathbf{x}) & \text{in } \Omega. \end{cases} \tag{3.24}$$

In view of (3.17) and of the compactness of the embedding from $C^{6+\beta,\frac{6+\beta}{2}}(\overline{Q}_T)$ into $C^{5+\beta,\frac{5+\beta}{2}}(\overline{Q}_T)$, there exist a function $\theta_\epsilon \in C^{6+\beta,\frac{6+\beta}{2}}(\overline{Q}_T)$ and a subsequence $(\hat{p}_\epsilon^{\delta_n})_{n>0}$, which we also denote by $(\hat{p}_\epsilon^{\delta_n})_{n>0}$, such that

$$p_\epsilon^{\delta_n} \xrightarrow[\delta_n\to 0]{} \theta_\epsilon \quad \text{in } C^{5+\beta,\frac{5+\beta}{2}}(\overline{Q}_T).$$

Letting δ_n tend to 0 in problem (3.24), we deduce that θ_ϵ satisfies the problem

$$\begin{cases} \dfrac{\partial}{\partial t}\varphi_\epsilon(\theta_\epsilon) = k\,\mathrm{div}\big(M(S_\epsilon)\nabla\theta_\epsilon\big) + \alpha_\epsilon(\overline{p}_\epsilon - \theta_\epsilon) & \text{in } \Omega \times [0,T], \\ \nabla\theta_\epsilon \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \times [0,T], \\ \theta_\epsilon(\mathbf{x},0) = p_{0,\epsilon}(\mathbf{x}) & \text{in } \Omega. \end{cases} \tag{3.25}$$

On the other hand, the function $\hat{p}_\epsilon = \mathcal{F}(p)$ satisfies problem (3.25) as well. By the uniqueness of the solution [6, Theorem 7.4, Chapter 5, p. 491] we deduce that

$$\theta_\epsilon = \mathcal{F}(p).$$

We conclude that the application $\mathcal{F}: \tilde{p} \to \hat{p}_\epsilon$ is continuous and compact for the topology of $C^{5+\beta, \frac{5+\beta}{2}}(\overline{Q}_T)$ from the convex closed set K_ϵ into itself. Therefore it follows from the Schauder fixed point theorem that there exists a pair of functions (S_ϵ, p_ϵ) satisfying Problem $(\mathcal{P}_\epsilon)$.

We formally showed above the correspondence between two formulations of Problem $(\mathcal{P})$; the regularity of the coefficients in (3.8)–(3.9) and (3.26)–(3.9) and the fact that

$$a_\epsilon^+(r) - a_\epsilon^-(r) = r, \quad \text{for all } r \in \mathbb{R},$$

allow us to obtain the following result.

Lemma 3.3. *The regularized Problem $(\mathcal{P}_\epsilon)$ is equivalent to the problem*

$$\begin{cases} \dfrac{\partial}{\partial t}\varphi_\epsilon(p) = k\,\mathrm{div}\big(M(S)\nabla p\big) + \alpha_\epsilon(\overline{p}_\epsilon - p), & \text{in } Q_T, \\ \dfrac{\partial}{\partial t}\big(\varphi_\epsilon(p)S\big) = k\Delta\psi_\epsilon(S) + k\,\mathrm{div}\big(f_w(S)M(S)\nabla p_\epsilon\big) & \\ \qquad + \alpha_\epsilon\big(f_w(S^*)a_\epsilon^+(\overline{p}_\epsilon - p) - f_w(S)a_\epsilon^-(\overline{p}_\epsilon - p)\big), & \text{in } Q_T, \end{cases} \tag{3.26}$$

with the boundary and initial conditions given in (3.9).

Lemma 3.4 *(A priori estimates for p_ϵ). Let (p_ϵ, S_ϵ) be a solution of the Problem $(\mathcal{P}_\epsilon)$; there exists a positive constant C independent of ϵ such that*

$$\text{(i)} \quad \|p_\epsilon\|_{L^2(0,T;H^1(\Omega))} \leqslant C,$$

$$\text{(ii)} \quad \big\|\varphi_\epsilon(p_\epsilon)\big\|_{L^2(0,T;H^1(\Omega))} \leqslant C,$$

$$\text{(iii)} \quad \left\|\frac{\partial}{\partial t}\varphi_\epsilon(p_\epsilon)\right\|_{L^2(0,T;(H^1(\Omega))')} \leqslant C. \tag{3.27}$$

Proof. (i) We multiply the second equation of problem (3.8) by p_ϵ and integrate on Q_T. We obtain:

$$\iint\limits_{Q_T} p_\epsilon \frac{\partial}{\partial t}\varphi_\epsilon(p_\epsilon)\,d\mathbf{x}\,dt + \iint\limits_{Q_T} kM(S_\epsilon)(\nabla p_\epsilon)^2\,d\mathbf{x}\,dt = \iint\limits_{Q_T} \alpha_\epsilon(\overline{p}_\epsilon - p_\epsilon)p_\epsilon\,d\mathbf{x}\,dt.$$

We define the function

$$\Phi_\epsilon(s) = \int_0^s r\varphi_\epsilon'(r)\,dr.$$

We have that

$$\iint\limits_{Q_T} p_\epsilon \frac{\partial}{\partial t}\varphi_\epsilon(p_\epsilon)\,d\mathbf{x}\,dt = \iint\limits_{Q_T} \frac{\partial}{\partial t}\Phi_\epsilon(p_\epsilon)\,d\mathbf{x}\,dt = \int\limits_{\Omega}\big(\Phi_\epsilon\big(p_\epsilon(.,T)\big) - \Phi_\epsilon(p_{0,\epsilon})\big)\,d\mathbf{x}.$$

Hence

$$\int\limits_{\Omega} \Phi_\epsilon\big(p_\epsilon(.,T)\big)\,d\mathbf{x} + kM_* \iint\limits_{Q_T} (\nabla p_\epsilon)^2\,d\mathbf{x}\,dt \leqslant \iint\limits_{Q_T} \alpha_\epsilon(\overline{p}_\epsilon - p_\epsilon)p_\epsilon\,d\mathbf{x}\,dt + \int\limits_{\Omega} \Phi_\epsilon(p_{0,\epsilon})\,d\mathbf{x},$$

which implies

$$kM_* \iint\limits_{Q_T} (\nabla p_\epsilon)^2\,d\mathbf{x}\,dt \leqslant \alpha^* T|\Omega|p^{*2} + |\Omega|p^{*2} \sup_{[p_*,p^*]} \varphi'(p).$$

(ii) In view of (3.1) we immediately deduce (ii) from (i).
(iii) Let $\phi \in L^2(0,T;H^1(\Omega))$, we have

$$\begin{aligned}\left|\int_0^T \left\langle \frac{\partial}{\partial t}\varphi_\epsilon(p_\epsilon),\phi\right\rangle\right| &\leqslant k\iint\limits_{Q_T} \big|M(S_\epsilon)\nabla p_\epsilon.\nabla\phi\big|\,d\mathbf{x}\,dt + \iint\limits_{Q_T} \big|\alpha_\epsilon(\overline{p}_\epsilon - p_\epsilon)\phi\big|\,d\mathbf{x}\,dt \\ &\leqslant kM^*\|\nabla p_\epsilon\|_{L^2(Q_T)}\|\nabla\phi\|_{L^2(Q_T)} + \alpha^* p^*|Q_T|\|\phi\|_{L^2(Q_T)} \\ &\leqslant C\|\phi\|_{L^2(0,T;H^1(\Omega))},\end{aligned}$$

which completes the proof of (iii). □

Lemma 3.5 *(A priori estimates for S_ϵ). Let (p_ϵ, S_ϵ) be a solution of Problem $(\mathcal{P}_\epsilon)$; there exists a positive constant C independent of ϵ such that*

(i) $$\iint\limits_{Q_T} \big|\nabla\mathcal{G}_\epsilon\big(S_\epsilon(\mathbf{x},t)\big)\big|^2\,d\mathbf{x}\,dt \leqslant C,$$

(ii) $$\iint\limits_{Q_T} \big|\nabla\psi_\epsilon\big(S_\epsilon(\mathbf{x},t)\big)\big|^2\,d\mathbf{x}\,dt \leqslant C,$$

(iii) $$\iint\limits_{Q_T} \big|\nabla\mathcal{K}\big(\varphi_\epsilon\big(p_\epsilon(\mathbf{x},t)\big)S_\epsilon(\mathbf{x},t)\big)\big|^2\,d\mathbf{x}\,dt \leqslant C,$$

(iv) $$\int_0^T\int\limits_{\Omega_\xi} \Big(\mathcal{K}\big(\varphi_\epsilon\big(p_\epsilon(\mathbf{x}+\xi,t)\big)S_\epsilon(\mathbf{x}+\xi,t)\big) - \mathcal{K}\big(\varphi_\epsilon\big(p_\epsilon(\mathbf{x},t)\big)S_\epsilon(\mathbf{x},t)\big)\Big)^2\,d\mathbf{x}\,dt \leqslant C|\xi|^2,$$

(v) $$\int_0^{T-\tau}\int\limits_{\Omega} \Big(\mathcal{K}\big(\varphi_\epsilon\big(p_\epsilon(\mathbf{x},t+\tau)\big)S_\epsilon(\mathbf{x},t+\tau)\big) - \mathcal{K}\big(\varphi_\epsilon\big(p_\epsilon(\mathbf{x},t)\big)S_\epsilon(\mathbf{x},t)\big)\Big)^2\,d\mathbf{x}\,dt \leqslant C\tau,$$

where

$$\mathcal{G}_\epsilon(S) = \int_0^S \sqrt{C_* r^{m-1} + \lambda_* \epsilon}\, dr, \qquad \mathcal{K}(r) = \frac{1}{m} r^m, \tag{3.28}$$

and

$$\Omega_\xi := \{\mathbf{x} \in \Omega \mid \mathbf{x} + r\xi \in \Omega \textit{ for all } 0 \leqslant r \leqslant 1\}.$$

Proof. (i) We multiply the second equation of (3.26) by $\varphi_\epsilon(p_\epsilon)S_\epsilon$ and integrate on Q_T. This yields

$$\mathcal{I}_0 + \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4 + \mathcal{I}_5 = 0, \tag{3.29}$$

where

$$\mathcal{I}_0 = \iint_{Q_T} \frac{1}{2} \frac{\partial}{\partial t} \big(\varphi_\epsilon(p_\epsilon)S_\epsilon\big)^2 \, d\mathbf{x}\, dt,$$

$$\mathcal{I}_1 = k \iint_{Q_T} \varphi_\epsilon(p_\epsilon)\psi_\epsilon'(S_\epsilon)(\nabla S_\epsilon)^2 \, d\mathbf{x}\, dt,$$

$$\mathcal{I}_2 = k \iint_{Q_T} S_\epsilon \psi_\epsilon'(S_\epsilon)\nabla S_\epsilon \nabla \varphi_\epsilon(p_\epsilon) \, d\mathbf{x}\, dt,$$

$$\mathcal{I}_3 = k \frac{\rho_w}{\mu_w} \iint_{Q_T} S_\epsilon^{a+1} \nabla p_\epsilon \nabla \varphi_\epsilon(p_\epsilon) \, d\mathbf{x}\, dt,$$

$$\mathcal{I}_4 = k \frac{\rho_w}{\mu_w} \iint_{Q_T} \varphi_\epsilon(p_\epsilon) S_\epsilon^a \nabla S_\epsilon \nabla p_\epsilon \, d\mathbf{x}\, dt,$$

$$\mathcal{I}_5 = -\iint_{Q_T} \alpha_\epsilon \big(f_w(S^*)a_\epsilon^+(\overline{p}_\epsilon - p_\epsilon) - f_w(S_\epsilon)a_\epsilon^-(\overline{p}_\epsilon - p_\epsilon)\big)\varphi_\epsilon(p_\epsilon)S_\epsilon \, d\mathbf{x}\, dt.$$

We have to estimate each of the quantities $\mathcal{I}_0$, $\mathcal{I}_1$ $\mathcal{I}_2$, $\mathcal{I}_3$, $\mathcal{I}_4$ and $\mathcal{I}_5$. We have that

$$\begin{aligned}\mathcal{I}_0 &= \int_\Omega \frac{1}{2}\big(\varphi_\epsilon\big(p_\epsilon(\mathbf{x}, T)\big)S_\epsilon(\mathbf{x}, T)\big)^2 \, d\mathbf{x} - \int_\Omega \frac{1}{2}\big(\varphi_\epsilon\big(p_{0,\epsilon}(\mathbf{x})\big)S_{\epsilon,0}(\mathbf{x})\big)^2 \, d\mathbf{x} \\ &\geqslant -\frac{1}{2}|\Omega|\varphi(p^*).\end{aligned} \tag{3.30}$$

For $\mathcal{I}_1$, we have

$$\begin{aligned}\mathcal{I}_1 &= k \int\int_{Q_T} \varphi_\epsilon(p_\epsilon)\big(\sqrt{\psi_\epsilon'(S_\epsilon)}\nabla S_\epsilon\big)^2 \, d\mathbf{x}\, dt \\ &\geqslant k\varphi(p_*) \iint_{Q_T} \Big(\big(\sqrt{C_* S_\epsilon^{m-1} + \lambda_* \epsilon}\big)\nabla S_\epsilon\Big)^2 \, d\mathbf{x}\, dt \\ &\geqslant k\varphi(p_*) \iint_{Q_T} \big(\nabla \mathcal{G}_\epsilon(S_\epsilon)\big)^2 \, d\mathbf{x}\, dt.\end{aligned} \tag{3.31}$$

Next we estimate $\mathcal{I}_2$,

$$
\begin{aligned}
|\mathcal{I}_2| &= k\Big|\iint\limits_{Q_T} S_\epsilon \psi_\epsilon' \nabla S_\epsilon \nabla\varphi_\epsilon(p_\epsilon)\,d\mathbf{x}\,dt\Big| \\
&\leqslant k\frac{c_1}{2}\iint\limits_{Q_T} S_\epsilon^2\big(\psi'(S_\epsilon)\big)^2(\nabla S_\epsilon)^2\,d\mathbf{x}\,dt + \frac{k}{2c_1}\iint\limits_{Q_T}\big(\nabla\varphi_\epsilon(p_\epsilon)\big)^2\,d\mathbf{x}\,dt \\
&\leqslant kc_1\iint\limits_{Q_T} S_\epsilon^2\big(C^{*2}S_\epsilon^{2(m-1)}+\lambda^{*2}\epsilon^2\big)(\nabla S_\epsilon)^2\,d\mathbf{x}\,dt + \frac{k}{2c_1}\iint\limits_{Q_T}\big(\nabla\varphi_\epsilon(p_\epsilon)\big)^2\,d\mathbf{x}\,dt.
\end{aligned}
$$

Choosing $\epsilon \leqslant \min(1, \frac{\lambda_* C^{*2}}{\lambda^{*2}C_*})$ and $S_\epsilon \leqslant 1$, we obtain

$$
\begin{aligned}
|\mathcal{I}_2| &\leqslant kc_1\frac{C^{*2}}{C_*}\iint\limits_{Q_T}\big(C_*S_\epsilon^{m-1}+\lambda_*\epsilon\big)(\nabla S_\epsilon)^2\,d\mathbf{x}\,dt + \frac{k}{2c_1}\iint\limits_{Q_T}\big(\nabla\varphi_\epsilon(p_\epsilon)\big)^2\,d\mathbf{x}\,dt \\
&\leqslant kc_1\frac{C^{*2}}{C_*}\iint\limits_{Q_T}\big(\nabla\mathcal{G}_\epsilon(S_\epsilon)\big)^2\,d\mathbf{x}\,dt + \frac{k}{2c_1}\iint\limits_{Q_T}\big(\nabla\varphi_\epsilon(p_\epsilon)\big)^2\,d\mathbf{x}\,dt. \qquad (3.32)
\end{aligned}
$$

For $\mathcal{I}_3$, we have

$$
|\mathcal{I}_3| \leqslant \frac{k}{2}\frac{\rho_w}{\mu_w}\iint\limits_{Q_T}(\nabla p_\epsilon)^2\,d\mathbf{x}\,dt + \frac{k}{2}\frac{\rho_w}{\mu_w}\iint\limits_{Q_T}\big(\nabla\varphi_\epsilon(p_\epsilon)\big)^2\,d\mathbf{x}\,dt. \qquad (3.33)
$$

For $\mathcal{I}_4$, we write

$$
|\mathcal{I}_4| \leqslant \frac{kc_2}{2}\frac{\varphi(p^*)\rho_w}{\mu_w}\iint\limits_{Q_T}\big(S_\epsilon^a\nabla S_\epsilon\big)^2\,d\mathbf{x}\,dt + \frac{k}{2c_2}\frac{\varphi(p^*)\rho_w}{\mu_w}\iint\limits_{Q_T}(\nabla p_\epsilon)^2\,d\mathbf{x}\,dt.
$$

In view of condition (2.15), we deduce the following inequalities:

$$
\begin{aligned}
\mathcal{I}_4 &\leqslant \frac{kc_2}{2}\frac{\rho_w}{\mu_w}\varphi(p^*)\iint\limits_{Q_T}S_\epsilon^{m-1}(\nabla S_\epsilon)^2\,d\mathbf{x}\,dt + \frac{k}{2c_2}\frac{\rho_w}{\mu_w}\varphi(p^*)\iint\limits_{Q_T}(\nabla p_\epsilon)^2\,d\mathbf{x}\,dt \\
&\leqslant \frac{kc_2}{2}\frac{\rho_w}{\mu_w}\frac{\varphi(p^*)}{C_*}\iint\limits_{Q_T}\big(C_*S_\epsilon^{m-1}+\lambda_*\epsilon\big)(\nabla S_\epsilon)^2\,d\mathbf{x}\,dt + \frac{k}{2c_2}\frac{\rho_w}{\mu_w}\varphi(p^*)\iint\limits_{Q_T}(\nabla p_\epsilon)^2\,d\mathbf{x}\,dt \\
&\leqslant \frac{kc_2}{2}\frac{\rho_w}{\mu_w}\frac{\varphi(p^*)}{C_*}\iint\limits_{Q_T}\big(\nabla\mathcal{G}_\epsilon(S_\epsilon)\big)^2\,d\mathbf{x}\,dt + \frac{k}{2c_2}\frac{\rho_w}{\mu_w}\varphi(p^*)\iint\limits_{Q_T}(\nabla p_\epsilon)^2\,d\mathbf{x}\,dt. \qquad (3.34)
\end{aligned}
$$

Finally, we have

$$
\mathcal{I}_5 \leqslant \alpha^*\varphi(p^*)|\Omega|T2(p^*-p_*). \qquad (3.35)
$$

We deduce from (3.29) that

$$
\mathcal{I}_0 = -\mathcal{I}_1 - \mathcal{I}_2 - \mathcal{I}_3 - \mathcal{I}_4 - \mathcal{I}_5,
$$

which combined with (3.30)–(3.35) implies that

$$
\begin{aligned}
& k\varphi(p_*) \iint\limits_{Q_T} \big(\nabla \mathcal{G}_\epsilon(S_\epsilon)\big)^2 \, d\mathbf{x}\, dt \\
& \leqslant \mathcal{I}_1 \leqslant -\mathcal{I}_0 + |\mathcal{I}_2| + |\mathcal{I}_3| + |\mathcal{I}_4| + |\mathcal{I}_5| \\
& \qquad \leqslant \frac{1}{2}|\Omega|\varphi(p^*) + \left(kc_1 \frac{C^{*2}}{C_*} + \frac{kc_2}{2}\frac{\rho_w}{\mu_w}\frac{\varphi(p^*)}{C_*}\right) \iint\limits_{Q_T} \big(\nabla \mathcal{G}_\epsilon(S_\epsilon)\big)^2 \, d\mathbf{x}\, dt \\
& \qquad\quad + \left(\frac{k}{2c_1} + \frac{k}{2}\frac{\rho_w}{\mu_w}\right) \iint\limits_{Q_T} \big(\nabla \varphi_\epsilon(p_\epsilon)\big)^2 \, d\mathbf{x}\, dt \\
& \qquad\quad + \frac{\rho_w}{\mu_w}\left(\frac{k}{2} + \frac{k}{2c_2}\varphi(p^*)\right) \iint\limits_{Q_T} (\nabla p_\epsilon)^2 \, d\mathbf{x}\, dt.
\end{aligned} \tag{3.36}
$$

Choosing suitable constants c_1 and c_2 and substituting Lemma 3.4 into (3.36) we deduce (i).

(ii) It follows from (3.6), (i) and (3.12) that for all $\epsilon \leqslant \min(1, \frac{\lambda_* C^{*2}}{\lambda^{*2} C_*})$,

$$
\begin{aligned}
\iint\limits_{Q_T} \big|\nabla \psi_\epsilon\big(S_\epsilon(\mathbf{x},t)\big)\big|^2 \, d\mathbf{x}\, dt &= \iint\limits_{Q_T} \big(C^* S_\epsilon^{m-1} + \lambda^* \epsilon\big)^2 (\nabla S_\epsilon)^2 \, d\mathbf{x}\, dt \\
&\leqslant 2 \iint\limits_{Q_T} \big(C^{*2}\big(S_\epsilon^{m-1}\big)^2 + \lambda^{*2}\epsilon^2\big)(\nabla S_\epsilon)^2 \, d\mathbf{x}\, dt \\
&\leqslant 2\frac{C^{*2}}{C_*} \iint\limits_{Q_T} \big(\nabla \mathcal{G}_\epsilon\big(S_\epsilon(\mathbf{x},t)\big)\big)^2 \, d\mathbf{x}\, dt \leqslant C.
\end{aligned}
$$

(iii) For all $\epsilon \leqslant \min(1, \frac{\lambda_* C^{*2}}{\lambda^{*2} C_*})$, we have

$$
\begin{aligned}
& \iint\limits_{Q_T} \big|\nabla \mathcal{K}\big(\varphi_\epsilon(p_\epsilon) S_\epsilon\big)\big|^2 \, d\mathbf{x}\, dt \\
& \quad \leqslant \varphi^{m-1}(p^*) \iint\limits_{Q_T} \big|\nabla \varphi_\epsilon(p_\epsilon)\big|^2 \, d\mathbf{x}\, dt + \varphi^{2m}(p^*) \iint\limits_{Q_T} \big|S_\epsilon^{m-1} \nabla S_\epsilon\big|^2 \, d\mathbf{x}\, dt \\
& \quad \leqslant \varphi^{m-1}(p^*) \iint\limits_{Q_T} \big|\nabla \varphi_\epsilon(p_\epsilon)\big|^2 \, d\mathbf{x}\, dt + \frac{\varphi^{2m}(p^*)}{C_*} \iint\limits_{Q_T} \big|\nabla \mathcal{G}_\epsilon(S_\epsilon)\big|^2 \, d\mathbf{x}\, dt.
\end{aligned}
$$

We deduce (iii) from Lemma 3.4(i) and (ii).

(iv) Using (iii), it follows that

$$
\begin{aligned}
& \int\limits_0^T \int\limits_{\Omega_\xi} \Big(\mathcal{K}\big(\varphi_\epsilon\big(p_\epsilon(\mathbf{x}+\xi,t)\big) S_\epsilon(\mathbf{x}+\xi,t)\big) - \mathcal{K}\big(\varphi_\epsilon\big(p_\epsilon(\mathbf{x},t)\big) S_\epsilon(\mathbf{x},t)\big)\Big)^2 \, d\mathbf{x}\, dt \\
& \quad = \int\limits_0^T \int\limits_{\Omega_\xi} \left(\int\limits_0^1 \nabla \mathcal{K}\big(\varphi_\epsilon\big(p_\epsilon(\mathbf{x}+r\xi,t)\big) S_\epsilon(\mathbf{x}+r\xi,t)\big).\xi \, dr \right)^2 d\mathbf{x}\, dt
\end{aligned}
$$

$$\leqslant \int_0^T \int_{\Omega_\xi} \left(\int_0^1 \Big(\nabla \mathcal{K}\big(\varphi_\epsilon\big(p_\epsilon(\mathbf{x}+\xi,t)\big) S_\epsilon(\mathbf{x}+\xi,t)\big)\Big)^2 dr \right) \left(\int_0^1 \xi^2 \, dr \right) d\mathbf{x}\, dt$$
$$\leqslant C_4 |\xi|^2.$$

(v) We set

$$\mathcal{J}_0 = \int_0^{T-\tau} \int_{\Omega} \Big(\mathcal{K}\big(\varphi_\epsilon\big(p_\epsilon(\mathbf{x},t+\tau)\big) S_\epsilon(\mathbf{x},t+\tau)\big) - \mathcal{K}\big(\varphi_\epsilon\big(p_\epsilon(\mathbf{x},t)\big) S_\epsilon(\mathbf{x},t)\big)\Big)^2 d\mathbf{x}\, dt,$$

which by (3.1) implies that

$$\begin{aligned}
\mathcal{J}_0 &\leqslant A^* \int_0^{T-\tau} \int_{\Omega} \Big(\mathcal{K}\big(\varphi_\epsilon\big(p_\epsilon(\mathbf{x},t+\tau)\big) S_\epsilon(\mathbf{x},t+\tau)\big) - \mathcal{K}\big(\varphi_\epsilon\big(p_\epsilon(\mathbf{x},t)\big) S_\epsilon(\mathbf{x},t)\big)\Big) \\
&\quad \times \big(\varphi_\epsilon\big(p_\epsilon(\mathbf{x},t+\tau)\big) S_\epsilon(\mathbf{x},t+\tau) - \varphi_\epsilon\big(p_\epsilon(\mathbf{x},t)\big) S_\epsilon(\mathbf{x},t)\big)\, d\mathbf{x}\, dt \\
&= A^* \int_0^{T-\tau} \int_{\Omega} \Big(\mathcal{K}\big(\varphi_\epsilon\big(p_\epsilon(\mathbf{x},t+\tau)\big) S_\epsilon(\mathbf{x},t+\tau)\big) - \mathcal{K}\big(\varphi_\epsilon\big(p_\epsilon(\mathbf{x},t)\big) S_\epsilon(\mathbf{x},t)\big)\Big) \\
&\quad \times \int_0^{\tau} \frac{\partial}{\partial t}\big(\varphi_\epsilon\big(p_\epsilon(\mathbf{x},t+r)\big) S_\epsilon(\mathbf{x},t+r)\big)\, dr\, d\mathbf{x}\, dt,
\end{aligned}$$

with $A^* = (\varphi(p^*)+1)^{m-1}$; hence, we deduce that

$$\mathcal{J}_0 \leqslant \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3,$$

with

$$\begin{aligned}
\mathcal{J}_1 &= \int_0^{\tau} \int_0^{T-\tau} \int_{\Omega} \Big(\mathcal{K}\big(\varphi_\epsilon\big(p_\epsilon(\mathbf{x},t+\tau)\big) S_\epsilon(\mathbf{x},t+\tau)\big) - \mathcal{K}\big(\varphi_\epsilon\big(p_\epsilon(\mathbf{x},t)\big) S_\epsilon(\mathbf{x},t)\big)\Big) \\
&\quad \times \Delta \psi_\epsilon\big(S_\epsilon(\mathbf{x},t+r)\big)\, d\mathbf{x}\, dt\, dr, \\
\mathcal{J}_2 &= \int_0^{\tau} \int_0^{T-\tau} \int_{\Omega} \Big(\mathcal{K}\big(\varphi_\epsilon\big(p_\epsilon(\mathbf{x},t+\tau)\big) S_\epsilon(\mathbf{x},t+\tau)\big) - \mathcal{K}\big(\varphi_\epsilon\big(p_\epsilon(\mathbf{x},t)\big) S_\epsilon(\mathbf{x},t)\big)\Big) \\
&\quad \times \operatorname{div}\big(S_\epsilon^a(\mathbf{x},t+r) \nabla p_\epsilon(\mathbf{x},t+r)\big)\, d\mathbf{x}\, dt\, dr, \\
\mathcal{J}_3 &= \int_0^{\tau} \int_0^{T-\tau} \int_{\Omega} \Big(\mathcal{K}\big(\varphi_\epsilon\big(p_\epsilon(\mathbf{x},t+\tau)\big) S_\epsilon(\mathbf{x},t+\tau)\big) - \mathcal{K}\big(\varphi_\epsilon\big(p_\epsilon(\mathbf{x},t)\big) S_\epsilon(\mathbf{x},t)\big)\Big) \\
&\quad \times \alpha_\epsilon(\mathbf{x}) \Big(f_w(S^*) a_\epsilon^+\big(\overline{p}_\epsilon(\mathbf{x}) - p_\epsilon(\mathbf{x},t+r)\big) \\
&\quad - f_w\big(S_\epsilon(\mathbf{x},t+r)\big) a_\epsilon^-\big(\overline{p}_\epsilon(\mathbf{x}) - p_\epsilon(\mathbf{x},t+r)\big)\Big)\, d\mathbf{x}\, dt\, dr.
\end{aligned}$$

We integrate $\mathcal{J}_1$ and $\mathcal{J}_2$ by parts to obtain on the one hand

$$\mathcal{J}_1 = -\int_0^{\tau}\int_0^{T-\tau}\int_{\Omega}\Big(\nabla\mathcal{K}\big(\varphi_\epsilon\big(p_\epsilon(\mathbf{x},t+\tau)\big)S_\epsilon(\mathbf{x},t+\tau)\big)-\nabla\mathcal{K}\big(\varphi_\epsilon\big(p_\epsilon(\mathbf{x},t)\big)S_\epsilon(\mathbf{x},t)\big)\Big)$$

$$\times\,\nabla\psi_\epsilon\big(S_\epsilon(\mathbf{x},t+r)\big)\,d\mathbf{x}\,dt\,dr$$

$$\leqslant 2\int_0^{\tau}\int_0^{T}\int_{\Omega}\Big(\nabla\mathcal{K}\big(\varphi_\epsilon\big(p_\epsilon(\mathbf{x},t)\big)S_\epsilon(\mathbf{x},t)\big)\Big)^2\,d\mathbf{x}\,dt\,dr+\int_0^{\tau}\int_0^{T}\int_{\Omega}\big(\nabla\psi_\epsilon\big(S_\epsilon(\mathbf{x},t)\big)\big)^2\,d\mathbf{x}\,dt\,dr$$

$$\leqslant C_1\tau$$

and on the other hand

$$\mathcal{J}_2 = -\int_0^{\tau}\int_0^{T-\tau}\int_{\Omega} S_\epsilon^a(\mathbf{x},t+r)\nabla p_\epsilon(\mathbf{x},t+r)$$

$$\times\Big(\nabla\mathcal{K}\big(\varphi_\epsilon\big(p_\epsilon(\mathbf{x},t+\tau)\big)S_\epsilon(\mathbf{x},t+\tau)\big)-\nabla\mathcal{K}\big(\varphi_\epsilon\big(p_\epsilon(\mathbf{x},t)\big)S_\epsilon(\mathbf{x},t)\big)\Big)\,d\mathbf{x}\,dt\,dr.$$

Applying the Cauchy–Schwarz inequality we obtain

$$\mathcal{J}_2 \leqslant 2M_*\int_0^{\tau}\int_0^{T}\int_{\Omega}\Big(\nabla\mathcal{K}\big(\varphi_\epsilon\big(p_\epsilon(\mathbf{x},t)\big)S_\epsilon(\mathbf{x},t)\big)\Big)^2\,d\mathbf{x}\,dt\,dr$$

$$+\,M_*\int_0^{\tau}\int_0^{T}\int_{\Omega}\big(\nabla p_\epsilon(\mathbf{x},t)\big)^2\,d\mathbf{x}\,dt\,dr$$

$$\leqslant C_2\tau,$$

whereas we easily get that

$$\mathcal{J}_3 \leqslant C_3\tau, \tag{3.37}$$

with $C_3 = \frac{2}{m}T|\Omega|\,\alpha^*(\varphi(p^*)+1)^m(p^*-p_*)$. Collecting the previous estimates, we deduce that

$$\mathcal{J}_0 \leqslant C\tau. \qquad \square$$

4. Convergence to a weak solution of Problem ($\mathcal{P}$)

Theorem 4.1. *There exists a weak solution $(p,S)\in(L^\infty(Q_T))^2$ of Problem ($\mathcal{P}$).*

Proof. Let (p_ϵ, S_ϵ) be a classical solution pair of the regularized problem ($\mathcal{P}_\epsilon$). We deduce from Lemma 3.4(ii) and (iii) that the family of the functions $\{\varphi_\epsilon(p_\epsilon)\}_\epsilon$ is relatively compact in $L^2(Q_T)$ [8]; consequently there exist a subsequence $\{\varphi_{\epsilon_n}(p_{\epsilon_n})\}_{\epsilon_n}$ and a function $\eta\in L^2(Q_T)$ such that

$$\varphi_{\epsilon_n}(p_{\epsilon_n}) \xrightarrow[n\to\infty]{} \eta \quad \text{in } L^2(Q_T) \text{ and a.e. in } Q_T. \tag{4.1}$$

Further we deduce from the bound $p_*\leqslant \hat{p}_\epsilon(\mathbf{x},t)\leqslant p^*$ that there exists a function $p\in L^2(Q_T)$ such that

$$p_{\epsilon_n} \xrightarrow[n\to\infty]{} p. \tag{4.2}$$

We have that

$$\left\|\varphi(p_{\epsilon_n})-\eta\right\|_{L^2(Q_T)} \leqslant \left\|\varphi(p_{\epsilon_n})-\varphi_{\epsilon_n}(p_{\epsilon_n})\right\|_{L^2(Q_T)}+\left\|\varphi_{\epsilon_n}(p_{\epsilon_n})-\eta\right\|_{L^2(Q_T)}. \tag{4.3}$$

Since φ_{ϵ_n} converge uniformly to φ in $[p_*, p^*]$ deduce that

$$\left\|\varphi(p_{\epsilon_n})-\varphi_{\epsilon_n}(p_{\epsilon_n})\right\|_{L^2(Q_T)} \to 0, \tag{4.4}$$

so that

$$\varphi(p_{\epsilon_n}) \underset{n\to\infty}{\longrightarrow} \eta \quad \text{in } L^2(Q_T). \tag{4.5}$$

Finally, (4.2), (4.5) and [3, Lemma 6.1] imply that

$$\varphi(p_{\epsilon_n}) \underset{n\to\infty}{\longrightarrow} \varphi(p) \quad \text{in } L^2(Q_T). \tag{4.6}$$

We deduce that there exists a subsequence $\{p_{\epsilon_n}\}_n$ such that

$$p_{\epsilon_n} \underset{n\to\infty}{\longrightarrow} p \quad \text{in } L^2(Q_T) \text{ and a.e. in } Q_T. \tag{4.7}$$

Moreover

$$p_* \leqslant p \leqslant p^* \quad \text{a.e. in } Q_T, \tag{4.8}$$

$$0 \leqslant S \leqslant 1 \quad \text{a.e. in } Q_T, \tag{4.9}$$

$$\nabla p_{\epsilon_n} \underset{n\to\infty}{\rightharpoonup} \nabla p \quad \text{in } L^2(Q_T). \tag{4.10}$$

Next we consider the sequence $\{S_{\epsilon_n}\}$. In view of the estimates (iv)–(v) of Lemma 3.5, (3.12) and (3.16), we can apply Fréchet–Kolmogorov theorem to show the compactness of the sequence functions $\{\mathcal{K}(\varphi_\epsilon(p_\epsilon)S_\epsilon)\}_{\epsilon>0}$ and thus deduce that there exist a subsequence $\{\mathcal{K}(\varphi_{\epsilon_n}(p_{\epsilon_n})S_{\epsilon_n})\}_{n\in\mathbb{N}}$ and a function $\chi \in L^2(Q_T)$ such that

$$\mathcal{K}(\varphi_{\epsilon_n}(p_{\epsilon_n})S_{\epsilon_n}) \underset{n\to\infty}{\longrightarrow} \chi \quad \text{in } L^2(Q_T) \text{ and a.e. in } Q_T. \tag{4.11}$$

We deduce from the strict monotony of the function $\mathcal{K}$ that,

$$\varphi_{\epsilon_n}(p_{\epsilon_n})S_{\epsilon_n} \underset{n\to\infty}{\longrightarrow} \mathcal{K}^{-1}(\chi) \quad \text{in } L^2(Q_T) \text{ and a.e. in } Q_T, \tag{4.12}$$

where $\mathcal{K}^{-1}$ is the inverse of the function $\mathcal{K}$. We deduce from (4.1), (4.6) and (4.12) that

$$S_{\epsilon_n} \underset{n\to\infty}{\longrightarrow} \frac{\mathcal{K}^{-1}(\chi)}{\varphi(p)} \quad \text{a.e. in } Q_T. \tag{4.13}$$

Since $0 \leqslant S_\epsilon \leqslant 1$, on the one hand there exists a function S in $L^2(Q_T)$ such that

$$S_{\epsilon_n} \underset{n\to\infty}{\rightharpoonup} S \quad \text{in } L^2(Q_T), \tag{4.14}$$

and on the other hand we deduce from Lebesgue dominated convergence theorem that

$$S_{\epsilon_n} \underset{n\to\infty}{\longrightarrow} \frac{\mathcal{K}^{-1}(\chi)}{\varphi(p)} \quad \text{in } L^2(Q_T). \tag{4.15}$$

We deduce that

$$S_{\epsilon_n} \underset{n\to\infty}{\longrightarrow} S \quad \text{in } L^2(Q_T) \text{ and a.e. in } Q_T. \tag{4.16}$$

Since

$$\begin{aligned}\left\|\psi_{\epsilon_n}(S_{\epsilon_n})-\psi(S)\right\|_{L^2(Q_T)} \quad &\leqslant \quad \left\|\psi_{\epsilon_n}(S_{\epsilon_n})-\psi(S_{\epsilon_n})\right\|_{L^2(Q_T)}+\left\|\psi(S_{\epsilon_n})-\psi(S)\right\|_{L^2(Q_T)}\\ &\xrightarrow[n\to\infty]{} 0,\end{aligned}$$

it follows that

$$\psi_{\epsilon_n}(S_{\epsilon_n}) \xrightarrow[n\to\infty]{} \psi(S) \quad \text{in } L^2(Q_T), \tag{4.17}$$

so that also by Lemma 3.5(ii)

$$\nabla\psi_{\epsilon_n}(S_{\epsilon_n}) \xrightarrow[n\to\infty]{} \nabla\psi(S) \quad \text{in } L^2(Q_T). \tag{4.18}$$

We will use below the uniform convergence of the sequences a_ϵ^+ and a_ϵ^- to a^+ and a^- respectively on the interval $[p_*-p^*, p^*-p_*]$ and the estimates

$$\left\|a_\epsilon^{+\prime}\right\|_{L^\infty(\mathbb{R})} \leqslant 1 \quad \text{and} \quad \left\|a_\epsilon^{-\prime}\right\|_{L^\infty(\mathbb{R})} \leqslant 1.$$

It follows from the convergence property (3.3) that

$$\begin{aligned}&\left\|a_\epsilon^+(\overline{p}_\epsilon-p_\epsilon)-(\overline{p}-p)^+\right\|_{L^2(Q_T)}\\ &\quad\leqslant \left\|a_\epsilon^+(\overline{p}_\epsilon-p_\epsilon)-(\overline{p}_\epsilon-p_\epsilon)^+\right\|_{L^2(Q_T)}+\left\|(\overline{p}_\epsilon-p_\epsilon)^+-(\overline{p}-p)^+\right\|_{L^2(Q_T)}\\ &\quad\xrightarrow[n\to\infty]{} 0\end{aligned} \tag{4.19}$$

and similarly that

$$\begin{aligned}&\left\|a_\epsilon^-(\overline{p}_\epsilon-p_\epsilon)-(\overline{p}-p)^-\right\|_{L^2(Q_T)}\\ &\quad\leqslant \left\|a_\epsilon^-(\overline{p}_\epsilon-p_\epsilon)-(\overline{p}_\epsilon-p_\epsilon)^-\right\|_{L^2(Q_T)}+\left\|(\overline{p}_\epsilon-p_\epsilon)^--(\overline{p}-p)^-\right\|_{L^2(Q_T)}\\ &\quad\xrightarrow[n\to\infty]{} 0.\end{aligned} \tag{4.20}$$

We multiply the first and second equation of system (3.26) by $v\in\mathcal{V}$ and integrate over Q_T. We obtain

$$\begin{aligned}&\iint_{Q_T} \varphi_{\epsilon_n}\big(p_{\epsilon_n}(\mathbf{x},t)\big)\frac{\partial}{\partial t}v(\mathbf{x},t)\,d\mathbf{x}\,dt\\ &\quad= k\iint_{Q_T} M(S_{\epsilon_n})\nabla p_{\epsilon_n}\nabla v\,d\mathbf{x}\,dt-\iint_{Q_T}\alpha_{\epsilon_n}(\overline{p}_{\epsilon_n}-p_{\epsilon_n})v\,d\mathbf{x}\,dt\\ &\qquad-\int_\Omega \varphi_{\epsilon_n}\big(p_{0,\epsilon_n}(\mathbf{x})\big)v(\mathbf{x},0)\,d\mathbf{x}\end{aligned} \tag{4.21}$$

and

$$\begin{aligned}&-\iint_{Q_T} \varphi_{\epsilon_n}(p_{\epsilon_n})S_{\epsilon_n}\frac{\partial}{\partial t}v\,d\mathbf{x}\,dt\\ &\quad= -k\iint_{Q_T}\nabla\psi_{\epsilon_n}(S_{\epsilon_n})\nabla v\,d\mathbf{x}\,dt-k\iint_{Q_T} f_w(S_{\epsilon_n})M(S_{\epsilon_n})\nabla p_{\epsilon_n}\nabla v\,d\mathbf{x}\,dt\\ &\qquad+\iint_{Q_T}\alpha_{\epsilon_n}\big(f_w(S^*)a_{\epsilon_n}^+(\overline{p}_{\epsilon_n}-p)-f_w(S_{\epsilon_n})a_{\epsilon_n}^-(\overline{p}_{\epsilon_n}-p)\big)v\,d\mathbf{x}\,dt\\ &\qquad+\int_\Omega \varphi_{\epsilon_n}(p_{0,\epsilon_n})S_{0,\epsilon_n}v(.,0)\,d\mathbf{x}.\end{aligned} \tag{4.22}$$

Because of the convergence properties established above, we can easily pass to the limit in (4.21) and (4.22) as $\epsilon_n \to 0$. We obtain

$$\iint_{Q_T} \varphi(p)\partial_t v \, d\mathbf{x}\, dt = \iint_{Q_T} \big(kM(S)\nabla p.\nabla v - \alpha(\overline{p} - p)v\big)\, d\mathbf{x}\, dt - \int_{\Omega} \varphi(p_0)v(0)\, d\mathbf{x}$$

and

$$\begin{aligned}\iint_{Q_T} \varphi(p)S\partial_t v \, d\mathbf{x}\, dt = \iint_{Q_T} &\big(-k\psi(S)\Delta v + kf_w(S)M(S)\nabla p.\nabla v \\ &- \alpha\big(f_w(S^*)(\overline{p} - p)^+ - f_w(S)(\overline{p} - p)^-\big)v\big)\, d\mathbf{x}\, dt \\ &- \int_{\Omega} \varphi(p_0)S_0 v(0)\, d\mathbf{x},\end{aligned}$$

which completes the proof. $\square$

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