Generalized probabilistic approach of uncertainties in computational dynamics

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GENERALIZED PROBABILISTIC APPROACH OF UNCERTAINTIES IN COMPUTATIONAL DYNAMICS

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ABSTRACT

A new generalized probabilistic approach of uncertainties is proposed for computational model in structural linear dynamics and can be extended without difficulty to computational linear vibroacoustics and to computational nonlinear structural dynamics. This method allows the prior probability model of each type of uncertainties (model-parameter uncertainties and modeling errors) to be separately constructed and identified. The modeling errors are not taken into account with the usual output-prediction-error method but with the nonparametric probabilistic approach of modeling errors recently introduced and based on the use of the random matrix theory. The theory, an identification procedure and a numerical validation are presented.

1 SETTING THE PROBLEM AND FUNDAMENTAL METHODOLOGIES

This paper is devoted to the presentation of a new generalized probabilistic approach which allows an independent modeling of both model-parameter uncertainties and modeling errors to be performed for computational dynamical models in structural dynamics, structural acoustics, vibration or vibroacoustics, for linear or nonlinear problems. This means that we consider a computational dynamical model of a *real system* as a selected class of mathematical models with an input and an output and depending on a *model parameter* and on a *design parameter*. Once the class of computational models has been selected, two types of uncertainties can be identified. The first type is related to the *model-parameter uncertainties* which mean that the model parameter is not exactly known and is uncertain. The second type is due to *modeling errors* which are introduced by the mechanical-mathematical process allowing the computational dynamical model to be constructed. The modeling errors introduce *model uncertainties* in the response predictions constructed with any model belonging to the selected class. This second type of uncertainties will be called *model uncertainties*. It should be noted that the robust design optimization consists in finding the optimal value of the design parameter which maximizes a cost function related to the model output predicted with the computational dynamical model for which uncertainties are modeled.

Various methods exist for assessing uncertainties in a model. These methods are either probabilistic or deterministic (see for instance [1]). The construction of the probabilistic model of uncertainties is a fundamental problem which must carefully carried out in order to improve the quality of the predictions of the computational model but also to solve robust design optimization in the best conditions (see for instance [2], [3], [4]).

1.1 Prior and posterior probabilistic models of uncertainties

A model chosen in the selected class will be called the *mean model*. For given nominal values of the model parameter, the mean model is often called the *nominal model*. The uncertainties are

then related to the mean model. For the construction of the probabilistic model of uncertainties, two main cases have to be considered. For the first one, no experiment of the real system is assumed to be available in order to update the model and to identify the prior probabilistic model of uncertainties. In this case, the prior probabilistic model must have the capability to take into account model uncertainties because there are no possibilities to improve the prior probabilistic model using experiments. In opposite, for the second one, some experiments are assumed to be available to update the mean model [5], [6], to identify the prior probabilistic model of uncertainties [7], [6], [8], [9] and to construct a posterior probabilistic model (see for instance [10], [7], [11]). Today, the first case is a frequent situation encountered for the design and the optimization of complex mechanical systems and has to be carefully analyzed. This means that no experiment is available and that there is neither possibility to update the mean model with experiments nor to identify the prior probability model of uncertainties using mathematical statistics such as the maximum likelihood method [12],[7] or the Bayesian method [13], [14], [15]. In these two cases, the prior probabilistic approach of uncertainties which has to be developed must have the capability to represent modeling errors for the family of models spanned when the design parameter runs through all its admissible set.

1.2 Model-parameter uncertainties

Concerning model-parameter uncertainties, the main method is based on the used of the *parametric probabilistic approach* which has extensively been developed in the last three decades, which is still in development and which allows the uncertain model parameters of the mean model to be taken into account through the introduction of a prior probability model of these model parameters (see for instance [16], [17], [18], [19], [20], [21]). Such an approach consists in modeling uncertain model parameters by a vector-valued random variable (this random vector can correspond to the finite approximation of a random field). The prior stochastic modeling of model-parameter uncertainties then consists (1) either in constructing an adapted representation based on a polynomial chaos decomposition (see for instance [16], [22], [23], [24], [25], [26], [18], [27], [8], [28]) (2) or in directly constructing the probability distribution of the random quantity using the available information and the Maximum Entropy Principle introduced by Jaynes [29] in the context of Information Theory developed by Shannon [30] (see for instance [31], [11] and for recent developments concerning the construction of probability distributions in high dimension using the maximum entropy principle and stochastic analysis [32]).

1.3 Modeling errors inducing model uncertainties

Concerning model uncertainties induced by modeling errors, it is well understood that prior and posterior probabilistic models of the uncertain model parameter are not sufficient and do not have the capability to take into account model uncertainties as explained in the context of computational mechanics (see for instance [10], [33], [34], [35]). Two main methods can be used to take into account model uncertainties (modeling errors).

1.3.1 Output-prediction-error probabilistic approach

The first one consists in introducing a probabilistic model of the *output prediction error* which is the difference between the real system output and the model output (note that such a probabilistic approach of model uncertainties is implemented at the output level of the mean model and not implemented at the operator level of the model). When experiments are available, the observed prediction error is then the difference between the measured real system output and the model output. A posterior probabilistic model can be constructed using, for instance, the Bayesian approach (see for instance [10], [13], [14], [15]). With such a method, it is usually assumed that

the measurement noise is negligible compared with the prediction error. The advantage of such an approach is its simplicity and its efficiency when simultaneously, a lot of experiments are available and the design parameters are fixed. However, such an approach is not really adapted if the design parameters are not fixed but have to run through an admissible set of values in the context of robust design optimization. Another reason is necessity that modeling errors be taken into account at the operator level of the mean model, for instance to take into account the mass and stiffness operators in order to analyze the generalized eigenvalue problem related to a dynamical system. In this case, the output-prediction-error method is not really adapted to take into account modeling errors.

1.3.2 Nonparametric probabilistic approach

The second one is based on the *nonparametric probabilistic approach* of model uncertainties (modeling errors) which has recently been proposed in [33] as another possible way to the use of the output-prediction-error method in order to take into account modeling errors. The nonparametric probabilistic approach consists in directly constructing the stochastic modeling of the operators of the mean computational model instead of introducing a probabilistic model of the prediction errors. The random matrix theory (see for instance [36] and [37]) is used to construct the prior probability distribution of the random matrices modeling the uncertain operators of the mean model are constructed using again the Maximum Entropy Principle for which the constraints are defined by the available information [33], [38], [39], [34], [35]. Since paper [33] were published, many works have been performed in order to validate the nonparametric probabilistic approach of model uncertainties with experimental results (see for instance [40], [35], [41], [42], [43], [9], [44],[45]) and to extend the theory, in particular, with the development of random impedance operators [46], [47], with the introduction of a new set of positive-definite random matrices yielding a more flexible description of the dispersion levels [48], with the analysis of the medium-frequency range for vibration analysis [49] and for complex vibroacoustic systems [43], [44], with the analysis of nonlinear dynamical systems for local nonlinear elements [50], [51] and for distributed nonlinear elements or nonlinear geometrical effects [52].

1.4 Generalized probabilistic approach of uncertainties

It should be noted that the major difference between the two approaches decribed in Sections 1.3.1 and 1.3.2 is due to the fact that the statistical fluctuations of responses generated by the output-prediction-error method are independent of the state variable of the dynamical system while the statistical fluctuations of responses generated by the nonparametric probabilistic approach depend on it. Approach defined in Sections 1.3.2 has been proposed to avoid the difficulty induced in the approach described in Sections 1.3.1. As it has been proven in [33], [34] and [35], the nonparametric probabilistic approach has the capability to simultaneously take into account both model-parameter uncertainties and modeling errors. With such an approach, for each random matrix of the stochastic reduced computational model such as the generalized mass matrix for instance, the level of uncertainties induced by both model-parameter uncertainties and modeling errors are controlled by only one dispersion parameter. Consequently, with such an approach, the level of uncertainties for the model parameter cannot be separated from the level of uncertainties induced by modeling errors. In addition, in the nonparametric probabilistic approach and by construction, the mean value of each random matrix is chosen as the matrix of the reduced mean computational model associated with the nominal value. Clearly, this point could be improved in choosing the mean value of each random matrix as the mean matrix of the stochastic computational model induced only by the parametric probabilistic approach of model-parameter uncertainties. Such a method should required again to separate the probabilistic model of model-parameter uncertainties and of modeling errors. This is the reason why, we propose a *generalized probabilistic approach of uncertainties* allowing both the model-parameter uncertainties and modeling errors to be simultaneously taken

into account but in a separate way. This new method allows the prior probability model of each type of uncertainties (model-parameter uncertainties and modeling errors) to be separately constructed and to be separately identified with experiments if experiments are available. In addition, this method will allow the biais of the mean values of the stochastic responses to be decreased.

1.5 Objectives of the paper

We present the theory, an identification procedure of the dispersion parameters controlling the level of each type of uncertainties when experimental data are available and a numerical validation. In order to simply explain this new approach, we have chosen to present the developments for the most simple context corresponding to structural linear dynamics. The extension to other cases is straightforward in particular (1) for computational vibroacoustics in low- and medium-frequency ranges with modeling errors in the structure, in the acoustic cavity, for the vibroacoustic coupling operator and for insulation schemes (see [43], [44]) and (2) for computational nonlinear structural dynamics with local or distributed nonlinear elements [50], [51] or with nonlinear geometrical effects in the general context of three-dimensional nonlinear electrodynamics [52].

2 FORMULATION OF A PRIOR GENERALIZED PROBABILISTIC APPROACH OF UNCERTAINTIES

2.1 Reduced mean computational model

We consider the reduced mean computational model of a linear dynamical system

$$\boldsymbol{y}(t) = [\,\phi(\boldsymbol{x})]\,\boldsymbol{q}(t) \quad ,$$

$$[M(\boldsymbol{x})]\,\ddot{\boldsymbol{q}}(t) + [D(\boldsymbol{x})]\,\dot{\boldsymbol{q}}(t) + [K(\boldsymbol{x})]\,\boldsymbol{q}(t) = \boldsymbol{f}(t;\boldsymbol{x}) \quad , \tag{1}$$

in which $\boldsymbol{y}(t) = (y_1(t),\, \ldots,\, y_m(t))$ is the displacement vector at time t (displacements and/or rotations), where $\boldsymbol{q}(t) = (q_1(t), \ldots, q_n(t)) \in \mathbb{R}^n$ is the vector of the generalized coordinates, in which the matrix of the normal modes $[\,\phi(\boldsymbol{x})]$, the reduced mass matrix $[M(\boldsymbol{x})]$, damping matrix $[D(\boldsymbol{x})]$, stiffness matrix $[K(\boldsymbol{x})]$ and generalized external force vector $\boldsymbol{f}(t;\boldsymbol{x})$ depend on an uncertain vector-valued parameter $\boldsymbol{x} = (x_1, \ldots, x_{n_p})$ belonging to an admissible subset $\mathcal{C}_{\text{par}}$ of $\mathbb{R}^{n_p}$.

2.2 Construction of the prior generalized probabilistic approach of uncertainties.

Let $(\Theta, \mathcal{T}, \mathcal{P})$ and $(\Theta', \mathcal{T}', \mathcal{P}')$ be two probability spaces. The first one will be devoted to the probabilistic model of model-parameter uncertainties using the parametric probabilistic approach and the second one to the probabilistic model of model uncertainties (modeling errors) using the nonparametric probabilistic approach. Let $\boldsymbol{X} = \{\theta \mapsto \boldsymbol{X}(\theta)\}$ be a random variable defined on $(\Theta, \mathcal{T}, \mathcal{P})$ and let $[\boldsymbol{G}] = \{\theta' \mapsto [\boldsymbol{G}(\theta')]\}$ be another random variable defined on $(\Theta', \mathcal{T}', \mathcal{P}')$. The two random variables $\boldsymbol{X}$ and $[\boldsymbol{G}]$ are then independent and their mathematical expectations are such that

$$E\{\boldsymbol{X}\} = \int_{\Theta} \boldsymbol{X}(\theta)\, d\mathcal{P}(\theta) \quad , \quad E\{[\boldsymbol{G}]\} = \int_{\Theta'} [\boldsymbol{G}(\theta')]\, d\mathcal{P}'(\theta') \quad . \tag{2}$$

If $\boldsymbol{Q} = \boldsymbol{h}(\boldsymbol{X}, [\boldsymbol{G}])$ is a random variable defined by a given deterministic transformation $\boldsymbol{h}$ of independent random variables $\boldsymbol{X}$ and $[\boldsymbol{G}]$, then the mathematical expectation of $\boldsymbol{Q}$ is

$$E\{\boldsymbol{Q}\} = \int_{\Theta} \int_{\Theta'} \boldsymbol{h}(\boldsymbol{X}(\theta), [\boldsymbol{G}(\theta')])\, d\mathcal{P}'(\theta')\, d\mathcal{P}(\theta) \quad . \tag{3}$$

(1) The first step of the generalized probabilistic approach of uncertainties consists in constructing the probabilistic model of model-parameter uncertainties for which parameter $\boldsymbol{x}$ is modeled by a random variable $\boldsymbol{X}$ defined on probability space $(\Theta, \mathcal{T}, \mathcal{P})$. Consequently, the normal modes matrix and the reduced matrices in Eq. (1) become random matrices $[\,\phi(\boldsymbol{X})]$ and $[M(\boldsymbol{X})]$, $[D(\boldsymbol{X})]$, $[K(\boldsymbol{X})]$ and, for all fixed t, the generalized external force $\boldsymbol{f}(t;\boldsymbol{x})$ becomes a random vector $\boldsymbol{f}(t;\boldsymbol{X})$. The mean values of the random reduced matrices are denoted by $[\underline{M}]$, $[\underline{D}]$ and $[\underline{K}]$. We then have

$$E\{[M(\boldsymbol{X})]\} = [\underline{M}] \quad , \quad E\{[D(\boldsymbol{X})]\} = [\underline{D}] \quad , \quad E\{[K(\boldsymbol{X})]\} = [\underline{K}] \quad . \tag{4}$$

It should be noted that the mean matrices $[\underline{M}]$, $[\underline{D}]$ and $[\underline{K}]$ are different from the matrices $[M(\underline{\boldsymbol{x}})]$, $[D(\underline{\boldsymbol{x}})]$ and $[K(\underline{\boldsymbol{x}})]$ of the nominal mean computational model corresponding to $\boldsymbol{x} = \underline{\boldsymbol{x}}$.

(2) The second step of the generalized probabilistic approach of uncertainties consists in constructing the probabilistic model of model uncertainties (modeling errors) in using the nonparametric probabilistic approach [33]. Therefore, for all $\boldsymbol{x}$ fixed in $\mathcal{C}_{\text{par}}$, the matrices $[M(\boldsymbol{x})]$, $[D(\boldsymbol{x})]$ and $[K(\boldsymbol{x})]$ are replaced by independent random matrices

$$[\boldsymbol{M}(\boldsymbol{x})] = \{\theta' \mapsto [\boldsymbol{M}(\theta';\boldsymbol{x})]\}\,,\, [\boldsymbol{D}(\boldsymbol{x})] = \{\theta' \mapsto [\boldsymbol{D}(\theta';\boldsymbol{x})]\}\,,\, [\boldsymbol{K}(\boldsymbol{x})] = \{\theta' \mapsto [\boldsymbol{K}(\theta';\boldsymbol{x})]\} \tag{5}$$

defined on probability space $(\Theta', \mathcal{T}', \mathcal{P}')$ and belonging to the set of random matrices introduced in [33] and [35], or in [48]. In order to simplify the presentation, we will limit the developments to the case for which these random matrices belong to the set SE^+ of random matrices introduced in [35]. The extension to the use of the set introduced in [48] is straightforward. The mathematical expectation of these random matrices must be such that

$$E\{[\boldsymbol{M}(\boldsymbol{x})]\} = [M(\boldsymbol{x})] \quad , \quad E\{[\boldsymbol{D}(\boldsymbol{x})]\} = [D(\boldsymbol{x})] \quad , \quad E\{[\boldsymbol{K}(\boldsymbol{x})]\} = [K(\boldsymbol{x})] \quad . \tag{6}$$

(3) The last step of the construction of the generalized probabilistic approach of uncertainties then consists in replacing in Eq. (5) $\boldsymbol{x}$ by $\boldsymbol{X}$ and in replacing the dependent random matrices $[M(\boldsymbol{X})]$, $[D(\boldsymbol{X})]$ and $[K(\boldsymbol{X})]$ by the dependent random matrices

$$\begin{aligned} [\boldsymbol{M}(\boldsymbol{X})] &= \{(\theta,\theta') \mapsto [\boldsymbol{M}(\theta';\boldsymbol{X}(\theta))]\} \\ [\boldsymbol{D}(\boldsymbol{X})] &= \{(\theta,\theta') \mapsto [\boldsymbol{D}(\theta';\boldsymbol{X}(\theta))]\} \\ [\boldsymbol{K}(\boldsymbol{X})] &= \{(\theta,\theta') \mapsto [\boldsymbol{K}(\theta';\boldsymbol{X}(\theta))]\} \end{aligned} \tag{7}$$

defined on the probability space $(\Theta \times \Theta', \mathcal{T} \otimes \mathcal{T}', \mathcal{P} \otimes \mathcal{P}')$. It can easily be deduced that

$$E\{[\boldsymbol{M}(\boldsymbol{X})]\} = [\underline{M}] \quad , \quad E\{[\boldsymbol{D}(\boldsymbol{X})]\} = [\underline{D}] \quad , \quad E\{[\boldsymbol{K}(\boldsymbol{X})]\} = [\underline{K}] \quad . \tag{8}$$

2.3 Stochastic reduced computation model generated by the prior generalized probabilistic approach of uncertainties.

The generalized probabilistic approach of uncertainties consists in replacing the mean computational model by the following stochastic reduced computational model,

$$\boldsymbol{Y}(t) = [\,\phi(\boldsymbol{X})]\,\boldsymbol{Q}(t) \quad , \tag{9}$$

$$[\boldsymbol{M}(\boldsymbol{X})]\,\ddot{\boldsymbol{Q}}(t) + [\boldsymbol{D}(\boldsymbol{X})]\,\dot{\boldsymbol{Q}}(t) + [\boldsymbol{K}(\boldsymbol{X})]\,\boldsymbol{Q}(t) = \boldsymbol{f}(t;\boldsymbol{X}) \quad , \tag{10}$$

in which for all fixed t, $\boldsymbol{Y}(t) = \{(\theta,\theta') \mapsto \boldsymbol{Y}(\theta,\theta';t)\}$ is an $\mathbb{R}^m$-valued random vector and $\boldsymbol{Q}(t) = \{(\theta,\theta') \mapsto \boldsymbol{Q}(\theta,\theta';t)\}$ is an $\mathbb{R}^n$-valued random vector defined on $(\Theta \times \Theta', \mathcal{T} \otimes \mathcal{T}', \mathcal{P} \otimes \mathcal{P}')$. Thus,

for any realization $\boldsymbol{X}(\theta)$ of random variable $\boldsymbol{X}$ with θ in Θ, and for any realization $[\boldsymbol{M}(\theta';\boldsymbol{x})]$, $[\boldsymbol{D}(\theta';\boldsymbol{x})]$, $[\boldsymbol{K}(\theta';\boldsymbol{x})]$ of independent random matrices $[\boldsymbol{M}(\boldsymbol{x})]$, $[\boldsymbol{D}(\boldsymbol{x})]$, $[\boldsymbol{K}(\boldsymbol{x})]$ for θ' in Θ' and $\boldsymbol{x}$ in $\mathcal{C}_{\text{par}}$, the realization $\boldsymbol{Y}(\theta,\theta';t)$ of the random variable $\boldsymbol{Y}(t)$ and the realization $\boldsymbol{Q}(\theta,\theta';t)$ of the random variable $\boldsymbol{Q}(t)$ verify the deterministic problem

$$\boldsymbol{Y}(\theta,\theta';t) = [\,\phi(\boldsymbol{X}(\theta)]\,\boldsymbol{Q}(\theta,\theta';t) \quad , \tag{11}$$

$$[\boldsymbol{M}(\theta';\boldsymbol{X}(\theta))]\,\ddot{\boldsymbol{Q}}(\theta,\theta';t) + [\boldsymbol{D}(\theta';\boldsymbol{X}(\theta))]\,\dot{\boldsymbol{Q}}(\theta,\theta';t) + [\boldsymbol{K}(\theta';\boldsymbol{X}(\theta))]\,\boldsymbol{Q}(\theta,\theta';t) = \boldsymbol{f}(t;\boldsymbol{X}(\theta))\,. \tag{12}$$

2.4 Construction of the prior probability model of model-parameter uncertainties.

The uncertain model parameter $\boldsymbol{x}$ (whose nominal value is $\underline{\boldsymbol{x}}$) is modeled by a random variable $\boldsymbol{X} = (X_1,\ldots,X_{n_p})$, defined on probability space $(\Theta,\mathcal{T},\mathcal{P})$, with values in $\mathbb{R}^{n_p}$. The unknown probability distribution of $\boldsymbol{X}$ is assumed to be defined by a probability density function $p_{\boldsymbol{X}}$ from $\mathbb{R}^{n_p}$ into $\mathbb{R}^+ = [0\,,+\infty[$ with respect to the Lebesgue measure $d\boldsymbol{x} = dx_1\ldots dx_{n_p}$. Under the assumption that no experimental data are available to construct $p_{\boldsymbol{X}}$, the prior model can be constructed using the maximum entropy principle [29] introduced in the context of Information Theory [30]. For such a construction, the available information has to be defined. Since $\boldsymbol{x}$ belongs to $\mathcal{C}_{\text{par}}$, the support of $p_{\boldsymbol{X}}$ is $\mathcal{C}_{\text{par}}$ and the normalization condition must be verified. Since the nominal value of $\boldsymbol{x}$ is $\underline{\boldsymbol{x}}$ in $\mathcal{C}_{\text{par}} \subset \mathbb{R}^{n_p}$, an additional available information consists in writing that the mean value $E\{\boldsymbol{X}\}$ of $\boldsymbol{X}$ is equal to $\underline{\boldsymbol{x}}$. In general, an additional available information can be deduced from the analysis of the mathematical properties of the solution of the stochastic reduced computational model under construction. Such a construction leads us to know the probability density function $\boldsymbol{x} \mapsto p_{\boldsymbol{X}}(\boldsymbol{x};\boldsymbol{\delta}_{\boldsymbol{X}})$ which depends on a vector-valued parameter $\boldsymbol{\delta}_{\boldsymbol{X}}$ belonging to an admissible subset $\mathcal{C}_{\boldsymbol{X}}$ of $\mathbb{R}^\mu$ with $\mu \geq 1$.

2.5 Construction of the prior probability model of model uncertainties (modeling errors).

Let A be the letter M, D or K relatively to the generalized mass, damping or stiffness matrix. In the nonparametric probabilistic approach of both model-parameter uncertainties and model uncertainties (modeling errors) introduced in [33], the probability model of the statistical fluctuations of the random matrix $[\boldsymbol{A}]$ has been constructed around the deterministic nominal value $[A(\underline{\boldsymbol{x}})]$ of the matrix in the nominal computational model. For the generalized approach of uncertainties which is proposed below, since the model-parameter uncertainties are taken into account by the parametric probabilistic approach in introducing the random matrix $[A(\boldsymbol{X})]$, only the model uncertainties (modeling errors) must be taken into account by the nonparametric probabilistic approach of uncertainties. This means that the probability model of the statistical fluctuations of the random matrix $[\boldsymbol{A}]$ due to the model uncertainties must be constructed around the random matrix $[A(\boldsymbol{X})]$ and not around the deterministic nominal value $[A(\underline{\boldsymbol{x}})]$. Such a construction of the probability model of modeling errors is performed below. Following the methodology of the construction of the nonparametric probabilistic approach (see [33], [38], [35]), for all $\boldsymbol{x}$ in $\mathcal{C}_{\text{par}}$, the construction of the probability model of random matrix $[\boldsymbol{A}(\boldsymbol{x})]$ defined on probability space $(\Theta',\mathcal{T}',\mathcal{P}')$, is based on the available information deduced from the fundamental properties of the dynamical system and from additional properties required in order that a second-order stochastic solution exists for Eq. (10) (see [38], [35]). For all $\boldsymbol{x}$ in $\mathcal{C}_{\text{par}}$, random matrix $[\boldsymbol{A}(\boldsymbol{x})]$ belongs to $\mathbb{M}_n^+(\mathbb{R})$ a.s., its mean value is defined by Eq. (6) and verifies the inequality $E\big\{\|[\boldsymbol{A}(\boldsymbol{x})]^{-1}\|_F^2\big\} < +\infty$ in which $\|A\|_F^2 = \text{tr}\{[A]^T[A]\}$. Since the deterministic matrix $[A(\boldsymbol{x})]$ is in $\mathbb{M}_n^+(\mathbb{R})$, there is an upper triangular matrix $[L_A(\boldsymbol{x})]$ in the set $\mathbb{M}_n(\mathbb{R})$ of all the square $(n\times n)$ real matrices, such that $[A(\boldsymbol{x})] = [L_A(\boldsymbol{x})]^T[L_A(\boldsymbol{x})]$. For all $\boldsymbol{x}$ in $\mathcal{C}_{\text{par}}$, the random matrix $[\boldsymbol{A}(\boldsymbol{x})]$ defined on probability space $(\Theta',\mathcal{T}',\mathcal{P}')$ belongs to the set SE^+ defined in [35] and is then written as $[\boldsymbol{A}(\boldsymbol{x})] = [L_A(\boldsymbol{x})]^T\,[\boldsymbol{G}_A]\,[L_A(\boldsymbol{x})]$ in which the random matrix $[\boldsymbol{G}_A]$ is the random germ which

is defined on probability space $(\Theta', \mathcal{T}', \mathcal{P}')$, which belongs to the set SG^+ defined in [35] and consequently, is a random matrix with values in $\mathbb{M}_n^+(\mathbb{R})$. In addition, the random matrices $[\boldsymbol{G}_M]$, $[\boldsymbol{G}_D]$ and $[\boldsymbol{G}_K]$ are statistically independent. Consequently, the prior probability distributions of random matrices $[\boldsymbol{M}(\boldsymbol{x})]$, $[\boldsymbol{D}(\boldsymbol{x})]$ and $[\boldsymbol{K}(\boldsymbol{x})]$ depend on the free vector-valued dispersion parameter $\boldsymbol{\delta_G} = (\delta_M, \delta_D, \delta_K)$ which is relative to random matrices $[\boldsymbol{G}_M]$, $[\boldsymbol{G}_D]$ and $[\boldsymbol{G}_K]$, which belongs to an admissible subset $\mathcal{C}_{\boldsymbol{G}}$ of $\mathbb{R}^3$ and which is independent of $\boldsymbol{x}$. Then the random matrices $[\boldsymbol{M}(\boldsymbol{X})]$, $[\boldsymbol{D}(\boldsymbol{X})]$ and $[\boldsymbol{K}(\boldsymbol{X})]$ introduced in Eq. (10) are written as $[\boldsymbol{M}(\boldsymbol{X})] = [L_M(\boldsymbol{X})]^T[\boldsymbol{G}_M][L_M(\boldsymbol{X})]$, $[\boldsymbol{D}(\boldsymbol{X})] = [L_D(\boldsymbol{X})]^T[\boldsymbol{G}_D][L_D(\boldsymbol{X})]$ and $[\boldsymbol{K}(\boldsymbol{X})] = [L_K(\boldsymbol{X})]^T[\boldsymbol{G}_K][L_K(\boldsymbol{X})]$.

3 ESTIMATION OF THE DISPERSION PARAMETERS OF THE PRIOR PROBABILITY MODELS OF THE GENERALIZED PROBABILISTIC APPROACH OF UNCERTAINTIES

The formulation introduces (1) a prior probability model of the model-parameter uncertainties depending on the dispersion parameter $\boldsymbol{\delta_X}$ belonging to an admissible subset $\mathcal{C}_{\boldsymbol{X}}$ of $\mathbb{R}^\mu$ and (2) a prior probability model depending on the dispersion parameter $\boldsymbol{\delta_G} = (\delta_M, \delta_D, \delta_K)$ belonging to an admissible subset $\mathcal{C}_{\boldsymbol{G}}$ of $\mathbb{R}^3$. If no experimental data are available, then the dispersions parameters $\boldsymbol{\delta_X}$ and $\boldsymbol{\delta_G}$ must be considered as parameters to perform a sensitivity analysis of the stochastic solution. Such a prior generalized probabilistic approach of uncertainties then allows the robustness of the solution to be analyzed in function of the level of model-parameter uncertainties controlled by $\boldsymbol{\delta_X}$ and of the level of model uncertainties (modeling errors) controlled by $\boldsymbol{\delta_G}$. For the particular case for which a few experimental data exist, we propose a methodology to estimate the dispersion parameters of the prior probability models of uncertainties.

3.1 Estimation of the dispersion parameter of the prior probability model of the uncertain model parameter

The first step of the method proposed consists in estimating the dispersion parameter $\boldsymbol{\delta_X} \in \mathcal{C}_{\boldsymbol{X}} \subset \mathbb{R}^\mu$ of the prior probability model of the uncertain model parameter in considering there is no modeling error and consequently in using the stochastic computational model with $\boldsymbol{\delta_G} = 0$ (no modeling errors). The estimation of $\boldsymbol{\delta_X}$ must then be performed with observations of the systems which are weakly sensitive to modeling errors and for which experimental data are available. There are several possibilities in the choice of such observations satisfying these criteria. Nevertheless, in order to limit the developments, we will propose only one of the possibilities which is in the framework of experimental modal analysis. Note that for a complex dynamical system, the first eigenfrequencies and the associated elastic modes can be experimentally measured. In addition, if the corresponding computational model is sufficiently large, the first eigenfrequencies predicted with the computational model depend on the model-parameter uncertainties but must not depend on modeling errors (it should be noted that if the fundamental eigenfrequency of the mean computational model cannot reasonably be predicted due to the presence of significant model errors, this means that the mean computational model cannot be considered as a correct model and should be be rebuilt). Let us assumed that the first $\widetilde{\mu}$ experimental eigenfrequencies $\nu_1^{\text{ref}}, \ldots, \nu_{\widetilde{\mu}}^{\text{ref}}$ are measured. Let $\lambda_\alpha^{\text{ref}} = (2\pi\nu_\alpha^{\text{ref}})^2$ be the corresponding experimental eigenvalues. Let $\Lambda_1, \ldots, \Lambda_{\widetilde{\mu}}$ be the corresponding eigenvalues of this stochastic computational model. Let $p_{\Lambda_1,\ldots,\Lambda_{\widetilde{\mu}}}(\lambda_1, \ldots, \lambda_{\widetilde{\mu}}; \boldsymbol{\delta_X})$ be the joint probability density function of the random variables $\Lambda_1, \ldots, \Lambda_{\widetilde{\mu}}$. The optimal value $\boldsymbol{\delta}_{\boldsymbol{X}}^{\text{opt}}$ of the dispersion parameter $\boldsymbol{\delta_X}$ can be estimated by maximizing the Neperian logarithm of the likelihood function (maximum likelihood method [12],[7]),

$$\boldsymbol{\delta}_{\boldsymbol{X}}^{\text{opt}} = \arg \max_{\boldsymbol{\delta_X} \in \mathcal{C}_{\boldsymbol{X}}} \{\ln(p_{\Lambda_1,\ldots,\Lambda_{\widetilde{\mu}}}(\lambda_1^{\text{ref}}, \ldots, \lambda_{\widetilde{\mu}}^{\text{ref}}; \boldsymbol{\delta_X}))\} \quad , \tag{13}$$

in which $p_{\Lambda_1,\ldots,\Lambda_{\widetilde{\mu}}}(\lambda_1^{\text{ref}}, \ldots, \lambda_{\widetilde{\mu}}^{\text{ref}}; \boldsymbol{\delta_X})$ is estimated using the stochastic computational model with $\boldsymbol{\delta_G} = 0$ and which is solved by the Monte Carlo numerical method.

3.2 Estimation of the dispersion parameters of the prior probability model of model uncertainties

The second step of the method proposed consists in estimating the dispersion parameter $\boldsymbol{\delta_G} \in \mathcal{C}_{\boldsymbol{G}} \subset \mathbb{R}^3$ of the prior probability model of the model uncertainties (modeling errors) in presence of the model-parameter uncertainties estimated in the first step using the stochastic computational model with $\boldsymbol{\delta_X} = \boldsymbol{\delta}_{\boldsymbol{X}}^{\text{opt}}$. For such an estimation, it is assumed that one experimental frequency response function is available for one or for a few observation points. We then consider the stochastic reduced computational model defined by Eqs. (11) and (12) rewritten in the frequency domain ω. It is assumed that n_{obs} observations are introduced and are such that $\boldsymbol{Y}^{obs}(\omega) = (|Y^n_{j_1}(\omega)|, \ldots, |Y^n_{j_{n_{obs}}}(\omega)|)$ are observed in n_{freq} frequencies $\omega_1, \ldots, \omega_{n_{freq}}$ in the frequency band of analysis B. We then introduce the random observed vector $\mathbb{Y} = (\boldsymbol{Y}^{obs}(\omega_1), \ldots, \boldsymbol{Y}^{obs}(\omega_{n_{freq}}))$ with values in $\mathbb{R}^s$ with $s = n_{obs} \times n_{freq}$. Let $\boldsymbol{y}^{\text{ref}}$ be the deterministic vector in $\mathbb{R}^s$ made up of the experimental data and corresponding to $\mathbb{R}^s$-valued random vector $\mathbb{Y}$ of the stochastic computational model. The estimation of $\boldsymbol{\delta_G}$ can be performed by using the maximum likelihood method. Since s can be very high, this method requires a big computational effort. We then use the method presented in [9]. For $\boldsymbol{\delta_X} = \boldsymbol{\delta}_{\boldsymbol{X}}^{\text{opt}}$ and for each fixed value of $\boldsymbol{\delta_G}$, let $\boldsymbol{m}_{\mathbb{Y}}(\boldsymbol{\delta_G})$ be the $\mathbb{R}^s$-valued mean vector and let $[C_{\mathbb{Y}}(\boldsymbol{\delta_G})]$ be the $(s \times s)$ covariance matrix of the random vector $\mathbb{Y}$, estimated by using the stochastic computational model and the Monte Carlo method. Let $\lambda_1(\boldsymbol{\delta_G}) \geq \lambda_2(\boldsymbol{\delta_G}) \geq \ldots$ be the positive eigenvalues and let $\boldsymbol{x}^1(\boldsymbol{\delta_G}), \boldsymbol{x}^2(\boldsymbol{\delta_G}), \ldots$ belong to $\mathbb{R}^s$ be the associated orthonormal eigenvectors of the eigenvalue problem $[C_{\mathbb{Y}}(\boldsymbol{\delta_G})]\, \boldsymbol{x}(\boldsymbol{\delta_G}) = \lambda(\boldsymbol{\delta_G})\, \boldsymbol{x}(\boldsymbol{\delta_G})$. Let $\mathbb{Y}^{\mu'}$ be the approximation of $\mathbb{Y}$ defined by $\mathbb{Y}^{\mu'} = \boldsymbol{m}_{\mathbb{Y}}(\boldsymbol{\delta_G}) + \sum_{j=1}^{\mu'} \sqrt{\lambda_j(\boldsymbol{\delta_G})}\, Z_j\, \boldsymbol{x}^j(\boldsymbol{\delta_G})$ in which μ' is an integer such that $1 \leq \mu' < s$, where $\boldsymbol{Z} = (Z_1, \ldots, Z_{\mu'})$ is a $\mathbb{R}^{\mu'}$-valued random variable such that $Z_j = \{\lambda_j(\boldsymbol{\delta_G})\}^{-1/2} < \mathbb{Y} - \boldsymbol{m}_{\mathbb{Y}}(\boldsymbol{\delta_G})\,, \boldsymbol{x}^j(\boldsymbol{\delta_G}) >$ where the bracket denotes the Euclidean inner product. It is known that the components of $\boldsymbol{Z}$ are second-order centered random variables which are uncorrelated. The order μ' of the statistical reduction is calculated in order to get an approximation with a given accuracy ε, independent of μ' and $\boldsymbol{\delta_G}$. The statistical reduction will be efficient if $\mu' \ll s$. Let $\boldsymbol{z}^{\text{ref}}(\boldsymbol{\delta_G}) = (z_1^{\text{ref}}(\boldsymbol{\delta_G}), \ldots, z_{\mu'}^{\text{ref}}(\boldsymbol{\delta_G})$ be the vector in $\mathbb{R}^{\mu'}$ corresponding to $\boldsymbol{Z}$ for the experimental data such that $z_j^{\text{ref}}(\boldsymbol{\delta_G}) = \{\lambda_j(\boldsymbol{\delta_G})\}^{-1/2} < \boldsymbol{y}^{\text{ref}} - \boldsymbol{m}_{\mathbb{Y}}(\boldsymbol{\delta_G})\,, \boldsymbol{x}^j(\boldsymbol{\delta_G}) >$. The estimation of $\boldsymbol{\delta_G}$ is performed using the maximum likelihood method for the random vector $\boldsymbol{Z} = (Z_1, \ldots, Z_{\mu'})$ whose components are centered and uncorrelated (but dependent) random variables. The Neperian logarithm of the likelihood function is then defined by $\mathcal{L}(\boldsymbol{\delta_G}) = \sum_{j=1}^{\mu'} \{\ln(p_{Z_j}(z_j^{\text{ref}}(\boldsymbol{\delta_G})\,; \boldsymbol{\delta}_{\boldsymbol{X}}^{\text{opt}}, \boldsymbol{\delta_G})$ in which, for all j in $\{1, \ldots \mu'\}$, the probability density function $z \mapsto p_{Z_j}(z\,; \boldsymbol{\delta}_{\boldsymbol{X}}^{\text{opt}}, \boldsymbol{\delta_G})$ depends on the known parameter $\boldsymbol{\delta}_{\boldsymbol{X}}^{\text{opt}}$ and on the unknown parameter $\boldsymbol{\delta_G}$ which must be estimated. This likelihood function is estimated with the stochastic reduced model and the Monte Carlo method. The optimal value $\boldsymbol{\delta}_{\boldsymbol{G}}^{\text{opt}}$ of $\boldsymbol{\delta_G}$ is then given as the solution of the following optimization problem,

$$\boldsymbol{\delta}_{\boldsymbol{G}}^{\text{opt}} = \arg \max_{\boldsymbol{\delta_G} \in \mathcal{C}_{\boldsymbol{G}}} \mathcal{L}(\boldsymbol{\delta_G}) \quad . \tag{14}$$

4 APPLICATION

We present an example of this generalized probabilistic approach of uncertainties. The designed system is a slender cylindrical elastic medium with length $10\ m$ and has a rectangular section with height $1.1\ m$ and width $1.6\ m$. The elastic medium is made of a composite material. The displacement field is zero on the part of the two end sections. The frequency band of analysis is $B =]0\,, 1200]\ Hz$. A point load is applied close to the middle of the slender cylinder with a flat spectrum on B. We are interested in the transversal displacement of the neutral line at an observation point belonging to the neutral line and close to the end section. A reference solution of the real system has been constructed in developing a 3D elastic model of the real system. The mean model is made up of a damped homogeneous Euler elastic beam with length $10\ m$ and simply supported.

From the Euler beam theory, the first eigenvalue (square of the first eigenfrequency) can be written $\lambda_1 = a_1\, \underline{x}$. The model-parameter uncertainties leads us to model $\underline{x}$ by a random variable X. The prior probability model of random variable X is constructed using the maximum entropy principle and yields a Gamma probability distribution for which the mean value is given and for which the dispersion is controlled by the coefficient of variation δ_X. The identification of parameter δ_X is performed using the maximum likelihood method for the lowest random eigenvalue Λ_1 and yields $\delta_X = 0.093$. We consider the stochastic reduced model constructed (1) with the parametric probabilistic approach of uncertain parameter X for which $\delta_X = 0.093$ and (2) with the nonparametric probabilistic approach of modeling errors for the mass and stiffness operators. Consequently, the dispersion parameters describing the statistical fluctuations induced by modeling errors are δ_M and δ_K and are estimated as explained in the previous section and yields $\delta_M = 0.9$ and $\delta_K = 0.15$. Figure 1 (left and right) displays the comparisons between the response of the mean model, the reference response of the real system and the confidence region of the random response (1) calculated with the parametric probabilistic approach with $\delta_X = 0.093$ (left figure) and (2) calculated with the generalized probabilistic approach, that is to say, with the parametric probabilistic approach of model-parameter uncertainties for $\delta_X = 0.093$ and with the nonparametric probabilistic approach of modeling errors for $\delta_M = 0.9$ and $\delta_K = 0.15$. These figures show that the coupling of the two probabilistic approaches for model-parameter uncertainties and modeling errors allow the quality of the prediction to be considerably improved. The method proposed allows the role played by each type of uncertainties to be separately quantified.

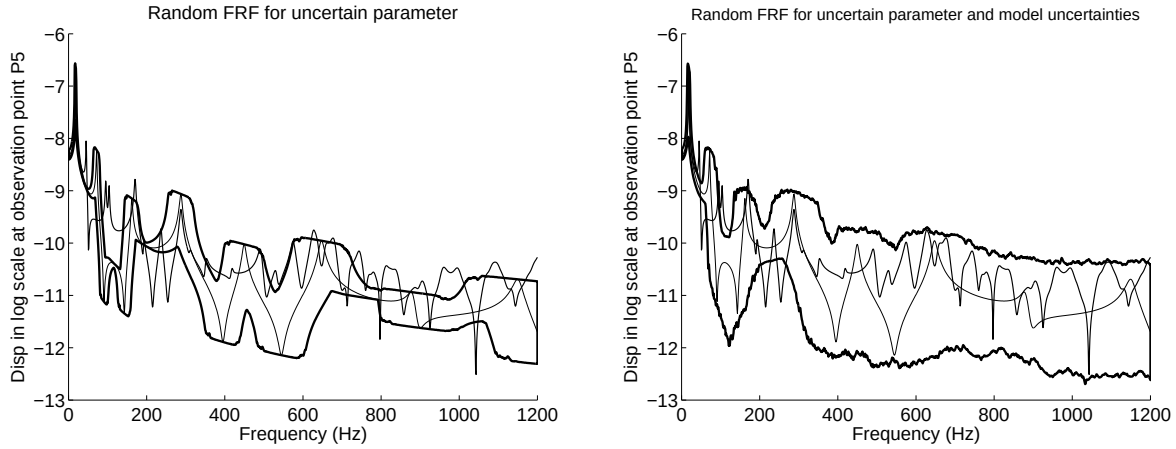

Figure 1: Confidence region of the response at 0.98 (between upper and lower thick solid lines), reference response (mid solid line), response of the mean model (thin solid line) at observation point. Left figure: parametric probabilistic approach. Right figure: generalized probabilistic approach.

5 CONCLUSION

A new generalized probabilistic approach to take into account model-parameter uncertainties and modeling errors in computational model for structural linear dynamics is proposed. This approach can easily be extended to computational linear vibroacoustics and to computational nonlinear structural dynamics. This method allows a prior probability model of model-parameter uncertainties and a prior probability model of modeling errors to be separately constructed. When a few experimental data are available, a procedure for the identification of the dispersion parameters of the prior probability models of uncertainties is proposed. As explained in [53], a chaos decomposition with random coefficients can also be used to represent the prior probabilistic model of random responses in separating the propagation of model-parameter uncertainties and the propagation of modeling errors in the computational model. Such a construction gives future perspectives to im-

prove the prior probability model of uncertainties in constructing a posterior probability model as soon as experimental data are available.

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