Stochastic modeling of uncertainties in computational dynamics and applications

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Abstract: Data uncertainties and model uncertainties in a predictive computational model of a real system are defined. The concept of the nonparametric probabilistic approach for random uncertainties due to model uncertainties and data uncertainties is introduced. A short overview of the main theoretical results of this nonparametric probabilistic approach based on the use of ensembles of random matrices constructed with the maximum entropy principle is given. The methodology of this nonparametric probabilistic approach is given. A numerical validation proving the capability of the nonparametric probabilistic approach to take into account model uncertainties is presented. Then an experimental validation is given for the dynamics of a composite sandwich panel. This nonparametric probabilistic modeling of random uncertainties has recently been applied to insdustrial applications in computational dynamics for linear and nonlinear dynamical systems, for complex structures and vibroacoustic systems.

Keywords: Comptational stochastic dynamics, model uncertainties, random matrices, probability

Introduction

The treatment of data uncertainties in structural mechanics has received a considerable attention these last decades. Data uncertainties affect the parameters of the mathematical-mechanical model such as the dimensions parameters, the parameters allowing the boundary conditions to be described, the constitutive equations, etc. Data uncertainties can clearly be taken into account by the parametric probabilistic approach. Such probabilistic analysis of data uncertainties performed with random variables modeling for relatively simple mechanical system can be found in many papers such as Shinozuka & Astill (1972), Chen & Soroka (1973), Prasthofer & Beadle (1975), Haug et al. (1986), Ibrahim (1987), Kotulski ans Sobczyk (1987), Shinozuka (1987), Jensen & Iwan (1992), Iwan & Jensen (1993), Lee & Singh (1994), Papadimitriou et al. (1995), Lin & Cai (1995), Micaletti et al. (1998), Schuëller (1997), Mignolet et al. (1998). Such a parametric probabilistic approach has also been developed with random field theory for data uncertainties and random media modeling, leading to the stochastic finite element method; see for instance Vanmarcke & Grigoriu (1983), Liu et al. (1986), Shinozuka & Deodatis (1988), Spanos & Ghanem (1989), Ghanem & Spanos (1991), Kleiber et al. (1992), Spanos and Zeldin (1994), Ditlevsen & Tarp-Johansen (1998). More recently, the parametric probabilistic approach for data uncertainties and for random media has been developed and applied to the computational mechanics of large and complex mechanical systems; see for instance Hien & Kleiber (1997), Ghanem & Dham (1998), Ghanem (1999), Székely & Schuëller (2001), Le Maître et al (2001) and (2002), Pradlwarter et al. (2002), Schuëller et al. (2003), Schenk & Schuëller (2003).
This paper does not deal with the above parametric probabilistic approach for data uncertainties or stochastic finite element method for random media, but rather with a nonparametric probabilistic approach of model uncertainties and data uncertainties for computational stochastic dynamics. This approach has been introduced by Soize (1999) and developed in the last five years. The main objectif of this paper is to present new validations of this approach and industrial applications in several fields for complex mechanical systems in computational stochastic mechanics for linear and nonlinear dynamical systems, for structural dynamics and vibroacoustic problems.

Notation. In this paper, $\mathbb{M}_n(\mathbb{R})$, $\mathbb{M}_n^S(\mathbb{R})$ and $\mathbb{M}_n^+(\mathbb{R})$ are the set of all $(n\times n)$ real matrices, the set of all symmetric $(n\times n)$ real matrices, and the set of all positive-definite symmetric $(n\times n)$ real matrices, respectively. We have $\mathbb{M}_n^+(\mathbb{R}) \subset \mathbb{M}_n^S(\mathbb{R}) \subset \mathbb{M}_n(\mathbb{R})$. If $[A]$ belongs to $\mathbb{M}_n(\mathbb{R})$, $\|[A]\|_F = (\text{tr}\{[A]\,[A]^T\})^{1/2}$ is the Frobenius norm of matrix $[A]$, where tr is the trace of the matrices, det is the determinant of the matrices and $[A]^T$ is the transpose of matrix $[A]$. The operator norm of a matrix $[A] \in \mathbb{M}_n(\mathbb{R})$ is defined as $\|A\| = \sup_{\|\mathbf{x}\|\leq 1} \|[A]\,\mathbf{x}\| \quad , \quad \mathbf{x} \in \mathbb{R}^n$ and is such that $\|[A]\,\mathbf{x}\| \leq \|A\|\,\|\mathbf{x}\|$, $\forall \mathbf{x} \in \mathbb{R}^n$. The indicatrix function $\mathbb{1}_{\mathcal{B}}(b)$ of any set $\mathcal{B}$ is such that $\mathbb{1}_{\mathcal{B}}(b)$ is equal to 1 if $b \in \mathcal{B}$ and is equal to zero if $b \notin \mathcal{B}$. All random variables are defined on a probability space $(\mathcal{A}, \mathcal{T}, \mathcal{P})$ and E is the mathematical expectation.

1. Uncertainties in a Predictive Model of Real System

In the context of engineering mechanics, the designed system is the mechanical system conceived by the designers and analysts. A designed system is defined by geometrical parameters, by the choice of materials, and by many other parameters. A designed system can be a very simple mechanical system e.g. an elastic bar or a very complex one such as an aircraft.
A real system is a manufactured version of a system realized from the designed system. Consequently, a real system is a man-made-physical system which is never exactly known (for instance, the geometry does not exactly coincide with the geometry of the designed system).
The objective of a predictive model is to predict the output $\mathbf{v}^{\text{exp}}$ of a real system to a given input $\mathbf{f}^{\text{exp}}$. Such predictive models are constructed by developing mathematical-mechanical model of the designed system for a given input (see Fig. 1). Consequently, the mean model has an input $\underline{\mathbf{f}}$ modeling $\mathbf{f}^{\text{exp}}$, an output $\underline{\mathbf{v}}$ modeling $\mathbf{v}^{\text{exp}}$ and exhibits a parameter $\underline{\mathbf{s}}$ for which data has to be given (it should be noted that the parameter can be a real number, a real vector, a real function, a field, a vector-valued function, etc.).

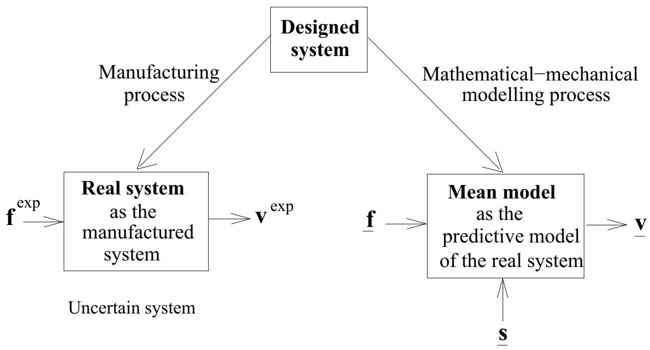

Figure 1. Designed system, real system and mean model as the predictive model of the real system.

(A) *Errors*: The errors are related to the construction of an approximation $\underline{\mathbf{v}}^n$ of the output $\underline{\mathbf{v}}$ of the mean model for given input $\underline{\mathbf{f}}$ and parameter $\underline{\mathbf{s}}$and have to be reduced and controled using adapted methods developed in applied mathematics and in numerical analysis. In general, these errors must not be considered as uncertainties (see below).

(B) *Uncertainties*: Below, the input $\underline{\mathbf{f}}$ and the parameter $\underline{\mathbf{s}}$ of the mean model will be hereafter referred to as the data of the mean model. The mathematical-mechanical modeling process of the designed system introduces two fundamental types of uncertainties: the data uncertainties and the model uncertainties.

(B.1) *Data uncertainties*: The input $\underline{\mathbf{f}}$ of the mean model does not exactly represent the input $\mathbf{f}^{\text{exp}}$ of the real system and there are also uncertainties on the parameter $\underline{\mathbf{s}}$ of the mean model.

Data uncertainties have to be taken into account for improving the predictability of the mean model. The best approach to take into account data uncertainties is the parametric probabilistic approach consisting in modeling the data of the mean model by random quantities.

(B.2) *Model uncertainties*: The mathematical-mechanical modeling process used for constructing the mean model induces model uncertainties with respect to the designed system. This type of uncertainties is mainly due to the introduction of simplifications in order to decrease the complexity of the mean model which is constructed. For instance, a slender cylindrical elastic medium will be modeled by using the beam theory (such as an Euler or a Timoshenko beam), a thick rectangular plate elastic medium will be modeled by a thick plate theory (such as the Midlin plate theory), a complex joint constituted of an assemblage of several plates attached together by lines of bolts will be modeled by an equivalent homogeneous orthotropic plate, etc. It is clear that the introduction of such simplified models yields a mean model whose variations of parameter $\underline{\mathbf{s}}$ do not allow the model uncertainties to be reduced. Model uncertainties have to be taken into account to improve the predictability of the mean model. As explained above, the parametric probabilistic approach cannot be used. This is the reason why a nonparametric probabilistic approach is proposed.

(C) *Predictability of the mean model*: The error between the prediction $\underline{\mathbf{v}}^n$ calculated with the mean model and the response $\mathbf{v}^{\text{exp}}$ of the real system can be measured by $\|\mathbf{v}^{\text{exp}} - \underline{\mathbf{v}}^n\|$. Clearly, the mean model can be considered as a predictive model if this error is sufficiently small. In general, due to data uncertainties and model uncertainties, this error is not sufficiently small and has to be reduced by taking into account data uncertainties and model uncertainties.

2. Nonparametric Probabilistic Approach of Random Uncertainties

2.1. Concept of the Nonparametric Probabilistic Approach

The following example will be used to clarify the concepts of the nonparametric approach that permits the consideration of model uncertainty. Let $\mathbf{s} \mapsto A(\mathbf{s})$ be a linear mapping from a space $\mathcal{S}$ into a space $\mathcal{A}$ of linear operators. The space $\mathcal{S}$ represents the set of all possible values of the vector-valued parameter $\mathbf{s}$ of the boundary value problem (for instance, geometric parameters, elastic properties, boundary conditions, etc). For $\mathbf{s}$ fixed in $\mathcal{S}$, operator $A(\mathbf{s})$ represents one operator of the boundary value problem (for instance, the stiffness operator which is assumed to be symmetric and positive, and in this case, any operator in $\mathcal{A}$ will be symmetric and positive). Let $R_{\text{par}} \subset \mathcal{A}$ be the range of the mapping $\mathbf{s} \mapsto A(\mathbf{s})$, i.e. the subset of $\mathcal{A}$ spanned by $A(\mathbf{s})$ when $\mathbf{s}$ runs through $\mathcal{S}$.

(A) *The operator of the real system*. It is assumed that the operator corresponding to the real system is $\mathbf{A}^{\text{exp}}$ belonging to $\mathcal{A}$.

(B) *The mean model of the operator*. If $\mathbf{s} = \underline{\mathbf{s}}$ is the nominal value, then $\underline{A} = A(\underline{\mathbf{s}}) \in R_{\text{par}}$ is the operator of the mean model.

(C) *Parametric probabilistic model of the operator*. The parametric probabilistic approach for the operator consists in modeling the parameter $\mathbf{s}$ by a vector valued random variable $\mathbf{S}$ whose probability distribution $P_{\mathbf{S}}(d\mathbf{s})$ has a support which is $\mathcal{S}$. Then the operator $\underline{A}$ of the mean model is replaced in the the BVP by the random operator $\mathbf{A}_{\text{par}}$ such that $\mathbf{A}_{\text{par}} = A(\mathbf{S})$. The probability distribution $P_{\mathbf{A}_{\text{par}}}$ of the random operator $\mathbf{A}_{\text{par}}$ is $P_{\mathbf{A}_{\text{par}}} = A(P_{\mathbf{S}})$ and its support is the set $R_{\text{par}} \subset \mathcal{A}$ (see Fig. 2). Clearly, the probability $P_{\mathbf{A}_{\text{par}}}$ on R_{par} allows data uncertainties to be taken into account, but $\mathbf{A}^{\text{exp}}$ may not be in R_{par} due to model uncertainties.

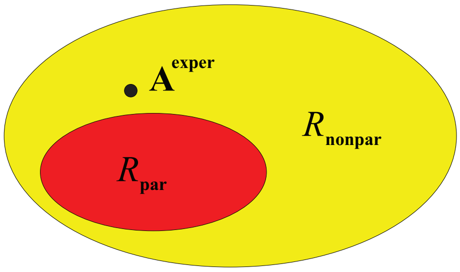

Figure 2. Parametric and nonparametric probabilistic approaches of random uncertainties.

(D) *Nonparametric probabilistic model of the operator*. The nonparametric probabilistic approach for the operator consists in replacing the operator $\underline{A}$ of the mean model by a random operator $\mathbf{A}_{\text{nonpar}}$ whose probability distribution $P_{\mathbf{A}_{\text{nonpar}}}$ has a support $R_{\text{nonpar}} = \mathcal{A}$. Since $\mathbf{A}^{\text{exp}}$ belongs to $\mathcal{A}$ and since the support of $P_{\mathbf{A}_{\text{nonpar}}}$ is also $\mathcal{A}$, model uncertainties can be taken into account by the proposed nonparametric approach (see Fig. 2). Of course, $P_{\mathbf{A}_{\text{nonpar}}}$ cannot be arbitrary chosen with support R_{nonpar}, but has to be constructed using the available information. Such a methodology has been developed (Soize 1999, 2000, 2001a and 2005b) by using the maximum entropy principle (Shannon 1948) and (Jaynes 1957).

2.2. Evolution of the Concepts: Some History

The fundamentals of the nonparametric approach for random uncertainties and the developments of the first ensemble of random matrices adapted to modeling random uncertainties in linear dynamical systems have been introduced by Soize (1999) and (2000). The algebraic closure of this theory and its convergence analysis as the dimension goes to infinity have studied by Soize (2001a) and (2001b) in the context of transient linear elastodynamics of stochastic systems. Others ensembles of random matrices adapted to modeling random uncertainties for coupled systems encountering vibroacoustics problems, have been introduced by Soize (2005b).

The extension of the theory to non homogeneous uncertainties in complex dynamical systems using substructuring techniques can be found, with experimental validations, in Chebli & Soize (2004), Duchereau & Soize (2005).

The identification of the parameters of the nonparametric probabilistic model from experiments is developed in Soize (2005a) and (2005b) and an experimental validation is given in Chen et al. (2006).

The extension of the theory to linear dynamical systems in the medium frequency range was achieved by Soize (2003b). The random eigenvalues for linear dynamical systems and the non adaptation of the Gaussian Orthogonal Ensemble (GOE) for low- and medium-frequency dynamics are analyzed in Soize (2003a).

The application of this nonparametric probabilistic approach for model uncertainties in nonlinear dynamical systems and transient nonlinear dynamics of stochastic systems have been studied in Soize (2001c), Desceliers et al. (2004).

Model uncertainties in dynamical systems with cyclic symmetry and applications to mistuned bladed disks have been developpped in Capiez-Lernout & Soize (2004) and Capiez-Lernout et al. (2005b).

Additional validations devoted to the capability of the nonparametric probabilistic approach to take into account model uncertainties is given in Soize (2005a) and Capiez-Lernout et al.

(2005b). The extension of the theory to vibroacoustic problems is presented in Durand et al. (2005).

3. Methodology of the Nonparametric Probabilistic Approach

The methodology of the nonparametric probabilistic approach of uncertainties in dynamical systems is as follows. (1) Developement of a mechanical-mathematical model, generally a finite element model, of the designed system. Such a model will be call the mean model (or the nominal model). (2) Construction of a reduced mean model from the mean model. (3) Construction of a stochastic reduced model from the reduced mean model using the nonparametric concept and the maximum entropy principle. In this fashion, the probability distribution of each random generalized matrix is constructed. (4) Construction and convergence analysis of the stochastic solution.

3.1. Mean Finite Element Model of the Dynamical System

The following presentation is limited to the nonlinear structural dynamics of a linear structure with localized nonlinearities but it can be extented to other more complex systems such as vibroacoustic systems. We consider a nonlinear dynamic system constituted of a three-dimensional, damped, fixed structure vibrating around a static equilibrium configuration considered as a natural state without prestresses. The struvture is subjected to an external load and does not display rigid body displacement. The basic finite element model of this nonlinear dynamic system is called the "mean finite element model" (the underlined quantities refer to this "mean finite element model") and leads to the following nonlinear differential equation,

$$[\underline{\mathbb{M}}]\,\ddot{\underline{\mathbf{y}}}(t) + [\underline{\mathbb{D}}]\,\dot{\underline{\mathbf{y}}}(t) + [\underline{\mathbb{K}}]\,\underline{\mathbf{y}}(t) + \mathbf{f}_{\mathrm{NL}}(\underline{\mathbf{y}}(t), \dot{\underline{\mathbf{y}}}(t)) = \mathbf{f}(t) \quad , \tag{1}$$

in which $\underline{\mathbf{y}} = (\underline{y}_1, \ldots, \underline{y}_m)$ is the unknown time response vector of the m degrees of freedom (DOF) (displacements and/or rotations); $\dot{\underline{\mathbf{y}}}$ and $\ddot{\underline{\mathbf{y}}}$ are the velocity and acceleration vectors respectively; $\mathbf{f}(t) = (f_1(t), \ldots, f_m(t))$ is the known external load vector of the m inputs (forces and/or moments); $[\underline{\mathbb{M}}]$, $[\underline{\mathbb{D}}]$ and $[\underline{\mathbb{K}}]$ are the mass, damping and stiffness matrices of the linear part of the model, which are positive-definite symmetric $(m \times m)$ real matrices; $(\mathbf{y}, \mathbf{z}) \mapsto \mathbf{f}_{\mathrm{NL}}(\mathbf{y}, \mathbf{z})$ is a nonlinear mapping from $\mathbb{R}^m \times \mathbb{R}^m$ into $\mathbb{R}^m$ modeling additional nonlinear damping and restoring forces such that $\mathbf{f}_{\mathrm{NL}}(0, 0) = 0$. The linear case can be derived from Eq. (1) by taking $\mathbf{f}_{\mathrm{NL}} = 0$.

3.2. Reduced Mean Model

Let $\{\underline{\varphi}_1, \ldots, \underline{\varphi}_m\}$ be an algebraic basis of $\mathbb{R}^m$. The reduced mean model of the dynamic system with mean finite element model given by Eq. (1) is obtained by projection of Eq. (1) on the subspace V_n of $\mathbb{R}^m$ spanned by $\{\underline{\varphi}_1, \ldots, \underline{\varphi}_n\}$ with $n \ll m$.

The construction of such an algebraic basis can be performed as follows.

(1) By using elastic modes of the underlying linear dynamical system. This choice is warranted for linear dynamical systems or for nonlinear dynamical systems with localized nonlinearities, when their responses are in the low-frequency range.

(2) By using eigenvectors of the mechanical energy operator. This solution is appropriate for a linear dynamical system in the medium-frequency range (Soize 1998a,1998b,1999,2003b), (Soize & Mziou 2003), (Ghanem & Sarkar 2003).

(3) By using adapted "nonlinear modes" for nonlinear dynamical systems with distributed nonlinearities.

Let $[\underline{\Phi}_n]$ be the $(m \times n)$ real matrix whose columns are the vectors $\{\underline{\varphi}_1, \ldots, \underline{\varphi}_n\}$. The generalized force $\mathbf{F}^n(t)$ is the $\mathbb{R}^n$-vector $\mathbf{F}^n(t) = [\underline{\Phi}_n]^T\, \mathbf{f}(t)$. The generalized mass, damping and stiffness matrices $[\underline{M}_n]$, $[\underline{D}_n]$ and $[\underline{K}_n]$ are the positive-definite symmetric $(n \times n)$ real matrices $[\underline{M}_n] = [\underline{\Phi}_n]^T[\underline{\mathbb{M}}][\underline{\Phi}_n]$, $[\underline{D}_n] = [\underline{\Phi}_n]^T[\underline{\mathbb{D}}][\underline{\Phi}_n]$, and $[\underline{K}_n] = [\underline{\Phi}_n]^T[\underline{\mathbb{K}}][\underline{\Phi}_n]$ which, in general, are full matrices. Consequently, the reduced mean model of the nonlinear dynamic system, written as the projection $\underline{\mathbf{y}}^n$ of $\underline{\mathbf{y}}$ on V_n, can be written as $\underline{\mathbf{y}}^n(t) = [\underline{\Phi}_n]\, \underline{\mathbf{q}}^n(t)$ in which the vector $\underline{\mathbf{q}}^n(t) \in \mathbb{R}^n$ of the generalized coordinates satisfies the mean nonlinear differential equation,

$$[\underline{M}_n]\, \ddot{\underline{\mathbf{q}}}^n(t) + [\underline{D}_n]\, \dot{\underline{\mathbf{q}}}^n(t) + [\underline{K}_n]\, \underline{\mathbf{q}}^n(t) + \mathbf{F}^n_{\mathrm{NL}}(\underline{\mathbf{q}}^n(t), \dot{\underline{\mathbf{q}}}^n(t)) = \mathbf{F}^n(t) \ , \ \forall t \geq 0 \quad , \quad (2)$$

where, for all $\mathbf{q}$ and $\mathbf{p}$ in $\mathbb{R}^n$, $\mathbf{F}^n_{\mathrm{NL}}(\mathbf{q}, \mathbf{p}) = [\underline{\Phi}_n]^T\, \mathbf{f}_{\mathrm{NL}}([\underline{\Phi}_n]\, \mathbf{q}, [\underline{\Phi}_n]\, \mathbf{p})$.

3.3. Stochastic Response of the Nonlinear Dynamical System

The principle of construction of the nonparametric model of random uncertainties for the dynamical system whose mean finite element model is defined by Eq. (1), consists in modeling the generalized mass, damping and stiffness matrices of the reduced mean model (see Eq. (2)) by random matrices $[\mathbf{M}_n]$, $[\mathbf{D}_n]$ and $[\mathbf{K}_n]$. If the nonlinear forces are uncertain, a usual parametric model can be used for these nonlinear forces. In this case, a mixed nonparametric-parametric formulation can easily be constructed. The stochastic transient response of the nonlinear dynamic system with a nonparametric probabilistic model of random uncertainties, with reduced mean model defined by Eq. (2), is the stochastic process $\mathbf{Y}^n(t)$, indexed by $\mathbb{R}^+$, with values in $\mathbb{R}^m$, such that

$$\mathbf{Y}^n(t) = [\underline{\Phi}_n]\, \mathbf{Q}^n(t) \quad . \quad (3)$$

In this equation, the stochastic process $\mathbf{Q}^n(t)$, indexed by $\mathbb{R}^+$, with values in $\mathbb{R}^n$, is such that

$$[\mathbf{M}_n]\, \ddot{\mathbf{Q}}^n(t) + [\mathbf{D}_n]\, \dot{\mathbf{Q}}^n(t) + [\mathbf{K}_n]\, \mathbf{Q}^n(t) + \mathbf{F}^n_{\mathrm{NL}}(\mathbf{Q}^n(t), \dot{\mathbf{Q}}^n(t)) = \mathbf{F}^n(t) \ , \ \forall t \geq 0 \quad , \quad (4)$$

with the initial conditions, $\mathbf{Q}^n(0) = 0$ and $\dot{\mathbf{Q}}^n(0) = 0$.

3.4. Construction of the Probability Model of the Random Matrices $[\mathbf{M}_n], [\mathbf{D}_n], [\mathbf{K}_n]$

The construction of the probability model of random matrices $[\mathbf{M}_n]$, $[\mathbf{D}_n]$ and $[\mathbf{K}_n]$ consists in taking these random matrices in ensemble SE$^+$ (Soize, 1999,2000,2001a,2001c) whose construction is recalled hereafter. (1) The random matrices $[\mathbf{M}_n]$, $[\mathbf{D}_n]$ and $[\mathbf{K}_n]$ are defined on the probability space $(\mathcal{A}, \mathcal{T}, \mathcal{P})$, with values in $\mathbb{M}^+_n(\mathbb{R})$. (2) The mean values of these random matrices are $E\{[\mathbf{M}_n]\} = [\underline{M}_n]$, $E\{[\mathbf{D}_n]\} = [\underline{D}_n]$ and $E\{[\mathbf{K}_n]\} = [\underline{K}_n]$. (3) These random matrices verify the following inequalities ensuring that Eq. (4) has a second-order stochastic solution, $E\{\|[\mathbf{M}_n]^{-1}\|^2_F\} < +\infty$, $E\{\|[\mathbf{D}_n]^{-1}\|^2_F\} < +\infty$ and $E\{\|[\mathbf{K}_n]^{-1}\|^2_F\} < +\infty$. (4) The random matrices $[\mathbf{M}_n]$, $[\mathbf{D}_n]$ and $[\mathbf{K}_n]$ are independent (because no available information is given concerning the correlation between these random matrices). (5) These random matrices can be normalized with respect to their mean values as follows:

$$[\mathbf{M}_n] = [\underline{L}_{M_n}]^T[\mathbf{G}_{M_n}][\underline{L}_{M_n}] \ , [\mathbf{D}_n] = [\underline{L}_{D_n}]^T[\mathbf{G}_{D_n}][\underline{L}_{D_n}] \ , [\mathbf{K}_n] = [\underline{L}_{K_n}]^T[\mathbf{G}_{K_n}][\underline{L}_{K_n}] \quad (5)$$

in which $[\underline{L}_{M_n}]$, $[\underline{L}_{D_n}]$ and $[\underline{L}_{K_n}]$ are the upper triangular real matrices in $\mathbb{M}_n(\mathbb{R})$ such that $[\underline{M}_n] = [\underline{L}_{M_n}]^T\, [\underline{L}_{M_n}]$, $[\underline{D}_n] = [\underline{L}_{D_n}]^T\, [\underline{L}_{D_n}]$ and $[\underline{K}_n] = [\underline{L}_{K_n}]^T\, [\underline{L}_{K_n}]$. The random matrices $[\mathbf{G}_{M_n}]$, $[\mathbf{G}_{D_n}]$ and $[\mathbf{G}_{K_n}]$ are in the ensemble SG$^+$ which is defined in Section 3.5. (6) The of ensemble SG$^+$ of random matrices is defined as follows.

3.5. Ensemble SG^+ of Random Matrices

(A) *Definition of ensemble* SG^+. This ensemble is defined and constructed (Soize 1999, 2000, 2001a) as the set of random matrices $[\mathbf{G}_n]$, defined on the probability space $(\mathcal{A}, \mathcal{T}, P)$, with values in $\mathbb{M}_n^+(\mathbb{R})$, whose probability distribution is constructed by using the entropy optimization principle (Shannon 1948), (Jaynes 1957), under the constraints (available information): (1) Any matrix $[\mathbf{G}_n]$ is symmetric positive-definite real random matrix, i.e. $[\mathbf{G}_n] \in \mathbb{M}_n^+(\mathbb{R})$ a.s. (2) Any matrix $[\mathbf{G}_n]$ is a second-order random variable, $E\{\|[\mathbf{G}_n]\|^2\} \leq E\{\|[\mathbf{G}_n]\|_F^2\} < +\infty$ and its mean value $[\underline{G}_n]$ is the $(n \times n)$ identity matrix $[I_n]$, $E\{[\mathbf{G}_n]\} = [\underline{G}_n] = [I_n] \in \mathbb{M}_n^+(\mathbb{R})$. (3) Any random matrix $[\mathbf{G}_n]$ is such that $E\{\ln(\det[\mathbf{G}_n])\} = v$ with $|v| < +\infty$. Constraint (3) implies the following fundamental property for random matrices in ensemble SG^+: $E\{\|[\mathbf{G}_n]^{-1}\|_F^2\} < +\infty$.

(B) *Dispersion parameter of a random matrix in ensemble* SG^+. Let δ be the real dispersion parameter defined by $\delta = \{E\{\| [\mathbf{G}_n] - [\underline{G}_n] \|_F^2\}/\| [\underline{G}_n] \|_F^2\}^{1/2} = \{n^{-1} E\{\| [\mathbf{G}_n] - [I_n] \|_F^2\}\}^{1/2}$. It controls the dispersion of the probability model of random matrix $[\mathbf{G}_n]$ provided that δ is independent of n and such that $0 < \delta < \sqrt{(n+1)(n+5)^{-1}}$.

(C) *Probability distribution of a random matrix in ensemble* SG^+. The probability distribution $P_{[\mathbf{G}_n]}$ of random matrix $[\mathbf{G}_n]$ is defined by a probability density function $[G_n] \mapsto p_{[\mathbf{G}_n]}([G_n])$ from $\mathbb{M}_n^+(\mathbb{R})$ into $\mathbb{R}^+ = [0\,, +\infty[$, with respect to the measure (volume element) $\widetilde{d}G_n = 2^{n(n-1)/4}\, \Pi_{1 \leq i \leq j \leq n}\, d[G_n]_{ij}$ on the set $\mathbb{M}_n^S(\mathbb{R})$. We then have $P_{[\mathbf{G}_n]} = p_{[\mathbf{G}_n]}([G_n])\, \widetilde{d}G_n$, with the normalization condition $\int_{\mathbb{M}_n^+(\mathbb{R})} p_{[\mathbf{G}_n]}([G_n])\, \widetilde{d}G_n = 1$. The probability density function $p_{[\mathbf{G}_n]}([G_n])$ is then written as

$$p_{[\mathbf{G}_n]}([G_n]) = \mathbb{1}_{\mathbb{M}_n^+(\mathbb{R})}([G_n]) \times C_{\mathbf{G}_n} \times \big(\det[G_n]\big)^{(n+1)\frac{(1-\delta^2)}{2\delta^2}} \exp\left\{ -\frac{(n+1)}{2\delta^2}\, \mathrm{tr}\,[G_n] \right\} \tag{6}$$

in which $\mathbb{1}_{\mathbb{M}_n^+(\mathbb{R})}([G_n])$ is equal to 1 if $[G_n] \in \mathbb{M}_n^+(\mathbb{R})$ and is equal to zero if $[G_n] \notin \mathbb{M}_n^+(\mathbb{R})$. Further, the positive constant $C_{\mathbf{G}_n}$ is such that

$$C_{\mathbf{G}_n} = \frac{(2\pi)^{-n(n-1)/4} \left(\frac{n+1}{2\delta^2}\right)^{n(n+1)(2\delta^2)^{-1}}}{\left\{\Pi_{j=1}^n \Gamma\left(\frac{n+1}{2\delta^2} + \frac{1-j}{2}\right)\right\}} \quad , \tag{7}$$

where $\Gamma(z)$ is the gamma function defined for $z > 0$ by $\Gamma(z) = \int_0^{+\infty} t^{z-1}\, e^{-t}\, dt$. Equation (6) shows that $\{[G_n]_{jk}, 1 \leq j \leq k \leq n\}$ are dependent random variables.

(D) *Algebraic representation of a random matrix in ensemble* SG^+. The following algebraic representation of a random matrix $[\mathbf{G}_n]$ of SG^+ allows the formulation of a procedure for the Monte Carlo numerical simulation of random matrix $[\mathbf{G}_n]$. With this procedure, the numerical cost induced by the simulation is a constant that depends on dimension n but that is independent of the dispersion parameter δ. Any random matrix $[\mathbf{G}_n]$ can be written as

$$[\mathbf{G}_n] = [\mathbf{L}_n]^T\, [\mathbf{L}_n] \quad , \tag{8}$$

in which $[\mathbf{L}_n]$ is an upper triangular random matrix with values in $\mathbb{M}_n(\mathbb{R})$ such that:

(1) The random variables $\{[\mathbf{L}_n]_{jj'}, j \leq j'\}$ are independent.

(2) For $j < j'$, real-valued random variable $[\mathbf{L}_n]_{jj'}$ can be written as $[\mathbf{L}_n]_{jj'} = \sigma_n U_{jj'}$ in which $\sigma_n = \delta(n+1)^{-1/2}$ and $U_{jj'}$ is a real-valued Gaussian random variable with zero mean and unit variance.

(3) For $j = j'$, the positive-valued random variable $[\mathbf{L}_n]_{jj}$ can be written as $[\mathbf{L}_n]_{jj} = \sigma_n \sqrt{2V_j}$ in which σ_n is defined above and V_j is a positive-valued gamma random variable with probability density function $p_{V_j}(v)$ with respect to dv

$$p_{V_j}(v) = \mathbb{1}_{\mathbb{R}^+}(v) \frac{1}{\Gamma\left(\frac{n+1}{2\delta^2} + \frac{1-j}{2}\right)} \; v^{\frac{n+1}{2\delta^2} - \frac{1+j}{2}} \; e^{-v} \quad . \tag{8}$$

(E) *Convergence property of a random matrix in ensemble* SG$^+$ *when dimension goes to infinity*. It can be shown that $\forall n \geq 2 \quad , \quad E\{\|[\mathbf{G}_n]^{-1}\|^2\} \leq C_\delta < +\infty$ in which C_δ is a positive finite constant that is independent of n but that depends on δ. This equation implies that $n \mapsto E\{\|[\mathbf{G}_n]^{-1}\|^2\}$ is a bounded function from $\{n \geq 2\}$ into $\mathbb{R}^+$.

3.6. Other Ensembles of Random Matrices

Five ensembles of random matrices have been developped which are useful for modeling data and model uncertainties in computational mechanics . These ensembles differ from the known ensemble of random matrices, e.g. found in Mehta (1991).

(1) The first ensemble, SG$^+$, of random matrices has been presented in Section 4.5 and is called the *the normalized positive-definite ensemble*. Then, a random matrix belonging to SG$^+$ is positive definite almost surely and its mean value is the identity matrix. This ensemble constitutes the main ensemble used for constructing the four other ensembles introduced below. Ensemble SG$^+$ differs from the GOE and from the other known ensembles of random matrix theory (Soize 2003a).

(2) The second ensemble, SE$^+$, of random matrices, herein called the *positive-definite ensemble*, has been constructed simultaneously with SG$^+$ and is used in Section 4.4. A random matrix belonging to SE$^+$ is positive definite almost surely and its mean value is a given positive-definite matrix. For instance, this ensemble is used for constructing probability model of positive operators such as the mass, damping or stiffness operators of a dynamical system.

(3) The construction of the third ensemble, SE^{+0}, has been introduced in Soize (1999) and is similar to the construction of ensemble SE$^+$. A random matrix belonging to this ensemble is semipositive definite almost surely instead of being positive definite almost surely. For instance, such an ensemble is useful for modeling uncertainties of the stiffness operator of dynamical systems for which there are rigid body displacement fields.

(4) The fourth ensemble of random matrices is the subset SE$^+_{\rm lf}$ of SE$^+$ introduced in Soize (2005b), constituted of random matrices in SE$^+$ for which a linear form on SE$^+$ is given. A particular case is the ensemble SE$^+_{\rm tr}$ for which the trace of the random matrix is given. For instance, such an ensemble is useful for modeling uncertainties of the mass operator of a dynamical system for which the spatial distibution of the mass is uncertain but for which the total mass is known.

(5) The fifth ensemble, SE$_{\rm inv}$, of random matrices, herein called the *the pseudo-inverse ensemble*, is a new ensemble introduced in Soize (2005b), constituted of rectangular random matrices having a mean-square pseudo-inverse. For instance, such an ensemble is useful for modeling uncertainties in the coupling operator between an elastic solid and an acoustic fluid (structural-acoustic system) and is used in the application presented in Section 7.4.

3.7. Construction and Convergence of the Stochastic Solution

(A) *Stochastic solution as a second-order stochastic process*. For any $T > 0$, it is proved in Soize (2001a,2001c) that, for all t in $[0\,,T]$, we have $E\{\|\mathbf{Y}^n(t)\|^2\} \leq C_1 < +\infty$ and

$E\{\|\dot{\mathbf{Y}}^n(t)\|^2\} \leq C_2 < +\infty$ under reasonable assumptions concerning the nonlinear damping and restoring forces and if $\int_0^T \|\mathbf{f}(t)\|^2 dt < +\infty$. Further, C_1 and C_2 are positive constants that are independent of n and t.

(B) *Construction of the stochastic solution*. The stochastic solution of Eq. (4) is constructed using the Monte Carlo numerical simulation with n_s realizations. For each realization, an implicit step-by-step integration method (Newmark method) is used for solving Eq. (4). The realizations of the random matrix $[\mathbf{A}_n]$, in which $[\mathbf{A}_n]$ represents $[\mathbf{M}_n]$, $[\mathbf{D}_n]$ or $[\mathbf{K}_n]$, are constructed using Section 3.5(D). It should be noted that the numerical cost is low with such a method because Eq. (4) corresponds to a stochastic reduced model with $n \ll m$.

(C) *Convergence analysis*. Using the usual estimation of the mathematical expectation operator E, the convergence with respect to dimension n of the stochastic reduced model and to the number n_s of realizations used in the Monte Carlo numerical method, is studied by constructing the function $(n_s, n) \mapsto \text{Conv}(n_s, n) = \{\frac{1}{n_s}\sum_{k=1}^{n_s}\int_0^T \|\mathbf{Q}^n(t,\theta_k)\|^2\, dt\}^{1/2}$, in which $\mathbf{Q}^n(t,\theta_1), \ldots, \mathbf{Q}^n(t,\theta_{n_s})$ are n_s realizations of $\mathbf{Q}^n(t)$.

4. Numerical Validation: Capability of the Proposed Approach

(A) *Designed system*. The designed system is a slender cylindrical elastic medium $\underline{\Omega}$ defined in a cartesian co-ordinate system $(Oxyz)$. The cylinder has a rectangular section and dimensions $\underline{h}_1 = 10\, m$, $\underline{h}_2 = 1\, m$ and $\underline{h}_3 = 1.5\, m$. The elastic medium is made of a composite material. This structure is simply supported on the lower edges. The other parts of the boundary $\partial\underline{\Omega}$ of domain $\underline{\Omega}$ are free.
(B) *Real system*. The real system corresponds to the designed system. There are uncertainties on the geometry due to manufacturing tolerances. The domain of the real system is Ω_{RS} which differs from $\underline{\Omega}$. The simply supported conditions are not exactly realized and the composite material does not exactly correspond to the given specifications of the designed system. This real system is excited by a frequency-dependent pressure field $p^{\text{exp}}(\omega)$ which is constant in space on the part Γ_{RS} of the boundary $\partial\Omega_{\text{RS}}$. We are interested in the dynamics of the real system in the frequency band $B =]0,\, 1000]\, Hz$ subjected to a pressure field excitation which is constant in space over Γ_{RS} and constant in frequency band B. The details of data can be found in (Soize, 2005a).
(C) *Mean model*. The mean model, as a predictive model of the real system, is constructed from the designed system. This mean model is constituted of a damped homogeneous Euler elastic beam with length $\underline{h}_1$, simply supported at $x = 0$ and $x = \underline{h}_1$. The mean model input is the point force located at $x_0 = 4.25\, m$ with an intensity $\underline{g}(\omega) = -\mathbb{1}_B(\omega)$. The composite material of the designed system is modeled by a homogeneous isotropic elastic material whose nominal parameters are $\underline{\mathcal{E}} = 10^{10}\, N/m^2$ (Young's modulus), $\underline{\rho} = 1700\, Kg/m^3$ (mass density) and $\underline{\xi} = 0.01$ (damping rates). The computed eigenfrequencies of the mean system are $\underline{\nu}_1 = 11,\ \underline{\nu}_2 = 44,\ \underline{\nu}_3 = 99,\ \underline{\nu}_4 = 176,\ \underline{\nu}_5 = 275, \underline{\nu}_6 = 396, \underline{\nu}_7 = 539, \underline{\nu}_8 = 704, \underline{\nu}_9 = 891,\ \underline{\nu}_{10} = 1100, \ldots, \underline{\nu}_{80} = 70385\, Hz$. For ω in B, this external force induces flexural vibrations in the plane (Oxy).
(D) *Numerical experiment of the real system*. A "numerical experimental" response of the real system is obtained by (1) constructing a 3D elastic model of the real system, (2) discretizing the real system by the finite element method, and (3) solving the equation by modal analysis. The material is taken as homogeneous and isotropic with a Young modulus of $10^{10}\, N/m^2$, a Poisson coefficient of 0.15, a mass density equal to $1700\, Kg/m^3$. The modal damping ratios are the realizations of a uniform random variable on $[0.009\,, 0.011]$ of mean value 0.01. The finite element mesh is constituted of $80 \times 8 \times 12 = 7680$ three-dimensional 8-nodes solid elements. There are 9477 nodes and a total of 28275 degrees of freedom (due to the boundary conditions, the displacement is zero for 2×26 nodes). A point force $(0, -\mathbb{1}_B(\omega), 0)$ is

applied to the node of co-ordinates $(4.25, 0.5, 0.75)$. The finite element approximation of the displacement field $(u^{\exp}, v^{\exp}, w^{\exp})$ is computed on frequency band B by using modal analysis with the first 150 elastic modes. There are 101 eigenfrequencies in band B and 49 eigenfrequencies in frequency band $[1000, 1197]\, Hz$. The fundamental eigenfrequency is $\nu_1^{\exp} = 16\, Hz$. There are 14 eigenfrequencies in frequency band $[0, 230]\, Hz$. The eigenfrequencies of the first 5 flexural modes corresponding to the first 5 elastic modes of the mean model (Euler beam) and having respectively 2 to 6 nodes (zero Oy-displacement) on the neutral fiber are $\nu_{j_1}^{\exp} = 16,\ \nu_{j_2}^{\exp} = 40,\ \nu_{j_3}^{\exp} = 91,\ \nu_{j_4}^{\exp} = 153,\ \nu_{j_5}^{\exp} = 220,\ Hz$ with $j_1 = 1, j_2 = 3, j_3 = 7, j_4 = 10, j_5 = 14$.

(E) *Estimation of the dispersion parameters for random uncertainties modeling*. Using the "numerical experiment" of the real system, an estimation of the dispersion parameters δ_M, δ_D and δ_K of the random generalized mass, damping and stiffness matrices is performed by using the method presented in Soize (2003a). Such an estimation yields $\delta_M = 0.29$, $\delta_D = 0.30$ and $\delta_K = 0.68$.

(F) *Prediction with random uncertainties and "experimental" comparisons*. In this section, we present (1) prediction with the nonparametric probabilistic model of random uncertainties and (2) comparisons with the mean model prediction and with the "experimental" response of the real system. The convergence with respect to n and n_s (dimension of the stochastic reduced model and number of realizations used in the Monte Carlo numerical simulation method) is first studied in Section 3.7. Figure 3 displays the graph of function $n_s \mapsto \text{Conv}(n_s, n)$ defined in Section 3.7 (C) for different values of n. This figure shows that a reasonable convergence is reached for $n \geq 80$ and $n_s \geq 1500$.

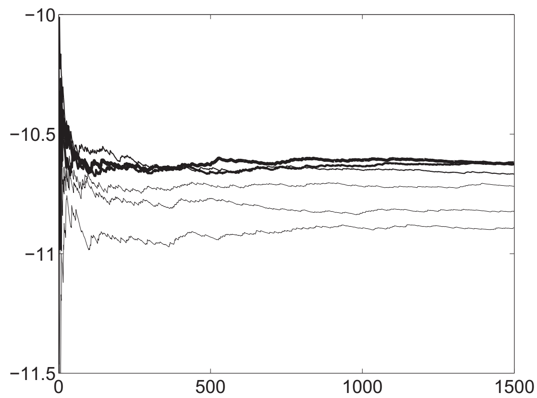

Figure 3. Statistical convergence: graphs of the function $n_s \mapsto \log_{10}\{\text{Conv}(n_s, n)\}$ for $n = 20$, $n = 30$ and 60 (three lower thin solid lines, for $n = 80$, $n = 120$ and $n = 160$ (three upper lines: $n = 80$ (thin solid line), $n = 120$ (mid solid line) and $n = 160$ (thick solid line). Horizontal axis n_s.

Let O_1 and O_2 be the observation points on the line $(x, 0, 0), x \in]0, \underline{h}_1[$ (neutral fiber) and located at $x_1 = 5.000\, m$ and $x_2 = 6.375\, m$ respectively. The confidence region of the modulus of the frequency response function at each observation point O_1 or O_2 is calculated by using the method presented in Section 3. The confidence region for frequency response at a given observation point is carried out with a probability level $P_c = 0.98$ and for $n = 80$ and $n_s = 3000$. For observation points O_1 and O_2, Figs. 4-a and 4-b, respectively, display the comparisons between the mean model response predictions, the "experimental" responses of the real system and the confidence region predictions of the stochastic system resulting from the use of the nonparametric probabilistic approach for random uncertainties.

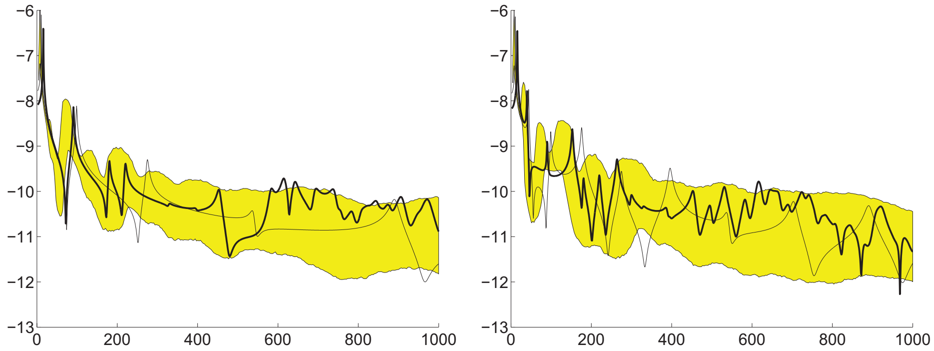

Figure 4. Confidence region prediction of the stochastic system with nonparametric approach at observation point O_1 (Fig. (a) left) and O_2 (Fig. (b) right) (grey region). Mean model response (thin solid line). "Numerical experiment" of the real system (thick solid line).

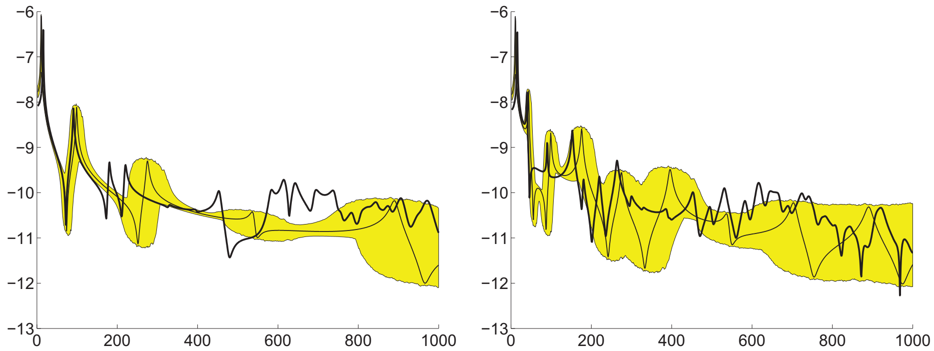

Figure 5. Confidence region prediction of the stochastic system with parametric approach at observation point O_1 (Fig. (a) left) and O_2 (Fig. (b) right)(grey region). Mean model response (thin solid line). "Numerical experiment" of the real system (thick solid line).

(G) *Lack of capability of the parametric probabilistic approach to take into account model uncertainties*. In this section, we presents the results obtained from the usual parametric probabilistic approach (data uncertainties). For such an approach, the random variables are the mass density ρ, the geometric parameters h_1, h_2 and h_3, the Young modulus $\mathcal{E}$ and the damping ratio ξ. These 6 random variables are assumed to be mutually independent. Positive-valued random variables h_1, h_2 and h_3 are uniformely distributed with known mean values $\underline{h}_1$, $\underline{h}_2$ and $\underline{h}_3$ and coefficients of variation δ_{h_1}, δ_{h_2} and δ_{h_3} to be identified (see below) (the coefficient of variation is the standard deviation divided by the mean value). In addition, it is assumed that $\delta_{h_1} = \delta_{h_2} = \delta_{h_3}$. Positive-valued random variables ρ, $\mathcal{E}$ and ξ are Gamma random variables with known mean values $\underline{\rho}$, $\underline{\mathcal{E}}$ and $\underline{\xi}$ and for coefficients of variation δ_ρ, $\delta_\mathcal{E}$ and δ_ξ to be identified (see below). Let Λ_1^{par} and $\Lambda_1^{\text{nonpar}}$ be the lowest random eigenfrequencies of the stochastic systems constructed with the parametric and nonparametric approach respectively. Let $\delta_{\Lambda_1^{\text{par}}}$ and $\delta_{\Lambda_1^{\text{nonpar}}}$ be the coefficients of variation of random variables Λ_1^{par} and $\Lambda_1^{\text{nonpar}}$. In order to compare comparable things, the coefficients of variation $\delta_{h_1} = \delta_{h_2} = \delta_{h_3}$, δ_ρ

and $\delta_{\mathcal{E}}$ of random variables h_1, h_2, h_3, ρ and $\mathcal{E}$ for the parametric probabilistic approach, were calculated to yield $\min\{(\delta_{\Lambda_1^{\text{par}}} - \delta_{\Lambda_1^{\text{nonpar}}})^2\}$ in which $\delta_{\Lambda_1^{\text{nonpar}}} = 0.076918$ is known. A solution is $\delta_{h_1} = \delta_{h_2} = \delta_{h_3} = 0.024$, $\delta_\rho = 0.03$, $\delta_{\mathcal{E}} = 0.1$ corresponding to $\delta_{\Lambda_1^{\text{par}}} = 0.076492$. The coefficient of variation δ_ξ is calculated by the equation $\delta_\xi = \delta_D$ which yields $\delta_\xi = 0.3$. For observation points O_1 and O_2, Figs. 5-a and 5-b display respectively the comparisons between the mean model response predictions, the "experimental" responses of the real system and the confidence region predictions of the stochastic system resulting from the use of the parametric probabilistic approach of random uncertainties.
(H) *Conclusion*. The comparaisons of Fig. 4-a with 5-a and 4-b with 5-b show that the two approaches yield similar results in the low-frequency range $[0\,,100]$ Hz (this band is relatively robust with respect to model uncertainties) are very different in the frequency band $[100, 1000]$ Hz. The parametric probabilistic approach allows data uncertainties to be taken into account but cannot address model uncertainties which become more significant for frequency increases. On the contrary, the nonparametric probabilistic approach allows model uncertainties to be taken into account.

5. Experimental Validation: Dynamics of a Composite Sandwich Panel

Two experimental validations of the proposed theory have been published by Chebli & Soize (2004) and Duchereau & Soize (2005) in the context of heterogeneous model uncertainties in vibration and transient dynamics of structures. In this section, we present an experimental validation in another context, i.e. the vibration of multilayer composites. In addition the experimental identification of the dispersion parameters controlling model uncertainties is presented. The details of the results summarized in this section can be found in Chen et al. (2006).
(A) *Designed panel*. The designed panel is a sandwich panel constituted of five layers four of which are thin carbon-resin unidirectional plies and one is a high stiffness closed-cell foam core (see Fig. 6). This panel is defined with respect to a Cartesian coordinate system $Oxyz$ and is $0.40\,m$ long(Ox axis), $0.30\,m$ wide (Oy axis) and $0.01068\,m$ thick (Oz axis). The middle plane of the sandwich panel is Oxy and the origine O is located in one corner. Each carbon layer is made of a thin carbon-resin ply with a thickness of $0.00017\,m$, a mass density $\rho = 1600\,Kg/m^3$ and whose elasticity constants are: $E_x = 101\,GPa$, $E_y = 6.2\,GPa$, $\nu_{xy} = 0.32$, $G_{xy} = G_{xz} = G_{yz} = 2.4\,GPa$. The first two layers are two carbon-resin unidirectional plies in a [-60/60] layup. The third layer is the closed-cell foam core with a thickness of $0.01\,m$, a mass density of $80\,Kg/m^3$ and elasticity constants: $E_x = E_y = 60\,MPa$, $\nu_{xy} = 0$, $G_{xy} = G_{xz} = G_{yz} = 30\,MPa$. The fourth and fifth layers are two carbon-resin unidirectional plies in a [60/-60] layup.

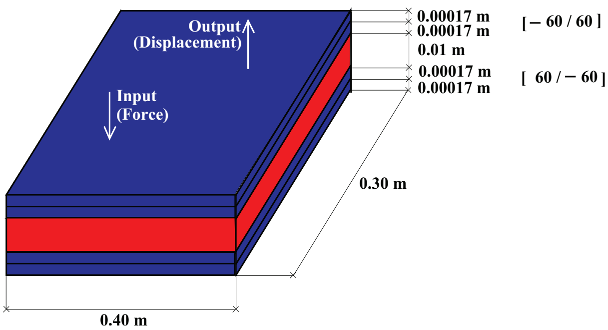

Figure 6. Composite sandwich panel.

(B) *Manufactured panels*. Eight sandwich panels were manufactured from the designed panel using the same process and the same materials. All sandwich panels were baked in the same batch to suppress the influence of variations in the baking conditions, e.g. time and temperature.

(C) *Experimental frequency response functions*. The dynamical testing of the eight sandwich panels was realized in the free-free condition. The middle plane of the sandwich panel was vertical and the panel was suspended with a very low eigenfrequency. The measurements were performed on the frequency band $B = [10\,,4500]\,Hz$. The input z-force was a point load applied to point $N0$ of coordinates $(0.187, 0.103, 0)\,m$. An electrodynamic shaker delivered a broad band signal. The output z-accelerations were measured at 25 points by accelerometers. For the sake of brevity, the presentation is limited to the point with coordinates $(0.337, 0.216, 0)\,m$. The experimental cross-frequency response functions were identified on frequency band B using the usual spectral analysis methods and signal processing.

(D) *Experimental modal analysis*. An experimental modal analysis was performed in the frequency band $[10\,,1550]\,Hz$ using the identified experimental frequency response functions. For each sandwich panel $r = 1, \ldots, 8$, eleven elastic modes were identified in this frequency band. For sandwich panel r, the following usual modal parameters of each experimental elastic mode α were identified: (1) the eigenfrequency $\omega_\alpha^{\rm exp}(\theta_r)$, (2) the damping ratio $\xi_\alpha^{\rm exp}(\theta_r)$, (3) the elastic mode shape $\psi_\alpha^{\rm exp}(\theta_r)$ and the corresponding generalized mass $\mu_\alpha^{\rm exp}(\theta_r)$. Let $\underline{\omega}_\alpha^{\rm exp} = (1/8)\sum_{r=1}^{8} \omega_\alpha^{\rm exp}(\theta_r)$ be the average experimental eigenfrequency α. Introducing $\underline{f}_\alpha^{\rm exp} = \underline{\omega}_\alpha^{\rm exp}/(2\,\pi)$, the results are $\underline{f}_1^{\rm exp} = 191.0\,Hz$, $\underline{f}_2^{\rm exp} = 329.5\,Hz$, $\underline{f}_3^{\rm exp} = 532.0\,Hz$ and $\underline{f}_4^{\rm exp} = 635.1\,Hz$. For $\alpha = 1, \ldots, 11$, let $\underline{\xi}_\alpha^{\rm exp} = (1/8)\sum_{r=1}^{8} \xi_\alpha^{\rm exp}(\theta_r)$ be the average experimental damping ratio α and let $\underline{\xi}^{\rm exp} = (1/11)\sum_{\alpha=1}^{11} \underline{\xi}_\alpha^{\rm exp}$ be the global average experimental damping ratio. The result is $\underline{\xi}^{\rm exp} = 0.01$.

(E) *Mean model and its updating*. The designed panel is modeled as a laminated composite thin plate for which each layer is an orthotropic elastic material in plane stress. Since we are only interested in the z-displacement of the middle plane of the sandwich panel in the bending mode and since the panel is a free structure, there are 3 rigid body modes. We are interested in the construction of the responses in the frequency domain over the frequency band of analysis B. The designed panel is modeled by using a regular finite element mesh constituted of 128×64 four-nodes finite elements for laminated plate bending. The number of DOF is $25\,155$. The damping of the structure is introduced by an arbitrary usual model controlled by the modal damping ratios deduced from the measurements. The mean model has been updated in average using the first 4 experimental eigenfrequencies for each panel (8 panels).

(F) *Reduced mean model*. The reduced mean model was constructed by using the first $n = 200$ elastic modes of the updated mean model including the 3 rigid body modes. Convergence was found to be reached for this number of modes.

(G) *FRF calculation with the reduced mean model and experimental comparisons*. The cross-frequency response function corresponding to the observation point is calcultated with the reduced mean model. Figure 7 displays, in log scale, the graphs of the modulus of the experimental and numerical cross-frequency response functions for which the input is the driven point and the output is the z-acceleration at the observation point. Note that there are 9 curves in the figure: 8 curves correspond to the experimental cross-frequency response functions of the 8 sandwich panels and 1 curve corresponds to the numerical cross-frequency response function computed with the reduced mean model. The comparison of the experimental cross-frequency response functions with the one constructed with the reduced mean model is reasonably good in the frequency band $[0,\ 1500]\,Hz$ and relatively poor in $[1500\,,4500]\,Hz$. In the frequency band $[1500,\ 4500]\,Hz$, the lack of predictability is increasing with the frequency and is mainly due to model uncertainties (modeling the sandwich panel by using the laminated composite thin plate theory) and to a lesser degree to data uncertainties (mechanical parameters).

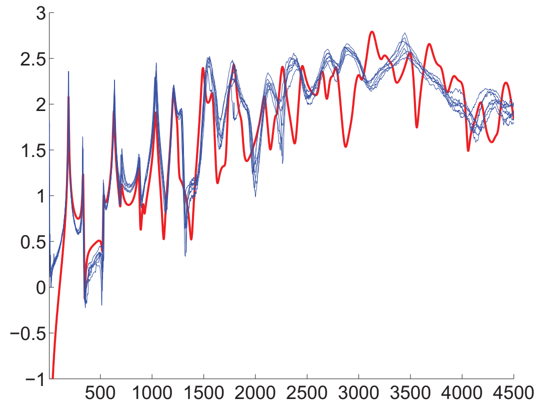

Figure 7. Graphs of the cross-FRF between driven point and observation point. Horizontal axis: frequency in Hertz. Vertical axis: $\log_{10}$ of the modulus of the acceleration in m/s^2. Experimental cross-FRF corresponding to the 8 panels (8 thin solid lines). Numerical cross-FRF calulated with the reduced mean model (thick solid line)

(H) *Experimental identification of the dispersion parameters of the nonparametric model*. Let δ_M, δ_D and δ_K be the dispersion parameters of the random generalized mass, damping and stiffness matrices. They were estimated by using the experimental generalized matrices corresponding to the 8 experimental sandwich panels, and for a dimension $\nu < n$. The dispersion parameters δ_M, δ_D and δ_K were estimated by (Chen et al. 2006) using the method presented in Soize (2003a and 2005b) and yields $\delta_M = 0.23$, $\delta_D = 0.43$ and $\delta_K = 0.25$ for random matrices $[\mathbf{M}_n]$, $[\mathbf{D}_n]$ and $[\mathbf{K}_n]$ (these values are independent of dimension n of the stochastic reduced model).
(I) *Confidence region prediction for the FRF and experimental comparisons*. Figure 8 displays the confidence region prediction for the random cross-frequency response functions between the driven point and the observation point, computed with $n_s = 2000$ realizations for the Monte Carlo numerical simulation and $n = 200$ (mean-square convergence is reached for these values).

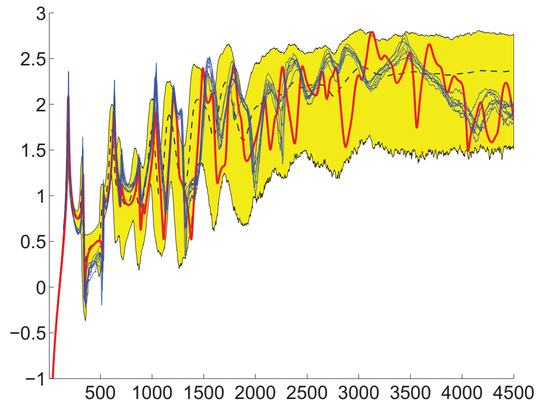

Figure 8. Confidence region prediction for the random cross-FRF. Horizontal axis: frequency in Hertz. Vertical axis: $\log_{10}$ of the modulus of the acceleration in m/s^2. Experimental cross-FRF corresponding to the 8 panels (8 thin solid lines). Numerical cross-FRF calculated with the reduced

mean model (thick solid line). Mean value of the random cross-FRF calculated with the nonparametric probabilistic model (thin dashed line). Confidence region of the random cross-FRF calculated with the nonparametric probabilistic model (grey region).

(J) *Conclusions*. Experimental results obtained for a set of 8 light sandwich panels show the sensitivity of the dynamical response of the panels in the medium-frequency range. The use of the classical laminated composite thin plate theory to construct the predictive mean model introduces significant model uncertainties in the medium-frequency range. Since such dynamical systems are very sensitive to uncertainties, the introduction of a probabilistic model of both data and model uncertainties is necessary to improve the predictability of the mean model. The prediction from the nonparametric model compared with the experiments is good.

6. Industrial Applications

The nonparametric probabilistic modeling of random uncertainties have recently be developed for industrial applications in computational dynamics for linear and nonlinear dynamical systems, for complex structures and vibroacoustic systems:

(1) The linear dynamics of a bladed disk mistuned due to manufacturing tolerances uncertainties (Capiez-Lernout et al. 2005a).

(2) The identification and quantification of the design margins in the nonlinear dynamics of a reactor coolant system (Desceliers et al. (2004).

(3) The robustness of the numerical simulation model with respect to model and data uncertainties in dynamics of a spatial structure (Capiez-Lernout et al. 2005b).

(4) The robustness of the numerical vibroacoustic FRF of cars with respect to model and data uncertainties (Durand et al. 2004 and 2005).

7. Conclusions

A nonparametric probabilistic approach has been proposed to take into account model uncertainties and data uncertainties in computational dynamics. It is shown with a simple example that the usual parametric probabilistic approach allows data uncertainties to be analyzed but does not allow model uncertainties to be taken into account. An additional experimental validation of the approach proposed is given. Such a therory has been applied to industrial applications.

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