A COSTA-HOFFMAN-MEEKS TYPE SURFACE IN $H^2 \times R$

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Abstract. We show the existence in the space $\mathbb{H}^2 \times \mathbb{R}$ of a family of embedded minimal surfaces of genus $1 \leqslant k < +\infty$ and finite total extrinsic curvature with two catenoidal type ends and one middle planar end. The proof is based on a gluing procedure.

1. Introduction

During recent years the study of the minimal surfaces in the product spaces $M \times \mathbb{R}$ with $M = \mathbb{H}^2, \mathbb{S}^2$ has become more and more active. The development of the theory of the minimal surfaces in these spaces started with [20] by H. Rosenberg and continued with [14] and [15] by W. H. Meeks and H. Rosenberg. In [17] B. Nelli and H. Rosenberg showed the existence in $\mathbb{H}^2 \times \mathbb{R}$ of a rich family of examples including helicoids, catenoids and, solving Plateau problems, of higher topological type examples inspired by the theory of minimal surfaces in $\mathbb{R}^3$. In [5] L. Hauswirth constructed and classified the minimal surfaces foliated by horizontal constant curvature curves in $M \times \mathbb{R}$, where M is $\mathbb{H}^2, \mathbb{R}^2$ or $\mathbb{S}^2$. Other examples of minimal surfaces of genus 0 in these product manifolds are described by R. Sa Earp and E. Toubiana in [21].

C. Costa in [1, 2] and D. Hoffman and W. H. Meeks in [7], [8] and [9] described a minimal surface in $\mathbb{R}^3$ of genus $1 \leqslant k < +\infty$ and finite total curvature with two ends asymptotic to the two ends of a catenoid and a middle end asymptotic to a plane. We will denote the Costa-Hoffman-Meeks surface of genus k by M_k.

The aim of this work is to show the existence in the space $\mathbb{H}^2 \times \mathbb{R}$ of a family of surfaces inspired by M_k. We will prove the following result.

Theorem 1.1. *For all $1 \leqslant k < +\infty$ there exists in $\mathbb{H}^2 \times \mathbb{R}$ a one-parameter family of embedded minimal surfaces of genus k and finite total extrinsic curvature with three horizontal ends: two catenoidal type ends and a middle planar end.*

We will observe that it is more convenient to construct a minimal surface enjoying the same properties mentioned in the statement of theorem in the Riemannian manifold $(\mathbb{D}^2 \times \mathbb{R}, g_{hyp})$ where $g_{hyp} = \frac{dx_1^2 + dx_2^2}{(1-x_1^2-x_2^2)^2} + dx_3^2$. It is usually denoted by $\mathbb{M}^2(-4) \times \mathbb{R}$, to point out that the sectional curvature of $\mathbb{D}^2 \times \{0\}$ endowed with the metric $\frac{dx_1^2 + dx_2^2}{(1-x_1^2-x_2^2)^2}$ equals -4. We observe that $\mathbb{H}^2 = \mathbb{M}^2(-1)$. Once having constructed this surface, it is easy to obtain by a diffeomorphism the wanted minimal surface in $\mathbb{H}^2 \times \mathbb{R}$.

The main result is proved by a gluing procedure (see for example [6]) usually adopted to construct in $\mathbb{R}^3$ new examples starting from known minimal surfaces. We consider a scaled version of a compact piece of a Costa-Hoffman-Meeks type surface, which is contained in a cylindrical neighbourhood of $\{0,0\}\times\mathbb{R}\subset\mathbb{M}^2(-4)\times\mathbb{R}$ of sufficiently small radius. Actually it's possible to prove that, in the same set, the mean curvature of such a surface with respect to the metric g_{hyp}, up to an infinitesimal term, equals the Euclidean one. We glue the surface described above along its three boundary curves to two minimal graphs that are respectively asymptotic to an upper half catenoid and a lower half catenoid defined in $\mathbb{M}^2(-4)\times\mathbb{R}$ and to a minimal graph over $\mathbb{M}^2(-4)\times\{0\}$ which goes to zero in a neighbourhood of $\partial_\infty\mathbb{M}^2(-4)\times\{0\}$. The existence of these surfaces is proved in sections 5 and 7.

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2. Preliminaries

In this work we will consider the unit disk model for $\mathbb{H}^2$. Let (x_1, x_2) denote the coordinates in the unit disk $\mathbb{D}^2$ and x_3 the coordinate in $\mathbb{R}$. Then the space $\mathbb{D}^2\times\mathbb{R}$ is endowed with the metric

$$g_{\mathbb{H}^2\times\mathbb{R}} = \frac{4(dx_1^2+dx_2^2)}{\left(1-x_1^2-x_2^2\right)^2}+dx_3^2.$$

As mentioned in the Introduction, one of the surfaces involved in the gluing procedure is a compact piece of a scaled version of the Costa-Hoffman-Meeks surface, that is, a minimal surface in $\mathbb{R}^3$ endowed with the Euclidean metric g_0. To simplify as much as possible the proof of the main theorem, it is convenient to consider a Riemannian manifold endowed with a metric more similar to g_0 than the standard metric of $\mathbb{H}^2\times\mathbb{R}$. The best choice is

$$g_{hyp} = \frac{dx_1^2+dx_2^2}{(1-x_1^2-x_2^2)^2}+dx_3^2,$$

because $g_{hyp}\to g_0$ if $(x_1,x_2)\to(0,0)$. This is the reason that induces us to give a proof of Theorem 1.1 working in the Riemannian manifold $\mathbb{M}^2(-4)\times\mathbb{R}$. Now we suppose to have shown the existence of a minimal surface in this Riemannian manifold. We need to show how it is possible to obtain a minimal surface in $\mathbb{H}^2\times\mathbb{R}$. Let $\bar{g}$ be the metric defined on $\mathbb{D}^2\times\mathbb{R}$ by

$$\bar{g} = 4g_{hyp} = \frac{4(dx_1^2+dx_2^2)}{(1-x_1^2-x_2^2)^2}+4dx_3^2.$$

We consider the map $f:(\mathbb{D}^2\times\mathbb{R}, g_{\mathbb{H}^2\times\mathbb{R}})\to(\mathbb{D}^2\times\mathbb{R},\bar{g})$ defined by

$$(x_1,x_2,x_3)\to\left(x_1,x_2,\frac{x_3}{2}\right). \tag{1}$$

It is easy to see that it is an isometric embedding, that is, the pull-back of the metric $\bar{g}$ by f equals $g_{\mathbb{H}^2\times\mathbb{R}}$. So if Σ is a minimal surface in $(\mathbb{D}^2\times\mathbb{R},\bar{g})$, then the image of Σ by f^{-1} is a minimal surface in $\mathbb{H}^2\times\mathbb{R}$.

Now we turn our attention to the Riemannian manifold $\mathbb{M}^2(-4)\times\mathbb{R}$, mentioned in the Introduction. In the following we will adopt the simplified notation $\mathbb{M}^2\times\mathbb{R}$. We recall that the metric $\bar{g}$ has been defined as $4g_{hyp}$, g_{hyp} being the metric of $\mathbb{M}^2\times\mathbb{R}$. As a consequence the mean curvature of a surface Σ in $\mathbb{M}^2\times\mathbb{R}$ equals the mean curvature of Σ in $(\mathbb{D}^2 \times \mathbb{R}, \bar{g})$ multiplied by 4. So if a surface is minimal in $\mathbb{M}^2 \times \mathbb{R}$, it is also minimal with respect to the metric $\bar{g}$.

We can conclude that if Σ is a minimal surface in $\mathbb{M}^2 \times \mathbb{R}$, then $f^{-1}(\Sigma)$ is a minimal surface in $\mathbb{H}^2 \times \mathbb{R}$.

Remark 2.1. To prove Theorem 1.1 we will need to consider spaces of functions invariant under the actions of the isometries of $\mathbb{R}^3$ which keep unchanged the Costa-Hoffman-Meeks surface (appropriate rotations about the vertical coordinate axis x_3, the reflection with respect to the horizontal plane $x_3 = 0$ and the vertical plane $x_2 = 0$). These are isometries of $\mathbb{M}^2 \times \mathbb{R}$ as well. So we will continue using the same language as if we are in $\mathbb{R}^3$.

3. Minimal graphs in $\mathbb{M}^2 \times \mathbb{R}$

We denote by H_u the mean curvature of the graph of the function u over a domain in $\mathbb{D}^2$. Its expression is

$$2H_u = F\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+F|\nabla u|^2}}\right), \tag{2}$$

where $F = \left(1 - x_1^2 - x_2^2\right)^2 = \left(1-r^2\right)^2$ and div denotes the divergence in $\mathbb{R}^2$. For the details of the computation, see subsection 12.3.

Let Σ_u be the graph of the function u. In this section we want to obtain the expression of the mean curvature of the surface Σ_{u+v}, that is, the graph of the function $u+v$. It can also be considered as the vertical graph of the function v over Σ_u. We will show how it follows from (2) that the linearized mean curvature, which we denote by L_u, is given locally by:

$$L_u v := F\mathrm{div}\left(\frac{\nabla v}{\sqrt{1+F|\nabla u|^2}} - F\nabla u \frac{\nabla u \cdot \nabla v}{\sqrt{(1+F|\nabla u|^2)^3}}\right). \tag{3}$$

Furthermore we will give the expression of H_{u+v}, the mean curvature of the graph of the function $u+v$, in terms of the mean curvature of Σ_u, that is, H_u. In the following we will restrict our attention to two cases: the plane (in section 5), that is, $u = 0$, and (in section 7) a piece of catenoid defined on the domain $\{(\theta, r) \in \mathbb{M}^2 \times \{0\} \,|\, r \in [r_\varepsilon, 1]\}$, where $r_\varepsilon = \varepsilon/2$.

Here we will show that:

$$2H_{u+v} = 2H_u + L_u v + FQ_u(\sqrt{F}\nabla v, \sqrt{F}\nabla^2 v), \tag{4}$$

where Q_u has bounded coefficients if $r \in [r_\varepsilon, 1]$ and it satisfies $Q_u(0,0) = 0$ and $\nabla Q_u(0,0) = 0$. To show this, we observe that

$$\frac{1}{\sqrt{1+F|\nabla(u+v)|^2}} = \frac{1}{\sqrt{1+F|\nabla u|^2}} - F\frac{\nabla u \cdot \nabla v}{\sqrt{(1+F|\nabla u|^2)^3}} + Q_{u,1}(v). \tag{5}$$

$Q_{u,1}(v)$ has the following expression:

$$\frac{-F|\nabla v|^2}{2\left(1+F|\nabla(u+\bar{t}v)|^2\right)^{3/2}} + \frac{3F^2\left(\nabla u \cdot \nabla v + \bar{t}|\nabla v|^2\right)^2}{2\left(1+F|\nabla(u+\bar{t}v)|^2\right)^{5/2}}, \tag{6}$$

with $\bar{t} \in (0,1)$, and it satisfies $Q_{u,1}(0) = 0$, $\nabla Q_{u,1}(0) = 0$. To prove (5) it's sufficient to set

$$f(t) = \frac{1}{\sqrt{1+F|\nabla(u+tv)|^2}}$$

and to write down the Taylor's series of order one of this function and to evaluate it in $t=1$. That is, $f(1)=f(0)+f'(0)+\frac{1}{2}f''(\bar{t})$, with $\bar{t}\in(0,1)$. We insert (5) in the expression that defines $2H_{u+v}$ to get

$$F\mathrm{div}\left(\frac{\nabla(u+v)}{\sqrt{1+F|\nabla u|^2}}-F\nabla(u+v)\frac{\nabla u\cdot\nabla v}{\sqrt{(1+F|\nabla u|^2)^3}}+\nabla(u+v)Q_{u,1}(v)\right)=$$
$$2H_u+F\mathrm{div}\left(\frac{\nabla v}{\sqrt{1+F|\nabla u|^2}}-F\nabla u\frac{\nabla u\cdot\nabla v}{\sqrt{(1+F|\nabla u|^2)^3}}\right)+FQ_u(\sqrt{F}\nabla v,\sqrt{F}\nabla^2 v).$$

Since we assume that Σ_u is a minimal surface, we will consider $H_u=0$.

Remark 3.1. The minimal surfaces in the families we will construct in sections 5 and 7 have finite total extrinsic curvature. These minimal surfaces are graphs over the domain $\{(\theta,r)\in\mathbb{M}^2\,|\,r\in[r_\varepsilon,1]\}$ of functions of class $\mathcal{C}^{2,\alpha}$. The total extrinsic curvature of the graph S of a function u defined on $\mathbb{M}^2$ is the integral of the extrinsic curvature, that is,

$$\int_S K_{ext}dA=\int_S\frac{II}{I}dA, \tag{7}$$

where I, II denote the determinants of the first and of the second fundamental form. It follows that $II=b_{11}b_{22}-b_{12}^2$, $I=g_{11}g_{22}-g_{12}^2$, $dA=\sqrt{I}$. For the expressions of the coefficients of the first and of the second differential form see subsection 12.3. Once their expressions have been replaced in (7), it is clear that, taking into account that u is a $\mathcal{C}^{2,\alpha}$ class function, $\int_S K_{ext}dA$ is bounded. This observation allows us to state that this property also holds for the surface obtained by a gluing procedure in section 11. In fact the total extrinsic curvature of this last surface equals the sum of the total extrinsic curvature of the surfaces glued together, that is, a compact piece of a Costa-Hoffman-Meeks type surface and three minimal graphs over the domain described above. Because of the compactness, the contribution to the total curvature of the piece of the Costa-Hoffman-Meeks type example is bounded. Then the result follows immediately, taking into account the observation made above concerning the graph of $\mathcal{C}^{2,\alpha}$ class functions over $\mathbb{M}^2$.

4. The mapping properties of the Laplace operator

Now we restrict our attention to the case of the minimal surfaces close to $\mathbb{M}^2\times\{0\}$, that is, the graph of the function $u=0$. In this case we obtain immediately from (3) that $L_{u=0}=F\Delta_0$, where Δ_0 denotes the Laplacian in the flat metric g_0 of the unit disk $\mathbb{D}^2$.

In this section we will study the mapping properties of Δ_0. In the sequel we will use the polar coordinates (θ,r).

In particular our aim is to solve in a unique way the problem:

$$\begin{cases}\Delta_0 w=f & \text{in } S^1\times[r_0,1],\\ w|_{r=r_0}=\varphi\end{cases}$$

with $r_0\in(0,1)$, considering a convenient normed functions space for w,f and φ, so that the norm of w is bounded by that of f.

Now we can give the definition of the space of functions we will consider.

Definition 4.1. Given $\ell \in \mathbb{N}$, $\alpha \in (0,1)$, and the closed interval $I \subset [0,1]$, we define

$$\mathcal{C}^{\ell,\alpha}(S^1 \times I)$$

to be the space of functions $w := w(\theta, r)$ in $\mathcal{C}^{\ell,\alpha}_{loc}(S^1 \times I)$ for which the norm

$$\|w\|_{\mathcal{C}^{\ell,\alpha}(S^1\times I)}$$

is finite and whose graph surfaces are invariant with respect to symmetry with respect to the $x_2 = 0$ plane, with respect to the rotation of an angle $\frac{2\pi}{k+1}$ about the vertical x_3-axis, with respect to the composition of a rotation of angle $\frac{\pi}{k+1}$ about the x_3-axis and the symmetry with respect to the $x_3 = 0$ plane.

We recall that one of the surfaces involved in the gluing procedure we will follow to prove the main theorem is a surface derived by the Costa-Hoffman-Meeks surface. This surface, as explained in subsection 9.1, enjoys many properties of symmetry that we want to be inherited by the surface obtained by the gluing procedure. This is the reason for which we have chosen the functional space described above.

Proposition 4.2. *Given $r_0 \in (0,1)$, there exists an operator*

$$\begin{array}{rccc} G_{r_0}: & \mathcal{C}^{0,\alpha}(S^1\times[r_0,1]) & \longrightarrow & \mathcal{C}^{2,\alpha}(S^1\times[r_0,1]) \\ & f & \longmapsto & w := G_{r_0}(f) \end{array}$$

satisfying the following statements:

(i) $\Delta_0\, w = f$ *on* $S^1 \times [r_0, 1]$,

(ii) $w = 0$ *on* $S^1 \times \{r_0\}$ *and* $S^1 \times \{1\}$,

(iii) $||w||_{\mathcal{C}^{2,\alpha}(S^1\times[r_0,1])} \leqslant c\,||f||_{\mathcal{C}^{0,\alpha}(S^1\times[r_0,1])}$, *for some constant $c > 0$ which does not depend on r_0, f and w.*

The proof of this result is contained in subsection 12.2.

5. A family of minimal surfaces close to $\mathbb{M}^2 \times \{0\}$

In this section we will show the existence of minimal graphs in $\mathbb{M}^2 \times \mathbb{R}$ over $\mathbb{D}^2 - B_{r_\varepsilon}$, having prescribed boundary and which are asymptotic to it. We recall that $r_\varepsilon = \varepsilon/2$. We will reformulate the problem to use the Schäuder fixed point theorem. We know already that the graph of a function v, denoted with Σ_v, is minimal, if and only if the function v is a solution of

$$F\left(\Delta_0\, v + Q_0\left(\sqrt{F}\nabla v, \sqrt{F}\nabla^2 v\right)\right) = 0. \tag{8}$$

This equation is a simplified version (since $u = 0$) of (4). The operator Q_0 has bounded coefficients for $r \in [r_\varepsilon, 1]$. Its expression is $\operatorname{div}(\nabla v\, Q_{0,1})$, where $Q_{0,1}$ is given by (6). To simplify the notation, in the sequel we will write $Q_0(\cdot)$ in place of $Q_0\left(\sqrt{F}\nabla\cdot, \sqrt{F}\nabla^2\cdot\right)$.

Now let's consider a function $\varphi \in \mathcal{C}^{2,\alpha}(S^1)$ which is even with respect to θ, collinear to $\cos(j(k+1)\theta)$ (for $k \geqslant 1$ fixed) with $j \geqslant 1$ and odd and such that

$$\|\varphi\|_{\mathcal{C}^{2,\alpha}} \leqslant \kappa\varepsilon^2. \tag{9}$$

We define

$$w_\varphi(\cdot,\cdot) := \mathcal{H}_{r_\varepsilon,\varphi}(\cdot,\cdot),$$

where $\mathcal{H}$ is the operator of harmonic extension introduced in Proposition 12.1. The particular choice of φ assures that its harmonic extension belongs to the functional space of Definition 4.1.

In order to solve equation (8), we look for v of the form $v = w_\varphi + w$, where $w \in \mathcal{C}^{2,\alpha}(S^1 \times [r_\varepsilon, 1])$ and $v = \varphi$ on $S^1 \times \{r_\varepsilon\}$. Using Proposition 4.2, we can rephrase this problem as a fixed point problem,

$$w = S(\varphi, w), \tag{10}$$

where the nonlinear mapping S which depends on ε and φ is defined by

$$S(\varphi, w) := -G_{r_\varepsilon}\left(Q_0\left(w_\varphi + w\right)\right),$$

and where the operator G_{r_ε} is defined in Proposition 4.2. To prove the existence of a fixed point for (10) we need the following result, which states that $S(\varphi, \cdot)$ is a contraction mapping:

Lemma 5.1. *Let $\varphi \in \mathcal{C}^{2,\alpha}(S^1)$ be a function satisfying (9) and enjoying the properties given above. There exist some constants $c_\kappa > 0$ and $\varepsilon_\kappa > 0$ such that*

$$\|S(\varphi, 0)\|_{\mathcal{C}^{2,\alpha}(S^1\times[r_\varepsilon,1])} \leqslant c_\kappa \varepsilon^4 \tag{11}$$

and, for all $\varepsilon \in (0, \varepsilon_\kappa)$,

$$\|S(\varphi, v_2) - S(\varphi, v_1)\|_{\mathcal{C}^{2,\alpha}(S^1\times[r_\varepsilon,1])} \leqslant \frac{1}{2}\|v_2 - v_1\|_{\mathcal{C}^{2,\alpha}(S^1\times[r_\varepsilon,1])},$$
$$\|S(\varphi_2, v) - S(\varphi_1, v)\|_{\mathcal{C}^{2,\alpha}(S^1\times[r_\varepsilon,1])} \leqslant c\varepsilon^2\, \|\varphi_2 - \varphi_1\|_{\mathcal{C}^{2,\alpha}(S^1)},$$

where c is a positive constant, for all $v_1, v_2 \in \mathcal{C}^{2,\alpha}(S^1 \times [r_\varepsilon, 1])$ such that $\|v_i\|_{\mathcal{C}^{2,\alpha}} \leqslant 2c_\kappa\varepsilon^4$ and for all boundary data $\varphi_1, \varphi_2 \in \mathcal{C}^{2,\alpha}(S^1)$ enjoying the same properties as φ.

Proof. We know from Proposition 4.2 that $\|G_{r_\varepsilon}(f)\|_{\mathcal{C}^{2,\alpha}} \leqslant c\|f\|_{\mathcal{C}^{0,\alpha}}$. Then

$$\|S(\varphi, 0)\|_{\mathcal{C}^{2,\alpha}} = \|G_{r_\varepsilon}(Q_0\left(w_\varphi\right))\|_{\mathcal{C}^{2,\alpha}} \leqslant c\|Q_0\left(w_\varphi\right)\|_{\mathcal{C}^{0,\alpha}}.$$

To find an estimate of the norm above we recall that $\|\varphi\|_{2,\alpha} \leqslant \kappa\varepsilon^2$ and thanks to Proposition 12.1 we obtain

$$\|w_\varphi\|_{\mathcal{C}^{2,\alpha}} \leqslant c\|\varphi\|_{\mathcal{C}^{2,\alpha}} \leqslant c_\kappa\varepsilon^2.$$

Then

$$\|Q_0\left(w_\varphi\right)\|_{\mathcal{C}^{0,\alpha}} \leqslant c\|w_\varphi\|^2_{\mathcal{C}^{2,\alpha}} \leqslant c\|\varphi\|^2_{\mathcal{C}^{2,\alpha}} \leqslant c_\kappa\varepsilon^4.$$

So we can conclude

$$\|S(\varphi, 0)\|_{\mathcal{C}^{2,\alpha}} \leqslant c_\kappa\varepsilon^4.$$

As for the second estimate, we observe that

$$\|S(\varphi, v_2) - S(\varphi, v_1)\|_{\mathcal{C}^{2,\alpha}} \leqslant c\|Q_0\left(w_\varphi + v_2\right) - Q_0\left(w_\varphi + v_1\right)\|_{\mathcal{C}^{0,\alpha}}.$$

Thanks to the considerations made above it follows that

$$\begin{aligned}||Q_0\left(w_\varphi + v_2\right) - Q_0\left(w_\varphi + v_1\right)||_{\mathcal{C}^{0,\alpha}} &\leqslant c||v_2 - v_1||_{\mathcal{C}^{2,\alpha}}\|w_\varphi\|_{\mathcal{C}^{2,\alpha}}\\ &\leqslant c_\kappa\varepsilon^2||v_2 - v_1||_{\mathcal{C}^{2,\alpha}}.\end{aligned}$$

Then

$$\|S(\varphi, v_2) - S(\varphi, v_1)\|_{\mathcal{C}^{2,\alpha}} \leqslant c_\kappa\varepsilon^2||v_2 - v_1||_{\mathcal{C}^{2,\alpha}}.$$

To show the third estimate we proceed as above:

$$\begin{aligned}\|S(\varphi_2, v) - S(\varphi_1, v)\|_{\mathcal{C}^{2,\alpha}} &\leqslant c\|Q_0\left(w_{\varphi_2} + v\right) - Q_0\left(w_{\varphi_1} + v\right)\|_{\mathcal{C}^{0,\alpha}}\\ &\leqslant c||w_{\varphi_2} - w_{\varphi_1}||_{\mathcal{C}^{2,\alpha}}\|v\|_{\mathcal{C}^{2,\alpha}} \leqslant c\varepsilon^2||\varphi_2 - \varphi_1||_{\mathcal{C}^{2,\alpha}}.\end{aligned}$$

$\square$

Theorem 5.2. *Let $B := \{w \in \mathcal{C}^{2,\alpha}(S^1 \times [r_\varepsilon, 1)) \,|\, ||w||_{\mathcal{C}^{2,\alpha}} \leqslant 2c_\kappa \varepsilon^4\}$ and φ be as above. Then the nonlinear mapping $S(\varphi, \cdot)$ defined above has a unique fixed point v in B.*

Proof. The previous lemma shows that, if ε is chosen small enough, the nonlinear mapping $S(\varphi, \cdot)$ is a contraction mapping from the ball B of radius $2c_\kappa\varepsilon^4$ in $\mathcal{C}^{2,\alpha}(S^1 \times [r_\varepsilon, 1])$ into itself. This value follows from the estimate of the norm of $S(\varphi, 0)$. Consequently, thanks to the Schäuder fixed point theorem, $S(\varphi, \cdot)$ has a unique fixed point v in this ball. □

We have proved the existence of a minimal surface with respect to the metric g_{hyp}, denoted by $S_m(\varphi)$, which is close to $\mathbb{D}^2 - B_{r_\varepsilon} \subset \mathbb{M}^2 \times \{0\}$, and close to its boundary is the vertical graph over the annulus $B_{2r_\varepsilon} - B_{r_\varepsilon}$ of a function which can be expanded as

$$\bar{U}_m(\theta, r) = \mathcal{H}_{r_\varepsilon,\varphi}(\theta, r) + \bar{V}_m(\theta, r), \quad \text{with} \quad ||\bar{V}_m||_{\mathcal{C}^{2,\alpha}} \leqslant c\varepsilon^2.$$

From the properties of the extension operator $\mathcal{H}_{r_\varepsilon,\varphi}$ (see Proposition 12.1) and Proposition 4.2 we can see that $\bar{U}_m(\theta, r)$ tends to 0 as $r \to 1$. In other terms $S_m(\varphi)$ is asymptotic to $\mathbb{M}^2 \times \{0\}$. Furthermore it is clear that $S_m(\varphi)$ is embedded in $\mathbb{M}^2 \times \mathbb{R}$.

The function $\bar{V}_m$ depends nonlinearly on ε, φ. Furthermore, as it is easy to prove thanks to the third estimate of Lemma 5.1, it satisfies

$$\|\bar{V}_m(\varepsilon, \varphi)(\cdot, r_\varepsilon \cdot) - \bar{V}_m(\varepsilon, \varphi')(\cdot, r_\varepsilon \cdot)\|_{\mathcal{C}^{2,\alpha}(\bar{B}_2 - B_1)} \leqslant c\varepsilon\|\varphi - \varphi'\|_{\mathcal{C}^{2,\alpha}(S^1)}. \tag{12}$$

6. The catenoid in $\mathbb{M}^2 \times \mathbb{R}$

The catenoid in the space $\mathbb{M}^2 \times \mathbb{R}$ can be obtained by revolution around the x_3-axis, $\{0, 0\} \times \mathbb{R}$, of an appropriate curve γ (see [17]). We consider a vertical geodesic plane containing the origin of $\mathbb{M}^2$ and the curve γ. Let r be the Euclidean distance between the point of γ at height t and the x_3-axis: we denote with $r = r(t)$ a parametrization of γ.

The surface obtained by revolution of γ is minimal with respect to the metric g_{hyp} if and only if $r = r(t)$ satisfies the following differential equation (see subsection 12.4):

$$r(t)\frac{\partial^2 r}{\partial t^2} - \left(\frac{\partial r}{\partial t}\right)^2 - (1 - r(t)^4) = 0. \tag{13}$$

A first integral for this equation is:

$$\left(\frac{\partial r}{\partial t}\right)^2 = Cr^2 - (1 + r^4) \tag{14}$$

with $C > 2$ and constant. By the resolution of equation $\left(\frac{\partial r}{\partial t}\right)^2 = 0$, it is easy to prove that the function $r(t)$ has a minimum value r_{min} given by:

$$r_{min} = \sqrt{\frac{C - \sqrt{C^2 - 4}}{2}} = \sqrt{\frac{C/2 + 1}{2}} - \sqrt{\frac{C/2 - 1}{2}} < 1.$$

Since we assume $C = \frac{1}{\varepsilon^4}$, we get

$$r_{min} = \sqrt{\frac{C/2+1}{2}} - \sqrt{\frac{C/2-1}{2}} = \frac{\sqrt{C}}{2}\left(1 + \frac{1}{C} - 1 + \frac{1}{C} + \mathcal{O}\left(\frac{1}{C^2}\right)\right)$$
$$= \frac{1}{\sqrt{C}} + \mathcal{O}\left(\frac{1}{C^{3/2}}\right) = \varepsilon^2 + \mathcal{O}(\varepsilon^6).$$

We denote with C_t and C_b, respectively, the piece of the catenoid contained in $\mathbb{M}^2 \times \mathbb{R}^+$ and $\mathbb{M}^2 \times \mathbb{R}^-$.

We set

$$t_\varepsilon = -\varepsilon^2 \ln \varepsilon.$$

We need to find the parametrization of C_t and C_b as graphs on the horizontal plane respectively for $t \in [t_\varepsilon - \varepsilon^2 \ln 2, t_\varepsilon + \varepsilon^2 \ln 2]$ and $t \in [-t_\varepsilon - \varepsilon^2 \ln 2, -t_\varepsilon + \varepsilon^2 \ln 2]$. We start by finding the expression of $r(t)$ for t in the interval specified before.

Lemma 6.1. *For $\varepsilon > 0$ small enough, we have*

$$r(t) = \varepsilon^2 \cosh \frac{t}{\varepsilon^2} + \mathcal{O}(\varepsilon^6 e^{\frac{t}{\varepsilon^2}}) \text{ and } \partial_t r(t) = \sinh \frac{t}{\varepsilon^2} + \mathcal{O}(\varepsilon^4 e^{\frac{t}{\varepsilon^2}})$$

for $t \in [0, t_\varepsilon + \varepsilon^2 \ln 2]$. Moreover if $t \in [t_\varepsilon - \varepsilon^2 \ln 2, t_\varepsilon + \varepsilon^2 \ln 2]$, we derive

$$r(t) \in [\frac{1}{4}\varepsilon + c_1\varepsilon^3, \varepsilon + c_2\varepsilon^3],$$

$$\partial_t r(t) \in [\frac{1}{4\varepsilon} - c_1'\varepsilon, \frac{1}{\varepsilon} - c_2'\varepsilon],$$

for some positive constants c_1, c_2, c_1', c_2'.

Proof. We define the function $v(t)$ in such a way that $r(t) = r(0)\cosh v(t)$, with $v(0) = 0$ and $r(0)$ the minimum for $r(t)$. It satisfies

$$Cr^2(0) - (1 + r^4(0)) = 0,$$

from which

$$1 = Cr^2(0) - r^4(0). \tag{15}$$

Plugging $r(t)$ in (14) and using (15), we have

$$(\partial_t v)^2 = C - r^2(0)(1 + \cosh^2 v(t))$$

and under the hypothesis

$$\frac{t}{\varepsilon^2} \leqslant v(t) \leqslant \frac{t}{\varepsilon^2} + 1$$

we obtain that $(\partial_t v)^2 = C + \mathcal{O}(\varepsilon^4 e^{\frac{2t}{\varepsilon^2}})$ and then $v(t) = \sqrt{C}t + \mathcal{O}(\varepsilon^6 e^{\frac{2t}{\varepsilon^2}})$. We remark a posteriori that $\frac{t}{\varepsilon^2} \leqslant v(t) \leqslant \frac{t}{\varepsilon^2} + 1$ holds for $t \in [0, t_\varepsilon + \varepsilon^2 \ln 2]$, $\varepsilon > 0$ small enough. Since $r(0) = r_{min} = \varepsilon^2 + \mathcal{O}(\varepsilon^6)$, we get

$$r(t) = r(0)\cosh v(t) = \varepsilon^2 \cosh\left(\frac{t}{\varepsilon^2}\right) + \mathcal{O}(\varepsilon^6 e^{\frac{t}{\varepsilon^2}}). \tag{16}$$

If $t \in [t_\varepsilon - \varepsilon^2 \ln 2, t_\varepsilon + \varepsilon^2 \ln 2]$, then we easily obtain $r(t) \in [\frac{1}{4}\varepsilon + c_1\varepsilon^3, \varepsilon + c_2\varepsilon^3]$, for some positive constants c_1, c_2. Using $\partial_t r(t) = \sinh\left(\frac{t}{\varepsilon^2}\right) + \mathcal{O}(\varepsilon^4 e^{\frac{t}{\varepsilon^2}})$, we find $\partial_t r(t) \in [\frac{1}{4\varepsilon} - c_1'\varepsilon, \frac{1}{\varepsilon} - c_2'\varepsilon]$ for some positive constants c_1', c_2'. □

Now we can prove a lemma that gives us the parametrization of the pieces of the catenoid whose height t belongs to a neighbourhood of t_ε and $-t_\varepsilon$.

Lemma 6.2. *For $\varepsilon > 0$ small enough and $t \in [t_\varepsilon - \varepsilon^2 \ln 2, t_\varepsilon + \varepsilon^2 \ln 2]$, the surface C_t can be seen as the graph, over the annulus $B_{2r_\varepsilon} - B_{r_\varepsilon/2}$, of the function $W_t(\theta, r)$ which satisfies*

$$W_t(\theta, r) = \varepsilon^2 \ln \frac{2r}{\varepsilon^2} + \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon^3). \tag{17}$$

Similarly if $t \in [-t_\varepsilon - \varepsilon^2 \ln 2, -t_\varepsilon + \varepsilon^2 \ln 2]$, the surface C_b can be seen as the graph over $B_{2r_\varepsilon} - B_{r_\varepsilon/2}$ of the function

$$W_b(\theta, r) = -\varepsilon^2 \ln \frac{2r}{\varepsilon^2} + \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon^3).$$

Proof. The first result easily follows from the hypothesis and equation (16). The second result can be shown by observing that C_b is the image of C_t by the reflection with respect to the $x_3 = 0$ plane. In other terms, $W_b(\theta, r) = -W_t(\theta, r)$. □

7. A family of minimal surfaces close to a catenoid on $S^1 \times [r_\epsilon, 1]$

In this section we want to show the existence of minimal graphs in $\mathbb{M}^2 \times \mathbb{R}$ over the parts of the surfaces C_t and C_b (described in the previous section) defined on $S^1 \times [r_\varepsilon, 1] \subset \mathbb{M}^2 \times \{0\}$ and asymptotic to them. We know that the graph of the function $u + v$ is minimal, u being the function whose graph is the catenoid, if and only if v is a solution of the equation

$$H_{u+v} = 0 \tag{18}$$

whose expression is given by (4). The explicit expression of $L_u v$ is

(19)

$$F\left(\frac{1}{\sqrt{A}}\Delta_0 v + \partial_r\left(\frac{1}{\sqrt{A}}\right)\partial_r v - \frac{1}{A^{\frac{3}{2}}}\partial_r u\, \partial_r\left(F\partial_r u\right)\partial_r v - F\partial_r u\, \partial_r\left(\frac{1}{A^{\frac{3}{2}}}\partial_r u\, \partial_r v\right)\right),$$

where $F = (1 - r^2)^2$,

$$A = 1 + F|\nabla u|^2 = \frac{(C-2)r^2}{Cr^2 - 1 - r^4}$$

and

$$\partial_r u = \pm \frac{1}{\sqrt{Cr^2 - 1 - r^4}},$$

as is easy to obtain using (14). It's useful to observe that since we assume $C = \frac{1}{\varepsilon^4}$ and $r_\varepsilon = \varepsilon/2$, we have that, for $r \in [r_\varepsilon, 1]$, $A = 1 + \mathcal{O}(\varepsilon^2)$, $\partial_r u = \mathcal{O}(\varepsilon)$,

$$\partial_r A = \frac{(2C-4)(-r + r^5)}{(Cr^2 - 1 - r^4)^2} = \mathcal{O}(\varepsilon)$$

and

$$\partial^2_{rr} u = \mp \frac{(Cr - 2r^3)}{\sqrt{(Cr^2 - 1 - r^4)^3}} = \mathcal{O}(1).$$

Taking into account these estimates, we can conclude that

(20)

$$\begin{aligned}\bar{L}_u v &:= \sqrt{A}\left(\partial_r\left(\frac{1}{\sqrt{A}}\right)\partial_r v - \frac{1}{A^{\frac{3}{2}}}\partial_r u\, \partial_r\left(F\partial_r u\right)\partial_r v - F\partial_r u\, \partial_r\left(\frac{1}{A^{\frac{3}{2}}}\partial_r u\, \partial_r v\right)\right)\\ &= l_1 \partial_r v + l_2 \partial^2_{rr} v,\end{aligned}$$

where $l_1, l_2 = \mathcal{O}(\varepsilon)$. Then we can write $\sqrt{A} L_u v = F\left(\Delta_0 v + \bar{L}_u v\right)$.

We remark that we have already studied the mapping properties of the operator Δ_0 in section 4.

Let Σ_u be the graph of the function u. Then the graph of a function v over Σ_u is minimal if and only if v is a solution of the following equation:

$$\Delta_0 v + \bar{L}_u v + \sqrt{A} Q_u(v) = 0, \tag{21}$$

where $Q_u(\cdot) := Q_u\left(\sqrt{F}\nabla\cdot, \sqrt{F}\nabla^2\cdot\right)$. Thanks to the observations on the functions A and $\partial_r u$, we can conclude that Q_u has bounded coefficients if $r \in [r_\varepsilon, 1]$.

Now we consider a function $\varphi \in \mathcal{C}^{2,\alpha}(S^1)$ which is even with respect to θ, collinear to $\cos(j(k+1)\theta)$ (for $k \geqslant 1$ fixed) and such that

$$\|\varphi\|_{\mathcal{C}^{2,\alpha}} \leqslant \kappa\varepsilon^2. \tag{22}$$

We define

$$w_\varphi(\cdot,\cdot) := \mathcal{H}_{r_\varepsilon,\varphi}(\cdot,\cdot),$$

where the operator $\mathcal{H}_{r_\varepsilon,\varphi}$ has been introduced in Proposition 12.1.

In order to solve equation (21), we look for v of the form $v = w_\varphi + w$, where $w \in \mathcal{C}^{2,\alpha}(S^1 \times [r_\varepsilon, 1])$ and $v = \varphi$ on $S^1 \times \{r_\varepsilon\}$. We can rephrase this problem as a fixed point problem, that is,

$$w = S(\varphi, w), \tag{23}$$

where the nonlinear mapping S is defined by

$$S(\varphi, w) := -G_{r_\varepsilon}\left(\bar{L}_u(w_\varphi + w) + \sqrt{A} Q_u\left(w_\varphi + w\right)\right),$$

and where the operator G_{r_ε} is defined in Proposition 4.2. To prove the existence of a solution for (23) we need the following result, which states that $S(\varphi,\cdot)$ is a contraction mapping.

Lemma 7.1. *Let $\varphi \in \mathcal{C}^{2,\alpha}(S^1)$ be a function satisfying (22) and enjoying the properties given above. There exist some constants $c_\kappa > 0$ and $\varepsilon_\kappa > 0$, such that*

$$\|S(\varphi, 0)\|_{\mathcal{C}^{2,\alpha}(S^1\times[r_\varepsilon,1])} \leqslant c_\kappa \varepsilon^3 \tag{24}$$

and, for all $\varepsilon \in (0, \varepsilon_k)$,

$$\begin{aligned}
\|S(\varphi, w_2) - S(\varphi, w_1)\|_{\mathcal{C}^{2,\alpha}(S^1\times[r_\varepsilon,1])} &\leqslant \frac{1}{2}\|w_2 - w_1\|_{\mathcal{C}^{2,\alpha}(S^1\times[r_\varepsilon,1])},\\
\|S(\varphi_2, w) - S(\varphi_1, w)\|_{\mathcal{C}^{2,\alpha}(S^1\times[r_\varepsilon,1])} &\leqslant c\varepsilon\,\|\varphi_2 - \varphi_1\|_{\mathcal{C}^{2,\alpha}(S^1)},
\end{aligned}$$

where c is a positive constant, for all $w_1, w_2 \in \mathcal{C}^{2,\alpha}(S^1 \times [r_\varepsilon, 1])$ such that $\|w_i\|_{\mathcal{C}^{2,\alpha}(S^1\times[r_\varepsilon,1])} \leqslant 2c_\kappa\varepsilon^4$ and for all boundary data $\varphi_1, \varphi_2 \in C^{2,\alpha}(S^1)$ enjoying the same properties as φ.

Proof. We know from Proposition 4.2 that $\|G_{r_\varepsilon}(f)\|_{\mathcal{C}^{2,\alpha}} \leqslant c\|f\|_{\mathcal{C}^{0,\alpha}}$. Then

$$\begin{aligned}
\|S(\varphi, 0)\|_{\mathcal{C}^{2,\alpha}} &\leqslant c\|\bar{L}_u w_\varphi + \sqrt{A} Q_u\left(w_\varphi\right)\|_{\mathcal{C}^{0,\alpha}}\\
&\leqslant c\left(\|\bar{L}_u w_\varphi\|_{\mathcal{C}^{0,\alpha}} + \|Q_u\left(w_\varphi\right)\|_{\mathcal{C}^{0,\alpha}}\right).
\end{aligned}$$

Here we have used the fact that $A = 1 + \mathcal{O}(\varepsilon^2)$.

So we need to find the estimates of each summand. We recall that $\|\varphi\|_{\mathcal{C}^{2,\alpha}} \leqslant \kappa\varepsilon$. Thanks to Proposition 12.1 we get that

$$\|w_\varphi\|_{\mathcal{C}^{2,\alpha}} \leqslant c\|\varphi\|_{\mathcal{C}^{2,\alpha}(S^1)} \leqslant c_\kappa\varepsilon^2.$$

We use (20) for finding the estimate of $\bar{L}_u w_\varphi$. We obtain

$$\|\bar{L}_u w_\varphi\|_{\mathcal{C}^{0,\alpha}} \leqslant c\varepsilon \|w_\varphi\|_{\mathcal{C}^{2,\alpha}} \leqslant c_\kappa \varepsilon^3.$$

The last term is estimated by observing that

$$\|Q_u\left(w_\varphi\right)\|_{\mathcal{C}^{0,\alpha}} \leqslant c\|w_\varphi\|^2_{\mathcal{C}^{2,\alpha}} \leqslant c_\kappa \varepsilon^4.$$

Putting together all these estimates we get

$$\|S(\varphi, 0)\|_{\mathcal{C}^{2,\alpha}} \leqslant c_\kappa \varepsilon^3.$$

As for the second estimate, we observe that

$$\begin{aligned} S(\varphi, w_2) - S(\varphi, w_1) &= -G_{r_\varepsilon}\left(\bar{L}_u(w_\varphi + w_2) + \sqrt{A} Q_u\left(w_\varphi + w_2\right)\right) \\ &\quad + G_{r_\varepsilon}\left(\bar{L}_u(w_\varphi + w_1) + \sqrt{A} Q_u\left(w_\varphi + w_1\right)\right). \end{aligned}$$

Consequently

$$\begin{aligned} &\|S(\varphi, w_2) - S(\varphi, w_1)\|_{\mathcal{C}^{2,\alpha}} \\ &\quad \leqslant c\|\bar{L}_u(w_\varphi + w_2) - \bar{L}_u(w_\varphi + w_1) + \sqrt{A} Q_u\left(w_\varphi + w_2\right) - \sqrt{A} Q_u\left(w_\varphi + w_1\right)\|_{\mathcal{C}^{0,\alpha}} \\ &\quad = c\|\bar{L}_u(w_2 - w_1) + \sqrt{A}\left(Q_u\left(w_\varphi + w_2\right) - Q_u\left(w_\varphi + w_1\right)\right)\|_{\mathcal{C}^{0,\alpha}} \\ &\quad \leqslant c\left(\|\bar{L}_u(w_2 - w_1)\|_{\mathcal{C}^{0,\alpha}} + ||Q_u\left(w_\varphi + w_2\right) - Q_u\left(w_\varphi + w_1\right)||_{\mathcal{C}^{0,\alpha}}\right). \end{aligned}$$

We observe that from the considerations above it follows that

$$\|\bar{L}_u(w_2 - w_1)\|_{\mathcal{C}^{0,\alpha}} \leqslant c\varepsilon ||w_2 - w_1||_{\mathcal{C}^{2,\alpha}}$$

and

$$\begin{aligned} &||Q_u\left(w_\varphi + w_2\right) - Q_u\left(w_\varphi + w_1\right)||_{\mathcal{C}^{0,\alpha}} \leqslant c||w_2 - w_1||_{\mathcal{C}^{2,\alpha}} \|w_\varphi\|_{\mathcal{C}^{2,\alpha}} \\ &\quad \leqslant c_\kappa \varepsilon ||w_2 - w_1||_{\mathcal{C}^{2,\alpha}}. \end{aligned}$$

Then

$$\|S(\varphi, w_2) - S(\varphi, w_1)\|_{\mathcal{C}^{2,\alpha}} \leqslant c\varepsilon ||w_2 - w_1||_{\mathcal{C}^{2,\alpha}}.$$

To show the third estimate we observe that

$$\begin{aligned} &\|S(\varphi_2, w) - S(\varphi_1, w)\|_{\mathcal{C}^{2,\alpha}} \\ &\quad \leqslant c\left(\|\bar{L}_u(w_{\varphi_2} - w_{\varphi_1})\|_{\mathcal{C}^{0,\alpha}} + ||Q_u\left(w_{\varphi_2} + w\right) - Q_u\left(w_{\varphi_1} + w\right)||_{\mathcal{C}^{0,\alpha}}\right) \\ &\quad \leqslant c\varepsilon \|\varphi_2 - \varphi_1\|_{\mathcal{C}^{2,\alpha}(S^1)} + \|w\|_{\mathcal{C}^{2,\alpha}} \|\varphi_2 - \varphi_1\|_{\mathcal{C}^{2,\alpha}(S^1)} \\ &\quad \leqslant c\varepsilon \|\varphi_2 - \varphi_1\|_{\mathcal{C}^{2,\alpha}(S^1)}. \end{aligned}$$

□

Theorem 7.2. *Let $B := \{w \in \mathcal{C}^{2,\alpha}(S^1 \times [r_\varepsilon, 1]) \,|\, ||w||_{\mathcal{C}^{2,\alpha}} \leqslant 2c_\kappa \varepsilon^3\}$. Then the nonlinear mapping $S(\varphi, \cdot)$ defined above has a unique fixed point v in B.*

Proof. The previous lemma shows that, if ε is chosen small enough, the nonlinear mapping $S(\varphi, \cdot)$ is a contraction mapping from the ball B of radius $2c_\kappa \varepsilon^3$ in $\mathcal{C}^{2,\alpha}(S^1 \times [r_\varepsilon, 1])$ into itself. This value follows from the estimate of the norm of $S(\varphi, 0)$. Consequently, thanks to the Schäuder fixed point theorem, $S(\varphi, \cdot)$ has a unique fixed point v in this ball. □

We have proved the existence of a minimal surface with respect to the metric g_{hyp}, $S_t(\varphi)$, which is close to the piece of catenoid C_t introduced in section 6 and close to its boundary is a graph over the annulus $B_{2r_\varepsilon} - B_{r_\varepsilon}$ of the function

$$\bar{U}_t(\theta, r) = \varepsilon^2 \ln \frac{2r}{\varepsilon^2} + \mathcal{H}_{r_\varepsilon,\varphi}(\theta, r) + \bar{V}_t(\theta, r),$$

with $\|\bar{V}_t\|_{\mathcal{C}^{2,\alpha}} \leqslant c\varepsilon^2$. From the properties of the extension operator $\mathcal{H}_{r_\varepsilon,\varphi}$ (see Proposition 12.1) and Proposition 4.2 we can see that $S_t(\varphi)$ is asymptotic to C_t if r tends to 1 and it is embedded in $\mathbb{M}^2 \times \mathbb{R}$.

The function $\bar{V}_t$ depends nonlinearly on ε, φ. Furthermore it satisfies

$$\|\bar{V}_t(\varepsilon, \varphi)(\cdot, r_\varepsilon \cdot) - \bar{V}_t(\varepsilon, \varphi')(\cdot, r_\varepsilon \cdot)\|_{\mathcal{C}^{2,\alpha}(\bar{B}_2 - B_1)} \leqslant c\varepsilon \|\varphi - \varphi'\|_{\mathcal{C}^{2,\alpha}(S^1)}. \tag{25}$$

This estimate follows from Lemma 7.1.

Now it is easy to show the existence of a minimal surface $S_b(\varphi)$, which is close to the part of the catenoid denoted by C_b introduced in section 6, and close to its boundary is a graph over the annulus $B_{2r_\varepsilon} - B_{r_\varepsilon}$. We start observing that C_b can be obtained by reflection of C_t with respect to the $x_3 = 0$ plane. So we can define $S_b(\varphi)$ as the image of $S_t(\varphi)$ by the composition of a rotation by an angle $\frac{\pi}{k+1}$ about the x_3-axis and the reflection with respect to the horizontal plane. This choice (in particular the apparently unnecessary rotation) is indispensable to assure that the surface we will construct by the gluing procedure in section 11 has the same properties of symmetry as the Costa-Hoffman-Meeks surface. See subsection 9.1 for more information.

It is clear that $S_b(\varphi)$ is the graph over the annulus $B_{2r_\varepsilon} - B_{r_\varepsilon}$ of the function

$$\bar{U}_b(\theta, r) = -\bar{U}_t\left(\theta - \frac{\pi}{k+1}, r\right).$$

8. The relation between the mean curvatures of a surface in $\mathbb{D}^2 \times \mathbb{R}$ with respect to two different metrics

In this section we want to express the mean curvature H_{hyp} of a surface in $\mathbb{D}^2 \times \mathbb{R}$ with respect to the metric g_{hyp} in terms of the mean curvature H_e of the same surface with respect to the Euclidean metric g_0.

We recall that, if x_1, x_2 denote the coordinates in $\mathbb{D}^2$ and x_3 the coordinate in $\mathbb{R}$, then

$$g_{hyp} = \frac{dx_1^2 + dx_2^2}{F} + dx_3^2, \quad \text{where} \quad F = \left(1 - x_1^2 - x_2^2\right)^2 = \left(1 - r^2\right)^2$$

and

$$g_0 = dx_1^2 + dx_2^2 + dx_3^2.$$

If N_{hyp} denotes the normal vector to a surface Σ with respect to the metric g_{hyp}, then its mean curvature with respect to the same metric is given by

$$H_{hyp}(\Sigma) := -\frac{1}{2} \operatorname{trace}\left(X \to [\bar{\nabla}_X N_{hyp}]^T\right),$$

where $[\cdot]^T$ denotes the projection on the tangent bunble $T\Sigma$ and $\bar{\nabla}$ is the Riemannian connection relative to g_{hyp}. The mean curvature of Σ with respect to g_0, denoted by $H_e(\Sigma)$, is given by

$$H_e(\Sigma) := -\frac{1}{2} \operatorname{trace}\left(X \to [\nabla_X N_e]^T\right),$$

where N_e denotes the normal vector to Σ with respect to the metric g_0 and ∇ is the flat Riemannian connection.

The Christoffel symbols, Γ_{ij}^k, associated to the metric g_{hyp} all vanish except

$$\Gamma_{11}^1 = \Gamma_{21}^2 = \Gamma_{12}^2 = -\Gamma_{22}^1 = \frac{2x_1}{\sqrt{F}},$$
$$\Gamma_{12}^1 = \Gamma_{22}^2 = \Gamma_{21}^1 = -\Gamma_{11}^2 = \frac{2x_2}{\sqrt{F}}.$$

Let $\partial_1 = \frac{\partial}{\partial x_1}$, $\partial_2 = \frac{\partial}{\partial x_2}$, $\partial_3 = \frac{\partial}{\partial x_3}$ be the elements of a basis of the tangent space. Now, if $X = \sum_i X^i \partial_i$ and $Y = \sum_j Y^j \partial_j$ are two tangent vector fields, the expression of the covariant derivative in $(\mathbb{D}^2 \times \mathbb{R}, g_{hyp})$ is given by

$$\bar{\nabla}_X Y = \sum_k \left(X(Y^k) + \sum_{i,j} X^i Y^j \Gamma_{ij}^k \right) \partial_k.$$

It is clear that

$$\bar{\nabla}_X Y = \nabla_X Y + \sum_{k=1}^{2} \sum_{i,j} X^i Y^j \Gamma_{ij}^k \partial_k. \tag{26}$$

We suppose that $N_{hyp} = (N^1, N^2, N^3)$. From (26) we get the relation

$$\bar{\nabla}_X N_{hyp} = \nabla_X N_{hyp} + \sum_{k=1}^{2} \sum_{i,j=1}^{2} X^i N^j \Gamma_{ij}^k \partial_k. \tag{27}$$

We start by evaluating the term $\nabla_X N_{hyp}$. We observe that the normal vector $N_e = (N_1, N_2, N_3)$ to Σ with respect to the metric g_0 does not coincide with N_{hyp}. But it is clear that

$$N_{hyp} = (N^1, N^2, N^3) = (\sqrt{F} N_1, \sqrt{F} N_2, N_3).$$

We observe that

$$\nabla_X N_{hyp} = \sum_{k=1}^{3} X(N^k)\partial_k = X(\sqrt{F} N_1)\partial_1 + X(\sqrt{F} N_2)\partial_2 + X(N_3)\partial_3.$$

We can write $X(\sqrt{F} N_k) = (1 - r^2) X(N_k) - X(r^2) N_k$, for $k = 1, 2$. Since

$$X(N_k) = \sum_l X^l \partial_{x_l} N_k \quad \text{and} \quad X(r^2) = 2x_1 X^1 + 2x_2 X^2,$$

it follows that

$$X(\sqrt{F} N_k) = X(N_k) - \left(2x_1 X^1 + 2x_2 X^2\right) N_k - r^2 \left(\sum_{l=1}^{3} X^l \partial_{x_l} N_k \right),$$

for $k = 1, 2$. We can conclude that $\nabla_X N_{hyp} = \sum_k X(N^k)\partial_k$ is given by

$$\sum_{k=1}^{3} X(N_k)\partial_k - \left(2x_1 X^1 + 2x_2 X^2\right) \sum_{k=1}^{2} N_k \partial_k - r^2 \sum_{k=1}^{2} \left(\sum_{l=1}^{3} X^l \partial_{x_l} N_k \right) \partial_k.$$

Inserting this equality into (26) and observing that $\sum_{k=1}^{3} X(N_k)\partial_k = \nabla_X N_e$, we obtain

$$\nabla_X N_{hyp} = \nabla_X N_e - \left(2x_1X^1 + 2x_2X^2\right)\sum_{k=1}^{2} N_k\partial_k - r^2\sum_{k=1}^{2}\left(\sum_{l=1}^{3} X^l\partial_{x_l}N_k\right)\partial_k.$$

Replacing this result into (27) we find the following expression of $\bar{\nabla}_X N_{hyp}$, which we will consider to compute the trace. We will assume X to be a vector field tangent to Σ. Then

$$\begin{array}{c}\bar{\nabla}_X N_{hyp} = \nabla_X N_e - \left(2x_1X^1 + 2x_2X^2\right)\sum_{k=1}^{2} N_k\partial_k \\ -r^2\sum_{k=1}^{2}\left(\sum_{l=1}^{3} X^l\partial_{x_l}N_k\right)\partial_k + \sum_{k=1}^{2}\sum_{i,j=1}^{2} X^iN^j\Gamma_{ij}^k\partial_k.\end{array} \tag{28}$$

We start to study the second summand. $\left(2x_1X^1 + 2x_2X^2\right)\sum_{k=1}^{2} N_k\partial_k$ is the vector whose components with respect to the basis $(\partial_1, \partial_2, \partial_3)$ are given by

$$\begin{bmatrix} 2x_1N_1 & 2x_2N_1 & 0 \\ 2x_1N_2 & 2x_2N_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} X^1 \\ X^2 \\ X^3 \end{bmatrix}.$$

So the trace of the mapping $\sum_{i=1}^{3} X^i\partial_i \to \left[\left(2x_1X^1 + 2x_2X^2\right)\sum_{k=1}^{2} N_k\partial_k\right]^T$ equals $2\mathcal{O}(x_1N_1 + x_2N_2)$.

The components of the vector

$$\sum_{k=1}^{2}\left(\sum_{l=1}^{3} X^l\partial_{x_l}N_k\right)\partial_k$$

with respect to the basis $(\partial_1, \partial_2, \partial_3)$ are given by

$$\begin{bmatrix} \partial_{x_1}N_1 & \partial_{x_2}N_1 & \partial_{x_3}N_1 \\ \partial_{x_1}N_2 & \partial_{x_2}N_2 & \partial_{x_3}N_2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} X^1 \\ X^2 \\ X^3 \end{bmatrix}.$$

So the trace of the mapping

$$\sum_{i=1}^{3} X^i\partial_i \to r^2\left[\sum_{k=1}^{2}\left(\sum_{l=1}^{3} X^l\partial_{x_l}N_k\right)\partial_k\right]^T$$

equals $\mathcal{O}\left(r^2\left(\partial_{x_1}N_1 + \partial_{x_2}N_2\right)\right)$.

As for the last term of (28), we can state that $\sum_{k=1}^{2}\sum_{i,j} X^iN^j\Gamma_{ij}^k\partial_k$ is the vector whose components with respect to the basis $(\partial_1, \partial_2, \partial_3)$ are given by

$$\begin{bmatrix} \frac{2x_1N^1}{\sqrt{F}} + \frac{2x_2N^2}{\sqrt{F}} & \frac{2x_2N^1}{\sqrt{F}} - \frac{2x_1N^2}{\sqrt{F}} & 0 \\ -\frac{2x_2N^1}{\sqrt{F}} + \frac{2x_1N^2}{\sqrt{F}} & \frac{2x_1N^1}{\sqrt{F}} + \frac{2x_2N^2}{\sqrt{F}} & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} X^1 \\ X^2 \\ X^3 \end{bmatrix}. \tag{29}$$

Taking into account the equalities $N^1 = \sqrt{F}N_1$ and $N^2 = \sqrt{F}N_2$, it is easy to conclude that the trace of the mapping

$$\sum_{i} X^i\partial_i \to \left[\sum_{k=1}^{2}\sum_{i,j} X^iN^j\Gamma_{ij}^k\partial_k\right]^T$$

equals $4\mathcal{O}\left(x_1 N_1 + x_2 N_2\right)$. From the definition of the mean curvatures it is easy to obtain the following relation:

$$H_{hyp}(\Sigma) = H_e(\Sigma) - \mathcal{O}(x_1 N_1 + x_2 N_2) + r^2 \mathcal{O}\left(\partial_{x_1} N_1 + \partial_{x_2} N_2\right). \tag{30}$$

We have proved the following result.

Proposition 8.1. *Let S be a surface in $\mathbb{D}^2 \times \mathbb{R}$ endowed with the metric g_{hyp}. If $H_{hyp}(\cdot)$ denotes the mean curvature with respect to the metric g_{hyp}, and if $H_e(\cdot)$ and (N_1, N_2, N_3) denote respectively the mean curvature and the normal vector to S with respect to g_0, then*

$$H_{hyp}(S) = H_e(S) - \mathcal{O}(x_1 N_1 + x_2 N_2) + r^2 \mathcal{O}\left(\partial_{x_1} N_1 + \partial_{x_2} N_2\right), \tag{31}$$

where x_1, x_2 are the Euclidean coordinates on $\mathbb{D}^2$ and $r^2 = x_1^2 + x_2^2$.

9. A scaled Costa-Hoffman-Meeks type surface

In this section we will describe the surface obtained by scaling of M_k, the Costa-Hoffmann-Meeks surface of genus $k \geqslant 1$ (see C. Costa [1], [2] and D. Hoffman and W. H. Meeks [8], [9]) and we will study the mapping properties of its Jacobi operator. We denote by $M_{k,\varepsilon}$ the image of M_k by a homothety of ratio ε^2. We will adapt to our situation some of the analytical tools used in [6] to show the existence of a family of minimal surfaces in $\mathbb{R}^3$ close to M_k with one planar end and two slightly bent catenoidal ends by an angle $\xi \in (-\xi_0, \xi_0)$, $\xi_0 > 0$ and small enough. We denote an element of this family by $M_k(\xi)$. Then $M_k(\xi)|_{\xi=0} = M_k$.

9.1. The Costa-Hoffman-Meeks surface.

We start by giving a brief description of the surface M_k. After suitable rotation and translation, M_k enjoys the following properties.

(1) It has one planar end E_m asymptotic to the $x_3 = 0$ plane, one top end E_t and one bottom end E_b that are respectively asymptotic to the upper end and to the lower end of a catenoid with x_3-axis of revolution. The planar end E_m is located between the two catenoidal ends.

(2) It is invariant under the action of the rotation of angle $\frac{2\pi}{k+1}$ about the x_3-axis, under the action of the symmetry with respect to the $x_2 = 0$ plane and under the action of the composition of a rotation of angle $\frac{\pi}{k+1}$ about the x_3-axis and the symmetry with respect to the $x_3 = 0$ plane.

(3) It intersects the $x_3 = 0$ plane in $k+1$ straight lines, which intersect themselves at the origin with angles equal to $\frac{\pi}{k+1}$. The intersection of M_k with the plane $x_3 = const\,(\neq 0)$ is a single Jordan curve. The intersection of M_k with the upper half space $x_3 > 0$ (resp. with the lower half space $x_3 < 0$) is topologically an open annulus.

We denote by X_i, with $i = t, b, m$, the parametrization of the end E_i and with $X_{i,\varepsilon}$ the parametrization of the corresponding end $E_{i,\varepsilon}$ of $M_{k,\varepsilon}$.

Now we give a local description of the surface $M_{k,\varepsilon}$ near its ends and we introduce coordinates that we will use.

The planar end. The planar end $E_{m,\varepsilon}$ of the surface $M_{k,\varepsilon}$ can be parametrized by

$$X_{m,\varepsilon}(x) := \left(\frac{\varepsilon^2 x}{|x|^2}, \varepsilon^2 u_m(x)\right) \in \mathbb{R}^3, \tag{32}$$

where $x \in \bar{B}_{\rho_0}(0) - \{0\} \subset \mathbb{R}^2$. Here $\rho_0 > 0$ is fixed small enough. In the sequel, where necessary, we will consider on $B_{\rho_0}(0)$ also the polar coordinates $(\theta, \rho) \in S^1 \times [0, \rho_0]$. The function u_m satisfies the minimal surface equation, which has the following form:

$$
2H_u = \frac{|x|^4}{\varepsilon^2} \operatorname{div} \left(\frac{\nabla u}{(1 + |x|^4 \, |\nabla u|^2)^{1/2}} \right) = 0. \tag{33}
$$

It can be shown (see [6]) that the function u_m can be extended at the origin continuously using the Weierstrass representation. In particular we can prove that $u_m \in \mathcal{C}^{2,\alpha}(\bar{B}_{\rho_0})$ and $u_m = \mathcal{O}_{C_b^{2,\alpha}}(|x|^{k+1})$, where the expression $\mathcal{O}_{C_b^{n,\alpha}}(g)$ denotes a function that, together with its partial derivatives of order less than or equal to $n + \alpha$, is bounded by a constant times g. Furthermore, taking into account the symmetries of the surface, it is possible to show that the function u_m, in polar coordinates, has to be collinear to $\cos(j(k+1)\theta)$, with $j \geqslant 1$ and odd.

If we linearize in $u = 0$ the nonlinear equation (33), we obtain the expression of an operator which is, up to a multiplication by ε^4, the Jacobi operator about the plane, that is, $\mathbb{L}_{\mathbb{R}^2} = |x|^4 \Delta_0$. To be more precise, the linearization of (33) gives

$$
L_u \, v = \frac{|x|^4}{\varepsilon^2} \operatorname{div} \left(\frac{\nabla v}{\sqrt{1 + |x|^4 |\nabla u|^2}} - |x|^4 \nabla u \frac{\nabla u \cdot \nabla v}{\sqrt{(1 + |x|^4 |\nabla u|^2)^3}} \right). \tag{34}
$$

We will give the expression of H_{u+v}, the mean curvature of the graph of the function $u + v$, in terms of the mean curvature of Σ_u, that is, H_u.

Here we will show that

$$
2H_{u+v} = 2H_u + L_u v + \frac{|x|^4}{\varepsilon^2} Q_u(|x|^2 \nabla v, |x|^2 \nabla^2 v), \tag{35}
$$

where Q_u satisfies

$$
Q_u(0,0) = 0, \nabla Q_u'(0,0) = 0.
$$

To show (35), we start by observing that

$$
\frac{1}{\sqrt{1 + |x|^4 |\nabla(u+v)|^2}} = \frac{1}{\sqrt{1 + |x|^4 |\nabla u|^2}} - |x|^4 \frac{\nabla u \cdot \nabla v}{\sqrt{(1 + |x|^4 |\nabla u|^2)^3}} + Q_{u,1}(v), \tag{36}
$$

where the function $Q_{u,1}$ satisfies $Q_{u,1}(0) = 0, \nabla Q_{u,1}(0) = 0$. The proof of that is very close to the one that appears in section 3: it's necessary only to replace F by $|x|^4$. So we can omit some details. Secondly we observe that $2H_{u+v}$ is given by

$$
\begin{aligned}
&\frac{|x|^4}{\varepsilon^2} \operatorname{div} \left(\frac{\nabla(u+v)}{\sqrt{1 + |x|^4 |\nabla u|^2}} - |x|^4 \nabla(u+v) \frac{\nabla u \cdot \nabla v}{\sqrt{(1 + |x|^4 |\nabla u|^2)^3}} + \nabla(u+v) Q_{u,1}(v) \right) \\
&\quad = 2H_u + \frac{|x|^4}{\varepsilon^2} \operatorname{div} \left(\frac{\nabla v}{\sqrt{1 + |x|^4 |\nabla u|^2}} - |x|^4 \nabla u \frac{\nabla u \cdot \nabla v}{\sqrt{(1 + |x|^4 |\nabla u|^2)^3}} \right) \\
&\quad + \frac{|x|^4}{\varepsilon^2} Q_u(|x|^2 \nabla v, |x|^2 \nabla^2 v).
\end{aligned}
$$

From this the wanted expression follows.

Since we assume that Σ_u is a minimal surface, we will consider $H_u = 0$.

Following what we have done in section 7 replacing F by $|x|^4$ we get:

$$\frac{|x|^4}{\varepsilon^2}\left(\Delta_0 v+\sqrt{1+|x|^4|\nabla u|^2}\left(\bar{L}_u v+Q_u(|x|^2\nabla v,|x|^2\nabla^2 v)\right)\right)=0, \tag{37}$$

where $\bar{L}_u v$ is a second order linear operator with coefficients in $\mathcal{O}_{C_b^{2,\alpha}}(|x|^{k+1})$.

It is important to remark that if the function v satisfies equation (37) with $u=u_m$, then the graph of the function $\varepsilon^2(u_m+v)$ is minimal. Now we are interested in finding the equation which a function w must satisfy in such a way that the surface parametrized by $X_{m,\varepsilon}+we_3$, that is, the graph of w over the middle end $E_{m,\varepsilon}$, is minimal, which is equivalent to requiring that the graph of $\varepsilon^2 u_m+w$ is minimal. Then we can obtain the wanted equation by replacing v by w/ε^2 in (37). So we get

$$\frac{|x|^4}{\varepsilon^2}\left(\frac{1}{\varepsilon^2}\Delta_0 w+\sqrt{1+|x|^4|\nabla u|^2}\left(\frac{1}{\varepsilon^2}\bar{L}_u w+Q_u\left(\frac{|x|^2}{\varepsilon^2}\nabla w,\frac{|x|^2}{\varepsilon^2}\nabla^2 w\right)\right)\right)=0. \tag{38}$$

If we set $Q_{\varepsilon,u}(\cdot):=\frac{|x|^4}{\varepsilon^2}\sqrt{1+|x|^4|\nabla u|^2}\,Q_u\left(\frac{|x|^2}{\varepsilon^2}\nabla\cdot,\frac{|x|^2}{\varepsilon^2}\nabla^2\cdot\right)$ to simplify the notation, we can write this equation in the following way:

$$\frac{|x|^4}{\varepsilon^4}\Delta_0 w+\frac{|x|^4}{\varepsilon^4}\sqrt{1+|x|^4|\nabla u|^2}\,\bar{L}_u w+Q_{\varepsilon,u}(w)=0. \tag{39}$$

The catenoidal ends. We denote by X_c the parametrization of the standard catenoid C whose axis of revolution is the x_3-axis. Its expression is

$$X_c(s,\theta):=(\cosh s\,\cos\theta,\cosh s\,\sin\theta,s)\in\mathbb{R}^3,$$

where $(s,\theta)\in\mathbb{R}\times S^1$. The unit normal vector field about C is given by

$$n_c(s,\theta):=\frac{1}{\cosh s}(\cos\theta,\sin\theta,-\sinh s).$$

The catenoid C may be divided into two pieces, denoted by $C_\pm$, which are defined as the image by X_c of $\mathbb{R}^\pm\times S^1$. For any $\varepsilon>0$, we define the catenoid C_ε as the image of C by a homothety of ratio ε^2. We denote by $X_{c,\varepsilon}:=\varepsilon^2 X_c$ its parametrization. Of course, by this transformation, two surfaces correspond to $C_\pm$. We denote them by $C_{\varepsilon,\pm}$.

Up to some dilation, we can assume that the two ends $E_{t,\varepsilon}$ and $E_{b,\varepsilon}$ of $M_{k,\varepsilon}$ are asymptotic to some translated copy of the catenoid parametrized by $X_{c,\varepsilon}$ in the vertical direction. Therefore, $E_{t,\varepsilon}$ and $E_{b,\varepsilon}$ can be parametrized, respectively, by

$$X_{t,\varepsilon}:=X_{c,\varepsilon}+w_t\,n_c+\sigma_{t,\varepsilon}\,e_3 \tag{40}$$

for $(s,\theta)\in(s_0,\infty)\times S^1$,

$$X_{b,\varepsilon}:=X_{c,\varepsilon}-w_b\,n_c-\sigma_{b,\varepsilon}\,e_3 \tag{41}$$

for $(s,\theta)\in(-\infty,-s_0)\times S^1$, where $\sigma_{t,\varepsilon},\sigma_{b,\varepsilon}\in\mathbb{R}$ and the functions w_t,w_b tend exponentially fast to 0 as s goes to $\pm\infty$, reflecting the fact that the ends are asymptotic to a catenoidal end. Furthermore, taking into account the symmetries of the surface, it is easy to show that the functions w_t,w_b, in terms of the (s,θ) coordinates, have to be collinear to $\cos(j(k+1)\theta)$, with $j\in\mathbb{N}$ and must satisfy $w_b(s,\theta)=-w_t(-s,\theta-\frac{\pi}{k+1})$. Furthermore we have $\sigma_{t,\varepsilon}=\sigma_{b,\varepsilon}$.

In section 3 of [12] the expression of the mean curvature operator of a surface close to a scaled standard catenoid is given. We can adapt this result to our situation.

We obtain that the surface parametrized by $X_{c,\varepsilon} + w\, n_c$ is minimal if and only if the function w satisfies the minimal surface equation

$$\frac{1}{\varepsilon^4}\,\mathbb{L}_C w + Q_\varepsilon(w) = 0, \tag{42}$$

$\mathbb{L}_C$ being the Jacobi operator about the catenoid, i.e.

$$\mathbb{L}_C w = \frac{1}{\cosh^2 s}\left(\partial^2_{ss} w + \partial^2_{\theta\theta} w + \frac{2w}{\cosh^2 s}\right),$$

and

$$Q_\varepsilon(w) = \frac{1}{\varepsilon^2\cosh^2 s} Q_{2,\varepsilon}\left(\frac{w}{\varepsilon^2\cosh s}\right) + \frac{1}{\varepsilon^2\cosh s} Q_{3,\varepsilon}\left(\frac{w}{\varepsilon^2\cosh s}\right). \tag{43}$$

Here Q_2, Q_3 are nonlinear second order differential operators which are bounded in $\mathcal{C}^k(\mathbb{R}\times\mathbb{S}^1)$, for every k, and satisfy $Q_2(0) = Q_3(0) = 0$, $\nabla Q_2(0) = \nabla Q_3(0) = 0$, $\nabla^2 Q_3(0) = 0$ together with:

$$\begin{aligned} &\|Q_j(v_2) - Q_j(v_1)\|_{\mathcal{C}^{0,\alpha}([s,s+1]\times S^1)} \\ &\quad\leqslant c\left(\sup_{i=1,2}\|v_i\|_{\mathcal{C}^{2,\alpha}([s,s+1]\times S^1)}\right)^{j-1}\|v_2 - v_1\|_{\mathcal{C}^{2,\alpha}([s,s+1]\times S^1)} \end{aligned} \tag{44}$$

for all $s\in\mathbb{R}$ and all v_1, v_2 such that $\|v_i\|_{\mathcal{C}^{2,\alpha}([s,s+1]\times S^1)} \leqslant 1$. The constant $c > 0$ does not depend on s.

For all $\rho < \rho_0$ and $s > s_0$, we define

(45)

$$M_{k,\varepsilon}(s,\rho) := M_{k,\varepsilon} - \left[X_{t,\varepsilon}((s,\infty)\times S^1)\cup X_{b,\varepsilon}((-\infty,-s)\times S^1)\cup X_{m,\varepsilon}(B_\rho(0))\right].$$

The parametrizations of the three ends of $M_{k,\varepsilon}$ induce a decomposition of $M_{k,\varepsilon}$ into slightly overlapping components: a compact piece $M_{k,\varepsilon}(s_0+1, \rho_0/2)$ and three noncompact pieces $X_{t,\varepsilon}((s_0,\infty)\times S^1)$, $X_{b,\varepsilon}((-\infty,-s_0)\times S^1)$ and $X_{m,\varepsilon}(\bar B_{\rho_0}(0))$.

We define a weighted space of functions on $M_{k,\varepsilon}$.

Definition 9.1. Given $\ell\in\mathbb{N}$, $\alpha\in(0,1)$ and $\delta\in\mathbb{R}$, the space $\mathcal{C}^{\ell,\alpha}_\delta(M_{k,\varepsilon})$ is defined to be the space of functions in $\mathcal{C}^{\ell,\alpha}_{loc}(M_{k,\varepsilon})$ for which the following norm is finite:

$$\begin{aligned} \|w\|_{\mathcal{C}^{\ell,\alpha}_\delta(M_{k,\varepsilon})} &:= \|w\|_{\mathcal{C}^{\ell,\alpha}(M_{k,\varepsilon}(s_0+1,\rho_0/2))} + \|\,w\circ X_{m,\varepsilon}\|_{\mathcal{C}^{\ell,\alpha}(\bar B_{\rho_0}(0))} \\ &\quad + \|w\circ X_{t,\varepsilon}\|_{\mathcal{C}^{\ell,\alpha}_\delta([s_0,+\infty)\times S^1)} + \|w\circ X_{b,\varepsilon}\|_{\mathcal{C}^{\ell,\alpha}_\delta((-\infty,-s_0]\times S^1)}, \end{aligned}$$

where

$$\begin{aligned} \|f\|_{\mathcal{C}^{\ell,\alpha}_\delta([s_0,+\infty)\times S^1)} &= \sup_{s\geqslant s_0}\left(e^{-\delta s}\,\|f\|_{\mathcal{C}^{\ell,\alpha}([s,s+1]\times S^1)}\right), \\ \|f\|_{\mathcal{C}^{\ell,\alpha}_\delta((-\infty,-s_0]\times S^1)} &= \sup_{s\leqslant -s_0}\left(e^{\delta s}\,\|f\|_{\mathcal{C}^{\ell,\alpha}([s-1,s]\times S^1)}\right) \end{aligned}$$

and which are invariant under the action of the symmetry with respect to the $x_2 = 0$ plane, that is, $w(p) = w(\bar p)$ for all $p\in M_{k,\varepsilon}$, where $\bar p := (x_1, -x_2, x_3)$ if $p = (x_1, x_2, x_3)$, invariant with respect to a rotation of angle $\frac{2\pi}{k+1}$ about the x_3-axis and to the composition of a rotation of angle $\frac{\pi}{k+1}$ about the x_3-axis and the symmetry with respect to the $x_3 = 0$ plane.

We remark that there is no weight on the middle end. In fact we compactify this end and we consider a weighted space of functions defined on a two-ended surface. We will perturb the surface $M_{k,\varepsilon}$ by the normal graph of a function $u\in\mathcal{C}^{2,\alpha}_\delta(M_{k,\varepsilon})$.

9.2. **The Jacobi operator.** The Jacobi operator about $M_{k,\varepsilon}$ is

$$\mathbb{L}_{M_{k,\varepsilon}} := \Delta_{M_{k,\varepsilon}} + |A_{M_{k,\varepsilon}}|^2,$$

where $|A_{M_{k,\varepsilon}}|$ is the norm of the second fundamental form on $M_{k,\varepsilon}$.

In the parametrization of the ends introduced above, the volume forms $dvol_{M_{k,\varepsilon}}$ can be written as $\gamma_t\, ds\, d\theta$ and $\gamma_b\, ds\, d\theta$ near the catenoidal type ends and as $\gamma_m\, dx_1\, dx_2$ near the middle end. Now we can define globally on $M_{k,\varepsilon}$ a smooth function

$$\gamma : M_{k,\varepsilon} \longrightarrow [0,\infty) \tag{46}$$

that is identically equal to ε^4 on $M_{k,\varepsilon}(s_0-1, 2\rho_0)$ and equal to γ_t (resp. γ_b, γ_m) on the end $E_{t,\varepsilon}$ (resp. $E_{b,\varepsilon}$, $E_{m,\epsilon}$). They are defined in such a way that on $X_{t,\varepsilon}((s_0,\infty)\times S^1)$ and on $X_{b,\varepsilon}((-\infty,-s_0)\times S^1)$ we have

$$\gamma\circ X_{t,\varepsilon}(s,\theta) \sim \varepsilon^4\cosh^2 s \qquad \text{and} \qquad \gamma\circ X_{b,\varepsilon}(s,\theta)\sim\varepsilon^4\cosh^2 s.$$

Finally on $X_{m,\varepsilon}(B_{\rho_0})$, we have

$$\gamma\circ X_{m,\varepsilon}(x) \sim \frac{\varepsilon^4}{|x|^4}.$$

It is possible to check that

$$\begin{array}{rccc} \mathcal{L}_{\varepsilon,\delta}: & \mathcal{C}^{2,\alpha}_\delta(M_{k,\varepsilon}) & \longrightarrow & \mathcal{C}^{0,\alpha}_\delta(M_{k,\varepsilon}) \\ & w & \longmapsto & \gamma\,\mathbb{L}_{M_{k,\varepsilon}}(w) \end{array}$$

is a bounded linear operator. The subscript δ is meant to keep track of the weighted space over which the Jacobi operator is acting. Observe that the function γ is here to counterbalance the effect of the conformal factor $\frac{1}{\sqrt{|g_{M_{k,\varepsilon}}|}}$ in the expression of the Laplacian in the coordinates we use to parametrize the ends of the surface $M_{k,\varepsilon}$. This is precisely what is needed to have the operator defined from the space $\mathcal{C}^{2,\alpha}_\delta(M_{k,\varepsilon})$ into the target space $\mathcal{C}^{0,\alpha}_\delta(M_{k,\varepsilon})$.

To have a better grasp of what is going on, let us linearize the nonlinear equation (42) at $w=0$. We get the expression of the Jacobi operator about the scaled catenoid C_ε:

$$\mathbb{L}_{C_\varepsilon} := \frac{1}{\varepsilon^4\cosh^2 s}\left(\partial_s^2+\partial_\theta^2+\frac{2}{\cosh^2 s}\right).$$

We can observe that the operator $\cosh^2 s\,\mathbb{L}_{C_\varepsilon}$ maps the space $(\cosh s)^\delta\,\mathcal{C}^{2,\alpha}((s_0,+\infty)\times S^1)$ into the space $(\cosh s)^\delta\,\mathcal{C}^{0,\alpha}((s_0,+\infty)\times S^1)$.

Similarly, if we linearize the nonlinear equation (33) at $u=0$, we obtain (see (3) with $u=0$), up to a multiplication by $1/\varepsilon^4$, the expression of the Jacobi operator about the plane:

$$\frac{1}{\varepsilon^4}\mathbb{L}_{\mathbb{R}^2} := \frac{|x|^4}{\varepsilon^4}\,\Delta_0.$$

Again, the operator $\gamma\frac{1}{\varepsilon^4}\,\mathbb{L}_{\mathbb{R}^2}=\Delta_0$ clearly maps the space $\mathcal{C}^{2,\alpha}(\bar B_{\rho_0})$ into the space $\mathcal{C}^{0,\alpha}(\bar B_{\rho_0})$. Now, the function γ plays, for the ends of the surface $M_{k,\varepsilon}$, the role played by the function $\cosh^2 s$ for the ends of the standard catenoid and the role played by the function $|x|^{-4}$ for the plane. Since the Jacobi operator about $M_{k,\varepsilon}$ is asymptotic to $\frac{1}{\varepsilon^4}\mathbb{L}_{\mathbb{R}^2}$ at $E_{m,\varepsilon}$ and is asymptotic to $\mathbb{L}_{C_\varepsilon}$ at $E_{t,\varepsilon}$ and $E_{b,\varepsilon}$, we conclude that the operator $\mathcal{L}_{\varepsilon,\delta}$ maps $\mathcal{C}^{2,\alpha}_\delta(M_{k,\varepsilon})$ into $\mathcal{C}^{0,\alpha}_\delta(M_{k,\varepsilon})$.

Now we recall the notion of nondegeneracy introduced in [6].

Definition 9.2. The surface $M_{k,\varepsilon}$ is said to be nondegenerate if $\mathcal{L}_{\varepsilon,\delta}$ is injective for all $\delta < -1$.

It is useful to observe that a duality argument in the weighted Lebesgue spaces implies that

$$(\mathcal{L}_{\varepsilon,\delta} \quad \text{is injective}) \quad \Leftrightarrow \quad (\mathcal{L}_{\varepsilon,-\delta} \quad \text{is surjective})$$

if $\delta \notin \mathbb{Z}$. See [13] and [10] for more details.

The nondegeneracy of $M_{k,\varepsilon}$ is related to the mapping properties of $\mathcal{L}_{\varepsilon,\delta}$ and to the kernel of this operator. From the observations made above, it follows that at the catenoidal type ends the Jacobi operators of $M_{k,\varepsilon}$ and M_k are respectively asymptotic to $\mathbb{L}_{C_\varepsilon}$ and $\mathbb{L}_C$ which coincide up to a multiplication by ε^4; at the middle end they are respectively asymptotic to $\frac{1}{\varepsilon^4}\mathbb{L}_{\mathbb{R}^2}$ and $\mathbb{L}_{\mathbb{R}^2}$. So we could transpose some of the results about the surface $M_k(0)$ contained in [6] related to the study of its mean curvature operator, to the surface $M_{k,\varepsilon}$, including nondegeneracy. The only difference is that here we work with spaces of functions invariant with respect to all of the symmetries of M_k.

The Jacobi fields. It is known that a smooth one-parameter group of isometries containing the identity generates a Jacobi field, that is, a solution of the equation $\mathbb{L}_{M_{k,\varepsilon}} u = 0$. The Jacobi fields of this type, which are invariant with respect to the mirror symmetry by the $x_2 = 0$ plane, the rotation by $\frac{2\pi}{k+1}$ about the x_3-axis, the composition of the rotation by $\frac{\pi}{k+1}$ about the x_3-axis and the mirror symmetry with respect to the $x_3 = 0$ plane, are generated by dilations. Of course the Jacobi equation has other solutions which are not taken into account because they are not invariant under the action of the symmetries listed above. See [6] for details.

The Killing vector field $\Xi(p) = p$, which is associated to the one-parameter group of dilations, generates the Jacobi field

$$\Phi(p) := n(p) \cdot p.$$

It is clear that $\Phi(p)$ grows linearly and so it is not bounded.

With this notation, we define the deficiency space

$$\mathcal{D} := \operatorname{Span}\{\chi_t\, \Phi, \chi_b\, \Phi\,\},$$

where χ_t is a cutoff function that is identically equal to 1 on $X_{t,\varepsilon}((s_0+1, +\infty) \times S^1)$, identically equal to 0 on $M_{k,\varepsilon} - X_{t,\varepsilon}((s_0, +\infty) \times S^1)$ and that is invariant under the action of the symmetries listed above. The cutoff function χ_b is obtained from χ_t by using the symmetries. Clearly, if $\delta < 0$, then

$$\begin{array}{rccc} \tilde{\mathcal{L}}_{\varepsilon,\delta} : & \mathcal{C}^{2,\alpha}_\delta(M_{k,\varepsilon}) \oplus \mathcal{D} & \longrightarrow & \mathcal{C}^{0,\alpha}_\delta(M_{k,\varepsilon}) \\ & w & \longmapsto & \gamma\, \mathbb{L}_{M_{k,\varepsilon}}\, w \end{array}$$

is a bounded linear operator. Thanks to a result of S. Nayatani about the dimension of the kernel of the Jacobi operator of M_k, shown in [18, 19] and extended in [16], we can state that there is not any bounded Jacobi field which is invariant with respect to the symmetries of $M_{k,\varepsilon}$.

So we get the following result about the operator $\mathcal{L}_{\varepsilon,\delta}$.

Proposition 9.3. *We choose $\delta \in (1, 2)$. Then the operator $\mathcal{L}_{\varepsilon,\delta}$ is surjective. Moreover, there exists $G_{\varepsilon,\delta}$, a right inverse for $\mathcal{L}_{\varepsilon,\delta}$ whose norm is bounded.*

This fact, together with an adaptation to our setting of the linear decomposition lemma proved in [11] for constant mean curvature surfaces (see also [10] for minimal hypersurfaces), allows us to prove the following result.

Proposition 9.4. *We choose $\delta \in (-2,-1)$. Then the operator $\tilde{\mathcal{L}}_{\varepsilon,\delta}$ is surjective.*

10. An infinite dimensional family of minimal surfaces which are close to a compact part of a scaled Costa-Hoffman-Meeks type surface in $\mathbb{M}^2 \times \mathbb{R}$

We recall that in section 8 we found that the mean curvature with respect to the metric g_{hyp} of a surface S in $\mathbb{M}^2 \times \mathbb{R}$ can be expressed in terms of the Euclidean mean curvature of S and the components of the normal vector to the same surface with respect to the flat metric g_0.

In this section we will apply this result to prove the existence of a family of embedded minimal surfaces with respect to the metric g_{hyp} which are close to the piece of the surface $M_{k,\varepsilon}$ contained in a cylindrical neighbourhood of radius $r_\varepsilon = \varepsilon/2$ of $\{0,0\} \times \mathbb{R}$.

We start by giving the statement of a result that can be easily obtained by [6], Lemma 2.2. It describes the region of the surface $M_{k,\varepsilon}$ which can be parametrized as a graph on an annular neighbourhood of r_ε contained in the $x_3 = 0$ plane.

Lemma 10.1. *There exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, an annular part of the ends $E_{t,\varepsilon}$, $E_{b,\varepsilon}$ and $E_{m,\varepsilon}$ of $M_{k,\varepsilon}$ can be written as vertical graphs over the annulus $B_{2r_\varepsilon} - B_{r_\varepsilon/2}$ of the functions*

$$Z_t(\theta, r) = \sigma_{t,\varepsilon} + \varepsilon^2 \ln\left(\frac{2r}{\varepsilon^2}\right) + \mathcal{O}_{\mathcal{C}^{2,\alpha}_b}(\varepsilon^3), \tag{47}$$

$$Z_b(\theta, r) = -Z_t\left(\theta - \frac{\pi}{k+1}, r\right). \tag{48}$$

As for the parametrization of the planar end, it satisfies

$$Z_m(\theta, r) = \mathcal{O}_{\mathcal{C}^{2,\alpha}_b}\left(\varepsilon^2 \left(\frac{r}{\varepsilon^2}\right)^{-(k+1)}\right). \tag{49}$$

Here (θ, r) are the polar coordinates in the $x_3 = 0$ plane. The functions $\mathcal{O}_{\mathcal{C}^{2,\alpha}_b}(f)$ are defined in the annulus $B_{2r_\varepsilon} - B_{r_\varepsilon/2}$ and are bounded in the $\mathcal{C}^{2,\alpha}_b$ topology by a constant (independent by f) multiplied by f, where the partial derivatives are computed with respect to the vector fields $r\,\partial_r$ and ∂_θ.

We set

$$s_\varepsilon = -\ln \varepsilon, \quad \rho_\varepsilon := 2\varepsilon$$

and we define $M^T_{k,\varepsilon}$ to be equal to $M_{k,\varepsilon}$, from which we have removed the image of $(s_\varepsilon, +\infty) \times S^1$ by $X_{t,\varepsilon}$, the image of $(-\infty, -s_\varepsilon) \times S^1$ by $X_{b,\varepsilon}$ and the image of $B_{\rho_\varepsilon}(0)$ by $X_{m,\varepsilon}$. The values of s_ε and ρ_ε have been chosen in such a way that the surface $M^T_{k,\varepsilon}$ is contained in a neighbourhood of radius $r_\varepsilon = \varepsilon/2$ of $\{0,0\} \times \mathbb{R}$. In this section we will prove the existence of a family of minimal surfaces close to $M^T_{k,\varepsilon}$. To this aim we will use Proposition 8.1 and we will follow the work [6].

First, we modify the parametrization of the ends $E_{t,\varepsilon}$, $E_{b,\varepsilon}$ and $E_{m,\varepsilon}$, for appropriate values of s and x, so that the images of $r = r_\varepsilon$ by

$$\theta \to (r\cos\theta, r\sin\theta, Z_b(\theta, r)), \quad \theta \to (r\cos\theta, r\sin\theta, Z_t(\theta, r)) \tag{50}$$

correspond, respectively, up to a vertical translation, to the horizontal curves at heights $\pm\varepsilon^2 \ln(2r_\varepsilon/\varepsilon^2)$.

The curve $\theta \to (r\cos\theta, r\sin\theta, Z_m(\theta, r))$, if $r = r_\varepsilon$, corresponds, up to a vertical translation, to a horizontal curve at height $\varepsilon^2 \left(r_\varepsilon/\varepsilon^2\right)^{-(k+1)}$.

The second step is the modification of the unit normal vector field on $M_{k,\varepsilon}$ into a transverse unit vector field $\tilde{n}_\varepsilon$ in such a way that it coincides with the normal vector field n_ε on $M_{k,\varepsilon}$, is equal to e_3 on the graph over $B_{2r_\varepsilon} - B_{r_\varepsilon/2}$ of the functions U_t and U_b and interpolates smoothly between the different definitions of $\tilde{n}_\varepsilon$ in different subsets of $M^T_{k,\varepsilon}$.

Finally we observe that close to $E_{t,\varepsilon}$, we can give the following estimate:

$$\left|\varepsilon^4 \cosh^2 s \left(\mathbb{L}_{M_{k,\varepsilon}} v - (\varepsilon^4 \cosh^2 s)^{-1} \left(\partial_{ss} v + \partial_{\theta\theta} v\right)\right)\right| \leqslant c \left|(\cosh^2 s)^{-1} v\right|. \tag{51}$$

This follows easily from (42) together with the fact that w_t decays at least like $(\cosh^2 s)^{-1}$ on $E_{t,\varepsilon}$. Similar considerations hold close to the bottom end $E_{b,\varepsilon}$. Near the middle planar end $E_{m,\varepsilon}$, we observe that the following estimate holds:

$$\left|\varepsilon^4 |x|^{-4} \left(\mathbb{L}_{M_{k,\varepsilon}} v - |x|^4 \varepsilon^{-4} \Delta_0 v\right)\right| \leqslant c \left||x|^{2k+3} \nabla v\right|. \tag{52}$$

This follows easily from (34) together with the fact that u_m decays at least like $|x|^{k+1}$ on $E_{m,\varepsilon}$.

The graph of a function u, Σ_u, using the vector field $\tilde{n}_\varepsilon$, is a minimal surface (with respect to the metric g_0) if and only if u is a solution of a second order nonlinear elliptic equation of the form

$$H_e(\Sigma_u) = \mathbb{L}_{M^T_{k,\varepsilon}} u - \tilde{L}_\varepsilon\, u - Q_\varepsilon\,(u) = 0,$$

where $\mathbb{L}_{M^T_{k,\varepsilon}}$ is the Jacobi operator about $M^T_{k,\varepsilon}$, Q_ε is a nonlinear second order differential operator and $\tilde{L}_\varepsilon$ is a linear operator which takes into account the change of the normal vector field (only for the top and bottom ends) n_ε into $\tilde{n}_\varepsilon$ and of the change of the parametrization.

This operator is supported in a neighbourhood of $\{\pm s_\varepsilon\}\times S^1$ and of $S^1\times\{\rho_\varepsilon\}$. It is possible to show that the coefficients of $\varepsilon^4 \tilde{L}_\varepsilon$ are uniformly bounded by a constant times ε^4. We start by noticing that the conformal factor $(\cosh^2 s)^{-1}$ contributes with a term equal to ε^2. Furthermore the fact that $\langle \tilde{n}_\varepsilon, n_\varepsilon\rangle = 1 + \mathcal{O}_{C^{2,\alpha}_b}(\varepsilon^2)$ in a neighbourhood of $\{\pm s_\varepsilon\}\times S^1$ and the result of [6], appendix B, show that the change of vector field induces a linear operator whose coefficients are bounded by a constant times ε^2. The contribution of the change of parametrization can be estimated as follows. As for the catenoidal ends, we determine the difference between the value of functions (47) and (48), for r in a neighbourhood of r_ε, and the height of the horizontal boundary curve (see (50)). We obtain it is bounded by a constant times ε^3. Using (49) we can show that a similar estimate holds for the planar end.

Now, we consider three functions $\varphi_t, \varphi_b, \varphi_m \in \mathcal{C}^{2,\alpha}(S^1)$ which are even, with respect to θ, φ_t is collinear to $\cos(j(k+1)\theta)$ ($k \geqslant 1$ fixed) with $j \geqslant 1$, $\varphi_b(\theta) = -\varphi_t(\theta - \frac{\pi}{k+1})$, while φ_m is collinear to $\cos(l(k+1)\theta)$, with $l \geqslant 1$ and odd. Assume that they satisfy

$$\|\varphi_t\|_{\mathcal{C}^{2,\alpha}} + \|\varphi_b\|_{\mathcal{C}^{2,\alpha}} + \|\varphi_m\|_{\mathcal{C}^{2,\alpha}} \leqslant \kappa\, \varepsilon^2. \tag{53}$$

We set $\Phi := (\varphi_t, \varphi_b, \varphi_m)$ and we define w_Φ to be the function equal to

(1) $\chi_+ \, H_{\varphi_t}(s_\varepsilon - s, \cdot)$ on the image of $X_{t,\varepsilon}$, where χ_+ is a cutoff function equal to 0 for $s \leqslant s_0 + 1$ and identically equal to 1 for $s \in [s_0 + 2, s_\varepsilon]$;

(2) $\chi_-\, H_{\varphi_b}(s+s_\varepsilon,\cdot)$ on the image of $X_{b,\varepsilon}$, where χ_- is a cutoff function equal to 0 for $s \geqslant -s_0-1$ and identically equal to 1 for $s\in[-s_\varepsilon,-s_0-2]$;
(3) $\chi_m\, \tilde H_{\rho_\varepsilon,\varphi_m}(\cdot,\cdot)$ on the image of $X_{m,\varepsilon}$, where χ_m is a cutoff function equal to 0 for $\rho\geqslant\rho_0$ and identically equal to 1 for $\rho\in[\rho_\varepsilon,\rho_0/2]$;
(4) zero on the remaining part of the surface $M^T_{k,\varepsilon}$.

The cutoff functions just introduced must have the same symmetry properties as the functions in $\mathcal{C}^{2,\alpha}_\delta(M_{k,\varepsilon})$. $\tilde H$ and H are harmonic extension operators introduced respectively in Propositions 12.3 and 12.2.

We would like to prove that, under appropriate hypotheses, the graph Σ_u over $M^T_{k,\varepsilon}$ of the function $u=w_\Phi+v$ is a minimal surface with respect to the metric g_{hyp}. We want to point out that to construct the graph of the function u, here we consider the normal vector field with respect to the Euclidean metric g_0. The equation to solve is:

$$H_{hyp}(\Sigma_u)=0.$$

If we denote by $N_u=(N_1(u),N_2(u),N_3(u))$ the unit normal vector to Σ_u, by equation (31) we can express $H_{hyp}(\Sigma_u)$ in terms of the Euclidean mean curvature and write the equation to solve as

$$H_e(\Sigma_u)-\mathcal{O}(x_1N_1(u)+x_2N_2(u))+r^2\mathcal{O}(\partial_{x_1}N_1(u)+\partial_{x_2}N_2(u))=0,$$

where x_1,x_2 are the coordinates on $\mathbb{D}^2$ and $r^2=x_1^2+x_2^2$. To simplify the notation we set $P(u):=\mathcal{O}(x_1N_1(u)+x_2N_2(u))-r^2\mathcal{O}(\partial_{x_1}N_1(u)+\partial_{x_2}N_2(u))$. Taking into account that $u=w_\Phi+v$, now the expression of the equation to solve is given by

$$\mathbb{L}_{M^T_{k,\varepsilon}}(w_\Phi+v)-\tilde L_\varepsilon(w_\Phi+v)-Q_\varepsilon(w_\Phi+v)-P(w_\Phi+v)=0.$$

The resolution of the previous equation is obtained by the following fixed point problem:

$$v=T(\Phi,v) \tag{54}$$

with

$$T(\Phi,v)=G_{\varepsilon,\delta}\circ\mathcal{E}_\varepsilon\;\left(\gamma\left(\tilde L_\varepsilon(w_\Phi+v)+P(w_\Phi+v)-\mathbb{L}_{M^T_{k,\varepsilon}}w_\Phi+Q_\varepsilon(w_\Phi+v)\right)\right),$$

where $\delta\in(1,2)$, the operator $G_{\varepsilon,\delta}$ is defined in Proposition 9.3 and $\mathcal{E}_\varepsilon$ is a linear extension operator such that

$$\mathcal{E}_\varepsilon:\mathcal{C}^{0,\alpha}_\delta(M^T_{k,\varepsilon})\longrightarrow\mathcal{C}^{0,\alpha}_\delta(M_{k,\varepsilon}),$$

where $\mathcal{C}^{0,\alpha}_\delta(M^T_{k,\varepsilon})$ denotes the space of functions of $\mathcal{C}^{0,\alpha}_\delta(M_{k,\varepsilon})$ restricted to $M^T_{k,\varepsilon}$. It is defined by $\mathcal{E}_\varepsilon v=v$ in $M^T_{k,\varepsilon}$, $\mathcal{E}_\varepsilon v=0$ in the image of $[s_\varepsilon+1,+\infty)\times S^1$ by $X_{t,\varepsilon}$, in the image of $(-\infty,-s_\varepsilon-1]\times S^1$ by $X_{b,\varepsilon}$ and in the image of $B_{\rho_\varepsilon/2}$ by $X_{m,\varepsilon}$. Finally $\mathcal{E}_\varepsilon v$ is an interpolation of these values in the remaining part of $M_{k,\varepsilon}$ such that

$$\begin{aligned}
(\mathcal{E}_\varepsilon v)\circ X_{t,\varepsilon}(s,\theta)&=(1+s_\varepsilon-s)(v\circ X_{t,\varepsilon}(s_\varepsilon,\theta)), \quad \text{for } (s,\theta)\in[s_\varepsilon,s_\varepsilon+1]\times S^1,\\
(\mathcal{E}_\varepsilon v)\circ X_{b,\varepsilon}(s,\theta)&=(1+s_\varepsilon+s)(v\circ X_{b,\varepsilon}(s_\varepsilon,\theta)), \quad \text{for } (s,\theta)\in[-s_\varepsilon-1,-s_\varepsilon]\times S^1,\\
(\mathcal{E}_\varepsilon v)\circ X_{m,\varepsilon}(\theta,\rho)&=\left(\frac{2}{\rho_\varepsilon}\rho-1\right)(v\circ X_{m,\varepsilon}(\theta,\rho_\varepsilon)) \quad \text{for } (\theta,\rho)\in S^1\times[\rho_\varepsilon/2,\rho_\varepsilon].
\end{aligned}$$

Remark 10.2. From the definition of $\mathcal{E}_\varepsilon$, if $\operatorname{supp} v \cap (B_{\rho_\varepsilon} - B_{\rho_\varepsilon/2}) \neq \emptyset$, then

$$\|(\mathcal{E}_\varepsilon v) \circ X_{m,\varepsilon}\|_{\mathcal{C}^{0,\alpha}(\bar{B}_{\rho_0})} \leqslant c\rho_\varepsilon^{-\alpha} \|v \circ X_{m,\varepsilon}\|_{\mathcal{C}^{0,\alpha}(B_{\rho_0} - B_{\rho_\varepsilon})}.$$

This phenomenon of explosion of the norm does not occur near the catenoidal type ends:

$$\|(\mathcal{E}_\varepsilon v) \circ X_{t,\varepsilon}\|_{\mathcal{C}^{0,\alpha}([s_0,+\infty)\times S^1)} \leqslant c\|v \circ X_{t,\varepsilon}\|_{\mathcal{C}^{0,\alpha}([s_0,s_\varepsilon]\times S^1)}.$$

A similar equation holds for the bottom end. In the following we will assume $\alpha > 0$ and close to zero.

The existence of a solution $v \in \mathcal{C}^{2,\alpha}_\delta(M^T_{k,\varepsilon})$ for equation (54) is a consequence of the following result, which proves that $T(\Phi, \cdot)$ is a contraction mapping.

Lemma 10.3. *Let $\delta \in (1, 2)$ and $\Phi = (\varphi_t, \varphi_b, \varphi_m) \in [\mathcal{C}^{2,\alpha}(S^1)]^3$ satisfy (53) and enjoy the properties given above. There exist constants $c_\kappa > 0$ and $\varepsilon_\kappa > 0$, such that*

$$\|T(\Phi, 0)\|_{\mathcal{C}^{2,\alpha}_\delta(M_{k,\varepsilon})} \leqslant c_\kappa\, \varepsilon^{5/2} \tag{55}$$

and, for all $\varepsilon \in (0, \varepsilon_\kappa)$, $\alpha \in (0, 1/2)$,

$$\|T(\Phi, v_2) - T(\Phi, v_1)\|_{\mathcal{C}^{2,\alpha}_\delta(M_{k,\varepsilon})} \leqslant \frac{1}{2}\, \|v_2 - v_1\|_{\mathcal{C}^{2,\alpha}_\delta(M_{k,\varepsilon})},$$
$$\|T(\Phi_2, v) - T(\Phi_1, v)\|_{\mathcal{C}^{2,\alpha}_\delta(M_{k,\varepsilon})} \leqslant c\varepsilon^{3/2}\, \|\Phi_2 - \Phi_1\|_{\mathcal{C}^{2,\alpha}(S^1)},$$

where c is a positive constant, with

$$\|\Phi_2 - \Phi_1\|_{\mathcal{C}^{2,\alpha}(S^1)} = \|\varphi_{t,2} - \varphi_{t,1}\|_{\mathcal{C}^{2,\alpha}(S^1)} + \|\varphi_{b,2} - \varphi_{b,1}\|_{\mathcal{C}^{2,\alpha}(S^1)} + \|\varphi_{m,2} - \varphi_{m,1}\|_{\mathcal{C}^{2,\alpha}(S^1)}$$

for all $v, v_1, v_2 \in \mathcal{C}^{2,\alpha}_\delta(M_{k,\varepsilon})$ and satisfying $\|v\|_{\mathcal{C}^{2,\alpha}_\delta} \leqslant 2\, c_\kappa\, \varepsilon^{5/2}$ and for all boundary data $\Phi_i = (\varphi_{t,i}, \varphi_{b,i}, \varphi_{m,i}) \in [\mathcal{C}^{2,\alpha}(S^1)]^3$, $i = 1, 2$, enjoying the same properties as Φ.

Proof. We recall that the Jacobi operator associated to $M_{k,\varepsilon}$, is asymptotic (up to a multiplication by $1/\varepsilon^4$) to the operator of the catenoid near the catenoidal ends, and it is asymptotic to the Laplacian near the planar end. The function w_Φ is identically zero far from the ends where the explicit expression of $\mathbb{L}_{M_{k,\varepsilon}}$ is not known: this is the reason for our particular choice in the definition of w_Φ. Then from the definition of w_Φ and thanks to Proposition 9.3 we obtain the estimate

$$\begin{aligned}
&\|\mathcal{E}_\varepsilon \left(\gamma \mathbb{L}_{M_{k,\varepsilon}}\, w_\Phi\right) \|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\varepsilon})} \\
&\quad = \left|\left| \left(\gamma \mathbb{L}_{M^T_{k,\varepsilon}} - \left(\partial_s^2 + \partial_\theta^2\right)\right) (w_\Phi \circ X_{t,\varepsilon}) \right|\right|_{\mathcal{C}^{0,\alpha}_\delta([s_0+1, s_\varepsilon]\times S^1)} \\
&\quad + \left|\left| \left(\gamma \mathbb{L}_{M^T_{k,\varepsilon}} - \left(\partial_s^2 + \partial_\theta^2\right)\right) (w_\Phi \circ X_{b,\varepsilon}) \right|\right|_{\mathcal{C}^{0,\alpha}_\delta([-s_\varepsilon, -s_0-1]\times S^1)} \\
&\quad + \rho_\varepsilon^{-\alpha} \left|\left| \left(\gamma \mathbb{L}_{M^T_{k,\varepsilon}} - \Delta_0\right) (w_\Phi \circ X_{m,\varepsilon}) \right|\right|_{\mathcal{C}^{0,\alpha}(S^1\times[\rho_\varepsilon, \rho_0])} \\
&\quad \leqslant c \left|\left| \cosh^{-2} s\, (w_\Phi \circ X_{t,\varepsilon}) \right|\right|_{\mathcal{C}^{0,\alpha}_\delta([s_0+1, s_\varepsilon]\times S^1)} \\
&\quad + c \left|\left| \cosh^{-2} s\, (w_\Phi \circ X_{b,\varepsilon}) \right|\right|_{\mathcal{C}^{0,\alpha}_\delta([-s_\varepsilon, -s_0-1]\times S^1)} \\
&\quad + c\varepsilon^{-\alpha} \left|\left| \rho^{2k+3} \nabla (w_\Phi \circ X_{m,\varepsilon}) \right|\right|_{\mathcal{C}^{0,\alpha}(S^1\times[\rho_\varepsilon, \rho_0])} \\
&\quad \leqslant c_\kappa\, \varepsilon^4 + c_\kappa\, \varepsilon^{5/2} \leqslant c_\kappa \varepsilon^{5/2}.
\end{aligned}$$

To obtain this estimate we used the following:

$$\sup_{[s_0+1,s_\varepsilon]\times S^1} e^{-\delta s}\|\cosh^{-2} s(w_\Phi \circ X_{t,\varepsilon/2})\|_{\mathcal{C}^{0,\alpha}([s,s+1]\times S^1)}$$
$$\leqslant c \sup_{[s_0+1,s_\varepsilon]\times S^1} e^{-\delta s} e^{-2(s_\varepsilon - s)} e^{-2s}\|\phi_t\|_{\mathcal{C}^{2,\alpha}(S^1)} \leqslant c\, e^{-2s_\varepsilon}\|\phi_t\|_{\mathcal{C}^{2,\alpha}(S^1)} \leqslant c_\kappa \varepsilon^4$$

(a similar estimate holds for the bottom end) and

$$\rho_\varepsilon^{-\alpha}\|\rho^{2k+3}\nabla(w_\Phi \circ X_{m,\varepsilon})\|_{C^{0,\alpha}(S^1\times[\rho_\varepsilon,\rho_0])} \leqslant c\varepsilon^{-\alpha}\rho_\varepsilon\|\phi_m\|_{\mathcal{C}^{2,\alpha}(S^1)} \leqslant c_\kappa \varepsilon^{5/2}$$

together with the fact that $s_\varepsilon = -\ln\varepsilon$ and $\rho_\varepsilon = 2\varepsilon$, from which $e^{-2s_\varepsilon} = \varepsilon^2$ and $\rho_\varepsilon^{-\alpha} = (2\varepsilon)^{-\alpha}$. Using the properties of $\tilde{L}_\varepsilon$ and the definition of γ (see (46)), we obtain

$$\begin{aligned}\|\mathcal{E}_\varepsilon\left(\gamma\tilde{L}_\varepsilon\, w_\Phi\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\varepsilon})} \leqslant & c\varepsilon^2\|w_\Phi \circ X_{t,\varepsilon}\|_{\mathcal{C}^{0,\alpha}_\delta([s_0+1,s_\varepsilon]\times S^1)} \\ & + c\varepsilon^2\|w_\Phi \circ X_{b,\varepsilon}\|_{\mathcal{C}^{0,\alpha}_\delta([-s_\varepsilon,-s_0-1]\times S^1)} \\ & + c\varepsilon^{2-\alpha}\|w_\Phi \circ X_{m,\varepsilon}\|_{\mathcal{C}^{0,\alpha}(S^1\times[\rho_\varepsilon,\rho_0/2])} \leqslant c_\kappa\varepsilon^{5/2}.\end{aligned}$$

The estimate of $\|\mathcal{E}_\varepsilon\left(\gamma P(w_\Phi)\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\varepsilon})}$ is related to the estimate of the horizontal components and their derivatives of order one of the normal vector to the surface and to the definition of the function γ on $M^T_{k,\varepsilon}$. It is convenient to recall that the operator $\mathcal{E}_\varepsilon$ smoothly extends a function $g \in \mathcal{C}^{0,\alpha}_\delta(M^T_{k,\varepsilon})$ to a function in $\mathcal{C}^{0,\alpha}_\delta(M_{k,\varepsilon})$, substantially leaving it unchanged on $M^T_{k,\varepsilon}$ and setting it equal to zero on the remaining part of $M_{k,\varepsilon}$. P keeps track of the difference of the mean curvatures of Σ_u computed with respect to two different metrics. It is sufficient to estimate the norm of $\gamma P(w_\Phi)$ only on $M^T_{k,\varepsilon}$. The function γ equals $\varepsilon^4\cosh^2 s$ at the catenoidal ends of $M^T_{k,\varepsilon}$, equals $\varepsilon^4/|x|^4$ at the middle end, and equals ε^4 far from the ends. We recall that $|s| \in [s_0, s_\varepsilon]$ and $|x| = \rho \in [\rho_\varepsilon, \rho_0]$. Furthermore it is easy to prove that the horizontal components N_1, N_2 of the normal vector (with respect to the metric g_0) to the graph of w_Φ over the middle end of $M^T_{k,\varepsilon}$, are, in absolute value, smaller than a constant times ε^2. Their derivatives $\partial_{x_1}N_1, \partial_{x_2}N_2$ are bounded by a constant times ε. As for the values of the coordinates x_1 and x_2, we recall that we are working inside a cylindrical neighbourhood of radius $r_\varepsilon = \varepsilon/2$. We get

$$\|\mathcal{E}_\varepsilon\left(\gamma P(w_\Phi)\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\varepsilon})} \leqslant c\,\varepsilon^{5/2}.$$

As for the last term, we recall that the expression of the operator Q_ε depends on the type of end we are considering (see equations (39) and (43)). It follows that

$$\|\mathcal{E}_\varepsilon\left(\gamma Q_\varepsilon\left(w_\Phi\right)\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\varepsilon})} \leqslant c_\kappa\varepsilon^{5/2}.$$

In fact

$$\begin{aligned}\|\mathcal{E}_\varepsilon\left(\gamma Q_\varepsilon\left(w_\Phi\right)\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\varepsilon})} & \leqslant c\varepsilon^2\left\|\frac{w_\Phi}{\varepsilon^2\cosh s}\circ X_{t,\varepsilon}\right\|^2_{\mathcal{C}^{2,\alpha}_{\delta/2}([s_0+1,s_\varepsilon]\times S^1)} \\ & + c\varepsilon^2\left\|\frac{w_\Phi}{\varepsilon^2\cosh s}\circ X_{b,\varepsilon}\right\|^2_{\mathcal{C}^{2,\alpha}_{\delta/2}([-s_\varepsilon,-s_0-1]\times S^1)} \\ & + c\varepsilon^{2(1-\alpha)}\left\|\frac{|x|^2}{\varepsilon^2}w_\Phi \circ X_{m,\varepsilon}\right\|^2_{\mathcal{C}^{2,\alpha}(S^1\times[\rho_\varepsilon,\rho_0/2])} \leqslant c_\kappa\varepsilon^{5/2}.\end{aligned}$$

As for the second estimate, we recall that

$$T(\Phi, v) := G_{\varepsilon,\delta}\circ\mathcal{E}_\varepsilon\left(\gamma\left(P(w_\Phi + v) + \tilde{L}_\varepsilon(w_\Phi + v) - \mathbb{L}_{M_{k,\varepsilon}}w_\Phi + Q_\varepsilon\left(w_\Phi + v\right)\right)\right).$$

Then

$$\begin{aligned}
&\|T(\Phi, v_2) - T(\Phi, v_1)\|_{\mathcal{C}^{2,\alpha}_\delta(M_{k,\varepsilon})} \\
&\quad \leqslant \|\mathcal{E}_\varepsilon\left(\gamma\left(P(w_\Phi + v_2) - P(w_\Phi + v_1)\right)\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\varepsilon})} \\
&\quad + \|\mathcal{E}_\varepsilon\left(\gamma \tilde{L}_\varepsilon(v_2 - v_1)\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\varepsilon})} \\
&\quad + \|\mathcal{E}_\varepsilon\left(\gamma\left(Q_\varepsilon\left(w_\Phi + v_1\right) - Q_\varepsilon\left(w_\Phi + v_2\right)\right)\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\varepsilon})}.
\end{aligned}$$

We observe that from the considerations above it follows that

$$\begin{aligned}
&\|\mathcal{E}_\varepsilon\left(\gamma\left(P(w_\Phi + v_2) - P(w_\Phi + v_1)\right)\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\varepsilon})} \leqslant c\varepsilon^{3/2}||v_2 - v_1||_{\mathcal{C}^{2,\alpha}_\delta(M_{k,\varepsilon})}, \\
&\|\mathcal{E}_\varepsilon\left(\gamma \tilde{L}_\varepsilon(v_2 - v_1)\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\varepsilon})} \leqslant c\varepsilon^2||v_2 - v_1||_{\mathcal{C}^{2,\alpha}_\delta(M_{k,\varepsilon})}
\end{aligned}$$

and

$$\begin{aligned}
&\|\mathcal{E}_\varepsilon\left(\gamma\left(Q_\varepsilon\left(w_\Phi + v_1\right) - Q_\varepsilon\left(w_\Phi + v_2\right)\right)\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\varepsilon})} \\
&\quad \leqslant c||v_2 - v_1||_{\mathcal{C}^{2,\alpha}_\delta(M_{k,\varepsilon})}\left(\varepsilon^2\|\frac{w_\Phi}{\varepsilon^2\cosh s} \circ X_{t,\varepsilon}\|_{\mathcal{C}^{0,\alpha}(E_{t,\varepsilon})}\right. \\
&\quad + \varepsilon^2\|\frac{w_\Phi}{\varepsilon^2\cosh s} \circ X_{b,\varepsilon}\|_{\mathcal{C}^{0,\alpha}(E_{b,\varepsilon})} \\
&\quad \left. + \varepsilon^{2-\alpha}\|\frac{|x|^2}{\varepsilon^2} w_\Phi \circ X_{m,\varepsilon}\|_{\mathcal{C}^{0,\alpha}(E_{m,\varepsilon})}\right) \\
&\quad \leqslant c_\kappa\varepsilon^2||v_2 - v_1||_{\mathcal{C}^{2,\alpha}_\delta(M_{k,\varepsilon})}.
\end{aligned}$$

Then

$$\|T(\Phi, v_2) - T(\Phi, v_1)\|_{\mathcal{C}^{2,\alpha}_\delta(M_{k,\varepsilon})} \leqslant c\varepsilon^{3/2}||v_2 - v_1||_{\mathcal{C}^{2,\alpha}_\delta(M_{k,\varepsilon})}.$$

To show the last estimate it is sufficient to observe that

$$\begin{aligned}
&\|T(\Phi_2, v) - T(\Phi_1, v)\|_{\mathcal{C}^{2,\alpha}_\delta(M_{k,\varepsilon})} \leqslant \|\mathcal{E}_\varepsilon\left(\gamma\left(P(w_{\Phi_2} + v) - P(w_{\Phi_1} + v)\right)\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\varepsilon})} \\
&\quad + \|\mathcal{E}_\varepsilon\left(\gamma \tilde{L}_\varepsilon(w_{\Phi_2} - w_{\Phi_1})\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\varepsilon})} \\
&\quad + \|\mathcal{E}_\varepsilon\left(\gamma\left(Q_\varepsilon\left(w_{\Phi_2} + v\right) - Q_\varepsilon\left(w_{\Phi_1} + v\right)\right)\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\varepsilon})} \\
&\quad \leqslant c\varepsilon^{3/2}\|\Phi_2 - \Phi_1\|_{\mathcal{C}^{2,\alpha}(S^1)} + c\|v\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\varepsilon})}\|\Phi_2 - \Phi_1\|_{\mathcal{C}^{2,\alpha}(S^1)} \\
&\quad \leqslant c\varepsilon^{3/2}\|\Phi_2 - \Phi_1\|_{\mathcal{C}^{2,\alpha}(S^1)}.
\end{aligned}$$

□

Theorem 10.4. *Let $B := \{w \in \mathcal{C}^{2,\alpha}_\delta(M_{k,\varepsilon}) \,|\, ||w||_{\mathcal{C}^{2,\alpha}} \leqslant 2c_\kappa\varepsilon^{5/2}\}$. Then the nonlinear mapping $T(\Phi, \cdot)$ defined above has a unique fixed point v in B.*

Proof. The previous lemma shows that, if ε is chosen small enough, the nonlinear mapping $T(\Phi, \cdot)$ is a contraction mapping from the ball B of radius $2c_\kappa\varepsilon^{5/2}$ in $\mathcal{C}^{2,\alpha}_\delta(M_{k,\varepsilon})$ into itself. This value follows from the estimate of the norm of $T(\Phi, 0)$. Consequently, thanks to the Schäuder fixed point theorem, $T(\Phi, \cdot)$ has a unique fixed point v in this ball. □

This argument provides an embedded minimal surface with respect to the metric g_{hyp}, $M^T_{k,\varepsilon}(\Phi)$, which is close to $M^T_{k,\varepsilon}$ and has three boundaries. This surface close to its upper and lower boundary is a vertical graph over the annulus $B_{r_\varepsilon} - B_{r_\varepsilon/2}$, with $r_\varepsilon = \varepsilon/2$, whose parametrization is, respectively, given by

$$U_t(\theta, r) = \sigma_{t,\varepsilon} + \varepsilon^2 \ln\left(\frac{2r}{\varepsilon^2}\right) + H_{\varphi_t}\left(s_\varepsilon - \ln\frac{2r}{\varepsilon^2}, \theta\right) + V_t(\theta, r),$$
$$U_b(\theta, r) = -U_t\left(\theta - \frac{\pi}{k+1}, r\right),$$

where $s_\varepsilon = -\ln \varepsilon$. Near the middle boundary, the surface is a vertical graph whose parametrization is

$$U_m(\theta, r) = \tilde{H}_{\rho_\varepsilon, \varphi_m}\left(\theta, \frac{\varepsilon^2}{r}\right) + V_m(\theta, r).$$

The boundaries of the surface correspond to $r = r_\varepsilon$. All the functions V_i, $i = t, m$, depend nonlinearly on ε, φ.

Lemma 10.5. *The function $V_i(\varepsilon, \varphi_i)$, for $i = t, b$, satisfies*

$$\|V_i(\varepsilon, \varphi_i)(\cdot, r_\varepsilon \cdot)\|_{\mathcal{C}^{2,\alpha}(\bar{B}_1 - B_{1/2})} \leqslant c\varepsilon^2$$

and

(56)
$$\|V_i(\varepsilon, \varphi_{i,2})(\cdot, r_\varepsilon \cdot) - V_i(\varepsilon, \varphi_{i,1})(\cdot, r_\varepsilon \cdot)\|_{\mathcal{C}^{2,\alpha}(\bar{B}_1 - B_{1/2})} \leqslant c\varepsilon^{3/2-\delta}\|\varphi_{i,2} - \varphi_{i,1}\|_{\mathcal{C}^{2,\alpha}(S^1)}.$$

The function $V_m(\varepsilon, \varphi)$ satisfies $\|V_m(\varepsilon, \varphi)(\cdot, \rho_\varepsilon \cdot)\|_{\mathcal{C}^{2,\alpha}(\bar{B}_1 - B_{1/2})} \leqslant c\varepsilon^2$ and

(57)
$$\|V_m(\varepsilon, \varphi_{m,2})(\cdot, \rho_\varepsilon \cdot) - V_m(\varepsilon, \varphi_{m,1})(\cdot, \rho_\varepsilon \cdot)\|_{\mathcal{C}^{2,\alpha}(\bar{B}_1 - B_{1/2})} \leqslant c\varepsilon^{3/2}\|\varphi_{m,2} - \varphi_{m,1}\|_{\mathcal{C}^{2,\alpha}(S^1)}.$$

Proof. We start by observing that the functions V_t, V_b, V_m are the restrictions to $E_{t,\varepsilon}, E_{b,\varepsilon}$, $E_{m,\varepsilon}$ of a fixed point for the operator $T(\Phi, \cdot)$. Then the second estimate follows from

$$\|V_i(\varepsilon, \varphi_{i,2})(\cdot, \cdot) - V_i(\varepsilon, \varphi_{i,2})(\cdot, \cdot)\|_{\mathcal{C}^{2,\alpha}(\bar{B}_{r_\varepsilon} - B_{r_\varepsilon/2})}$$
$$\leqslant ce^{\delta s_\varepsilon}\|(T(\Phi_2, V_i) - T(\Phi_1, V_i)) \circ X_{i,\varepsilon}\|_{\mathcal{C}^{2,\alpha}_\delta(\Omega_i \times S^1)},$$

for $i = t, b$, with $\Omega_t = [s_0, s_\varepsilon]$ and $\Omega_b = [-s_\varepsilon, -s_0]$. The third estimate comes from

$$\|V_m(\varepsilon, \varphi_{m,2})(\cdot, \cdot) - V_m(\varepsilon, \varphi_{m,2})(\cdot, \cdot)\|_{\mathcal{C}^{2,\alpha}(\bar{B}_{r_\varepsilon} - B_{r_\varepsilon/2})}$$
$$\leqslant c\|(T(\Phi_2, V_m) - T(\Phi_1, V_m)) \circ X_{m,\varepsilon}\|_{\mathcal{C}^{2,\alpha}(S^1 \times [\rho_\varepsilon, \rho_0])}.$$

□

11. The matching of Cauchy data

In this section we will complete the proof of Theorem 1.1.

Using the result of section 7, we obtain two minimal surfaces that are perturbations of two parts of the catenoid defined in $\mathbb{M}^2 \times \mathbb{R}$. The first surface, which we denote by $S_{t,d_t}(\varphi_t)$, after a translation by d_t along the x_3-axis, can be parameterized in $B_{2r_\varepsilon} - B_{r_\varepsilon}$ as the vertical graph of

$$\bar{U}_t(\theta, r) = \varepsilon^2 \ln\left(\frac{2r}{\varepsilon^2}\right) + d_t + \mathcal{H}_{r_\varepsilon, \varphi_t}(\theta, r) + \mathcal{O}_{C^{2,\alpha}_b}(\varepsilon^2).$$

The second surface, which we denote by $S_{b,d_b}(\varphi_b)$, where $\varphi_b(\theta) = \varphi_t(\theta - \frac{\pi}{k+1})$ and $d_b = -d_t$, can be parameterized in $B_{2r_\varepsilon} - B_{r_\varepsilon}$ as the vertical graph of

$$\bar{U}_b(\theta, r) = -\bar{U}_t\left(\theta - \frac{\pi}{k+1}, r\right). \tag{58}$$

Using the result of section 5, we can construct the minimal graph $S_m(\varphi_m)$. It can be parameterized in $B_{2r_\varepsilon} - B_{r_\varepsilon}$ as the vertical graph of

$$\bar{U}_m(\theta, r) = \mathcal{H}_{r_\varepsilon, \varphi_m}(\theta, r) + \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon^2).$$

By the result of section 10, we can obtain a minimal surface $M_{k,\varepsilon}^T(\Psi)$, with $\Psi = (\psi_t, \psi_b, \psi_m)$, where $\psi_b(\theta) = \psi_t(\theta - \frac{\pi}{k+1})$, which is close to a truncated and scaled genus k Costa-Hoffman-Meeks surface and has three boundaries. This surface is close to its upper and lower boundary, and is a vertical graph over the annulus $B_{r_\varepsilon} - B_{r_\varepsilon/2}$, whose parametrization is, respectively, given by

$$\begin{aligned} U_t(\theta, r) &= \sigma_{t,\varepsilon} + \varepsilon^2 \ln\left(\frac{2r}{\varepsilon^2}\right) + H_{\psi_t}\left(s_\varepsilon - \ln\frac{2r}{\varepsilon^2}, \theta\right) + \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon^2), \\ U_b(\theta, r) &= -U_t\left(\theta - \frac{\pi}{k+1}, r\right), \end{aligned} \tag{59}$$

where $s_\varepsilon = -\ln\varepsilon$. Near the middle boundary, the surface is a vertical graph whose parametrization is

$$U_m(\theta, r) = \tilde{H}_{\rho_\varepsilon, \psi_m}\left(\theta, \frac{\varepsilon^2}{r}\right) + \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon^2).$$

We assume that the parameters and the boundary functions are chosen so that

$$\begin{aligned} &|\eta_t| + |\eta_b| + \|\varphi_t\|_{C^{2,\alpha}(S^1)} + \|\varphi_m\|_{C^{2,\alpha}(S^1)} \\ &\quad + \|\psi_t\|_{C^{2,\alpha}(S^1)} + \|\psi_m\|_{C^{2,\alpha}(S^1)} \leqslant \kappa\varepsilon^2, \end{aligned}$$

where $\eta_t = d_t - \sigma_{t,\varepsilon}$, and the constant $\kappa > 0$ is fixed large enough. The functions $\mathcal{O}_{C_b^{2,\alpha}}(\varepsilon^2)$ replace the functions $V_t, V_m, \bar{V}_t, \bar{V}_m$ that appear at the end of sections 5, 7 and 10. They depend nonlinearly on the different parameters and boundary data functions but they are bounded by a constant (independent of κ and ε) times ε^2 in the $C_b^{2,\alpha}$ topology, where partial derivatives are taken with respect to the vector fields $r\partial_r$ and ∂_θ. It remains to show that, for all ε small enough, it is possible to choose the parameters and boundary functions in such a way that the surface

$$S_{t,d_t}(\varphi_t) \cup S_{b,d_b}(\varphi_b) \cup S_m(\varphi_m) \cup \bar{M}_{k,\varepsilon}^T(\Psi)$$

is a C^1 surface across the boundaries of the different summands. Regularity theory will then ensure that this surface is in fact smooth and by construction it has the desired properties. We point out that, in this way, we will have constructed a one-parameter family of minimal surfaces: each one of them is determined by a different value of the parameter ε. Therefore this will complete the proof of the main theorem.

It is necessary to fulfill the following system of equations:

$$\left\{ \begin{array}{c} U_t(\cdot, r_\varepsilon) = \bar{U}_t(\cdot, r_\varepsilon) \\ U_b(\cdot, r_\varepsilon) = \bar{U}_b(\cdot, r_\varepsilon) \\ U_m(\cdot, r_\varepsilon) = \bar{U}_m(\cdot, r_\varepsilon) \\ \partial_r U_b(\cdot, r_\varepsilon) = \partial_r \bar{U}_b(\cdot, r_\varepsilon) \\ \partial_r U_t(\cdot, r_\varepsilon) = \partial_r \bar{U}_t(\cdot, r_\varepsilon) \\ \partial_r U_m(\cdot, r_\varepsilon) = \partial_r \bar{U}_m(\cdot, r_\varepsilon) \end{array} \right. \tag{60}$$

on S^1. The first three equations together with the following identities,

$$H_{\psi_t}\left(s_\varepsilon-\ln(2r)/\varepsilon^2,\theta\right)_{|r=r_\varepsilon}=\psi_t(\theta),\quad \tilde H_{\rho_\varepsilon,\psi_m}\left(\theta,\varepsilon^2/r\right)_{|r=r_\varepsilon}=\psi_m(\theta),$$
$$\mathcal{H}_{r_\varepsilon,\varphi_t}(\theta,r)_{|r=r_\varepsilon}=\varphi_t(\theta),\quad \mathcal{H}_{r_\varepsilon,\varphi_m}(\theta,r)_{|r=r_\varepsilon}=\varphi_m(\theta),$$

lead to the system

$$\left\{\begin{array}{c}\eta_t+\varphi_t-\psi_t=\mathcal{O}_{C^{2,\alpha}_b}(\varepsilon^2)\\ \varphi_m-\psi_m=\mathcal{O}_{C^{2,\alpha}_b}(\varepsilon^2),\end{array}\right. \tag{61}$$

where $\eta_t=d_t-\sigma_{t,\varepsilon}$. The relations (58) and (59) allowed us to omit one equation. The last three equations give the system (we applied Lemmas 12.4 and 12.5)

$$\left\{\begin{array}{c}\partial^*_\theta(\varphi_t+\psi_t)=\mathcal{O}_{C^{1,\alpha}_b}(\varepsilon^2)\\ \partial^*_\theta(\varphi_m+\psi_m)=\mathcal{O}_{C^{1,\alpha}_b}(\varepsilon^2).\end{array}\right. \tag{62}$$

Here ∂^*_θ denotes the operator which associates to $\phi=\sum_{i\geqslant1}\phi_i\cos(i\theta)$ the function $\partial^*_\theta\phi=\sum_{i\geqslant1}i\phi_i\cos(i\theta)$. The functions $\mathcal{O}_{C^{l,\alpha}}(\varepsilon^2)$ in the above expansions depend nonlinearly on the different parameters and boundary data functions, but they are bounded by a constant (independent of κ and ε) times ε^2 in the $C^{l,\alpha}$ topology. Projecting every equation of this system over the L^2-orthogonal complement of Span$\{1\}$, we obtain the system

$$\left\{\begin{array}{c}\varphi_t-\psi_t=\mathcal{O}_{C^{2,\alpha}_b}(\varepsilon^2)\\ \varphi_m-\psi_m=\mathcal{O}_{C^{2,\alpha}_b}(\varepsilon^2)\\ \partial^*_\theta\varphi_t+\partial^*_\theta\psi_t=\mathcal{O}_{C^{1,\alpha}_b}(\varepsilon^2)\\ \partial^*_\theta\varphi_m+\partial^*_\theta\psi_m=\mathcal{O}_{C^{1,\alpha}_b}(\varepsilon^2).\end{array}\right. \tag{63}$$

Lemma 11.1. *The operator h, defined by*

$$\begin{array}{c}C^{2,\alpha}(S^1)\to C^{1,\alpha}(S^1)\\ \varphi\to\partial^*_\theta\varphi,\end{array}$$

acting on functions that are orthogonal to the constant function in the L^2-sense and are even, is invertible.

Proof. We observe that if we decompose $\varphi=\sum_{j\geqslant1}\varphi_j\cos(j\theta)$, then

$$h(\varphi)=\sum_{j\geqslant1}\varphi_j\cos(j\theta),$$

which is clearly invertible from $H^1(S^1)$ into $L^2(S^1)$. Now elliptic regularity theory implies that this is also the case when this operator is defined between Hölder spaces. □

Using this result, the system (63) can be rewritten as

$$(\varphi_t,\varphi_m,\psi_t,\psi_m)=\mathcal{O}_{C^{2,\alpha}}(\varepsilon^2). \tag{64}$$

Recall that the right hand side depends nonlinearly on $\varphi_t,\varphi_b,\psi_t,\psi_m$ and also on the parameter η_t. We look at this equation as a fixed point problem and fix κ large enough. Thanks to the estimates (12), (25), (56) with $\delta\in(1,3/2)$, and (57), we can use a fixed point theorem for contraction mappings in the ball of radius $\kappa\varepsilon^2$ in $(C^{2,\alpha}(S^1))^4$ to obtain, for all ε small enough, a solution $(\varphi_t,\varphi_m,\psi_t,\psi_m)$ of (64). Since the right hand side of (64) is continuous with respect to all data, we see that the fixed point $(\varphi_t,\varphi_m,\psi_t,\psi_m)$ depends continuously (and in fact smoothly) on the parameter η_t. Inserting the found solution into (61) and (62), we see that it remains to solve an equation that can be rewritten as

$$\eta_t = \mathcal{O}(\varepsilon^2), \tag{65}$$

where this time, the right hand side depends nonlinearly on η_t. Now, provided κ has been fixed large enough, we can use the Schäuder fixed point theorem in the ball of radius $\kappa\varepsilon^2$ in $\mathbb{R}$ to solve (65), for all ε small enough. This provides a set of parameters and boundary data such that (61) and (62) hold. Equivalently we have proved the existence of a solution of system (60). So the proof of Theorem 1.1 is complete.

12. Appendix

12.1. **Harmonic extension operators.** The results contained in this section are about the existence of some harmonic extension operators. The first one gives the harmonic extension of a function defined on ∂B_{r_0} to $\mathbb{D}^2 \setminus B_{r_0}$.

Proposition 12.1. *There exists an operator*

$$\mathcal{H}_{r_0} : C^{2,\alpha}(S^1) \longrightarrow C^{2,\alpha}(S^1 \times [r_0, 1]),$$

such that for every even function $\varphi(\theta) \in C^{2,\alpha}(S^1)$, the function $w_\varphi = \mathcal{H}_{r_0,\varphi}$ solves

$$\begin{cases} \Delta_0 w_\varphi = 0 & on \quad S^1 \times [r_0, 1] \\ w_\varphi = \varphi & on \quad S^1 \times \{r_0\} \\ w_\varphi = 0 & on \quad S^1 \times \{1\}. \end{cases}$$

Moreover,

$$||\mathcal{H}_{r_0,\varphi}||_{C^{2,\alpha}(S^1 \times [r_0,1])} \leqslant c\,||\varphi||_{C^{2,\alpha}(S^1)}, \tag{66}$$

for some constant $c > 0$.

$S^1 \times [r_0, 1]$ being a compact domain, the existence of the solution of the Dirichlet problem above is a classical result; for example, see Theorem 2.14 of [4].

The uniqueness of the solution follows from the maximum principle. To get the wanted estimate of the norm of the solution, we observe that the maximum principle also implies that $\sup |w_\varphi| \leqslant c \sup |\varphi|$ and that we can obtain similar estimates (involving the derivatives of φ) for the norm of the derivatives of w_φ from their harmonicity. ☐

Now we give the statement of a result whose proof is contained in [3].

Proposition 12.2. *There exists an operator*

$$H : \mathcal{C}^{2,\alpha}(S^1) \longrightarrow \mathcal{C}^{2,\alpha}_{-2}([0, +\infty) \times S^1),$$

such that for all $\varphi \in \mathcal{C}^{2,\alpha}(S^1)$, even functions and orthogonal to e_i, $i = 0, 1$, in the L^2-sense, the function $w = H_\varphi$ solves

$$\begin{cases} (\partial_s^2 + \partial_\theta^2)\, w &= & 0 & in & S^1 \times [0, +\infty) \\ w &= & \varphi & on & S^1 \times \{0\}. \end{cases}$$

Moreover

$$\|H_\varphi\|_{\mathcal{C}^{2,\alpha}_{-2}(S^1 \times [0,+\infty))} \leqslant c\,\|\varphi\|_{\mathcal{C}^{2,\alpha}(S^1)},$$

for some constant $c > 0$.

The following result gives a harmonic extension of a function on $\mathbb{R}^2 \setminus D_{\bar\rho}$.

Proposition 12.3. *There exists an operator*

$$\tilde{H}_{\bar{\rho}} : C^{2,\alpha}(S^1) \longrightarrow C^{2,\alpha}(S^1 \times [\bar{\rho}, +\infty)),$$

such that for each even function $\varphi(\theta) \in C^{2,\alpha}(S^1)$, *which is* L^2*-orthogonal to the constant function, then* $w_\varphi = \tilde{H}_{\bar{\rho},\varphi}$ *solves*

$$\begin{cases} \Delta w_\varphi = 0 & on \quad S^1 \times [\bar{\rho}, +\infty) \\ w_\varphi = \varphi & on \quad S^1 \times \{\bar{\rho}\}. \end{cases}$$

Moreover,

$$||\tilde{H}_{\bar{\rho},\varphi}||_{C^{2,\alpha}(S^1\times[\bar{\rho},+\infty))} \leqslant c\,||\varphi||_{C^{2,\alpha}(S^1)}, \tag{67}$$

for some constant $c > 0$.

Proof. We consider the decomposition of the function φ with respect to the basis $\{\cos(i\theta)\}$, that is,

$$\varphi = \sum_{i=1}^{\infty} \varphi_i \cos(i\theta).$$

Then the solution w_φ is given by

$$w_\varphi(\theta, \rho) = \sum_{i=1}^{\infty} \left(\frac{\bar{\rho}}{\rho}\right)^i \varphi_i \cos(i\theta).$$

Since $\frac{\bar{\rho}}{\rho} \leqslant 1$, then $\left(\frac{\bar{\rho}}{\rho}\right)^i \leqslant \left(\frac{\bar{\rho}}{\rho}\right)$, we can conclude that $|w(\theta,\rho)| \leqslant c|\varphi(\theta)|$ and then $||w_\varphi||_{C^{2,\alpha}} \leqslant c||\varphi||_{C^{2,\alpha}}$. □

Lemma 12.4. *Let* $u(\theta, r)$ *be the harmonic extension defined on* $S^1 \times [r_0, +\infty)$ *of the even function* $\varphi = \sum_{i\geqslant 0} \varphi_i \cos(i\theta) \in \mathcal{C}^{2,\alpha}(S^1)$ *and such that* $u(\theta, r_0) = \varphi(\theta)$. *Then*

$$\partial_\theta^* \varphi(\theta) = r_0 \partial_r u(\theta, r)_{|r=r_0}.$$

Proof. If $\varphi(\theta) = \sum_{i\geqslant 0} \varphi_i \cos(i\theta)$, then the function u is given by

$$u(\theta, r) = \sum_{i\geqslant 0} \varphi_i \left(\frac{r}{r_0}\right)^i \cos(i\theta).$$

Then

$$\partial_r u(\theta, r) = \sum_{i\geqslant 1} \varphi_i \left(\frac{r}{r_0}\right)^i \frac{i\cos(i\theta)}{r}.$$

Consequently

$$\partial_\theta^* \varphi(\theta) = r_0 \partial_r u(\theta, r)_{|r=r_0}.$$ □

Lemma 12.5. *Let* $u(\theta, r)$ *be the harmonic extension defined on* $S^1 \times [0, r_0]$ *of the even function* $\varphi \in \mathcal{C}^{2,\alpha}(S^1)$ *and such that* $u(\theta, r_0) = \varphi(\theta)$. *Then*

$$\partial_\theta^* \varphi(\theta) = -r_0 \partial_r u(\theta, r)_{|r=r_0}.$$

Proof. If $\varphi(\theta) = \sum_{i\geqslant 0} \varphi_i \cos(i\theta)$, then the function u is given by

$$u(\theta, r) = \sum_{i\geqslant 0} \varphi_i \left(\frac{r_0}{r}\right)^i \cos(i\theta).$$

Then

$$\partial_r u(\theta, r) = -\sum_{i \geqslant 1} \varphi_i \left(\frac{r_0}{r}\right)^i \frac{i\cos(i\theta)}{r}.$$

Consequently

$$\partial_\theta \varphi^*(\theta) = -r_0 \partial_r u(\theta, r)_{|r=r_0}.$$

□

12.2. **The proof of Proposition 4.2.** We start by giving the statement of a classical result about the injectivity of Δ_0.

Lemma 12.6. *Given $0 < r_0 < r_1 \leqslant 1$, let w be a solution of $\Delta_0 w = 0$ on $S^1 \times [r_0, r_1]$ such that $w(\cdot\,, r_0) = w(\cdot\,, r_1) = 0$. Then $w = 0$.*

As a consequence of Lemma 12.6, the operator Δ_0 is injective. Hence, the Fredholm alternative assures that there exists a unique $w \in \mathcal{C}^{2,\alpha}(S^1 \times [r_0, 1])$, with $w(\theta, r)$ satisfying:

$$\left\{ \begin{array}{l} \Delta_0\, w = f \quad \text{on } S^1 \times [r_0, 1] \\ w(\cdot, r_0) = w(\cdot, 1) = 0. \end{array} \right. \tag{68}$$

We want to prove the following assertion.

Assertion 12.7. *For every $0 < r_0 < 1$, $f \in \mathcal{C}^{0,\alpha}(S^1 \times [r_0, 1])$ and $w \in \mathcal{C}^{2,\alpha}(S^1 \times [r_0, 1])$ satisfying (68) there exists a constant c such that*

$$\|w\|_{\mathcal{C}^{0,\alpha}(S^1 \times [r_0,1])} \leqslant c\, \|f\|_{\mathcal{C}^{0,\alpha}(S^1 \times [r_0,1])}.$$

We suppose by contradiction that the assertion 12.7 is false; that is, there does not exist a universal constant for which the previous estimate holds. Then, for each $n \in \mathbb{N}$, there exist $r_{0,n}$ and f_n, w_n satisfying (68) (with $r_{0,n}, f_n, w_n$ instead of r_0, f, w) such that

$$\sup_{S^1 \times [r_{0,n},1]} |f_n| = 1 \quad \text{and} \quad A_n := \sup_{S^1 \times [r_{0,n},1]} |w_n| \to +\infty \quad \text{as} \quad n \to \infty.$$

Since $S^1 \times [r_{0,n}, 1]$ is a compact set, A_n is achieved at a point $(\theta_n, r_n) \in S^1 \times [r_{0,n}, 1]$.

The sequence of sets $I_n = [\frac{r_{0,n}}{r_n}, \frac{1}{r_n}]$ converges (up to some subsequence) to a set that we denote by I_∞. We will show that it is nonempty and contains 1. If $r_{0,n} < r' < r'' < 1$, elliptic estimates allow us to conclude that

$$\sup_{S^1 \times [r_{0,n}, r']} |\nabla w_n| \leqslant c(\sup_{S^1 \times [r_{0,n}, r'']} |f_n| + \sup_{S^1 \times [r_{0,n}, r'']} |w_n|) \leqslant c(1 + A_n),$$

where c is a constant independent of n.

Then, if $n \to +\infty$, $\frac{r_{0,n}}{r_n} \to R_1 < 1$ and $\frac{1}{r_n} \to R_2 > 1$. The fact that $R_1 < 1$ follows from the above estimate for the gradient of w_n near $r = r_{0,n}$. That implies that the supremum A_n cannot be achieved at a point which is too close to $r_{0,n}$, that is, the point where w_n vanishes. In other terms, the quotient $\frac{r_{0,n}}{r_n}$ remains bounded away from 1. Using similar arguments it is possible to show that ∇w_n is bounded near 1 and consequently $\frac{1}{r_n}$ also remains bounded away from 1. Then we can conclude that I_∞ is not empty. We set $I_\infty = [R_1, R_2]$, where $0 \leqslant R_1 < 1 < R_2 < +\infty \in \mathbb{R}$.

We define

$$\widetilde{w}_n(\theta, r) := \frac{1}{A_n}\, w_n(\theta, r r_n) \quad \text{and} \quad \widetilde{f}_n(\theta, r) := \frac{1}{A_n}\, f_n(\theta, r r_n),$$

for all $(\theta, r) \in S^1 \times I_n$, with $I_n = [r_{0,n}/r_n, 1/r_n]$. These functions satisfy $r_n^2 \Delta \widetilde{w}_n = \widetilde{f}_n$.

From the definition of $\widetilde{w}_n$, we obtain that

$$\nabla \widetilde{w}_n = \frac{1}{A_n} \nabla w_n(\theta, r r_n).$$

Then

$$|\nabla \widetilde{w}_n| \leqslant c \frac{1 + A_n}{A_n} < 2c.$$

Since the sequences $(\tilde{w}_n)_n$ and $(\nabla \tilde{w}_n)_n$ are uniformly bounded, the Ascoli-Arzelà theorem assures that a subsequence of $(\tilde{w}_n)_n$ converges on compact sets of $S^1 \times I_\infty$ to a nonzero function w_∞ that vanishes on $S^1 \times \partial I_\infty$. The function w_∞ inherits the properties of $\tilde{w}_n$. In particular,

$$\sup_{S^1 \times I_\infty} |w_\infty| = 1. \tag{69}$$

In the same way it's possible to prove that a subsequence of $(\tilde{f}_n)_n$ converges on compact sets of $S^1 \times I_\infty$ to the function $f_\infty \equiv 0$ since, if $n \to \infty$,

$$\sup_{S^1 \times I_n} |\widetilde{f}_n| \to 0.$$

Then the limit function w_∞ must satisfy the differential equation

$$\Delta_0 w_\infty = 0$$

on $S^1 \times I_\infty$ with null boundary conditions on ∂I_∞. So we can conclude that $\forall r \in I_\infty$, $w_\infty(\theta, r) = 0$. This function does not satisfy (69), a contradiction. This proves the assertion 12.7.

The elliptic estimate,

$$|\nabla w| \leqslant c \left(\sup_{S^1 \times [r_0, 1]} |f| + \sup_{S^1 \times [r_0, 1]} |w| \right),$$

allows us to get a uniform estimate of ∇w. This proves the existence of a solution of $\Delta_0 w = f$ defined on $S^1 \times [r_0, 1]$ for which

$$\|w\|_{\mathcal{C}^{0,\alpha}(S^1 \times [r_0, 1])} \leqslant c \, \|f\|_{\mathcal{C}^{0,\alpha}(S^1 \times [r_0, 1])}.$$

Now it is sufficient again to use elliptic estimates to obtain the estimates for the derivatives.

12.3. **Minimal graphs in $(\mathbb{D}^2 \times \mathbb{R}, g_{hyp})$.** In this section, following [17], we will find the condition to be satisfied such that the graph Σ of a function defined on $\mathbb{D}^2$ is minimal with respect to the metric $g_{hyp} = \frac{dx_1^2 + dx_2^2}{F} + dx_3^2$, where $F = (1 - x_1^2 - x_2^2)^2$. We assume that the immersion of Σ in $\mathbb{D}^2 \times \mathbb{R}$ is given by

$$(x_1, x_2) \to (x_1, x_2, u(x_1, x_2)).$$

The Christoffel symbols, Γ_{ij}^k, associated to g_{hyp} all vanish except

$$\Gamma_{11}^1 = \Gamma_{21}^2 = \Gamma_{12}^2 = -\Gamma_{22}^1 = \frac{2x_1}{\sqrt{F}},$$

$$\Gamma_{12}^1 = \Gamma_{22}^2 = \Gamma_{21}^1 = -\Gamma_{11}^2 = \frac{2x_2}{\sqrt{F}}.$$

Let e_1, e_2, e_3 be the canonical basis of $\mathbb{R}^3$. Then $\varepsilon_1 = \sqrt{F}e_1, \varepsilon_2 = \sqrt{F}e_2, \varepsilon_3 = e_3$ is an orthonormal basis for $\mathbb{M}^2 \times \mathbb{R}$. The coordinate vector fields on Σ are

$$X_1 = \frac{\varepsilon_1}{\sqrt{F}} + u'_{x_1}\varepsilon_3, \quad X_2 = \frac{\varepsilon_2}{\sqrt{F}} + u'_{x_2}\varepsilon_3, \quad N = \frac{1}{W}\left(-u'_{x_1}\sqrt{F}\varepsilon_1 - u'_{x_2}\sqrt{F}\varepsilon_2 + \varepsilon_3\right),$$

with $W = \sqrt{1 + F|\nabla u|^2}$. The induced metric on Σ is defined by

$$g_{11} = \frac{1}{F} + (u'_{x_1})^2, \quad g_{22} = \frac{1}{F} + (u'_{x_2})^2, \quad g_{12} = u''_{x_1x_2}.$$

If $\bar{\nabla}$ denotes the Riemannian connection of the metric g_{hyp}, then the coefficients of the second fundamental form are

$$\begin{aligned}
b_{11} = \langle \nabla_{X_1} X_1, N\rangle &= \frac{1}{W}\left(-\frac{2x_1}{\sqrt{F}}u'_{x_1} + \frac{2x_2}{\sqrt{F}}u'_{x_2} + u''_{x_1x_1}\varepsilon_3\right), \\
b_{22} = \langle \nabla_{X_2} X_2, N\rangle &= \frac{1}{W}\left(\frac{2x_1}{\sqrt{F}}u'_{x_1} - \frac{2x_2}{\sqrt{F}}u'_{x_2} + u''_{x_2x_2}\varepsilon_3\right), \\
b_{12} = \langle \nabla_{X_1} X_2, N\rangle &= \frac{1}{W}\left(-\frac{2x_2}{\sqrt{F}}u'_{x_1} - \frac{2x_1}{\sqrt{F}}u'_{x_2} + u''_{x_1x_2}\varepsilon_3\right),
\end{aligned}$$

where we used the following identities:

$$\begin{aligned}
&\nabla_{X_1} X_1 = 2x_1\varepsilon_1 - 2x_2\varepsilon_2 + u''_{x_1x_1}\varepsilon_3, \quad \nabla_{X_2} X_2 = -2x_1\varepsilon_1 + 2x_2\varepsilon_2 + u''_{x_2x_2}\varepsilon_3, \\
&\nabla_{X_1} X_2 = 2x_2\varepsilon_1 + 2x_1\varepsilon_2 + u''_{x_1x_2}\varepsilon_3.
\end{aligned}$$

The mean curvature of Σ with respect to g_{hyp} is given by

$$H(\Sigma) = \frac{1}{2}\frac{b_{11}g_{22} + b_{22}g_{11} - 2b_{12}g_{12}}{g_{11}g_{22} - g_{12}^2}.$$

Using the expressions of the coefficients of the first and second fundamental form, we find that

$$H(\Sigma) = \frac{F}{2}\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + F|\nabla u|^2}}\right).$$

12.4. **Minimal surfaces of rotation in** $(\mathbb{D}^2 \times \mathbb{R}, g_{hyp})$**.** In this section, following [17], we will find the condition to be satisfied such that a surface of revolution Σ in $\mathbb{D}^2 \times \mathbb{R}$ is minimal with respect to the metric $g_{hyp} = \frac{dx_1^2 + dx_2^2}{F} + dx_3^2$, where $F = (1 - x_1^2 - x_2^2)^2$. We will assume that the immersion of Σ in $\mathbb{D}^2 \times \mathbb{R}$ is given, in terms of the cylindrical coordinates (r, θ, z), by:

$$(\theta, z) \to (r(z), \theta, z),$$

where $r(z)$ is a function of z.

It is convenient to express the metric g_{hyp} in terms of the new coordinates. We find

$$g_{hyp} = \frac{dr^2 + r^2 d\theta^2}{F} + dz^2,$$

with $F = (1 - r^2)^2$. The Christoffel symbols, Γ^k_{ij}, associated to g_{hyp} all vanish except

$$\Gamma^1_{11} = \frac{2r}{1 - r^2}, \quad \Gamma^1_{22} = -\frac{r(1 + r^2)}{1 - r^2}, \quad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1 + r^2}{r(1 - r^2)}.$$

Let e_1, e_2, e_3 be the canonical basis of $\mathbb{R}^3$. Then the coordinate vector fields on Σ are

$$X_1 = r'(z)e_1 + e_3 \quad X_2 = e_2, \quad N = \frac{-e_1 + r'(z)e_3}{R},$$

with $R = \sqrt{\frac{1}{F} + (r'(z))^2}$. The induced metric on Σ is defined by

$$g_{11} = \frac{(r'(z))^2}{F} + 1, \quad g_{22} = \frac{r^2(z)}{F}, \quad g_{12} = 0.$$

If $\bar{\nabla}$ denotes the Riemannian connection of the metric g_{hyp}, then the coefficients of the second fundamental form are

$$\begin{aligned}
b_{11} &= \langle \bar{\nabla}_{X_1} X_1, N \rangle = -\frac{1}{RF}\left(\frac{2r(r')^2}{1-r^2} + r''\right),\\
b_{22} &= \langle \bar{\nabla}_{X_2} X_2, N \rangle = \frac{1}{RF}\left(\frac{r(1+r^2)}{1-r^2}\right),\\
b_{12} &= \langle \bar{\nabla}_{X_1} X_2, N \rangle = 0,
\end{aligned}$$

where we used the following identities:

$$\begin{aligned}
&\bar{\nabla}_{X_1} X_1 = \left(\frac{2r(r')^2}{1-r^2} + r''\right) e_1, \quad \bar{\nabla}_{X_2} X_2 = -\frac{r(1+r^2)}{1-r^2} e_1,\\
&\bar{\nabla}_{X_1} X_2 = \frac{r'(1+r^2)}{r(1-r^2)} e_2.
\end{aligned}$$

The mean curvature of Σ with respect to g_{hyp} is given by

$$H(\Sigma) = \frac{1}{2}\frac{b_{11}g_{22} + b_{22}g_{11} - 2b_{12}g_{12}}{g_{11}g_{22} - g_{12}^2}.$$

Using the expressions of the coefficients of the first and second fundamental form, we find that $H(\Sigma) = 0$ if the function $r(z)$ satisfies the following differential equation:

$$r(z)r''(z) - (r'(z))^2 - (1 - r(z)^4) = 0.$$

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