Uncertain non-linear dynamical system submitted to uncertain stochastic loads

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7th European Conference on Structural Dynamics
EURODYN 2008
7-9 July 2008 Southampton

UNCERTAIN NON-LINEAR DYNAMICAL SYSTEM SUBMITTED TO UNCERTAIN STOCHASTIC LOADS.

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Keywords: non-linear stochastic dynamics, uncertainties, identification of stochastic process

ABSTRACT

The problem presented deals with tubes bundles in Pressurized Water Reactors. The final objective is to identify a model of the external loads applied to theses tubes bundles through the knowledge of dynamical responses. In complex dynamical systems, such an identification is difficult due to the size of the computational model and due to the high number of parameters to be identified. As a consequence, the computational model is simplified implying a loss of accuracy and of predictability due to model uncertainties induced by the simplification introduced. We are first interested in the implementation (modelling and identification) of a probabilistic approach of uncertainties in the mean computational model using the non-parametric probabilistic approach for data uncertainties and model uncertainties. In addition, a probabilistic model for the stochastic loads is constructed to take into account model uncertainties in the probabilistic model of the stochastic loads. Finally, the non-linear stochastic dynamical system submited to the uncertain stochastic loads is used to identify the probability model of its uncertainties. In a first part, the theory is presented. The second part is devoted to the validation of the theory in presenting an application.

1. INTRODUCTION

The present research has been developed in the context of the dynamical analysis of the tubes bundles in Pressurized Water Reactors. The tubes are excited by a turbulent flow which induces a non-linear dynamical response of the dynamical system made up of a structure coupled with the fluid. The final objective is to identify a mathematical model of the stochastic loads applied to the tubes bundles and induced by the turbulent flow using both experimental responses of the real non-linear dynamical system and a simplified computational non-linear dynamical model. The real system under consideration is made up of several hundreds of tubes and several grids. The dynamical behaviour of any tube in a free-free configuration is linear. The non-linearities are induced by the shocks between the tubes and the grids. In the frequency band of analysis for which the stochastic loads have to be identified, there are between ten thousand and twenty thousand elastic modes for the linearized dynamical system. In this condition a simplified computational non-linear dynamical model has to be introduced for that the identification of the stochastic loads can effectively be done. For the identification of the stochastic loads, the use of a simplified computational model induces model uncertainties. A non-parametric probabilistic approach is then used to take into account both data uncertainties and model uncertainties

The real dynamical system (real tubes bundles) is replaced by a reference non-linear dynamical system made up of five tubes and three grids. This reference system is representative of the real system and a computational model, defined as the reference computational model, is developed. This reference computational model allows the responses of the reference system to be simulated. In a first step, the probability model of uncertainties is identified in the simplified computational model using the reference computational model and the maximum likelihood method. We then deduce a stochastic simplified computational model which allows a robust identification of stochastic loads to be carried out with respect to uncertainties in the dynamical system.

The second step is devoted to the stochastic inverse problem consisting in identifying the stochastic loads. The stochastic loads used in the simplified computational model are represented by a vector-valued centred stationary Gaussian stochastic process. Such a stochastic process is then completely defined by a matrix-valued spectral density function. The use of a rough spatial discretization of the random field in the simplified computational model introduces uncertainties in the stochastic process which models the stochastic loads. These uncertainties are then taken into account in introducing a probabilistic model for the matrix-valued spectral density function which becomes a random quantity which has to be constructed and identified. The identification of the stochastic loads then corresponds (1) to the identification of the mean value of the random matrix-valued spectral density function in the frequency band of analysis and (2) to the identification of the dispersion parameter introduced in the probability model of uncertainties and allowing the level of uncertainties to be controlled.

2. REFERENCE COMPUTATIONAL MODEL

In this section, we introduce a reference computational model for which the responses will be considered as the experimental responses and will be used to identify the stochastic simplified computational model that will be introduced in the next session.

2.1 Transient dynamical response of the reference computational model

Let Ω be the domain of a three dimensional damped structure having a non-linear behaviour (the non-linearities are not distributed but are localized). The structure is fixed on the part Γ_0 of the boundary Γ of Ω. Let $\mathbf{u}^{ref}(t)$ be the vector of the n degrees of freedom at time t. Let $[\underline{\mathbb{M}}^{ref}]$, $[\underline{\mathbb{D}}^{ref}]$ and $[\underline{\mathbb{K}}^{ref}]$ be respectively the mass, damping and stiffness matrices of the linear part of the finite element model. Since there are no rigid body displacements, these three matrices are positive definite. Let $\mathbf{f}^{ref}(t)$ be the vector of the external loads applied to the structure and let $\mathbf{f}^{NL}(\mathbf{u}^{ref}(t), \dot{\mathbf{u}}^{ref}(t))$ be the vector of the non-linear forces induced by the localized non-linearities. Let $[\mathbf{\Phi}]$ be the $(n \times m)$ matrix whose columns are the m structural modes $\varphi_1, ..., \varphi_m$

of the linear structure without the non-linearities corresponding to the m eigenfrequencies $0 < \omega_1 \leq ... \leq \omega_m$. The non-linear dynamical equation is projected on the the basis represented by $[\mathbf{\Phi}]$. Therefore, the displacement vector at time t is written as $\mathbf{u}^{ref}(t) = [\mathbf{\Phi}]\mathbf{q}^{ref}(t)$ in which $\mathbf{q}^{ref}(t)$ is the vector of the m generalized coordinates. Such a projection introduces the positive definite matrices $[\underline{\mathbf{M}}^{ref}] = [\underline{\mathbf{\Phi}}]^T[\underline{\mathbb{M}}^{ref}][\underline{\mathbf{\Phi}}]$, $[\underline{\mathbf{D}}^{ref}] = [\underline{\mathbf{\Phi}}]^T[\underline{\mathbb{D}}^{ref}][\underline{\mathbf{\Phi}}]$ and $[\underline{\mathbf{K}}^{ref}] = [\underline{\mathbf{\Phi}}]^T[\underline{\mathbb{K}}^{ref}][\underline{\mathbf{\Phi}}]$. The function $t \mapsto \mathbf{q}^{ref}(t)$ is the solution of the following non-linear dynamical equation,

$$[\underline{\mathbf{M}}^{ref}]\ddot{\mathbf{q}}^{ref}(t) + [\underline{\mathbf{D}}^{ref}]\dot{\mathbf{q}}^{ref}(t) + [\underline{\mathbf{K}}^{ref}]\mathbf{q}^{ref}(t) + \tilde{\mathbf{f}}^{NL}(\mathbf{u}^{ref}(t), \dot{\mathbf{u}}^{ref}(t)) = \tilde{\mathbf{f}}^{ref}(t) \ , \tag{1}$$

where $\tilde{\mathbf{f}}^{NL}(\mathbf{u}^{ref}(t), \dot{\mathbf{u}}^{ref}(t)) = [\underline{\mathbf{\Phi}}]^T\mathbf{f}^{NL}(\mathbf{u}^{ref}(t), \dot{\mathbf{u}}^{ref}(t))$ and $\tilde{\mathbf{f}}^{ref}(t) = [\underline{\mathbf{\Phi}}]^T\mathbf{f}^{ref}(t)$.

2.2 Decomposition in one linear subsystem and one non-linear subsystem

The domain Ω is decomposed in two subdomains, the subdomain Ω^A_{ref} which corresponds to a non-linear subsystem made up of one part of the structure containing the localized non-linearities and the subdomain Ω^B_{ref} which corresponds to a linear subsystem made up of the second part of the structure and which has a linear behaviour. Each uncoupled subsystem is considered as fixed and therefore does not have rigid body displacement. These two subsytems are coupled on the coupling interface Γ_C . The finite element model of the linear subsytem Ω^B_{ref} is analyzed in the frequency band of analysis $\mathcal{B} = [-\omega_{max}, \omega_{max}]$. Let $[\underline{\mathbb{A}}^{B,ref}(\omega)] = -\omega^2[\underline{\mathbb{M}}^{B,ref}] + i\omega[\underline{\mathbb{D}}^{B,ref}] + [\underline{\mathbb{K}}^{B,ref}]$ be the dynamic stiffness matrix of this linear subsystem with free coupling interface, where $[\underline{\mathbb{M}}^{B,ref}]$, $[\underline{\mathbb{D}}^{B,ref}]$ and $[\underline{\mathbb{K}}^{B,ref}]$ are the mass, damping and stiffness matrices which are positive definite. Introducing the vector $\mathbf{u}_p^{B,ref}(\omega)$ of the n_p internal DOF and the vector $\mathbf{u}_c^{B,ref}(\omega)$ of the n_c coupling DOF on the interface. A reduction of the linear subsystem is performed using the Craig Bampton method [4]. The block decomposition of the reduced dynamical stiffness matrix related to the generalized coordinates $\mathbf{y}^B(\omega)$ and the coupling DOF $\mathbf{u}_c^B(\omega)$ is written as

$$[\underline{\boldsymbol{A}}^{B,ref}(\omega)] = \begin{bmatrix} \underline{\boldsymbol{A}}_{yy}^{B,ref}(\omega) & \underline{\boldsymbol{A}}_{yc}^{B,ref}(\omega) \\ \underline{\boldsymbol{A}}_{cy}^{B,ref}(\omega) & \underline{\boldsymbol{A}}_{cc}^{B,ref}(\omega) \end{bmatrix} . \tag{2}$$

In order to perform the identification of stochastic simplified computational model, for the reference computational model, we introduce an observation related to $[\underline{\boldsymbol{Z}}^{B,ref}(\omega)] = [\underline{\boldsymbol{A}}_{cc}^{B,ref}(\omega)] - [\underline{\boldsymbol{A}}_{cy}^{B}(\omega)][\underline{\boldsymbol{A}}_{yy}^{B,ref}(\omega)]^{-1}[\underline{\boldsymbol{A}}_{yc}^{B,ref}(\omega)]$ which corresponds to the condensed dynamical stiffness matrix of the linear subsystem on the coupling interface of the reference computational model. For all ω fixed in the frequency band $\mathcal{B}$, the matrix $[\underline{\boldsymbol{Z}}^{B,ref}(\omega)]$ is invertible. The observation is then the finite positive real number J^{ref} defined by

$$J^{ref} = \int_{\mathcal{B}} \|[\underline{\boldsymbol{Z}}^{B,ref}(\omega)]^{-1}\|_F^2 d\omega \quad . \tag{3}$$

3. STOCHASTIC NON-LINEAR SIMPLIFIED COMPUTATIONAL MODEL, INCLUDING SYSTEM UNCERTAINTIES, AND IDENTIFICATION

In this part, the mean model of the non-linear simplified computational model system is introduced. Then, the probabilistic nonparametric approach will be used to take into account data uncertainties and model uncertainties in the linear subsystem of the simplified computational model. The dispersion parameters controlling the dispersion on the mass, damping and stiffness matrices will be identified using the maximum likelihood method.

3.1 Mean reduced linear subsystem of the simplified computational model

The simplified computational model is constructed from the reference computational model. Indeed, the linear subsytem of the simplified model is derived from the linear part of the reference model. The non-linear subsystems of the two models are the same (see Fig. 1). And consequently, the degrees of freedom on the coupling interface are the same.

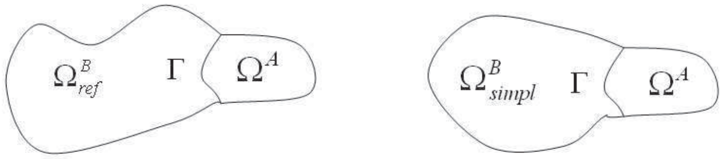

Figure 1. Reference model (left) and simplified model (right).

Using the same reduction method as the one used for the linear subsystem of the reference model, we obtain the following mean reduced dynamical stiffness matrix for the linear subsystem of the simplified model $[\underline{\boldsymbol{A}}^B(\omega)] = -\omega^2[\underline{\boldsymbol{M}}^B] + i\omega[\underline{\boldsymbol{D}}^B] + [\underline{\boldsymbol{K}}^B]$, where $[\underline{\boldsymbol{M}}^B]$, $[\underline{\boldsymbol{D}}^B]$ and $[\underline{\boldsymbol{K}}^B]$ are respectively the mean reduced positive definite mass, damping and stiffness matrices of the linear subsystem of the mean reduced computational simplified model. Let $\underline{\mathbf{y}}^B(\omega)$ be the vector of the N mean generalized coordinates and $\underline{\mathbf{u}}_c^B(\omega)$ is the vector of the n_c mean coupling DOFs. Then, the block decomposition of the reduced stiffness matrix related to the generealized coordinates $\underline{\mathbf{y}}^B(\omega)$ and the coupling DOFs $\underline{\mathbf{u}}_c^B(\omega)$ is written as

$$[\underline{\boldsymbol{A}}^B(\omega)] = \begin{bmatrix} \underline{\boldsymbol{A}}^B_{yy}(\omega) & \underline{\boldsymbol{A}}^B_{yc}(\omega) \\ \underline{\boldsymbol{A}}^B_{cy}(\omega) & \underline{\boldsymbol{A}}^B_{cc}(\omega) \end{bmatrix} . \tag{4}$$

3.2 System uncertainties modeling using the non-parametric probabilistic approach

The non-parametric probabilistic approach is used to take into account both model uncertainties and data uncertainties in the dynamical system. This approach has recently been introduced (see [1], [2]) and consists in replacing the matrices of reduced mean model by random matrices for which the probability distributions are constructed by using the maximum entropy principle with constraints defined by the available information. Such an approach has been validated for different cases. Therefore, the mean reduced dynamical stiffness matrix $[\underline{A}^B(\omega)]$ is replaced by the complex random matrix $[\mathbf{A}^B(\omega)]$ written as

$$[\mathbf{A}^B(\omega)] = -\omega^2[\mathbf{M}^B] + i\omega[\mathbf{D}^B] + [\mathbf{K}^B] \quad , \tag{5}$$

in which the matrices $[\underline{\boldsymbol{M}}^B]$, $[\underline{\boldsymbol{D}}^B]$ and $[\underline{\boldsymbol{K}}^B]$ of the reduced mean system are replaced by the random matrices $[\mathbf{M}^B]$, $[\mathbf{D}^B]$ and $[\mathbf{K}^B]$ defined on a probability space $(\Theta, \mathcal{T}, \mathcal{P})$ and whose probability distributions depend respectively on the dispersion parameters δ^B_M, δ^B_D and δ^B_K.

3.3 Identification of the dispersion parameters

The dispersion parameters are identified using the reference computational model. The observation of the stochastic simplified computational model is defined similarly to the observation defined by Eq. (3) for the reference computational model. We then introduce the condensed dynamical stiffness matrix $[\mathbf{Z}^B(\omega)] = [\mathbf{A}^B_{cc}(\omega)] - [\mathbf{A}^B_{cy}(\omega)][\mathbf{A}^B_{yy}(\omega)]^{-1}[\mathbf{A}^B_{yc}(\omega)]$ of the linear subsystem on the coupling interface. Taking into account the properties of the probabilistic model, it can be shown that for all ω fixed in the frequency band $\mathcal{B}$, the random matrix $[\mathbf{Z}^B(\omega)]$ is invertible almost surely and that the random variable $J(\boldsymbol{\delta})$ defined by

$$J(\delta) = \int_{\mathcal{B}} \|[\mathbf{Z}^B(\omega)]^{-1}\|_F^2 d\omega \quad , \tag{6}$$

exists and has a finite mean value. It should be noted that the random variable J depends on $\boldsymbol{\delta}$ because the probability distributions of the random matrices $[\mathbf{M}^B]$, $[\mathbf{D}^B]$ and $[\mathbf{K}^B]$ depend on $\boldsymbol{\delta}$. Let $x \mapsto p_J(x, \boldsymbol{\delta})$ be the probabilty distribution of the random variable $J(\boldsymbol{\delta})$ with respect to dx. For any x fixed in $[0, +\infty[$ and for any value of the vector $\boldsymbol{\delta}$ in the admissible space $\mathcal{C}_{ad}$ of the dispersion parameters, the value $p_J(x, \boldsymbol{\delta})$ of the probability density function is estimated by using the above proabilistic model and the Monte Carlo simulation. Note that the corresponding deterministic value of $J(\boldsymbol{\delta})$ for the reference computational model is denoted by J^{ref} given by Eq. (3). The method used to identify vector $\boldsymbol{\delta}$ is the maximum likelihood method (see for instance [3]) for the random observation $J(\boldsymbol{\delta})$. We then have to solve the following optimisation problem

$$\boldsymbol{\delta}^{opt} = \arg\max_{\boldsymbol{\delta} \in \mathcal{C}_{ad}} (p_J(J^{ref}; \boldsymbol{\delta})) \quad , \tag{7}$$

in which $\boldsymbol{\delta}^{opt}$ is the identified value of vector $\boldsymbol{\delta}$.

3.4 Random dynamical transient response of the stochastic non-linear simplified computational model

For the stochastic system, the displacement vector of the stochastic linear subsystem Ω^B_{simpl} is denoted by $(\mathbf{U}^B_p(t), \mathbf{U}^B_c)(t)$. The displacement vector of the stochastic non-linear subsystem Ω^A is denoted by $(\mathbf{U}^A_p(t), \mathbf{U}^A_c(t))$. Then, the random variable $\mathbf{Q}(t) = (\mathbf{Y}^A(t), \mathbf{Y}^B(t), \mathbf{U}^c(t))$ which is composed of the random generalized coordinates $\mathbf{Y}^A(t)$ of the random non-linear subsystem, of the random generalized coordinates $\mathbf{Y}^B(t)$ of the linear subsystem and of the random coupling DOF $\mathbf{U}_c(t)$, is solution of the random non-linear dynamical system

$$[\tilde{\mathbf{M}}]\ddot{\mathbf{Q}}(t) + [\tilde{\mathbf{D}}]\dot{\mathbf{Q}}(t) + [\tilde{\mathbf{K}}]\mathbf{Q}(t) + \mathbf{f}^{NL}(\mathbf{Q}(t), \dot{\mathbf{Q}}(t)) = \tilde{\mathbf{f}}(t) \quad . \tag{8}$$

4. PROBABILISTIC MODEL OF THE STOCHASTIC LOAD WITH MODEL UNCERTAINTIES

The vector of the non zero components of the transient load is denoted by $\mathbf{f}(t)$. This load is, for instance, due to the turbulent flow which then induces a stochastic load. Consequently, such a load is modelled by a stochastic process $\{\mathbf{F}(t), t \in \mathbb{R}\}$. Since the probabilistic model developed below will only be a simple representation of the real stochastic load applied to the structure, model uncertainties must be introduced in order to improve the efficiency of the representation which will be used for the identification of this stochastic load. We then take into account these uncertainties in introducing an additional probabilistic model of uncertainties for this stochastic process $\{\mathbf{F}(t), t \in \mathbb{R}\}$ which is then rewritten as $\{\mathbf{F}^{\text{unc}}(t), t \in \mathbb{R}\}$.

4.1 Construction of the stochastic load $\mathbf{F}(t)$

The stochastic load is modelled by a $\mathbb{R}^m$-valued second order stationary Gaussian stochastic process $\{\mathbf{F}(t), t \in \mathbb{R}\}$ defined on a probability space $(\Theta', \mathcal{T}', \mathcal{P}')$ different from the probability space $(\Theta, \mathcal{T}, \mathcal{P})$, indexed by $\mathbb{R}$, centred, mean square continous on $\mathbb{R}$, physically realizable (causal) and whose matrix-valued autocorrelation function $[\underline{R}_{\mathbf{F}}(\tau)]$ is integrable on $\mathbb{R}$. This stochastic process is then completely defined by its matrix-valued spectral density function $[\underline{\mathbf{S}}_{\mathbf{F}}(\omega)]$. In addition, we will assume that for all ω in $\mathbb{R}$, the matrix $[\underline{\mathbf{S}}_{\mathbf{F}}(\omega)]$ is with values in the set $\mathbb{M}_m^+(\mathbb{C})$ of all the positive definite $(m \times m)$ hermitian matrices.

In addition, for all ω in $\mathcal{B} \subset \mathbb{R}$, the matrix valued spectral density function being positive definite, the Hermitian matrix $[\underline{\mathbf{S}}_{\mathbf{F}}(\omega)]$ is invertible and its Cholesky decomposition yields $[\underline{\mathbf{S}}_{\mathbf{F}}(\omega)] = [\underline{\mathbf{L}}_{\mathbf{F}}(\omega)]^* [\underline{\mathbf{L}}_{\mathbf{F}}(\omega)]$, in which $[\underline{\mathbf{L}}_{\mathbf{F}}(\omega)]$ is an upper triangular matrix in $\mathbb{M}_m(\mathbb{C})$.

4.2 Defining stochastic load $\mathbf{F}^{\mathrm{unc}}(t)$ including a probabilistic model of uncertainties

Model uncertainties are introduced using the probabilistic parametric approach of data uncertainties consisting in modelling the deterministic function $[\underline{\mathbf{S}}_{\mathbf{F}}] = \{[\underline{\mathbf{S}}_{\mathbf{F}}(\omega)], \omega \in \mathbb{R}\}$ by a random function with values in $\mathbb{M}_m^+(\mathbb{C})$ and denoted by $[\mathbf{S}_{\mathbf{F}}] = \{[\mathbf{S}_{\mathbf{F}}(\omega)], \omega \in \mathbb{R}\}$, defined on a probability space $(\Theta", \mathcal{T}", \mathcal{P}")$. Consequently, the stochastic process $\{\mathbf{F}^{\mathrm{unc}}(t), t \in \mathbb{R}\}$ indexed by $\mathbb{R}$ with values in $\mathbb{R}^m$, defined on the probability space $(\Theta', \mathcal{T}', \mathcal{P}') \times (\Theta'', \mathcal{T}'', \mathcal{P}'')$, is such that, for all $\theta' \in \Theta'$ and $\theta" \in \Theta"$, $\mathbf{F}^{\mathrm{unc}}(t, \theta', \theta'') = \mathbf{F}(t, \theta'; [\mathbf{S}_{\mathbf{F}}](\theta''))$.

4.3 Construction of the random function $[\mathbf{S}_{\mathbf{F}}]$

The random function $[\mathbf{S}_{\mathbf{F}}]$ is constructed using the information theory using the maximum entropy principle (Shannon 48). The available information concerning the random function $[\mathbf{S}_{\mathbf{F}}(\omega)], \omega \in \mathbb{R}$ is the following. For all ω in $\mathbb{R}$, $[\mathbf{S}_{\mathbf{F}}(\omega)]$ is a random matrix with values in $\mathbb{M}_m^+(\mathbb{C})$ and by construction, it is written that for all ω in $\mathbb{R}$, $E\{[\mathbf{S}_{\mathbf{F}}(\omega)]\} = [\underline{\mathbf{S}}_{\mathbf{F}}(\omega)]$. Consequently, for all ω in $\mathbb{R}$, the random matrix $[\mathbf{S}_{\mathbf{F}}(\omega)]$ is invertible almost surely, which means that for $\mathcal{P}"$-almost $\theta"$ in $\Theta"$, the matrix $[\mathbf{S}_{\mathbf{F}}(\omega, \theta")]^{-1}$ exists. By construction we will impose that the random matrix $[\mathbf{S}_{\mathbf{F}}(\omega)]^{-1}$ is a second order random variable which means that $E\{\|[\mathbf{S}_{\mathbf{F}}(\omega)]^{-1}\|^2\} < +\infty$. For all ω in $\mathcal{B}$, the random matrix $[\mathbf{S}_{\mathbf{F}}(\omega)]$ is normalized as $[\mathbf{S}_{\mathbf{F}}(\omega)] = [\underline{\mathbf{L}}_{\mathbf{F}}(\omega)]^* [\mathbf{G}_m] [\underline{\mathbf{L}}_{\mathbf{F}}(\omega)]$, in which the random matrix $[\mathbf{G}_m]$ is defined on the probability space $(\Theta'', \mathcal{T}'', \mathcal{P}'')$ and belongs to the *normalized positive-definite ensemble* denoted by SG^+ (see [2]). This random matrix which is independent of ω is such that $[\mathbf{G}_m] \in \mathbb{M}_m^+(\mathbb{R})$, $E\{[\mathbf{G}_m]\} = [\underline{\mathbf{I}}_m]$ and $E\{\|[\mathbf{G}_m]^{-1}\|_F^2\} < +\infty$.The dispersion of $[\mathbf{S}_{\mathbf{F}}]$ is independant of ω and is controlled by the dispersion parameter δ_F which is such that $\delta_F = \{(1/m)E\{\|[\mathbf{G}_m] - [\underline{\mathbf{I}}_m]\|\}_F^2\}^{1/2}$. In addition, it can be proved that the stochastic process $\mathbf{F}^{\mathrm{unc}}(t)$ is a second order stationary stochastic process, centred, mean square continous on $\mathbb{R}$, physically realizable whose matrix-valued spectral density function $[\underline{\mathbf{S}}_{\mathbf{F}^{\mathrm{unc}}}(\omega)]$ is such that $[\underline{\mathbf{S}}_{\mathbf{F}^{\mathrm{unc}}}(\omega)] = [\underline{\mathbf{S}}_{\mathbf{F}}(\omega)]$ for all $\omega \in \mathbb{R}$. Nevertheless, it can be proved that the stochastic process $\mathbf{F}^{\mathrm{unc}}(t)$ is not Gaussian.

5. IDENTIFICATION OF THE STOCHASTIC LOAD

This section is devoted to the identification of the stochastic load $\{\mathbf{F}^{\mathrm{unc}}(t), t \in \mathbb{R})\}$ defined and studied in Section 4. using the random responses of the stochastic simplified model of the structure excited by this stochastic loads and defined in Section 3.. This identification consists in identifying the mean value $[\underline{\mathbf{S}}_{\mathbf{F}}(\omega)]$ of the matrix-valued spectral density function and the parameter δ_F that controls the level of uncertainties. In practice, the parameter $[\underline{\mathbf{S}}_{\mathbf{F}}]$ which has to be identified is in fact a function $\omega \mapsto [\underline{\mathbf{S}}_{\mathbf{F}}(\omega)]$. This identification will be performed in introducing a parametric representation of this function which is written as $[\underline{\mathbf{S}}_{\mathbf{F}}(\omega)] = [\underline{\mathbf{S}}(\omega, \mathbf{r})]$ for all $\omega \in \mathbb{R}$. Let $\mathcal{C}_r \subset \mathbb{R}^{\nu_r}$ be the admissible set of the parameter $\mathbf{r}$. The vector $\mathbf{r}$ and the dispersion parameter δ_F are identified separetly using two different cost functions.

5.1 Identification of the parameter r

Such an identification is performed using the stochastic equation deduced from Eqs. (8) in which the deterministic load $\mathbf{f}(t)$ is replaced by the stochastic load $\mathbf{F}(t; [\underline{S}_{\mathbf{F}}])$. We then extract the $\mathbb{R}^{\mu}$-valued random variable $\mathbf{Z}_s(t) = (Z_{s,1}(t), ..., Z_{s,\mu}(t))$ which represents the observations of the simplified stochastic model. Therefore, for all $\theta \in \Theta$, the matrix-valued spectral density function $\{[\mathbf{S}_{\mathbf{Z}_s}(\omega, \theta)], \omega \in \mathbb{R}\}$ of the stationary stochastic process $\{\mathbf{Z}_s(t, \theta), t \in \mathbb{R}\}$ can be estimated. Generating ν_θ independent realizations of the random matrices $[\tilde{\mathbf{M}}]$, $[\tilde{\mathbf{D}}]$ and $[\tilde{\mathbf{K}}]$, the matrix-valued spectral density function $[\underline{\mathbf{S}}_{\mathbf{Z}_s}]$ is estimated by the Monte Carlo simulation method, i.e., for all $\omega \in \mathbb{R}$, one has

$$[\underline{\mathbf{S}}_{\mathbf{Z}_s}(\omega)] = \frac{1}{\nu_\theta} \sum_{i=1}^{\nu_\theta} [\mathbf{S}_{\mathbf{Z}_s}(\omega, \theta_i)] \quad . \tag{9}$$

Let $\{\mathbf{Z}_s^{exp}(t) = (Z_{s,1}^{exp}(t), ..., Z_{s,\mu}^{exp}(t)), t \in \mathbb{R}\}$ be the $\mathbb{R}^{\mu}$-valued stationary stochastic process which is measured for the manufactured real system and corresponding to the observation stochastic process $\{\mathbf{Z}_s(t), t \in \mathbb{R}\}$. The matrix-valued spectral density function $\{[\underline{\mathbf{S}}_{\mathbf{Z}_s^{exp}}(\omega)], \omega \in \mathbb{R}\}$ of this stochastic process is estimated using the periodogram method. The parameter $\mathbf{r}$ is then estimated minimizing the distance $D(\mathbf{r}) = \int_{\mathcal{B}} \| [\underline{\mathbf{S}}_{\mathbf{Z}_s}(\omega, \mathbf{r})] - [\underline{\mathbf{S}}_{\mathbf{Z}_s^{exp}}(\omega)] \|_F^2 \, d\omega$ between the matrix-valued spectral density function calculated with the stochastic simplified model and the experimental matrix-valued spectral density function. We then have to solve the following optimization problem

$$\mathbf{r}_{opt} = \underset{\mathbf{r} \in \mathcal{C}_r}{\mathrm{argmin}} \, D(\mathbf{r}) \quad , \tag{10}$$

in which $\mathbf{r}_{opt}$ is the identified value of the vector $\mathbf{r}$.

5.2 Identification of the dispersion parameter δ_F

This identification is performed using the stochastic equation deduced from Eqs. (8) in which the deterministic load $\mathbf{f}(t)$ is replaced by the stochastic load $\mathbf{F}^{\mathrm{unc}}(t) = \mathbf{F}(t; [\mathbf{S}_{\mathbf{F}}])$ (including uncertainties). We then extract the $\mathbb{R}^{\mu}$-valued random variable $\mathbf{Z}'_s(t) = (Z'_{s,1}(t), ..., Z'_{s,\mu}(t))$. For all $(\theta, \theta'') \in \Theta \times \Theta''$, the matrix-valued spectral density function $\{[\mathbf{S}_{\mathbf{Z}'_s}(\omega, \theta, \theta'')], \omega \in \mathbb{R}\}$ of the stationary stochastic process $\{\mathbf{Z}'_s(t, \theta, \theta''), t \in \mathbb{R}\}$ is estimated. One then define the random variable J_s such that $J_s(\theta, \theta'') = \int_{\mathcal{B}} \| [\mathbf{S}_{\mathbf{Z}'_s}(\omega, \theta, \theta'')] \|_F^2 d\omega$. Generating ν_θ independent realizations of the random matrices $[\tilde{\mathbf{M}}]$, $[\tilde{\mathbf{D}}]$ and $[\tilde{\mathbf{K}}]$ and $\nu_{\theta''}$ independent realizations of the random function $\{[\mathbf{S}_{\mathbf{F}}(\omega)], \omega \in \mathbb{R}\}$, the probabilty distribution $x \mapsto p_{J_s}(x)$ of the random variable J_s with respect to dx is estimated by the Monte Carlo simulation method.From the matrix-valued spectral density function $\{[\underline{\mathbf{S}}_{\mathbf{Z}_s^{exp}}(\omega)], \omega \in \mathbb{R}\}$, the variable J_s^{exp} is calculated such that $J_s^{\mathrm{exp}} = \int_{\mathcal{B}} \| [\underline{\mathbf{S}}_{\mathbf{Z}_s^{exp}}(\omega)] \|_F^2 d\omega$. Then, dispersion parameter δ_F is estimated using the maximum likelihood method on the random varaiable J_s, i.e,

$$\delta_F^{opt} = \arg \max_{\delta_F \in \mathcal{C}_{\delta_F}} \left(p_{J_s}(J_s^{\mathrm{exp}}, \delta_F) \right) \quad , \tag{11}$$

in which δ_F^{opt} is the identified value of the variable δ_F and $\mathcal{C}_{\delta_F}$ is the admissible set for the dispersion parameter δ_F.

6. APPLICATION

6.1 Data for the reference computational model

The reference computational model is made up of one linear subsystem and one non-linear subsystem. The linear subsystem is made up of four parallel Euler beams fixed at their ends. The non-linear subsytem is made up of a beam fixed at its ends, parallel to the other beams and with one transverse symmetric elastic stop (two identical tranverse stops). The five beams are linked by three transversal grids, each grid being modelled by four transverval springs (see Fig. 2). Therefore, the coupling interface between the two subsytems is composed of three points located in the neutral fiber of the beam of the non-linear subsystem. Each beam has a constant circular section with radius $0.5\ m$, thickness $0.2\ m$, length $16\ m$, mass density $250\ kg/m^3$, Young's modulus $450\ N/mm^2$ and dampding rate 0.02. The Young's modulus of the beam of the non-linear subsystem is $750\ N/mm^2$.The elastic stops are localized at $6\ m$ from the left fixed end, the gap of each stop is $1.5.10^{-6}\ m$ and the shock stiffness is $10^8\ N/m$. The stiffness of each spring of the tranversal grid is $4.10^7\ N/m$. Each beam is modelled by eight beam Euler finite elements of equal lengths and nine nodes. The DOF of the two nodes at the ends of the beam are locked. The twelve springs in the three tranversal grids are modelled by twelve spring elements. We are only interested in the transversal displacements in direction y for the plane xy of the beam of the non-linear subsystem (see Fig. 2). Consequently, each beam has 14 DOF of y translation and z rotation. The beam of

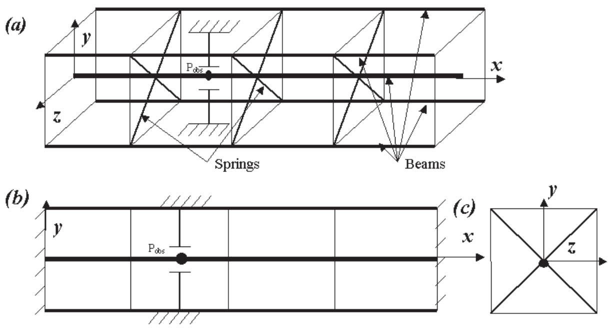

Figure 2: (a) Reference model:3D view.(b) Transversal view.(c) Tranversal view in the plane of one grid:the 6 diagonal lines represent the 12 springs.

the nonlinear subsystem is exited by 7 transversal forces applied following the y direction. The vector of these 7 forces are denoted by $\mathbf{f}^{\text{ref}}$. Then $\{\mathbf{f}^{\text{ref}}(t), t \in \mathbb{R}\}$ is modeled by a second-order centred stationary Gaussian stochastic process for which its matrix-valued spectral density function $[S_{\mathbf{f}^{\text{ref}}}(\omega)]$ is such that (1) for all i in $\{1, \ldots, 7\}$, $[S_{\mathbf{f}^{\text{ref}}}(\omega)]_{ii}$ is a constant equal to $1.3N^2/Hz$ on the frequency band of analysis $\mathcal{B} = [-100, 100]\ Hz$ and (2) for all i and j in $\{1, \ldots, 7\}$, $|[S_{\mathbf{f}^{\text{ref}}}(\omega)]_{ij}|^2 = \gamma_{ij}(\omega)[S_{\mathbf{f}^{\text{ref}}}(\omega)]_{ii}[S_{\mathbf{f}^{\text{ref}}}(\omega)]_{jj}$ where $\gamma_{ij}(\omega) = exp(-|x_i - x_j|/\lambda)$ in which $|x_i - x_j|$ is the distance between the two excited points and $\lambda = 4\,m$ is a reference lenght related to the correlation lengh. In the frequency band of analysis $\mathcal{B}$, there are 21 eigenfrequencies for the linear system made up of the two coupled subsystems without the stops. The first three eigenfrequencies are 5.78 Hz , 15.9 Hz and 31.1 Hz and correspond to the eigenmodes for which all the transversal displacements of the five beams are in phase.

6.2 Data for the simplified computational model

This part is devoted to the construction of a simplified computational model that will be used to identify the stochastic loads. The simplified computational model consists in replacing the linear substem composed of 4 beams by a linear subsystem composed of an equivalent Euler beam (see Fig. 3). The non-linear subsystem is the same for the 2 models. The section of the

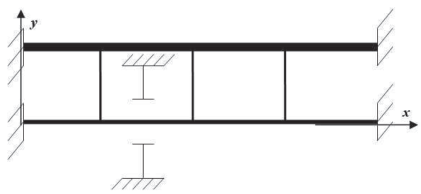

Figure 3. Simplified model.

equivalent beam is arbitrarily defined and is chosen as a constant circular section with radius $0.5\,m$, thickness $0.2\,m$, length $16\,m$. Its Young's modulus and its the mass density are identified so that the three first eigenfrequency of the simplified computational model are the same as the three first eigenfrequency of the reference computational model. It should be noted that this choice of simplified model as an equivalent beam does not allow several eigenfrequencies to be correctly fitted. After identification, the equivalent beam has a mass density $4 * 250 = 1000\,kg/m^3$ and a Young's modulus $4*450 = 1800\,N/mm^2$. In the frequency band of analysis $\mathcal{B}$, there are 10 eigenfrequencies for the linear system made up of the two coupled subsystems without the stops. The first three eigenfrequencies of the mean simplified computational model are 5.74 Hz , 15.3 Hz and 30.8 Hz which are to compare to 5.78 Hz , 15.9 Hz and 31.1 Hz of the reference computational model.

6.3 Comparison between the dynamical response of the reference computational model and the dynamical response of the mean simplified computational model.

For the two models, the stationary stochastic response is calculated in the time interval $[0, 220]\ s$ using an explicit Euler integration scheme for which the time step is $3\ ms$. Let P_{obs} be the impact point of the non-linear subsystem. The power spectral density function of the stochastic transversal displacement and the stochastic rotation responses in point P_{obs} (see Fig. 4) is estimated using the periodogram method. It can be seen that the prevision given by the mean simplified computational model is good in the frequency band $[0, 35]\ Hz$. Nevertheless, there are significant differences in the frequency band $[35, 100]\ Hz$ induced by model uncertainties. This is the reason why the model uncertainties are taken into account in order to extend the domain of validity of the mean simplified computational model in the frequency band $[35, 100]\ Hz$.

6.4 System uncertainties modelling and dispersion parameter identification.

The non-parametric probabilistic model of model uncertainites introduced in Section 3.2 is used for stiffness part of the linear subsystem of the mean simplified computational model. We then have to identify the dispersion parameter $\boldsymbol{\delta} = (\delta_K^B)$. The estimation of each probability

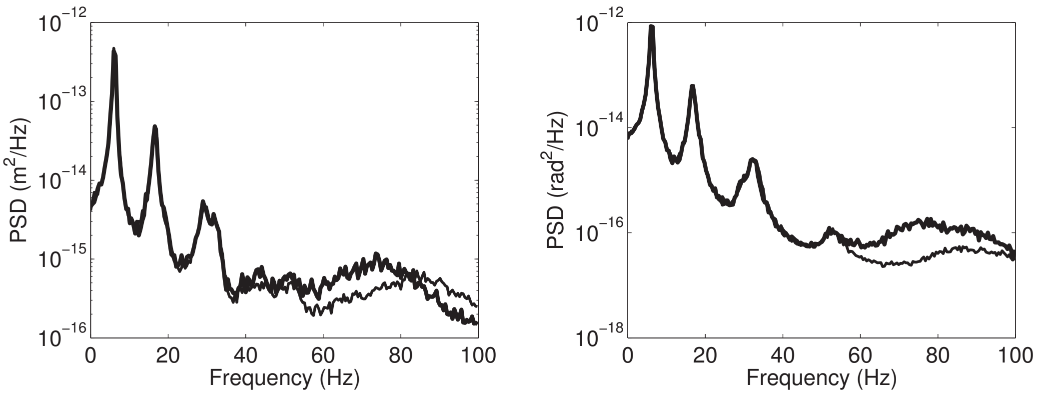

Figure 4: Power spectral density function (PSD) for the transversal displacement (left figure) and the rotation (right figure) responses. Comparison between the reference computational model in point $Pobs$ (thin line) and the response given by the mean simplified computational model (thick line).

density function in Eq. (7) is carried out with 200 realizations for the Monte Carlo simulation in order to solve stochastic simplified computational model. Fig. 5 shows the likelihood function calculated using Eq. (7) with $\mathcal{C}_{ad} = [0, \sqrt{23/27}]$. The maximum is reached for $\boldsymbol{\delta}^{opt} = 0.45$. Using Eq. (8) the confidence region associated with a probability level $P_c = 0.95$ of the reponse

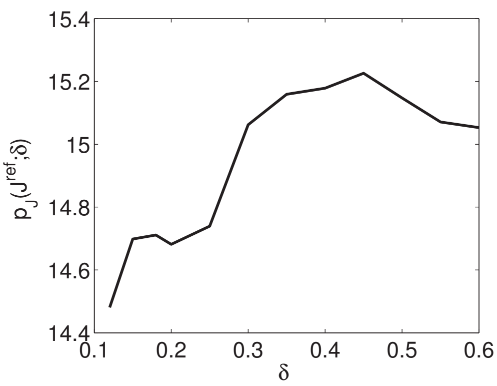

Figure 5. Graph of function $\delta \mapsto p_J(J^{ref}; \boldsymbol{\delta})$.

of the stochastic simplified computational model can then be estimated. The calculations are carried out with 100 simulations. The comparison between the reference solution with the response constructed with the stochastic simplified computational model is given in Fig. 6. This figure displays the confidence region of the power spectral density function of the stochastic transversal displacement and the stochastic rotation in point P_{obs}.

6.5 Probabilistic model of the stochastic load $\mathbf{F}(t)$.

We recall that the real model of the stochastic load used to construct the experimental responses in Section 6.4 is now assumed unknown and has to be identified using the stochastic simplified computational model. Consequently, we then have to define a model as simple as possible for the stochastic load $\mathbf{F}(t)$ introduced in Section 4.1. We have then chosen to model $\mathbf{F}(t)$ as

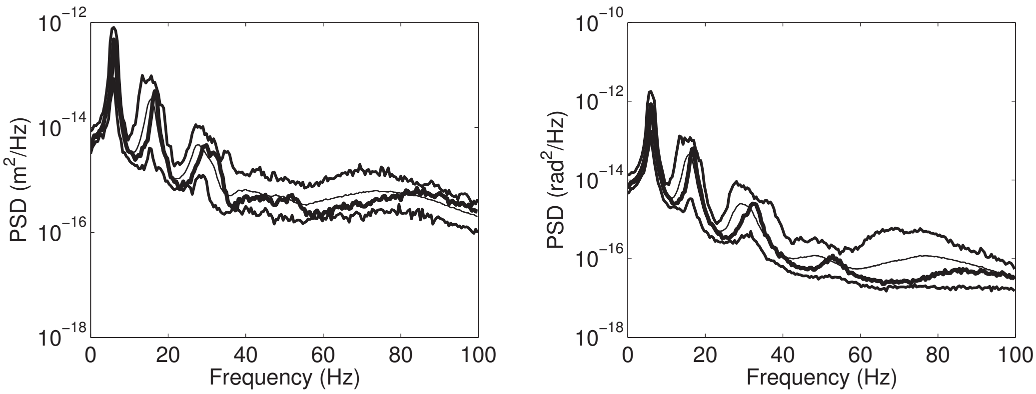

Figure 6: Power spectral density function (PSD) for the stochastic transversal displacement (left figure) and stochastic rotation (right figure) in point $Pobs$: confidence region calculated with the simplified model (thin line) and reference model (thick line).

$\{\mathbf{F}(t) = (T(t), M(t)), t \in \mathbb{R}\}$ in which $T(t)$ is a tranversal force and $M(t)$ a moment applied to the middle of the nonlinear beam (see Fig. 7). This force and this moment are second order

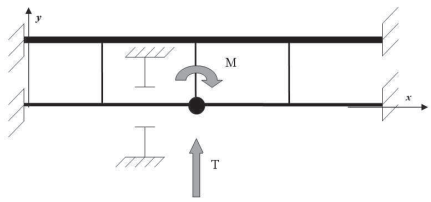

Figure 7. Definition of the stochastic load.

centred stationary Gaussian independent stochastic processes. So, they are both completely defined by their power spectral density functions $\underline{S}_{\mathbf{T}}(\omega)$ and $\underline{S}_{\mathbf{M}}(\omega)$. The matrix-valued spectral density function of the stochastic process $\{\mathbf{F}(t), t \in \mathbb{R}\}$ is then defined by

$$[\underline{\mathbf{S}}_{\mathbf{F}}(\omega)] = \begin{bmatrix} \underline{S}_{\mathbf{T}}(\omega) & 0 \\ 0 & \underline{S}_{\mathbf{M}}(\omega) \end{bmatrix} \quad , \quad \omega \in \mathbb{R} \quad . \tag{12}$$

It is assumed that the function $\omega \mapsto [\underline{\mathbf{S}}_{\mathbf{F}}(\omega)]$ is constant in the frequency band of analysis $\mathcal{B}$. As explained in Section 4.2, the stochastic process $\{\mathbf{F}^{\text{unc}}(t), t \in \mathbb{R}\}$ including the probabilistic model of uncertainties is contructed from the stochastic process $\{\mathbf{F}(t), t \in \mathbb{R}\}$.

6.6 Identification of the stochastic load $\mathbf{F}^{\text{unc}}(t)$.

The function $\omega \mapsto [\underline{\mathbf{S}}_{\mathbf{F}}(\omega)]$ which is a constant diagonal hermitian matrix over the frequency band of analysis $\mathcal{B}$ can then be rewritten as

$$[\underline{\mathbf{S}}_{\mathbf{F}}(\omega)] = [\underline{\mathbf{S}}(\omega, \mathbf{r})] = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix} \quad , \quad \omega \in \mathcal{B} \quad , \quad \mathbf{r} \in \mathcal{C}_r \quad , \tag{13}$$

in which the admissible space $\mathcal{C}_r = \{\mathbf{r} = (r_1, r_2); r_1 > 0, r_2 > 0\}$. This vector $\mathbf{r}$ is identified using the trial method, consisting in calculating the cost function $D(\mathbf{r})$ for 100 values of the vector $\mathbf{r}$. Then, the optimal value $\mathbf{r}_{opt}$ defined by Eq. (10) is such that $\mathbf{r}_{opt} =$

$(18\,N^2/Hz, 20\,(N/m)^2/Hz)$. The dispersion paramater δ_F is identified using Eq. (11) with $\mathcal{C}_{\delta_F} =]0, \sqrt{3/7}[$. The maximum of the likelihood function $\delta_F \mapsto p_{J_s}(J_s^{\exp}; \delta_F)$ is reached for $\delta_F^{opt} = 0.04$.

7. CONCLUSIONS

The problem under consideration is the identification of a stochastic load applied to a structure through the knowledge of dynamical responses of the structure which has a non-linear behaviour. This identification is performed using a computational non-linear dynamical model of the structure. Since there are both data uncertainties and model uncertainties in the computational model used to perform the identification, the first step of the development consists in introducing a probabilistic model of uncertainties in the structure. In addition, the identification of the stochastic load is carried out using a parametric representation of the stochastic process in order that the optimization problem relative to this inverse problem be feasible. The introduction of such a parametric representation induces again model uncertainties on the stochastic loads. The second step of the development then consists in introducing a probabilistic model of uncertainties concerning the stochastic loads. We have then presented a complete methodology to identify the stochastic load taking into account uncertainites in the computational model and in the stochastic loads representation. With respect to the state of the art, this work proposes a new way to perform the experimental identification of a stochastic load with a robust method. The robustness is introduced in taking into account (1) uncertainties in the simplified computational non-linear dynamical model used to carry out this identification and (2) uncertainties in the mathematical representation of the stochastic process which models the loads to be identified. The application presented is representative of a real industrial system and validates the methodology proposed.

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