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# TRANSIENT ELASTIC WAVES IN FLUID-STRUCTURE MULTILAYER SYSTEMS WITH A PROBABILISTIC MODEL OF STRUCTURAL UNCERTAINTIES

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**ABSTRACT.** This paper deals with the development of a computational model to predict transient elastic waves in fluid-structure multilayer systems for which the elasticity constants of the structure are uncertain. The fluid-structure system is a three layers system make up of an elastic solid layer sandwiched between two acoustic fluid layers and excited by an acoustic line source located in one of the two acoustic fluid layers. The mean model of the elastic solid layer is represented by a transverse isotropic material. The elasticity tensor of the solid layer is modeled by a random tensor for which the probabilistic model is constructed using the information theory. A Monte Carlo stochastic numerical solver is used in order to solve the stochastic boundary value problem. A numerical application is presented.

**KEYWORDS:** Uncertainties, Wave propagation, multilayer

## 1 INTRODUCTION

The analysis of wave phenomena [1,2] in layered elastic [3,4] and acoustic media plays a fundamental role in the fields of non-destructive testing, geophysics and seismology [2,5-10]. This paper deals with the development of a computational model to predict transient elastic waves in fluid-structure multilayer systems for which the elasticity tensor of the structure is uncertain (e.g [9]). The fluid-structure system is a three layers system constituted of a homogeneous elastic solid layer sandwiched between two acoustic fluid layers and excited by an acoustic line source located in one of the two acoustic fluid layers. The mean model of the elastic solid layer is represented by a transverse isotropic material. Due to uncertainties in the solid layer induced by heterogeneities in the material, this solid layer is modeled by a stochastic homogeneous anisotropic material for which the mean value is the mean model. A parametric probabilistic approach is used to take into account uncertainties in the dynamical system. The elasticity tensor of the solid layer is modeled by a random tensor for which the probabilistic model is constructed using the information theory. The Monte-Carlo numerical method is used to solve the stochastic boundary value problem. For each realization of the random elastic tensor, the transient elastic waves are calculated in the coupled system by using a hybrid method [11] based on a time-domain formulation associated with the space Fourier transform for the infinite dimension and using a finite element approximation [12,13] for the finite dimension.

A completed numerical application concerning the cortical bone excited with a transient acoustic line source whose central frequency is 1MHz is presented showing the propagation of uncertainties in the fluid-structure dynamical system.

## 2 MEAN 3D BOUNDARY VALUE PROBLEMS IN THE 3D SPACE-DOMAIN WITH A TIME-DOMAIN FORMULATION

We consider a three-dimensional multilayer medium composed of one solid layer sandwiched between two fluid layers (see Figure 1). Let  $\mathbf{R} = (O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$  be the reference Cartesian frame

where  $O$  is the origin of the space and  $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$  is an orthogonal basis for this space. Let  $(x_1, x_2, x_3)$  be the coordinates of a generic point  $\mathbf{x}$  in  $\mathbf{R}$ . The thicknesses of the layers are denoted as  $h_1, h$  and  $h_2$ . Thus,  $h_1$  is the thickness of the first fluid layer,  $h$  is the solid layer thickness and  $h_2$  is the thickness of the second fluid layer. The first fluid layer occupies the unbounded domain  $\Omega_1$ , the solid elastic layer occupies the domain  $\Omega$  and the second acoustic fluid layer occupies the domain  $\Omega_2$ . Let  $\Gamma_1, \Gamma_0, \Gamma$  and  $\Gamma_2$  be the planes defined by

$$\Gamma_1 = \{x_1 \in \mathbb{R}, \quad x_2 \in \mathbb{R}, x_3 = z_1\}$$

$$\Gamma_0 = \{x_1 \in \mathbb{R}, \quad x_2 \in \mathbb{R}, x_3 = 0\}$$

$$\Gamma = \{x_1 \in \mathbb{R}, \quad x_2 \in \mathbb{R}, x_3 = z\}$$

$$\Gamma_2 = \{x_1 \in \mathbb{R}, \quad x_2 \in \mathbb{R}, x_3 = z_2\}$$

in which  $z_1 = h_1, z = -h$  and  $z_2 = -(h + h_2)$ . Then, the boundaries of domains  $\Omega_1, \Omega$  and  $\Omega_2$  are respectively  $\partial\Omega_1 = \Gamma_1 \cup \Gamma_0, \partial\Omega = \Gamma_0 \cup \Gamma, \partial\Omega_2 = \Gamma \cup \Gamma_2$ . Therefore, domains  $\Omega_1, \Omega$  and  $\Omega_2$  are unbounded along the transversal directions  $\mathbf{e}_1$  and  $\mathbf{e}_2$  whereas they are bounded along the direction  $\mathbf{e}_3$ .

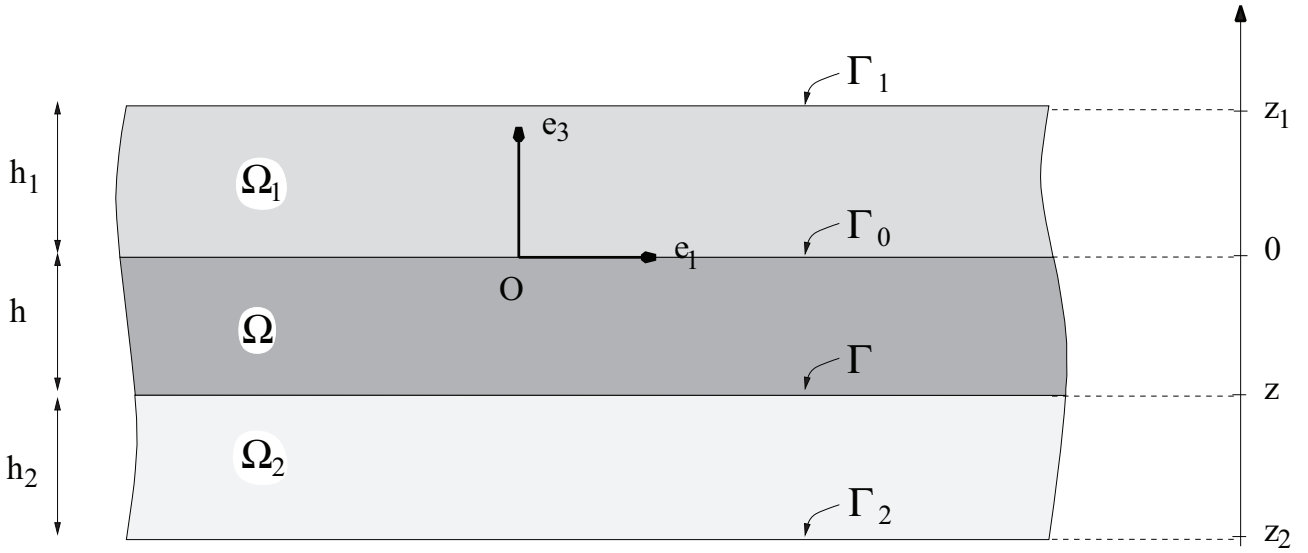


Fig 1. Geometric configuration

Let  $p_1(\mathbf{x}, t)$  and  $p_2(\mathbf{x}_t)$  be the disturbance of the pressure of the fluid layer at time  $t > 0$  for  $\mathbf{x}$  belonging to respectively  $\Omega_1$  and  $\Omega_2$ . The mean boundary value problems for the two fluid layers are written as

$$\frac{1}{k_1} \frac{\partial^2 p_1}{\partial t^2} - \frac{1}{\rho_1} \Delta p_1 = \frac{1}{\rho_1} \frac{\partial Q}{\partial t}, \quad \mathbf{x} \in \Omega_1 \quad (1)$$

$$p_1 = 0, \quad \mathbf{x} \in \Gamma_1 \quad (2)$$

$$\frac{\partial p_1}{\partial x_3} = -\rho_1 \frac{\partial^2 u_3}{\partial t^2}, \quad \mathbf{x} \in \Gamma_0 \quad (3)$$

$$\frac{1}{k_2} \frac{\partial^2 p_2}{\partial t^2} - \frac{1}{\rho_2} \Delta p_2 = 0, \quad \mathbf{x} \in \Omega_2 \quad (4)$$

$$p_2 = 0, \quad \mathbf{x} \in \Gamma_2 \quad (5)$$

$$\frac{\partial p_2}{\partial x_3} = -\rho_2 \frac{\partial^2 u_3}{\partial t^2}, \quad \mathbf{x} \in \Gamma \quad (6)$$

in which  $k_1 = \rho_1 c_1^2$  where  $c_1$  and  $\rho_1$  are, respectively, the wave velocity and the mass density at equilibrium of the first fluid occupying domain  $\Omega_1$ ;  $k_2 = \rho_2 c_2^2$  where  $c_2$  and  $\rho_2$  are, respectively, the wave velocity and the mass density at equilibrium of the second fluid occupying domain  $\Omega_2$ ;  $\Delta$  is the Laplacian operator with respect to  $\mathbf{x}$  and  $Q(\mathbf{x}, t)$  is an acoustic source density at point  $\mathbf{x} = (x_1, x_2, x_3)$

and at time  $t > 0$ . Acoustic source density  $Q(\mathbf{x}, t)$  is such that

$$\frac{\partial Q}{\partial t}(\mathbf{x}, t) = \rho_1 F(t) \delta_0(x_1 - x_1^S) \delta_0(x_3 - x_3^S) \quad , \quad (7)$$

where  $x_3^S$  is a given parameter in  $[0, h_1]$  and where  $x_1^S$  is a given parameter in  $\mathbb{R}$ . Thus, Eq. (7) describes an impulse line source parallel to  $(O; \mathbf{x}_2)$ , placed in the fluid  $\Omega_1$  at a given distance from the interface  $\Gamma_0$ .

Let  $\mathbf{u}(\mathbf{x}, t)$  be the displacement of a particle located in point  $\mathbf{x}$  of  $\Omega$  at time  $t > 0$  and verifying the following boundary value problem,

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} - \text{div} \boldsymbol{\sigma} = \mathbf{0} \quad , \quad \mathbf{x} \in \Omega \quad (8)$$

$$\boldsymbol{\sigma} \mathbf{n} = -p_1 \mathbf{n} \quad , \quad \mathbf{x} \in \Gamma_0 \quad (9)$$

$$\boldsymbol{\sigma} \mathbf{n} = -p_2 \mathbf{n} \quad , \quad \mathbf{x} \in \Gamma \quad (10)$$

in which  $\rho$  is the mass density and  $\boldsymbol{\sigma}(\mathbf{x}, t)$  is the Cauchy stress tensor of the elastic medium at point  $\mathbf{x}$  and at time  $t > 0$ ,  $\mathbf{n}$  is the outward unit normal to domain  $\Omega$  and  $\text{div}$  is the divergence operator with respect to  $\mathbf{x}$ . The constitutive equation of the solid elastic medium is written as

$$\boldsymbol{\sigma}(\mathbf{x}, t) = \sum_{i,j,k,h=1}^3 c_{ijkl} \varepsilon_{kh}(\mathbf{x}, t) \mathbf{e}_i \otimes \mathbf{e}_j \quad (11)$$

in which  $\sum_{i,j,k,h=1}^3 c_{ijkl} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_h$  is the elasticity tensor of the medium and  $\varepsilon_{kh} = \frac{1}{2} \left( \frac{\partial u_k}{\partial x_h} + \frac{\partial u_h}{\partial x_k} \right)$  is the linearized strain tensor. Finally, the system is at rest at time  $t = 0$ . Consequently, we have

$$p_1(\mathbf{x}, 0) = 0 \quad , \quad \mathbf{x} \in \Omega_1 \cup \partial\Omega_1 \quad (12)$$

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{0} \quad , \quad \mathbf{x} \in \Omega \cup \partial\Omega \quad (13)$$

$$p_2(\mathbf{x}, 0) = 0 \quad , \quad \mathbf{x} \in \Omega_2 \cup \partial\Omega_2 \quad (14)$$

### 3 MEAN ELASTICITY MATRIX FOR AN ISOTROPIC TRANSVERSE MATERIAL

Let  $[C]$  be the elasticity matrix whose components are the coefficients of elasticity tensor  $c_{ijkl}$  such that

$$[C] = \begin{pmatrix} c_{1111} & c_{1122} & c_{1133} & \sqrt{2}c_{1123} & \sqrt{2}c_{1131} & \sqrt{2}c_{1112} \\ c_{2211} & c_{2222} & c_{2233} & \sqrt{2}c_{2223} & \sqrt{2}c_{2231} & \sqrt{2}c_{2213} \\ c_{3311} & c_{3322} & c_{3333} & \sqrt{2}c_{3323} & \sqrt{2}c_{3331} & \sqrt{2}c_{3312} \\ \sqrt{2}c_{2311} & \sqrt{2}c_{2322} & \sqrt{2}c_{2333} & 2c_{2323} & 2c_{2331} & 2c_{2312} \\ \sqrt{2}c_{3111} & \sqrt{2}c_{3122} & \sqrt{2}c_{3133} & 2c_{3123} & 2c_{3131} & 2c_{3112} \\ \sqrt{2}c_{1211} & \sqrt{2}c_{1222} & \sqrt{2}c_{1233} & 2c_{1223} & 2c_{1231} & 2c_{1212} \end{pmatrix}. \quad (15)$$

Let  $c_{ij} = [C]_{ij}$  be the components of matrix  $[C]$ . If the elasticity tensor is modeling an isotropic homogeneous medium, then all the components  $c_{ij}$  are zeros except the following

$$c_{11} = \frac{e_L^2(1 - \nu_T)}{(e_L - e_L\nu_T - 2e_T\nu_L^2)}, \quad c_{22} = c_{33} = \frac{e_T(e_L - e_T\nu_L^2)}{(1 + \nu_T)(e_L - e_L\nu_T - 2e_T\nu_L^2)}$$

$$c_{12} = c_{13} = c_{21} = c_{31} = \frac{e_T e_L \nu_L}{(e_L - e_L\nu_T - 2e_T\nu_L^2)}, \quad c_{23} = c_{32} = \frac{e_T(e_L\nu_T + e_T\nu_L^2)}{(1 + \nu_T)(e_L - e_L\nu_T - 2e_T\nu_L^2)}$$

$$c_{44} = g_T = \frac{e_T}{2(1 + \nu_T)}, \quad c_{55} = c_{66} = g_L$$

in which  $e_L$  and  $e_T$  are the longitudinal and transversal Young moduli,  $g_L$  and  $g_T$  are the longitudinal and transversal shear moduli, respectively;  $\nu_L$  and  $\nu_T$  are the longitudinal and transversal Poisson coefficients, respectively.

#### 4 MEAN WEAK FORMULATION IN THE 1D-SPECTRAL DOMAIN WITH A TIME-DOMAIN FORMULATION

Due to the nature of the source and to the geometrical configuration, the transverse waves polarized in the  $(\mathbf{e}_1, \mathbf{e}_2)$  plane are not excited. Then, the present study can be conducted in the plane  $(O; \mathbf{e}_1, \mathbf{e}_3)$  and the mean 3D boundary value problem is independent of  $x_2$ .

For all  $x_3$  fixed in  $]z_2, z_1[$ , the 1D-Fourier transform of an integrable function  $x_1 \mapsto f(x_1, x_3, t)$  on  $\mathbb{R}$  is defined by

$$\widehat{f}(k_1, x_3, t) = \int_{\mathbb{R}} f(x_1, x_3, t) e^{ik_1 x_1} dx_1 \quad .$$

Let  $\widehat{p}_1, \widehat{\mathbf{u}}$  and  $\widehat{p}_2$  be the 1D-Fourier transforms of functions  $p_1, \mathbf{u}$  and  $p_2$ . Let  $C_1$  and  $C_2$  be the function spaces constituted of all the sufficiently differentiable complex-valued functions  $x_3 \mapsto \delta p_1(x_3)$  and  $x_3 \mapsto \delta p_2(x_3)$  respectively, defined on  $]0, z_1[$  and  $]z_2, z_1[$ . We introduce the admissible function spaces  $C_{1,0} \subset C_1$  and  $C_{2,0} \subset C_2$  such that

$$\begin{aligned} C_{1,0} &= \{\delta p_1 \in C_1; \quad \delta p_1(z_1) = 0\} \\ C_{2,0} &= \{\delta p_2 \in C_2; \quad \delta p_2(z_2) = 0\} \end{aligned}$$

Let  $C$  be the admissible function space constituted of all the sufficiently differentiable functions  $x_3 \mapsto \delta \mathbf{u}(x_3)$  from  $]z, 0[$  into  $\mathbb{C}^2$  where  $\mathbb{C}$  is the set of all the complex numbers. The weak formulation of the present problem is written as : for all  $k_1$  fixed in  $\mathbb{R}$  and for all fixed  $t$ , find  $\widehat{p}_1(k_1, \cdot, t) \in C_{1,0}$ ,  $\widehat{\mathbf{u}}(k_1, \cdot, t) \in C$  and  $\widehat{p}_2(k_1, \cdot, t) \in C_{2,0}$  such that, for all  $\delta p_1 \in C_{1,0}$ ,  $\delta \mathbf{u} \in C$  and  $\delta p_2 \in C_{2,0}$ ,

$$\begin{aligned} a_1 \left( \frac{\partial^2 \widehat{p}_1}{\partial t^2}, \delta p_1 \right) + k_1^2 c_1^2 a_1(\widehat{p}_1, \delta p_1) + b_1(\widehat{p}_1, \delta p_1) + r_1 \left( \frac{\partial^2 \widehat{\mathbf{u}}}{\partial t^2}, \delta p_1 \right) &= f(\delta p_1; t), \\ m \left( \frac{\partial^2 \widehat{\mathbf{u}}}{\partial t^2}, \delta \mathbf{u} \right) + s_1(\widehat{\mathbf{u}}, \delta \mathbf{u}) + k_1^2 s_2(\widehat{\mathbf{u}}, \delta \mathbf{u}) - ik_1 s_3(\widehat{\mathbf{u}}, \delta \mathbf{u}) + \overline{r_2(\delta \mathbf{u}, \widehat{p}_2)} - \overline{r_1(\delta \mathbf{u}, \widehat{p}_1)} &= 0, \\ a_2 \left( \frac{\partial^2 \widehat{p}_2}{\partial t^2}, \delta p_2 \right) + k_1^2 c_2^2 a_2(\widehat{p}_2, \delta p_2) + b_2(\widehat{p}_2, \delta p_2) - r_2 \left( \frac{\partial^2 \widehat{\mathbf{u}}}{\partial t^2}, \delta p_2 \right) &= 0, \end{aligned}$$

in which the positive-definite and definite sesquilinear forms  $a_1$  and  $b_1$  defined on  $C_1 \times C_1$ , the sesquilinear form  $r_1$  defined on  $C \times C_1$ , the antilinear form  $f_1$  defined on  $C_1$ , the sesquilinear forms positive-definite and positive  $a_2$  and  $b_2$  defined on  $C_2 \times C_2$ , the sesquilinear form  $r_2$  defined on  $C \times C_2$ , the positive-definite sesquilinear form  $a$  defined on  $C \times C$  and finally, the sesquilinear form  $b$  defined on  $C \times C$  which are presented in Appendix and where

$$s_1(\widehat{\mathbf{u}}, \delta \mathbf{u}) = \int_z^0 \langle [D_1] \frac{\partial \widehat{\mathbf{u}}}{\partial x_3}, \overline{\frac{\partial \delta \mathbf{u}}{\partial x_3}} \rangle dx_3 \quad (16)$$

$$s_2(\widehat{\mathbf{u}}, \delta \mathbf{u}) = \int_z^0 \langle [D_2] \widehat{\mathbf{u}}, \overline{\delta \mathbf{u}} \rangle dx_3 \quad (17)$$

$$s_3(\widehat{\mathbf{u}}, \delta \mathbf{u}) = \int_z^0 \left( \langle [D_3] \widehat{\mathbf{u}}, \overline{\frac{\partial \delta \mathbf{u}}{\partial x_3}} \rangle - \langle [D_3] \overline{\delta \mathbf{u}}, \frac{\partial \widehat{\mathbf{u}}}{\partial x_3} \rangle \right) dx_3 \quad (18)$$

in which  $\langle \cdot, \cdot \rangle$  means the usual Euclidean inner product on  $\mathbb{R}^2$  extended to  $\mathbb{C}^2$  and where

$$[D_1] = \begin{bmatrix} c_{55}/2 & c_{53}/\sqrt{2} \\ c_{35}/\sqrt{2} & c_{33} \end{bmatrix}, \quad [D_2] = \begin{bmatrix} c_{11} & c_{15}/\sqrt{2} \\ c_{51}/\sqrt{2} & c_{55}/2 \end{bmatrix}, \quad [D_3] = \begin{bmatrix} c_{51}/\sqrt{2} & c_{55}/2 \\ c_{31} & c_{35}/\sqrt{2} \end{bmatrix} \quad . \quad (19)$$

It should be noted that only  $s_1(\widehat{\mathbf{u}}, \delta \mathbf{u})$ ,  $s_2(\widehat{\mathbf{u}}, \delta \mathbf{u})$  and  $s_3(\widehat{\mathbf{u}}, \delta \mathbf{u})$  depend on components of elasticity matrix  $[C]$ .

## 5 MEAN FINITE ELEMENT MODEL IN THE 1D-SPECTRAL DOMAIN WITH A TIME-DOMAIN FORMULATION

We introduce a finite element mesh of domain  $[z_2, z] \cup [z, 0] \cup [0, z_1]$  which is constituted of  $\nu_{tot}$  nodes. The finite elements used are Lagrangian 1D-finite element with 3 nodes. Let  $\widehat{\mathbf{p}}_1(k_1, t)$ ,  $\widehat{\mathbf{v}}(k_1, t)$  and  $\widehat{\mathbf{p}}_2(k_1, t)$  be the complex vectors of the nodal values of the functions  $x_3 \mapsto \widehat{p}_1(k_1, x_3, t)$ ,  $x_3 \mapsto \widehat{\mathbf{u}}(k_1, x_3, t)$  and  $x_3 \mapsto \widehat{p}_2(k_1, x_3, t)$ . Let  $\widehat{\mathbf{f}}(k_1, t)$  be the complex vector in  $\mathbb{C}^{\nu_1}$  where  $\nu_1$  is the number of degree of freedom related to the mesh of domain  $[0, z_1]$ , corresponding to the finite element approximation of the antilinear form  $f(\delta p_1; t)$ . For all  $k_1$  fixed in  $\mathbb{R}$  and for all fixed  $t$ , the finite element approximation of the weak formulation of the 1D boundary value problem yields the following linear system of equations

$$[A_1] \ddot{\widehat{\mathbf{p}}}_1 + (k_1^2 c_1^2 [A_1] + [B_1]) \widehat{\mathbf{p}}_1(k_1, t) + [R_1] \widehat{\mathbf{v}}(k_1, t) = \widehat{\mathbf{f}}(k_1, t) \quad (20)$$

$$[M] \ddot{\widehat{\mathbf{v}}}(k_1, t) + ([S_1] - ik_1 [S_3] + k_1^2 [S_2]) \widehat{\mathbf{v}}(k_1, t) + [R_2]^T \widehat{\mathbf{p}}_2(k_1, t) - [R_1]^T \widehat{\mathbf{p}}_1(k_1, t) = 0 \quad (21)$$

$$[A_2] \ddot{\widehat{\mathbf{p}}}_2(k_1, t) + (k_1^2 c_2^2 [A_2] + [B_2]) \widehat{\mathbf{p}}_2(k_1, t) - [R_2] \widehat{\mathbf{v}}(k_1, t) = 0 \quad (22)$$

in which the double dots means the second partial derivative with respect to  $t$ . Each of Eqs. (20), (21) and (22) form linear systems whose the square matrices are respectively of dimensions  $\nu_1 \times \nu_1$ ,  $\nu \times \nu$  and  $\nu_2 \times \nu_2$ . The integer numbers  $\nu$  and  $\nu_2$  are respectively the number of degree of freedom related to the meshes of domains  $[z, 0]$  and  $[z_2, z]$ . Moreover, the components of these matrices are complex numbers. These three equations can be rewritten as

$$[M] \ddot{\widehat{\mathbf{v}}}(k_1, t) + ([K_1] - ik_1 [K_2] + k_1^2 [K_3]) \widehat{\mathbf{v}}(k_1, t) = \widehat{\mathbf{f}}(k_1, t) \quad (23)$$

in which the vectors  $\widehat{\mathbf{v}}(k_1, t) = (\widehat{\mathbf{p}}_1(k_1, t), \widehat{\mathbf{v}}(k_1, t), \widehat{\mathbf{p}}_2(k_1, t))$  and  $\widehat{\mathbf{f}}(k_1, t) = (\widehat{\mathbf{f}}(k_1, t), 0, 0)$  belong to  $\mathbb{C}^{\nu_1 + \nu + \nu_2}$  and where

$$[M] = \begin{bmatrix} [A_1] & [R_1] & 0 \\ 0 & [M] & 0 \\ 0 & -[R_2] & [A_2] \end{bmatrix}, \quad [K_1] = \begin{bmatrix} [B_1] & 0 & 0 \\ -[R_1]^T & [S_1] & [R_2]^T \\ 0 & 0 & [B_2] \end{bmatrix}, \quad [K_2] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & [S_3] & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [K_3] = \begin{bmatrix} c_1^2 [A_1] & 0 & 0 \\ 0 & [S_2] & 0 \\ 0 & 0 & c_2^2 [A_2] \end{bmatrix}$$

where upper-script  $T$  denotes the transpose matrix. It should be noted that matrices  $[S_1]$ ,  $[S_2]$  and  $[S_3]$  correspond to the finite element approximations of sesquilinear forms  $s_1$ ,  $s_2$  and  $s_3$ . Consequently, matrices  $[S_1]$ ,  $[S_2]$  and  $[S_3]$  depend on components of elasticity matrix  $[C]$  (see Eqs (16) to (19)) and there exist mappings  $\mathfrak{g}_1$ ,  $\mathfrak{g}_2$  and  $\mathfrak{g}_3$  such that  $[S_1] = \mathfrak{g}_1([C])$ ,  $[S_2] = \mathfrak{g}_2([C])$  and  $[S_3] = \mathfrak{g}_3([C])$ .

## 6 PROBABILISTIC MODEL OF STRUCTURAL UNCERTAINTIES

This section is devoted to the construction of a probabilistic model of uncertainties in the solid layer. It is assumed that uncertainties are only related to the components of elasticity tensor  $c_{ijkl}$ . The stochastic finite element model is constructed by substituting matrix  $[C]$  in the mean finite element model with a random matrix  $[\mathbf{C}]$  whose probabilistic model is constructed using the information theory. The available information on  $[\mathbf{C}]$  is defined as follows: (1) the mean value of random matrix  $[\mathbf{C}]$  is the mean elasticity matrix  $[C]$  of the mean model; (2) random matrix  $[\mathbf{C}]$  is a second-order random variable with values in  $\mathbb{M}_n^+(\mathbb{R})$  with  $n = 6$  where  $\mathbb{M}_n^+(\mathbb{R})$  is the set of all the  $(n \times n)$  real symmetric positive-definite matrices; (3) the inverse matrix of  $[\mathbf{C}]$  which exists almost surely is assumed to be a second-order random variable. Thus, random matrix  $[\mathbf{C}]$  belongs to the set  $\text{SE}^+$  (see [13]) and is written as

$$[\mathbf{C}] = [L]^T [\mathbf{G}] [L] \quad (24)$$

in which the  $(6 \times 6)$  upper triangular matrix  $[L]$  corresponds to the Cholesky factorization of matrix  $[C]$  and where random matrix  $[G]$  belongs to the set  $SG^+$  defined in [13]. The probability density function  $p_{[G]}$  of random matrix  $[G]$  is written as

$$p_{[G]}([G]) = \mathbb{1}_{\mathbb{M}_n^+(\mathbb{R})}([G]) \times c_n \times (\det[G])^{b_n} \times \exp\{-a_n \text{tr}[G]\} \quad , \quad (25)$$

with  $n = 6$  and where  $a_n = (n + 1)/(2\delta^2)$ ,  $b_n = a_n(1 - \delta^2)$ ,  $\mathbb{1}_{\mathbb{M}_n^+(\mathbb{R})}([G])$  is equal to 1 if  $[G]$  belongs to  $\mathbb{M}_n^+(\mathbb{R})$  and is equal to zero if  $[G]$  does not belong to  $\mathbb{M}_n^+(\mathbb{R})$ ,  $\text{tr}[G]$  is the trace of a matrix  $[G]$  and where positive constant  $c_n$  is such that

$$c_n = \frac{(2\pi)^{-n(n-1)/4} a_n^{na_n}}{\prod_{j=1}^n \Gamma(\alpha_j)} \quad , \quad (26)$$

in which  $\alpha_j = (n + 1)/(2\delta^2) + (1 - j)/2$  and  $\Gamma$  is the Gamma function. Parameter  $\delta$  allows the dispersion of the stochastic model to be controlled. It should be noted that such a probabilistic model takes into account any anisotropic perturbation of the elasticity tensor with respect to a mean elasticity tensor of a simplified elasticity model such as, for instance, an isotropic transverse solid. Note that the components  $C_{ij} = [C]_{ij}$  of random matrix  $[C]$  are statistically dependent random variables with values in  $\mathbb{R}$  and depends on dispersion parameter  $\delta$ .

## 7 STOCHASTIC FINITE ELEMENT MODEL IN THE 1D-SPECTRAL DOMAIN WITH A TIME-DOMAIN FORMULATION

The stochastic finite element model of the system is constructed substituting  $[S_1]$ ,  $[S_2]$  and  $[S_3]$  in Eqs. (20) to (22) with random matrices  $[S_1]$ ,  $[S_2]$  and  $[S_3]$  by  $[S_1] = \mathfrak{g}_1([C])$ ,  $[S_2] = \mathfrak{g}_2([C])$  and  $[S_3] = \mathfrak{g}_3([C])$  where mappings  $\mathfrak{g}_1$ ,  $\mathfrak{g}_2$  and  $\mathfrak{g}_3$  are introduced in Section 5. Consequently, for all time  $t$  fixed in  $\mathbb{R}$  and for all  $k_1$  fixed in  $\mathbb{R}$ , the solution of the stochastic finite element model is a random vector  $\widehat{\mathbf{V}}(k_1, t) = (\widehat{\mathbf{P}}_1(k_1, t), \widehat{\mathbf{V}}(k_1, t), \widehat{\mathbf{P}}_2(k_1, t))$  such that

$$[A_1] \ddot{\widehat{\mathbf{P}}}_1 + (k_1^2 c_1^2 [A_1] + [B_1]) \widehat{\mathbf{P}}_1(k_1, t) + [R_1] \ddot{\widehat{\mathbf{V}}}(k_1, t) = \widehat{\mathbf{f}}(k_1, t) \quad (27)$$

$$[M] \ddot{\widehat{\mathbf{V}}}(k_1, t) + ([S_1] - ik_1 [S_3] + k_1^2 [S_2]) \widehat{\mathbf{V}}(k_1, t) + [R_2]^T \widehat{\mathbf{P}}_2(k_1, t) - [R_1]^T \widehat{\mathbf{P}}_1(k_1, t) = 0 \quad (28)$$

$$[A_2] \ddot{\widehat{\mathbf{P}}}_2(k_1, t) + (k_1^2 c_2^2 [A_2] + [B_2]) \widehat{\mathbf{P}}_2(k_1, t) - [R_2] \ddot{\widehat{\mathbf{V}}}(k_1, t) = 0 \quad (29)$$

These three equations can be rewritten as

$$[M] \ddot{\widehat{\mathbf{V}}}(k_1, t) + ([\mathbf{K}_1] - ik_1 [\mathbf{K}_2] + k_1^2 [\mathbf{K}_3]) \widehat{\mathbf{V}}(k_1, t) = \widehat{\mathbf{F}}(k_1, t) \quad (30)$$

in which, matrix  $[M]$  and vector  $\widehat{\mathbf{F}}(k_1, t)$  are defined in Section 5 and where random matrices  $[\mathbf{K}_1]$ ,  $[\mathbf{K}_2]$  and  $[\mathbf{K}_3]$  are such that

$$[\mathbf{K}_1] = \begin{bmatrix} [B_1] & 0 & 0 \\ -[R_1]^T & [S_1] & [R_2]^T \\ 0 & 0 & [B_2] \end{bmatrix} \quad , \quad [\mathbf{K}_2] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & [S_3] & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad , \quad [\mathbf{K}_3] = \begin{bmatrix} c_1^2 [A_1] & 0 & 0 \\ 0 & [S_2] & 0 \\ 0 & 0 & c_2^2 [A_2] \end{bmatrix}$$

## 8 STOCHASTIC SOLVER

By construction, for all  $k_1$  fixed in  $\mathbb{R}$  and for all fixed time  $t > 0$ , random vector  $\widehat{\mathbf{P}}_1(k_1, t)$  in Eqs. (27) to (29) is the finite element approximation of random field  $\widehat{P}_1(k_1, \cdot, t)$  indexed by  $[0, h_1]$  associated with the deterministic field  $\widehat{p}_1(k_1, \cdot, t)$ . The inverse 1D-Fourier transform in  $k_1$  of  $\widehat{P}_1$  is denoted as  $P_1$  and is a random field indexed by  $\mathbb{R} \times [0, h_1] \times [0, +\infty[$  modeling the random disturbance of the pressure in the first fluid layer due to uncertainties in the solid layer (see Section 6). Then, there exists a deterministic mapping  $\mathfrak{g}_{P_1}$  such that  $P_1 = \mathfrak{g}_{P_1}(\widehat{\mathbf{P}}_1)$ . Let the random arrival time  $T$  be the first



local maximum of stochastic field  $\{P_1(x_1^R, x_3^R, t)\}_{t>0}$  in which  $x_1^R$  and  $x_3^R$  are given parameters in  $\mathbb{R}$  and  $[0, h_1]$ , respectively. Let  $\mathfrak{G}_T$  be the mapping defined as  $T = \mathfrak{G}_T(P_1)$ .

The stochastic solver used in order to construct statistical estimations of  $T$  and  $P_1$  is based on the Monte-Carlo numerical simulation. For each realization  $[\mathbf{C}(\theta)]$  of random matrix  $[\mathbf{C}]$ , realization  $[\mathbf{S}_1(\theta)] = \mathfrak{G}_1([\mathbf{C}(\theta)])$ ,  $[\mathbf{S}_2(\theta)] = \mathfrak{G}_2([\mathbf{C}(\theta)])$  and  $[\mathbf{S}_3(\theta)] = \mathfrak{G}_3([\mathbf{C}(\theta)])$  of random matrices  $[\mathbf{S}_1]$ ,  $[\mathbf{S}_2]$  and  $[\mathbf{S}_3]$  are constructed. Then, for all  $k_1$  fixed in  $\mathbb{R}$  and for all fixed time  $t > 0$ , the realization  $\widehat{\mathbf{P}}_1(k_1, t, \theta)$  of random vector  $\widehat{\mathbf{P}}_1(k_1, t)$  is calculated solving the deterministic equation associated with stochastic Eq. (30) using an implicit time integration scheme. Then, the realization  $P_1(\theta) = \mathfrak{G}_{P_1}(\widehat{\mathbf{P}}_1(\theta))$  of random field  $P_1$  is calculated. Finally, the realization  $T(\theta) = \mathfrak{G}_T(P_1(\theta))$  of random arrival time  $T$  can be calculated.

## 9 NUMERICAL APPLICATION

For the numerical application presented in this section, the fluid layer  $\Omega_1$  is excited by a line source located at  $x_1^S = 0$  and  $x_3^S = 2 \times 10^{-3}$  m with a time-history defined with the function  $F$  in Eq. (7) such that

$$F(t) = F_1 \sin(2\pi f_c t) e^{-4(t f_c - 1)^2} \quad ,$$

where  $f_c = 1$  MHz is the center frequency and  $F_1 = 100$  m.s<sup>-2</sup> is an amplitude factor. Figure 2 shows the power spectrum of  $F$  (left) and the graph of function  $t \mapsto F(t)$  (right). The thicknesses of the three layers are  $h_1 = 2 \times 10^{-3}$  m,  $h = 4 \times 10^{-3}$  m and  $h_2 = 10^{-2}$  m. The mechanical parameters of the first fluid layer are  $\rho_1 = 1000$  kg/m<sup>3</sup> and  $c_1 = 1500$  m/s. For the second fluid layer, the mechanical parameters are  $\rho_2 = 1000$  kg/m<sup>3</sup> and  $c_2 = 1500$  m/s. Finally, for the elastic solid layer we will use the longitudinal and transversal Young moduli  $e_L = 16.6$  GPa and  $e_T = 9.5$  GPa, respectively; the longitudinal and transversal shear moduli  $g_L = 4.7$  GPa and  $g_T = 3.3$  GPa, respectively; the longitudinal and transversal Poisson coefficients  $\nu_L = 0.38$  and  $\nu_T = 0.44$ , respectively.

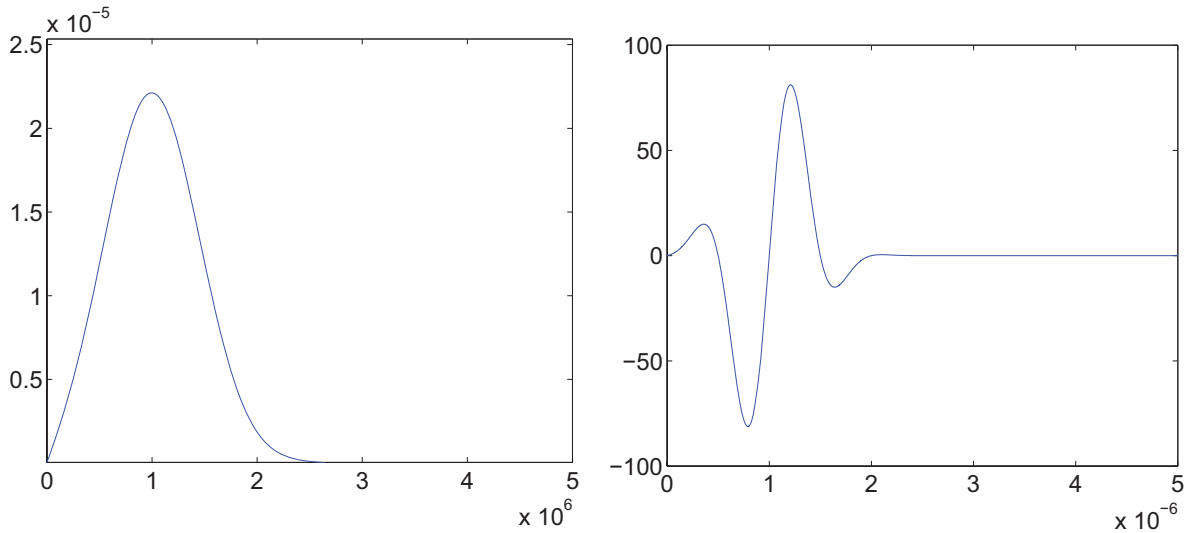


Fig 2. Definition of the function  $F$ . Graphs of the power spectrum of  $F$  (left) and function  $t \mapsto F(t)$  (right). Vertical axis: power spectrum (left) and  $F(t)$  (right). Horizontal axis: frequency (left) and  $t$  (right).

Figure 3 shows the graph of the confidence region of stochastic field  $P_1(x_1^R, x_3^R, \cdot)$  indexed by  $[0, +\infty[$  for a probability level  $P_c = 0.95$  and with a dispersion parameter  $\delta = 0.2$ ,  $x_1^R = 2 \times 10^{-3}$  m and  $x_3^R = 2 \times 10^{-3}$  m. Figure 4 shows the graph of the density probability function of random arrival time  $T$  with  $\delta = 0.2$  and  $x_1^R = 2 \times 10^{-3}$  m and  $x_3^R = 2 \times 10^{-3}$  m.



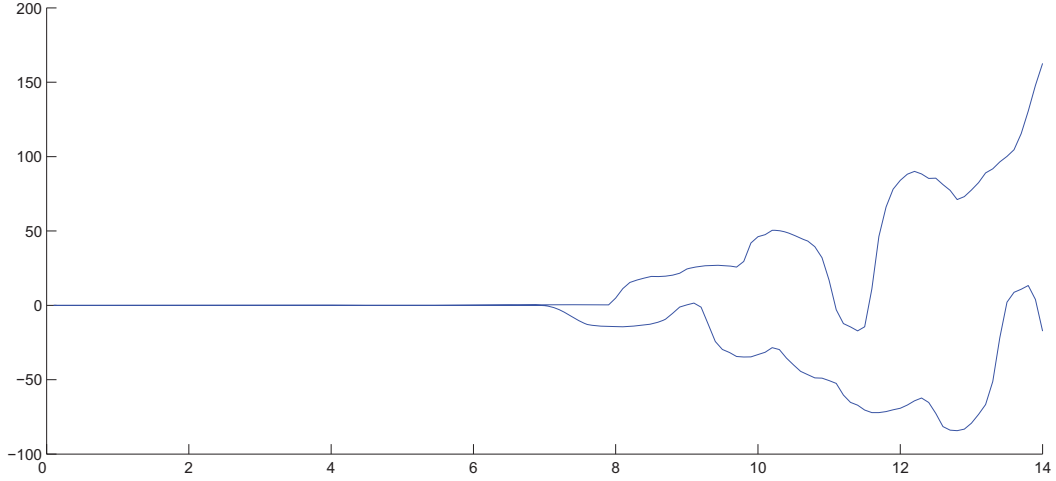


Fig 3. Confidence region of the stochastic process  $\{P_1(x_1^R, x_3^R, t)\}_{t>0}$  with a probability level  $P_c = 0.95$   $\delta = 0.2$ ,  $x_1^R = 2 \times 10^{-3}\text{m}$  and  $x_3^R = 2 \times 10^{-3}\text{m}$ . Vertical axis: disturbance of the pressure in the first fluid layer. Horizontal axis: time  $t$  ( $10^{-6}\text{s}$ ).

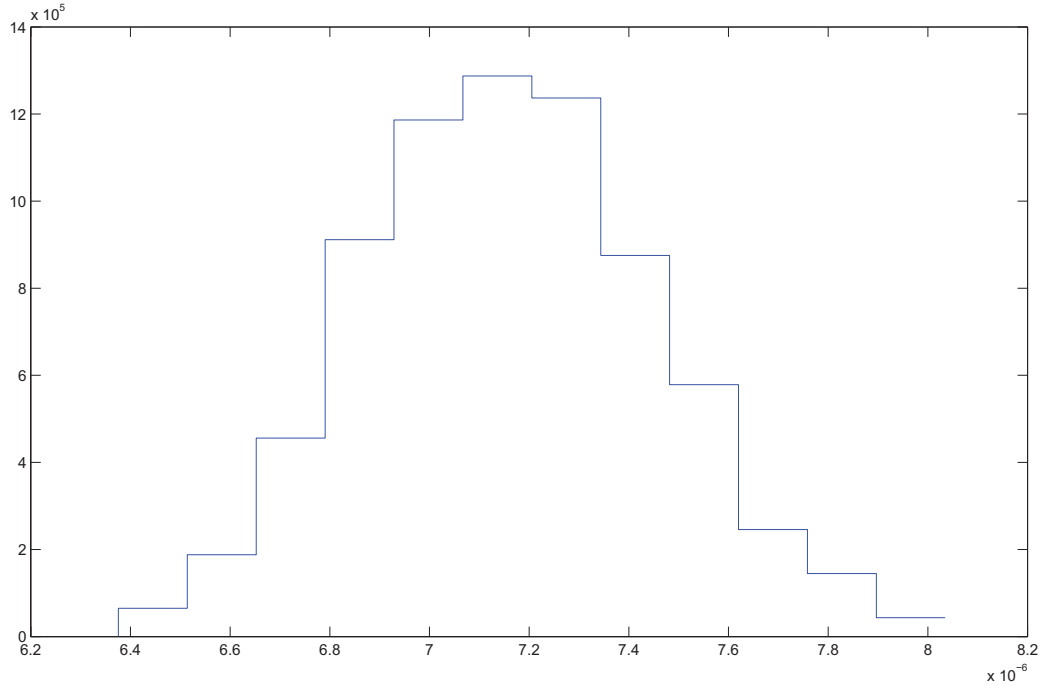


Fig 4. Probability density function of random arrival time  $T$  with  $\delta = 0.2$ ,  $x_1^R = 2 \times 10^{-3}\text{m}$  and  $x_3^R = 2 \times 10^{-3}\text{m}$ . Vertical axis: probability density. Horizontal axis: arrival time (s).

## 10 CONCLUSION

We have presented a probabilistic model to predict the transient elastic wave propagation in a multilayer unbounded media with uncertainties in the solid layer. Uncertainties are taken into account with a probabilistic model. Thanks to the introduction of an efficient numerical solver, the Monte-Carlo numerical method can be used as solver of the stochastic equations. The numerical application devoted to the cortical bone shows the interest of such an approach.

## 11 APPENDIX

The different quantities introduced in Section 4 are defined below

$$a_1(\widehat{p}_1, \delta p_1) = \frac{1}{K_1} \int_0^{z_1} \widehat{p}_1 \overline{\delta p_1} dx_3 \quad (31)$$

$$b_1(\widehat{p}_1, \delta p_1) = \frac{1}{\rho_1} \int_0^{z_1} \frac{\partial \widehat{p}_1}{\partial x_3} \frac{\partial \delta p_1}{\partial x_3} dx_3 \quad (32)$$

$$r_1(\widehat{\mathbf{u}}, \delta p_1) = \widehat{u}_3(0) \overline{\delta p_1(0)} \quad (33)$$

$$f(\delta p_1; t) = F(t) e^{ik_1 x_1^s} \overline{\delta p_1(x_3^s)} \quad (34)$$

$$m(\widehat{\mathbf{u}}, \delta \mathbf{u}) = \int_z^0 \rho \langle \widehat{\mathbf{u}}, \delta \widehat{\mathbf{u}} \rangle dx_3 \quad (35)$$

$$a_2(\widehat{p}_2, \delta p_2) = \frac{1}{K_2} \int_{z_2}^z \widehat{p}_2 \overline{\delta p_2} dx_3 \quad (36)$$

$$b_2(\widehat{p}_2, \delta p_2) = \frac{1}{\rho_2} \int_{z_2}^z \frac{\partial \widehat{p}_2}{\partial x_3} \frac{\partial \delta p_2}{\partial x_3} dx_3 \quad (37)$$

$$r_2(\widehat{\mathbf{u}}, \delta p_2) = \widehat{u}_3(z) \overline{\delta p_2(z)} \quad (38)$$

## REFERENCES

- [1] J. D. Achenbach, *Wave Propagation in Elastic Solids*. (North-Holland/American Elsevier, 1987)
- [2] K. Aki, P.G. Richard, *Quantitative Seismology: Theory and Methods*. (1980)
- [3] L.M. Brekhovskikh, *Waves in Layered Media. Applied Mathematics and mechanics*. (San Diego/Academic Press, 1973)
- [4] K. F. Graff *Wave Motion in Elastic Solids*, (Oxford University Press, 1975)
- [5] L. Cagniard, *Reflection and Refraction of Progressive Seismic Waves*. 1962.
- [6] D. Clouteau, M. Arnst, T.M. Al-Hussainia, G. Degrande, "Freefield vibrations due to dynamic loading on a tunnel embedded in a stratified medium", *Journal of Sound and Vibration*, **283**(1-2), 173-199 (2005)
- [7] J.H.M.T. van der Hijden, *Propagation of transient elastic waves in stratified anisotropic media*, 1987
- [8] B.L.N. Kennett, *Seismic wave propagation in stratified media*. (1983)
- [9] K. Macocco, Q. Grimal, S. Naili, C. Soize, "Elastoacoustic model with uncertain mechanical properties for ultrasonic wave velocity prediction: Application to cortical bone evaluation", *J. Acoust. Soc. Am.*, **119**(2), 729–740 (2006)
- [10] E. Savin, D. Clouteau, "Elastic wave propagation in a 3-D unbounded random heterogeneous medium coupled with a bounded medium. Application to seismic soil-structure interaction (SSSI)", *International Journal for numerical Methods in Engineering* **54**(4), 607-630 (2002)
- [11] C. Desceliers, C. Soize, Q. Grimal, G. Haiat, S. Naili, "Three dimensional transient elastic waves in multilayer semi-infinite media solved by a time-space-spectral numerical method", submitted to *Wave Motion* in September 2006.
- [12] K. J. Bathe, E. L. Wilson, *Numerical Methods in Finite Element Analysis*, (New York/Prentice-Hall, 1976)
- [13] C. Zienkiewicz, R. L. Taylor, *The Finite Element Method*, (McGraw-Hill Book Company 2 volumes, 1991)
- [14] C. Soize, "Random matrix theory for modeling uncertainties in computational mechanics", *Comput. Methods Appl. Mech. Engrg.* 194 (2005) 1333-1366