# Stochastic reduced-order model in low frequency dynamics in presence of numerous local elastic modes

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# STOCHASTIC REDUCED-ORDER MODEL IN LOW-FREQUENCY DYNAMICS IN PRESENCE OF NUMEROUS LOCAL ELASTIC MODES

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## ABSTRACT

This paper concerns the non usual case in structural dynamics for which a complex structure exhibits both the usual global elastic modes and numerous local elastic modes in the low-frequency range. Despite the presence of these local elastic modes, we are interested in constructing a stochastic reduced-order model using only the global modes and in taking into account the local elastic modes with a probabilistic approach. The ”global elastic modes” and the ”local elastic modes” are calculated separately using a new formulation solving two generalized eigenvalue problems. The union of these two families constitutes a basis of the admissible space. Then, the reduced-order model is constructed by projection of the dynamical equation on the global elastic modes. The apparent damping generated in this reduced-order model by the power flow of the mechanical energy from the global modes to the local modes is constructed by a statistical approach. The theory is presented and is validated through an application.

## 1. INTRODUCTION

This paper is devoted to the construction of stochastic reduced-order models in linear structural dynamics for complex structures in the low-frequency range. In general, the low-frequency range (for which the modal density is very small and for which the resonances are isolated) can clearly be separated from the medium-frequency range (for which the modal density is larger but not uniform in frequency). The low-frequency dynamical analysis of the structure can be carried out using the first global elastic modes which are calculated with a computational model and a reduced-order model is constructed with the modal analysis in order to calculate the dynamical responses. Sometimes, a complex structure for which the low-frequency range must be analyzed as explained above, can exhibit both the global elastic modes (which characterize this low-frequency range) and numerous local elastic modes. This situation appears for complex heterogeneous structures presenting stiff parts and flexible parts such as an automotive vehicle. The problem which occurs is induced by the existence of an efficient sorting method which could be used to select the elastic modes as global elastic modes or as local elastic modes.

In addition, although the reduced-order model must be constructed with respect to the global elastic modes, this reduced-order model must have the capability to predict the amplitudes of the responses of the structure in this low-frequency range. Since there are local elastic modes in the frequency band, a part of the mechanical energy is transferred from the global elastic modes to the local elastic modes which store this energy and then which induces an apparent damping at the resonances associated with the global elastic modes. This mechanism is exactly the phenomenon described and modeled by the fuzzy structure theory introduced in [1].

The objective of this paper is double: (1) The first one is to develop a method to compute separately the "global elastic modes" and the "local elastic modes" by solving two generalized eigenvalue problems. (2) The second one is to construct a reduced-order model with the global elastic modes but in taking into account the effects of the local elastic modes, in order to correctly predict the frequency response functions in the low-frequency range.

A spatial filtering of short wavelengths, achieved by regularization, has been proposed to extract (from the measured frequency response functions) the left eigenvectors in the context of the experimental modal analysis [3], [4]. This work is not adapted to the first objective of the present paper for which a complete basis of the admissible space, made up of the two families (global and local elastic modes), has to be constructed with an algorithm which is not intrusive. Another approach consists in approximating the consistent mass matrix of the computational model by a lumped mass matrix allowing some local elastic modes to be filtered [5, 6]. When such a procedure of lumping mass is used, it is then assumed that the acceleration are approximatively constant in the neighborhood of the node in which the point mass is concentrated. Although properties of solutions obtained with lumped mass matrix have been studied (see for instance [7–9]), such a filter would depend on the mesh and the bandpass filter could not be selected. In addition, it would seem that no way exists to complete the construction of the basis including the local elastic modes.

In this paper, we summarize the theory developed and detailed in [2].

## 2. REFERENCE REDUCED MATRIX MODEL

We are interested in predicting the frequency response functions of a three dimensional linear damped structure, occupying a bounded domain $\Omega$, in the frequency band of analysis $\mathcal{B} = [\omega_{\text{min}}, \omega_{\text{max}}]$ with $0 < \omega_{\text{min}}$. The complex vector $\mathbb{U}(\omega)$ of the $m$ DOF of the computational model constructed by the finite element method is solution of the following complex matrix equation,

$$(-\omega^2[\mathbb{M}] + i\omega[\mathbb{D}] + [\mathbb{K}])\mathbb{U}(\omega) = \mathbb{F}(\omega) \quad , \tag{1}$$

in which $[\mathbb{M}]$, $[\mathbb{D}]$ and $[\mathbb{K}]$ are respectively the $(m \times m)$ positive-definite symmetric real mass, damping and stiffness matrices and where $\mathbb{F}(\omega)$ is relative to the discretization of the external forces. The eigenfrequencies and the elastic modes of the associated conservative dynamical system consists in finding $\lambda$ and $\boldsymbol{\varphi}$ in $\mathbb{R}^m$ such that

$$[\mathbb{K}]\,\boldsymbol{\varphi} = \lambda\,[\mathbb{M}]\,\boldsymbol{\varphi}\,. \tag{2}$$

Using the modal method, the approximation $\mathbb{U}_n(\omega)$ at order $n$ of $\mathbb{U}(\omega)$ is written as

$$\mathbb{U}_n(\omega) = \sum_{\alpha=1}^{n} q_\alpha(\omega)\,\boldsymbol{\varphi}_\alpha = [\Phi]\,\boldsymbol{q}\,, \tag{3}$$

in which $\boldsymbol{q} = (q_1, \ldots, q_n)$ is the complex vector of the $n$ generalized coordinates and where $[\Phi] = [\boldsymbol{\varphi}_1 \ldots \boldsymbol{\varphi}_n]$ is the $(m \times n)$ real matrix of the elastic modes associated with the $n$ first eigenvalues.

## 3. DECOMPOSITION OF THE MASS MATRIX

In this section, we introduce a decomposition of the mass matrix which is adapted to the calculation of the global elastic modes in the low-frequency band of analysis in which there are also a large number of local elastic modes. We first introduce a decomposition of the domain $\Omega$ and a projection operator. The details of the methodology for the discrete and the continuous cases are presented in [2].

### 3.1 Decomposition of domain $\Omega$

Domain $\Omega$ is partitioned into $n_J$ subdomains $\Omega_j^\epsilon$ such that, for $j$ and $k$ in $\{1, \ldots, n_J\}$,

$$\Omega = \bigcup_{j=1}^{n_J} \Omega_j^\epsilon \quad , \quad \Omega_j^\epsilon \cap \Omega_k^\epsilon = \emptyset \, . \tag{4}$$

The parameter $\epsilon$ is the characteristic length of the subdomains. The choice of $\epsilon$ is related to the smallest "wavelength" of the global elastic modes that we want to extract in presence of numerous local modes.

### 3.2 Projection operator

Let $\boldsymbol{u} \mapsto h_\epsilon^r(\boldsymbol{u})$ be the linear operator defined by

$$\{h_\epsilon^r(\boldsymbol{u})\}(\boldsymbol{x}) = \sum_{j=1}^{n_J} \mathbb{1}_{\Omega_j^\epsilon}(\boldsymbol{x}) \, \frac{1}{m_j} \int_{\Omega_j^\epsilon} \rho(\boldsymbol{x}) \, \boldsymbol{u}(\boldsymbol{x}) \, d\boldsymbol{x} \, , \tag{5}$$

in which $\boldsymbol{x} \mapsto \mathbb{1}_{\Omega_j^\epsilon}(\boldsymbol{x}) = 1$ if $\boldsymbol{x}$ is in $\Omega_j^\epsilon$ and $= 0$ otherwise. The local mass $m_j$ is defined, for all $j$ in $\{1, \ldots, n_J\}$, by $m_j = \int_{\Omega_j^\epsilon} \rho(\boldsymbol{x}) \, d\boldsymbol{x}$, where $\boldsymbol{x} \mapsto \rho(\boldsymbol{x})$ is the mass density. Let $\boldsymbol{u} \mapsto h_\epsilon^c(\boldsymbol{u})$ be the linear operator defined by

$$h_\epsilon^c(\boldsymbol{u}) = \boldsymbol{u} - h_\epsilon^r(\boldsymbol{u}) \, . \tag{6}$$

Function $h_\epsilon^r(\boldsymbol{u})$ will also be denoted by $\boldsymbol{u}^r$ and function $h_\epsilon^c(\boldsymbol{u})$ by $\boldsymbol{u}^c$. We then have $\boldsymbol{u} = h_\epsilon^r(\boldsymbol{u}) + h_\epsilon^c(\boldsymbol{u})$ that is to say, $\boldsymbol{u} = \boldsymbol{u}^r + \boldsymbol{u}^c$. Let $[H_\epsilon^r]$ be the $(m \times m)$ matrix relative to the finite element discretization of the projection operator $h_\epsilon^r$ defined by Eq. (5). Therefore, the finite element discretization $\mathbb{U}$ of $\boldsymbol{u}$ can be written as $\mathbb{U} = \mathbb{U}^r + \mathbb{U}^c$, in which $\mathbb{U}^r = [H_\epsilon^r] \, \mathbb{U}$ and $\mathbb{U}^c = [H_\epsilon^c] \, \mathbb{U} = \mathbb{U} - \mathbb{U}^r$ which shows that $[H_\epsilon^c] = [I_m] - [H_\epsilon^r]$. Then, the $(m \times m)$ reduced mass matrix $[\mathbb{M}^r]$ is constructed such that $[\mathbb{M}^r] = [\mathbb{M}][H_\epsilon^r] = [H_\epsilon^r]^T[\mathbb{M}] = [H_\epsilon^r]^T[\mathbb{M}][H_\epsilon^r]$ and where the $(m \times m)$ complementary mass matrix $[\mathbb{M}^c]$ is constructed such that $[\mathbb{M}^c] = [\mathbb{M}] - [\mathbb{M}^r]$.

## 4. GLOBAL AND LOCAL ELASTIC MODES

Two methods are proposed to calculate the global and the local elastic modes that will be used to reduce the matrix equation.

### 4.1 Direct method

In such a method, the global and the local elastic modes are directly calculated using the decomposition of the mass matrix $[\mathbb{M}]$. The global elastic modes $\boldsymbol{\phi}^g$ in $\mathbb{R}^m$ are solution of the generalized eigenvalue problem

$$[\mathbb{K}]\boldsymbol{\phi}^g = \lambda^g [\mathbb{M}^r] \boldsymbol{\phi}^g \, . \tag{7}$$

This generalized eigenvalue problem admits an increasing sequence of $3n_J$ positive eigenvalues $0 < \lambda^g_1 \leq \ldots \leq \lambda^g_{3n_J}$, associated with the finite family of algebraically independent functions $\{\boldsymbol{\phi}^g_1, \ldots, \boldsymbol{\phi}^g_{3n_J}\}$. The family $\{\boldsymbol{\phi}^g_1, \ldots, \boldsymbol{\phi}^g_{3n_J}\}$ is defined as the family of the global elastic modes although these functions are not, in general, some elements of the family of the elastic modes, but represent some approximations of such elements. The local elastic modes $\boldsymbol{\phi}^\ell$ in $\mathbb{R}^m$ are solution of the generalized eigenvalue problem

$$[\mathbb{K}]\boldsymbol{\phi}^\ell = \lambda^\ell [\mathbb{M}^c]\boldsymbol{\phi}^\ell \,. \tag{8}$$

This generalized eigenvalue problem admits an increasing sequence of positive eigenvalues $0 < \lambda^\ell_1 \leq \ldots \leq \lambda^\ell_{m-3n_J}$, associated with the infinite family of functions $\{\boldsymbol{\phi}^\ell_1, \ldots, \boldsymbol{\phi}^\ell_{m-3n_J}\}$. The family $\{\boldsymbol{\phi}^\ell_1, \ldots, \boldsymbol{\phi}^\ell_{m-3n_J}\}$ is defined as the family of the local elastic modes although these functions are not, in general, some elements of the family of the elastic modes, but represent some approximations of such elements. Matrices $[\mathbb{M}^r]$ and $[\mathbb{M}^c]$ are symmetric and positive but are not positive definite (positive semi-definite matrices). The number of zero eigenvalues for matrices $[\mathbb{M}^r]$ and $[\mathbb{M}^c]$ are respectively $n_J$ and $m - 3n_J$.

### 4.2 Double projection method

This method is less intrusive with respect to the commercial software and less time-consuming than the direct method. The solutions of the generalized eigenvalue problems defined by Eqs. (7) and (8) are then written, for $n$ sufficiently large, as

$$\boldsymbol{\phi}^g = [\Phi]\,\widetilde{\boldsymbol{\phi}}^g \quad , \quad \boldsymbol{\phi}^\ell = [\Phi]\,\widetilde{\boldsymbol{\phi}}^\ell \,. \tag{9}$$

The generalized global elastic modes are the solutions of the generalized eigenvalue problem

$$[\widetilde{K}]\,\widetilde{\boldsymbol{\phi}}^g = \lambda^g\,[\widetilde{M^r}]\,\widetilde{\boldsymbol{\phi}}^g \,, \tag{10}$$

in which $[\widetilde{M^r}] = [\Phi^r_\epsilon]^T\,[\mathbb{M}]\,[\Phi^r_\epsilon]$ and $[\widetilde{K}] = [\Phi]^T\,[\mathbb{K}]\,[\Phi]$, and where the $(m \times n)$ real matrix $[\Phi^r_\epsilon]$ is such that $[\Phi^r_\epsilon] = [H^r_\epsilon]\,[\Phi]$. The generalized local elastic modes are the solutions of the generalized eigenvalue problem

$$[\widetilde{K}]\,\widetilde{\boldsymbol{\phi}}^\ell = \lambda^\ell [\widetilde{M^c}]\,\widetilde{\boldsymbol{\phi}}^\ell \,, \tag{11}$$

in which $[\widetilde{M^c}] = [\Phi^c_\epsilon]^T\,[\mathbb{M}]\,[\Phi^c_\epsilon]$ and where the $(m \times n)$ real matrix $[\Phi^c_\epsilon]$ is such that $[\Phi^c_\epsilon] = [H^c_\epsilon]\,[\Phi] = [\Phi] - [\Phi^r_\epsilon]$.

## 5. MEAN REDUCED MATRIX MODEL

It is proven in [2] that the family $\{\boldsymbol{\phi}^g_1, \ldots, \boldsymbol{\phi}^g_{3n_J}, \boldsymbol{\phi}^\ell_1, \ldots, \boldsymbol{\phi}^\ell_{m-3n_J}\}$ is a basis of $\mathbb{R}^m$. The mean reduced matrix model is obtained using the projection of $\mathbb{U}(\omega)$ on the subspace of $\mathbb{C}^m$ spanned by the family $\{\boldsymbol{\phi}^g_1, \ldots, \boldsymbol{\phi}^g_{n_g}, \boldsymbol{\phi}^\ell_1, \ldots, \boldsymbol{\phi}^\ell_{n_\ell}\}$ of real vectors associated with the $n_g$ first global elastic modes such that $n_g \leq 3n_J \leq m$ and with the $n_\ell$ first local elastic modes such that $n_\ell \leq m$. It should be noted that, if the double projection method is used, then we must have $n_g \leq n$, $n_\ell \leq n$ and $n_t \leq n$ in which $n_t = n_g + n_\ell$. Then, the approximation $\mathbb{U}_{n_g,n_\ell}(\omega)$ of $\mathbb{U}(\omega)$ at order $(n_g, n_\ell)$ is written as

$$\mathbb{U}_{n_g,n_\ell}(\omega) = \sum_{\alpha=1}^{n_g} q^g_\alpha(\omega)\,\boldsymbol{\phi}^g_\alpha + \sum_{\beta=1}^{n_\ell} q^\ell_\beta(\omega)\,\boldsymbol{\phi}^\ell_\beta \,. \tag{12}$$

Let $\boldsymbol{q}(\omega) = (\boldsymbol{q}^g(\omega)\,, \boldsymbol{q}^\ell(\omega))$ be the vector in $\mathbb{C}^{n_t}$ of all the generalized coordinates such that $\boldsymbol{q}^g(\omega) = (q_1^g(\omega), \ldots, q_{n_g}^g(\omega))$ and $\boldsymbol{q}^\ell(\omega) = (q_1^\ell(\omega), \ldots, q_{n_\ell}^\ell(\omega))$. Consequently, vector $\boldsymbol{q}(\omega)$ is solution of the following mean reduced matrix equation such that

$$(-\omega^2[M] + i\omega[D] + [K])\,\boldsymbol{q}(\omega) = \boldsymbol{\mathcal{F}}(\omega)\,, \tag{13}$$

where $[M]$, $[D]$ and $[K]$ are the $(n_t \times n_t)$ mean generalized mass, damping and stiffness matrices defined by blocks as

$$[M] = \begin{bmatrix} M^{gg} & M^{g\ell} \\ (M^{g\ell})^T & M^{\ell\ell} \end{bmatrix}, [D] = \begin{bmatrix} D^{gg} & D^{g\ell} \\ (D^{g\ell})^T & D^{\ell\ell} \end{bmatrix}, [K] = \begin{bmatrix} K^{gg} & K^{g\ell} \\ (K^{g\ell})^T & K^{\ell\ell} \end{bmatrix}. \tag{14}$$

Let $A$ (or $\mathbb{A}$) be denoting $M$, $D$ or $K$ (or $\mathbb{M}$, $\mathbb{D}$ or $\mathbb{K}$). Therefore, the block matrices are defined by

$$[A]^{gg}_{\alpha\beta} = (\boldsymbol{\phi}^g_\alpha)^T[\mathbb{A}]\boldsymbol{\phi}^g_\beta\,,\ [A]^{g\ell}_{\alpha\beta} = (\boldsymbol{\phi}^g_\alpha)^T[\mathbb{A}]\boldsymbol{\phi}^\ell_\beta\,,\ [A]^{\ell\ell}_{\alpha\beta} = (\boldsymbol{\phi}^\ell_\alpha)^T[\mathbb{A}]\boldsymbol{\phi}^\ell_\beta\,. \tag{15}$$

The matrices $[K]^{gg}$ and $[K]^{\ell\ell}$ are diagonal. The generalized force is a vector in $\mathbb{C}^{n_t}$ which is written as $\boldsymbol{\mathcal{F}}(\omega) = (\boldsymbol{\mathcal{F}}^g(\omega)\,, \boldsymbol{\mathcal{F}}^\ell(\omega))$ in which

$$\boldsymbol{\mathcal{F}}^g_\alpha(\omega) = (\boldsymbol{\phi}^g_\alpha)^T\,\mathbb{F}(\omega) \quad , \quad \boldsymbol{\mathcal{F}}^\ell_\alpha(\omega) = (\boldsymbol{\phi}^\ell_\alpha)^T\,\mathbb{F}(\omega)\,. \tag{16}$$

Then, for all $\omega$ fixed in $\mathcal{B}$, the generalized coordinates are calculated by inverting Eq. (13) and the response $\mathbb{U}_{n_g,n_\ell}(\omega)$ is calculated using Eq. (12).

## 6. PROBABILISTIC MODEL OF UNCERTAINTIES FOR THE LOCAL ELASTIC MODES

In the low-frequency range, the first global elastic modes are not really sensitive to uncertainties introduced in the computational model. Nevertheless, we have assumed that the structure under consideration had also local elastic modes in the same low-frequency band. It is well known that the modal density of such local modes increases rapidly with the frequency and that, in addition, the local modes are sensitive both to the system parameters uncertainties and to the model errors which induce model uncertainties. In order to improve the predictability of the computational model, the nonparametric probabilistic approach (see [12]) is used to take into account uncertainties for the local-elastic-modes contribution.

### 6.1 Random reduced matrix model

The nonparametric probabilistic approach consists in replacing the matrices of the reduced mean matrix model by random matrices for which the probability distributions are constructed by using the maximum entropy principle with the constraints defined by the available information. Then, the random generalized mass, damping and stiffness matrices are written as

$$[\boldsymbol{M}] = \begin{bmatrix} M^{gg} & M^{g\ell} \\ (M^{g\ell})^T & \boldsymbol{M}^{\ell\ell} \end{bmatrix}, [\boldsymbol{D}] = \begin{bmatrix} D^{gg} & D^{g\ell} \\ (D^{g\ell})^T & \boldsymbol{D}^{\ell\ell} \end{bmatrix}, [\boldsymbol{K}] = \begin{bmatrix} K^{gg} & K^{g\ell} \\ (K^{g\ell})^T & \boldsymbol{K}^{\ell\ell} \end{bmatrix}, \tag{17}$$

in which the random matrices $[\boldsymbol{M}^{\ell\ell}]$, $[\boldsymbol{D}^{\ell\ell}]$ and $[\boldsymbol{K}^{\ell\ell}]$ are with values in the set of all the positive-definite symmetric $(n_\ell \times n_\ell)$ real matrices, for which their mean values are such that $E\{[\boldsymbol{M}^{\ell\ell}]\} = [M^{\ell\ell}]$, $E\{[\boldsymbol{D}^{\ell\ell}]\} = [D^{\ell\ell}]$ and $E\{[\boldsymbol{K}^{\ell\ell}]\} = [K^{\ell\ell}]$, and finally, verify the following inequalities $E\{\|[\boldsymbol{M}^\ell]^{-1}\|_F^2\} < +\infty$ , $E\{\|[\boldsymbol{D}^{\ell\ell}]^{-1}\|_F^2\} < +\infty$ and $E\{\|[\boldsymbol{K}^{\ell\ell}]^{-1}\|_F^2\} < +\infty$ which assure that there exists a second-order random solution to the stochastic reduced-order equation. The probability distribution of each random matrix $[\boldsymbol{M}^{\ell\ell}]$, $[\boldsymbol{D}^{\ell\ell}]$ or $[\boldsymbol{K}^{\ell\ell}]$ depend on the mean value $[M^{\ell\ell}]$, $[D^{\ell\ell}]$ or $[K^{\ell\ell}]$ and on a dispersion parameter $\delta_M$, $\delta_D$ or $\delta_K$ defined by

$$\delta_A^2 = \frac{E\{\|[\boldsymbol{A}^{\ell\ell}] - [A^{\ell\ell}]\|_F^2\}}{\|[A^{\ell\ell}]\|_F^2}\,, \tag{18}$$

in which $A$ (or $\boldsymbol{A}$) is $M$, $D$ or $K$ (or, $\boldsymbol{M}$, $\boldsymbol{D}$ or $\boldsymbol{K}$). The dispersion parameters allow the level of uncertainties to be controlled.

### 6.2 Random frequency responses

The random response $\boldsymbol{U}_{n_g,n_l}(\omega)$ is then written as

$$\boldsymbol{U}_{n_g,n_\ell}(\omega) = \sum_{\alpha=1}^{n_g} Q_\alpha^g(\omega)\,\boldsymbol{\phi}_\alpha^g + \sum_{\beta=1}^{n_\ell} Q_\beta^\ell(\omega)\,\boldsymbol{\phi}_\beta^\ell\,, \tag{19}$$

in which the random vector $\boldsymbol{Q}(\omega) = (\boldsymbol{Q}^g(\omega)\,,\boldsymbol{Q}^\ell(\omega))$ with valued in $\mathbb{C}^{n_t}$ of all the generalized coordinates is such that $\boldsymbol{Q}^g(\omega) = (Q_1^g(\omega),\ldots,Q_{n_g}^g(\omega))$ and $\boldsymbol{Q}^\ell(\omega) = (Q_1^\ell(\omega),\ldots,Q_{n_\ell}^\ell(\omega))$. Consequently, vector $\boldsymbol{Q}(\omega)$ is solution of the following stochastic reduced matrix equation such that

$$(-\omega^2[\boldsymbol{M}] + i\omega[\boldsymbol{D}] + [\boldsymbol{K}])\,\boldsymbol{Q}(\omega) = \boldsymbol{\mathcal{F}}(\omega)\,. \tag{20}$$

This equation is solved using the Monte Carlo simulation method.

## 7. APPLICATION AND VALIDATION

### 7.1 Mean Finite Element Model

The dynamical system is made up of 12 flexible panels and of a stiff structure (see Fig. 1). Each

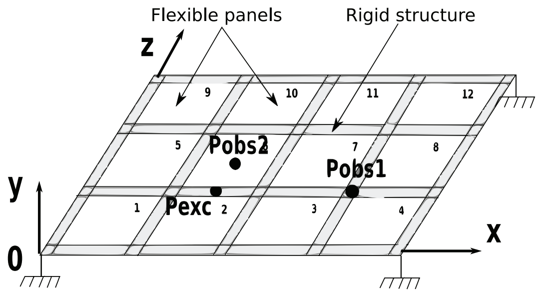


Figure 1. Geometry of the dynamical system.

panel is a rectangular, homogeneous, isotropic, thin plate with constant thickness $0.002\ m$, width $4.0\ m$, length $4.0\ m$, mass density $7,800\ kg/m^3$, Poisson ratio $0.29$. The Young modulus of the 12 panels are respectively, $1.31\times10^{12}$, $1.47\times10^{12}$, $1.54\times10^{12}$, $1.74\times10^{12}$, $1.47\times10^{12}$, $1.71\times10^{12}$, $0.49\times10^{12}$, $1.09\times10^{12}$, $1.74\times10^{12}$, $1.34\times10^{12}$, $1.68\times10^{12}$ and $0.61\times10^{12}\ N/m^2$. The rigid structure is composed of rectangular, homogeneous, isotropic, thin plates with a constant thickness $0.017\ m$, width $2.0\ m$, mass density $9,800\ kg/m^3$, Poisson ratio $0.29$, Young's modulus $2.1 \times 10^{12}\ N/m^2$. The coordinate system $(Oxyx)$ is shown in Fig. 1. The four nodes corresponding to the four corners $(0, 0, 0)$, $(26, 0, 0)$, $(26, 0, 20)$ and $(0, 0, 20)$ are fixed. The frequency band of analysis is $\mathcal{B} = 2\pi\times]0\,,11]\ rad/s$. The finite element model is made up of $64$ Kirchhoff plate elements for each panel and $456$ Kirchhoff plate elements for the rigid structure. The structure has $m = 13,014$ DOF.

### 7.2 Modal analysis, global and local elastic modes

In a first step, the elastic modes are calculated with the finite element model defined by Eq. (2). There are $86$ eigenfrequencies in the frequency band of analysis $\mathcal{B}$ and $n = 120$ eigenfrequencies in the band $]0\,,13.2]\ Hz$. The first elastic mode $\boldsymbol{\phi}_1$ and the second elastic mode $\boldsymbol{\phi}_2$ are displayed in Fig. 2 which shows that $\boldsymbol{\phi}_1$ is a local elastic mode while $\boldsymbol{\phi}_2$ is a global elastic mode with an important local displacement. In a second step, the global and local elastic modes are

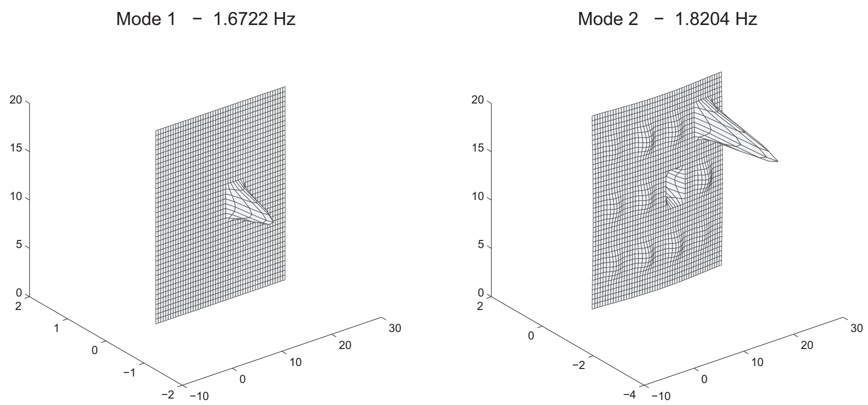


Figure 2. First elastic (left) and second elastic mode (right).

constructed using the double projection method. The matrix $[H_\epsilon^r]$ relative to the discretization of the operator $h_\epsilon^r$ is constructed for $\epsilon = 3.0\ m$. In band $]0\,,13.2]\ Hz$, there are $n_g = 8$ global elastic modes and $n_\ell = 112$ local elastic modes. Fig. 3 displays the distribution of the number of eigenfrequencies for the global elastic modes and for the local elastic modes. It can be seen that there are numerous local modes intertwined with the global elastic modes. The two first global elastic modes $\boldsymbol{\phi}_1^g$, $\boldsymbol{\phi}_2^g$ and the two first local elastic modes $\boldsymbol{\phi}_1^l$ and $\boldsymbol{\phi}_2^l$ are shown in Fig. 4.

### 7.3 Frequency responses calculated with the mean model

For all $\omega \in \mathcal{B}$, the structure is subjected to an external point load equal to $1\ N$ applied to the node whose coordinates are $(10, 0, 7)$ located in the stiff part. The mean damping matrix is constructed using a Rayleigh model corresponding to a damping rate $\xi = 0.04$ for the eigenfrequency $f_1 = 1.67\ Hz$ and for the eigenfrequency $f_{120} = 13.2\ Hz$. The response is calculated at two observation points, the point $\text{Pobs}_1$ located in the stiff part at the node whose coordinates are $(19, 0, 7)$ and the point $\text{Pobs}_2$ located in the flexible part at the node whose coordinates are $(10, 0, 10)$ (see Fig. 1). The response is calculated for different projections associated with the different bases: for the initial elastic modes with Eq. (3) ($n = 120$), for global elastic modes with Eq. (12) ($n_g = 8$ and $n_\ell = 0$), for local elastic modes with Eq. (12) ($n_g = 0$ and $n_\ell = 112$) and finally, for global and local elastic modes with Eq. (12) ($n_g = 8$ and $n_\ell = 112$). The

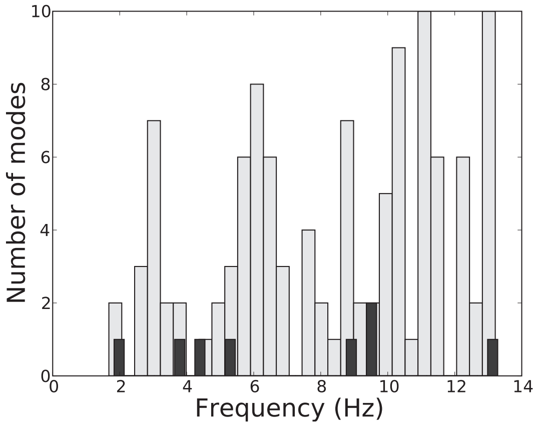


Figure 3: Distribution of the number of eigenfrequencies for the global elastic modes (black histogram) and for the local elastic modes (grey histogram) as a function of the frequency in Hz).

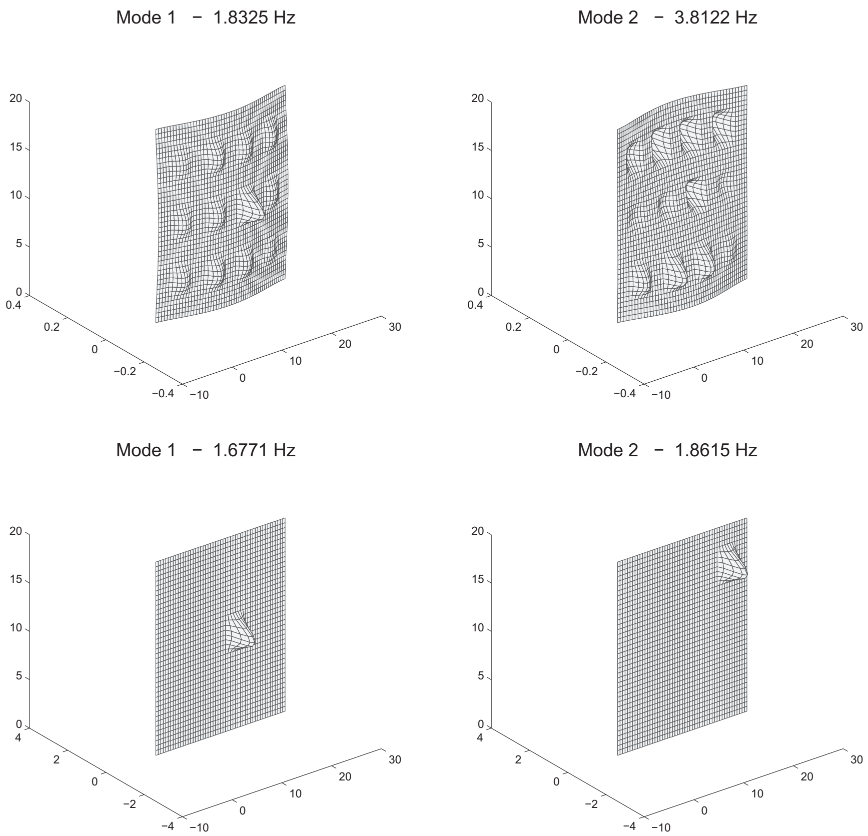


Figure 4. Two first global elastic modes (top) and two first local elastic modes (bottom).

modulus in log scale of the responses are displayed in Fig. 5. It can be seen that the responses calculated using global and local elastic modes are exactly the same that the response calculated using the initial elastic modes. For point $\text{Pobs}_1$ in the stiff part, the contribution of the global elastic modes is preponderant in the very low-frequency band but the contribution of the local elastic modes becomes not negligible in the high part of the low-frequency band. For point

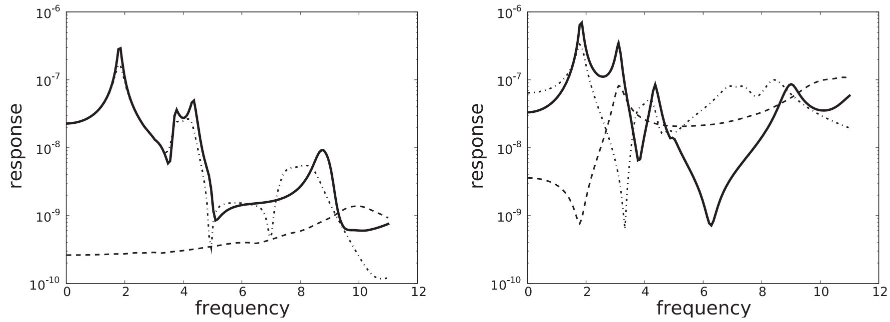


Figure 5: Modulus in log scale of the frequency response for $\mathrm{Pobs}_1$ (left) and $\mathrm{Pobs}_2$ (right). Comparisons between different projection bases: initial elastic modes (solid thick line), global elastic modes only (mixed line), local elastic modes only (dashed line), global and local elastic modes (solid thin line superimposed to the solid thick line).

$\mathrm{Pobs}_2$ in the flexible part, the contribution of the local elastic modes is important except for the first resonance corresponding to the first global elastic mode (because the flexible plates follow the stiff part in its displacement).

### 7.4 Random frequency responses calculated with the stochastic model

The nonparametric probabilistic approach is used for the contribution of the local elastic modes as explained in Section 6.. The dispersion parameters are chosen as $\delta_M = 0.1$, $\delta_D = 0.1$ and $\delta_K = 0.1$. The Monte Carlo simulation method is carried out with $200$ simulations. The confidence regions corresponding to a probability level $P_c = 0.99$ are presented in Fig. 6. As it

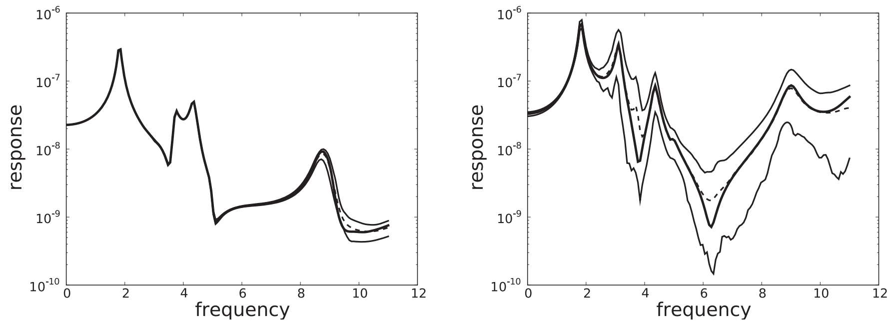


Figure 6: Random frequency response functions for $\mathrm{Pobs}_1$ (left) and $\mathrm{Pobs}_1$ (right). Confidence region (lower and upper lines), mean response (middle line), deterministic response calculated with the initial elastic modes (solid thick line).

can be seen in these two figures, the sensitivity to uncertainties of local elastic modes is higher than for the global elastic modes. For observation $\mathrm{Pobs}_1$ which is located in the stiff part of the structure and then, which is mainly driven by the global elastic modes, the main resonances are robust with respect to uncertainties. For observation $\mathrm{Pobs}_2$ which is located in the flexible part of the structure and then, which is mainly driven by the local elastic modes, the responses are not robust with respect to uncertainties.

## 8. CONCLUSIONS

A general method has been developed and validated to construct a stochastic reduced-order model in low-frequency dynamics in presence of numerous local elastic modes. The projection basis is made up of two families of vector bases: the global elastic modes and the local elastic modes which are separately computed. The effects of uncertainties on the local modes are taken into account with the nonparametric probabilistic approach. The double projection method proposed is not really intrusive and can easily be implemented in commercial software.

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