Non Gaussian matrix-valued random fields for nonparametric probabilistic modeling of elliptic stochastic partial differential operators

Christian Soize

Non Gaussian Matrix-Valued Random Fields for Nonparametric Probabilistic Modeling of Elliptic Stochastic Partial Differential Operators

C. Soize
University of Marne-la-Vallée, Paris, France
soize@univ-mlv.fr

Abstract

This paper deals with the construction of a non Gaussian positive-definite matrix-valued random field whose mathematical properties allow elliptic stochastic partial differential operators to be modeled. Such a matrix-valued random field can directly be used for modeling random uncertainties in computational sciences with a stochastic model having a small number of parameters. For instance, in three-dimensional linear elasticity, the fourth-order elasticity tensor of a random non homogeneous anisotropic elastic material is constituted of 21 dependent random fields which have to be such that the positive-definiteness property of this fourth-order tensor be verified in a given probabilistic sense. If the usual parametric probabilistic approach is used, then the identification of such a probabilistic model by using experimental data seems to be difficult. The non Gaussian positive-definite matrix-valued random field presented in this paper allows such a probabilistic model of the fourth-order tensor-valued random field to be constructed and depends only of 4 scalar parameters: three spatial correlation lengths and one parameter allowing the level of the random fluctuations to be controlled. Such a model can directly be used in the stochastic finite element method.

1. Introduction

A great challenge is the construction of construct stochastic representations for uncertain parameters for which probabilistic data are known and can be identified by using experimental data. Such a probabilistic model is useful in computational sciences and in particular for stochastic finite elements (Kleiber *et al.*, 1992; Ghanem and Spanos, 2003). For instance, consider the following deterministic elliptic partial differential operator $\underline{A}$ on a bounded open domain Ω of $\mathbb{R}^3$, related to the three-dimensional linear elasticity for a non homogeneous anisotropic elastic material,

$$\underline{A}\,\underline{\mathbf{u}} = -\sum_{i=1}^{3} \mathbf{e}^i \sum_{j=1}^{3} \frac{\partial}{\partial x_j}\{\sum_{k,h=1}^{3} \underline{c}_{ijkh}(\mathbf{x})\,\varepsilon_{kh}(\underline{\mathbf{u}})\} \quad , \tag{1}$$

in which $\mathbf{x} = (x_1, x_2, x_3) \in \Omega \subset \mathbb{R}^3$, where $\mathbf{e}^1 = (1,0,0)$, $\mathbf{e}^2 = (0,1,0)$ and $\mathbf{e}^3 = (0,0,1)$ are the vectors of the canonical basis of $\mathbb{R}^3$ and where $\mathbf{x} \mapsto \underline{\mathbf{u}}(\mathbf{x}) = (\underline{u}_1(\mathbf{x}), \underline{u}_2(\mathbf{x}), \underline{u}_3(\mathbf{x}))$ is a twice differentiable function from Ω into $\mathbb{R}^3$. The second-order strain tensor is such that $\varepsilon_{kh}(\underline{\mathbf{u}}) = (1/2)\,(\partial\underline{u}_k/\partial x_h + \partial\underline{u}_h/\partial x_k)$. The fourth-order elasticity tensor $\underline{c}_{ijkh}(\mathbf{x})$ has to verify the symmetry property $\underline{c}_{ijkh}(\mathbf{x}) = \underline{c}_{jikh}(\mathbf{x}) = \underline{c}_{ijhk}(\mathbf{x}) = \underline{c}_{khij}(\mathbf{x})$ and, for all symmetric second-order real tensors $\{z_{ij}\}_{ij}$, has to verify the positive-definiteness property, $\sum_{i,j,k,h=1}^{3} \underline{c}_{ijkh}(\mathbf{x}) z_{kh} z_{ij} \;\geq\; \underline{c}_0 \sum_{i,j=1}^{3} z_{ij}^2$, in which $\underline{c}_0$ is a positive constant independent of $\mathbf{x}$. For a random medium, for all $\mathbf{x}$ fixed in Ω, tensor $\{\underline{c}_{ijkh}(\mathbf{x})\}_{ijkh}$ is replaced by a fourth-order tensor-valued random variable $\{C_{ijkh}(\mathbf{x})\}_{ijkh}$ whose mean value is $\{\underline{c}_{ijkh}(\mathbf{x})\}_{ijkh}$ and which has to verify the symmetry and the positive-definiteness properties in a probabilistic sense which has to be defined. Nevertheless, for the random case, the deterministic constant $\underline{c}_0$ (introduced above) cannot generally be justified from a probabilistic modeling point of view. Finally, $\mathbf{x} \mapsto \{C_{ijkh}(\mathbf{x})\}_{ijkh}$ is a fourth-order tensor-valued random field indexed by Ω, constituted of 21 mutually dependent random fields and the stochastic partial differential operator $\mathbf{A}$ associated with operator $\underline{A}$ written as

$$\mathbf{A}\,\mathbf{U} = -\sum_{i}^{3} \mathbf{e}^{i} \sum_{j=1}^{3} \frac{\partial}{\partial x_j}\{ \sum_{k,h=1}^{3} C_{ijkh}(\mathbf{x})\, \varepsilon_{kh}(\mathbf{U})\} \quad . \tag{2}$$

It should be noted that the probability distribution of this fourth-order tensor-valued random field (that is to say the system of the marginal distributions) is required because the unknown solution of the stochastic boundary value problem is a nonlinear mapping of random field $\mathbf{x} \mapsto \{C_{ijkh}(\mathbf{x})\}_{ijkh}$. If the usual parametric probabilistic approach is used, then the identification of this probability model by using experimental data seems to be difficult. This paper deals with a nonparametric construction of a random field such as $\mathbf{x} \mapsto \{C_{ijkh}(\mathbf{x})\}_{ijkh}$. For that, an ensemble of non Gaussian positive-definite matrix-valued random fields is constructed and studied which allows, for instance, the fourth-order tensor-valued random field $\mathbf{x} \mapsto \{C_{ijkh}(\mathbf{x})\}_{ijkh}$ to be modeled. Then, such a tensor-valued random field will depend only on 4 scalar parameters: three spatial correlation lengths and one parameter allowing the level of the random fluctuations to be controlled. With such a model, the inverse problem related to the experimental identification seems to be more feasible.

The following algebraic notations are used. Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a vector in $\mathbb{R}^n$. The Euclidean space $\mathbb{R}^n$ is equipped with the usual inner product $(\mathbf{x}, \mathbf{y}) \mapsto < \mathbf{x}\,, \mathbf{y} > = \sum_{j=1}^{n} x_j y_j$ and the associated norm $\|\mathbf{x}\| \;= < \mathbf{x}\,, \mathbf{x} >^{1/2}$. Let $\mathbb{M}_{n,m}(\mathbb{R})$ be the set of all the $(n \times m)$ real matrices, $\mathbb{M}_n(\mathbb{R}) = \mathbb{M}_{n,n}(\mathbb{R})$ be the set of all the square $(n \times n)$ real matrices, $\mathbb{M}_n^S(\mathbb{R})$ be the set of all the $(n \times n)$ real symmetric matrices and $\mathbb{M}_n^+(\mathbb{R})$ be the set of all the $(n \times n)$ real symmetric positive-definite matrices. We then have $\mathbb{M}_n^+(\mathbb{R}) \subset \mathbb{M}_n^S(\mathbb{R}) \subset \mathbb{M}_n(\mathbb{R})$. We denote (i) the trace of the matrix $[A] \in \mathbb{M}_n(\mathbb{R})$ as $\mathrm{tr}[A] = \sum_{j=1}^{n} [A]_{jj}$; (ii) the transpose of $[A] \in \mathbb{M}_{n,m}(\mathbb{R})$ as $[A]^T \in \mathbb{M}_{m,n}(\mathbb{R})$; (iii) the operator norm of the matrix $[A] \in \mathbb{M}_{n,m}(\mathbb{R})$ as $\|A\| = \sup_{\|\mathbf{x}\| \leq 1} \|[A]\,\mathbf{x}\|$, $\mathbf{x} \in \mathbb{R}^m$, which is such that $\|[A]\,\mathbf{x}\| \leq \|A\|\,\|\mathbf{x}\|$, $\forall \mathbf{x} \in \mathbb{R}^m$, and if $m = n$, then $\|A\| = |\lambda_n|$, in which $|\lambda_n|$ is the largest modulus of the eigenvalues of $[A]$; (iv) for $[A] \in \mathbb{M}_{n,m}(\mathbb{R})$, we note $\|A\|_F^2 = \mathrm{tr}\{[A]^T[A]\} = \sum_{j=1}^{n} \sum_{k=1}^{m} [A]_{jk}^2$ and for $[A]$ in $\mathbb{M}_n(\mathbb{R})$, we have $\|A\| \leq \|A\|_F \leq \sqrt{n}\|A\|$.

2. Construction and properties of the ensemble SFG^+ of homogeneous and normalized non Gaussian positive-definite matrix-valued random fields

2.1. Random field U as the germ of ensemble SFG^+

Definition. Let $d \geq 1$ be an integer. Let $\mathbf{x} \mapsto U(\mathbf{x})$ be a second-order centered homogeneous Gaussian random field, defined on probability space $(\Theta, \mathcal{T}, P)$, indexed by $\mathbb{R}^d$, with values in $\mathbb{R}$. Let $L_1, \ldots, L_d$ be positive real numbers. Its autocorrelation function $R_U(\boldsymbol{\eta}) = E\{U(\mathbf{x} + \boldsymbol{\eta})\, U(\mathbf{x})\}$, defined for all $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_d)$ in $\mathbb{R}^d$, is written as $R_U(\boldsymbol{\eta}) = \rho_1(\eta_1) \times \ldots \times \rho_d(\eta_d)$ in which, for all $j = 1, \ldots, d$, we have $\rho_j(0) = 1$ and $\rho_j(\eta_j) = 4L_j^2/(\pi^2\eta_j^2)\ \sin^2(\pi\eta_j/(2L_j))$ for $\eta_j \neq 0$.

Properties. For all $\mathbf{x}$ in $\mathbb{R}^d$, $E\{U(\mathbf{x})\} = 0$ and $E\{U(\mathbf{x})^2\} = 1$. The random field U is mean-square continuous on $\mathbb{R}^d$. Let $\mathbf{k} = (k_1, \ldots, k_d)$ be a point in $\mathbb{R}^d$ and let $d\mathbf{k} = dk_1 \ldots dk_d$ be the Lebesgue measure. Then, there is a power spectral density function $\mathbf{k} \mapsto S_U(\mathbf{k})$ from $\mathbb{R}^d$ into $\mathbb{R}^+$, integrable, such that $\forall \boldsymbol{\eta} \in \mathbb{R}^d$, $R_U(\boldsymbol{\eta}) = \int_{\mathbb{R}^d} e^{i<\boldsymbol{\eta},\mathbf{k}>}\, S_U(\mathbf{k})\, d\mathbf{k}$ which can be written as $S_U(\mathbf{k}) = s_1(k_1) \times \ldots \times s_d(k_d)$ in which, for all $j = 1, \ldots, d$, the function $k_j \mapsto s_j(k_j)$ from $\mathbb{R}$ into $\mathbb{R}^+$ is defined by $s_j(k_j) = (L_j/\pi)\, q(k_j L_j/\pi)$. The function $\tau \mapsto q(\tau)$ is continuous from $\mathbb{R}$ into $\mathbb{R}^+$, has a compact support $[-1\,, 1]$ and is such that $q(0) = 1$, $q(-\tau) = q(\tau)$ and $q(\tau) = 1 - \tau$ for $\tau \in [0\,, 1]$. This means that

S_U has a compact support. Introducing L_j^U as the spatial correlation length relative to coordinate x_j and defined by $L_j^U = \int_0^{+\infty} |R_U(0,\ldots,0,\eta_j,0,\ldots,0)|\, d\eta_j$, it can easily be deduced that $L_j^U = L_j$. Consequently, parameters $L_1,\ldots,L_d$ represent the spatial correlation lengths of random field U.

Representation of the random field U adapted to its numerical simulation. The spatial discretization of this random field will directly be related to the spatial discretization of the elliptic stochastic partial differential operator for which germ U will be used. In general, the problem is setted on an arbitrary bounded domain Ω of $\mathbb{R}^d$ and the finite element method is utilized. Consequently, U has to be simulated in N given points $\mathbf{x}^1,\ldots,\mathbf{x}^N$ in $\Omega \subset \mathbb{R}^d$ (for instance, located in the integrating points of the finite elements of the finite element mesh of domain Ω). We then have to simulate realizations of the random vector $\mathbf{U} = (U(\mathbf{x}^1),\ldots,U(\mathbf{x}^N))$. A first representation adapted to a large value of N is based on the usual numerical simulation of homogeneous Gaussian vector-valued random field U constructed with the stochastic integral representation of homogeneous stochastic fields. A second representation adapted to a small or moderate value of N consists in writing $\mathbf{U} = [L_\mathbf{U}]^T \mathbf{V}$ in which $\mathbf{V} = (V_1,\ldots,V_N)$ is an $\mathbb{R}^N$-valued random variable whose components $V_1,\ldots,V_N$ are N independent normalized Gaussian random variables ($E\{V_j\} = 0$ and $E\{V_j^2\} = 1$ for $j = 1,\ldots,N$) and where $[L_\mathbf{U}]$ is the upper real triangular matrix corresponding to the Chowlesky factorization $[C_\mathbf{U}] = [L_\mathbf{U}]^T [L_\mathbf{U}]$ of the covariance matrix $[C_\mathbf{U}]$ in $\mathbb{M}_N^+(\mathbb{R})$ such that $[C_\mathbf{U}]_{ij} = R_U(\mathbf{x}^i - \mathbf{x}^j)$.

2.2. *Ensemble* SFG$^+$

Defining the family of functions $\{u \mapsto h(\alpha,u)\}_{\alpha>0}$. Let α be a positive real number. The function $u \mapsto h(\alpha,u)$ from $\mathbb{R}$ into $]0\,,+\infty[$ is such that $\Gamma_\alpha = h(\alpha,U)$ is a gamma random variable with parameter α while U is a normalized Gaussian random variable ($E\{U\} = 0$ and $E\{U^2\} = 1$). Consequently, for all u in $\mathbb{R}$, we have $h(\alpha,u) = F_{\Gamma_\alpha}^{-1}(F_U(u))$ in which $u \mapsto F_U(u) = P(U \leq u)$ is the cumulative distribution function of the normalized Gaussian random variable U. The function $p \mapsto F_{\Gamma_\alpha}^{-1}(p)$ from $]0\,,1[$ into $]0\,,+\infty[$ is the reciprocical function of the cumulative distribution function $\gamma \mapsto F_{\Gamma_\alpha}(\gamma)$ from $]0\,,+\infty[$ into $]0\,,1[$ of the gamma random variable Γ_α with parameter α, which is such that, for all γ in $\mathbb{R}^+$, $F_{\Gamma_\alpha}(\gamma) = \int_0^\gamma \frac{1}{\Gamma(\alpha)}\, t^{\alpha-1}\, e^{-t}\, dt$ in which $\Gamma(\alpha)$ is the gamma function.

Defining the ensemble SFG$^+$ *of the random field $\mathbf{x} \mapsto [\mathbf{G}_n(\mathbf{x})]$.* The ensemble SFG$^+$ is defined as the set of all the random fields $\mathbf{x} \mapsto [\mathbf{G}_n(\mathbf{x})]$, defined on the probability space $(\Theta,\mathcal{T},P)$, indexed by $\mathbb{R}^d$ where $d \geq 1$ is a fixed integer, with values in $\mathbb{M}_n^+(\mathbb{R})$ where $n \geq 2$ is another fixed integer, and defined as follows: (i) Let $\{U_{jj'}(\mathbf{x}), \mathbf{x} \in \mathbb{R}^d\}_{1\leq j \leq j' \leq n}$ be $n(n+1)/2$ independent copies of the random field $\{U(\mathbf{x}), \mathbf{x} \in \mathbb{R}^d\}$. Consequently, for $1 \leq j \leq j' \leq n$, we have $E\{U_{jj'}(\mathbf{x})\} = 0$ and $E\{U_{jj'}(\mathbf{x})^2\} = 1$ and the random field $\mathbf{x} \mapsto U_{jj'}(\mathbf{x})$ is completely defined. (ii) Let δ be the real number, independent of $\mathbf{x}$ and n, such that $0 < \delta < \sqrt{(n+1)(n+5)^{-1}} < 1$. This parameter will allow the dispersion of the random field to be controlled. (iii) For all $\mathbf{x}$ in $\mathbb{R}^d$, $[\mathbf{G}_n(\mathbf{x})] = [\mathbf{L}_n(\mathbf{x})]^T\, [\mathbf{L}_n(\mathbf{x})]$ in which $[\mathbf{L}_n(\mathbf{x})]$ is the upper $(n \times n)$ real triangular random matrix defined as follows. The $n(n+1)/2$ random fields $\mathbf{x} \mapsto [\mathbf{L}_n(\mathbf{x})]_{jj'}$ for $1 \leq j \leq j' \leq n$, are independent. For $j < j'$, the real-valued random field $\mathbf{x} \mapsto [\mathbf{L}_n(\mathbf{x})]_{jj'}$, indexed by $\mathbb{R}^d$, is defined by $[\mathbf{L}_n(\mathbf{x})]_{jj'} = \sigma_n U_{jj'}(\mathbf{x})$ in which σ_n is such that $\sigma_n = \delta\,(n+1)^{-1/2}$. For $j = j'$, the positive-valued random field $\mathbf{x} \mapsto [\mathbf{L}_n(\mathbf{x})]_{jj}$, indexed by $\mathbb{R}^d$, is defined by $[\mathbf{L}_n(\mathbf{x})]_{jj} = \sigma_n \sqrt{2\, h(\alpha_j, U_{jj}(\mathbf{x}))}$ in which, for $j = 1,\ldots,n$, $\alpha_j = (n+1)/(2\delta^2) + (1-j)/2$.

Basic properties. $\mathbf{x} \mapsto [\mathbf{G}_n(\mathbf{x})]$ is a homogeneous second-order mean-square continuous random field indexed by $\mathbb{R}^d$ with values in $\mathbb{M}_n^+(\mathbb{R})$. In addition, the trajectories of random field $\mathbf{x} \mapsto [\mathbf{G}_n(\mathbf{x})]$ are continuous from $\mathbb{R}^d$ into $\mathbb{M}_n^+(\mathbb{R})$ almost surely. For all $\mathbf{x} \in \mathbb{R}^d$, we have $E\{\|[\mathbf{G}_n(\mathbf{x})]\|_F^2\} < +\infty$ and $E\{[\mathbf{G}_n(\mathbf{x})]\} = [I_n]$. The parameter δ is such that $\delta = \left\{\frac{1}{n}E\{\| [\mathbf{G}_n(\mathbf{x})] - [I_n] \|_F^2\}\right\}^{1/2}$ which shows that $E\{\| [\mathbf{G}_n(\mathbf{x})] \|_F^2\} = n\,(\delta^2 + 1)$. For all $\mathbf{x}$ fixed in $\mathbb{R}^d$, the probability distribution on $\mathbb{M}_n^+(\mathbb{R})$ of random matrix $[\mathbf{G}_n(\mathbf{x})]$ is explicitly calculated in (Soize, 2001) and shows that, for all $\mathbf{x}$ in $\mathbb{R}^d$, the random variables $\{[\mathbf{G}_n(\mathbf{x})]_{ij}, 1 \le i \le j \le n\}$ are mutually dependent. The system of the marginal probability distributions of random field $\mathbf{x} \mapsto [\mathbf{G}_n(\mathbf{x})]$ is well defined but cannot be explicitly calculated. Random field $\mathbf{x} \mapsto [\mathbf{G}_n(\mathbf{x})]$ is non Gaussian. There exists a positive constant c_0 independent of n and independent of $\mathbf{x}$, but depending on δ, such that $E\{\|[\mathbf{G}_n(\mathbf{x})]^{-1}\|^2\} \le c_0 < +\infty$ for all $n \ge 2$ and for all $\mathbf{x} \in \mathbb{R}^d$. We then have $E\{\|[\mathbf{G}_n(\mathbf{x})]^{-1}\|_F^2\} \le c_n < +\infty$ for all $n \ge 2$ and for all $\mathbf{x} \in \mathbb{R}^d$ in which $c_n = n\,c_0$. It should be noted that, since $[\mathbf{G}_n(\mathbf{x})]$ belongs to $\mathbb{M}_n^+(\mathbb{R})$ almost surely, then $[\mathbf{G}_n(\mathbf{x})]^{-1}$ exists almost surely. However, since almost sure convergence does not yield mean-square convergence, the previous result cannot simply be deduced (see Soize, 2001).

Fundamental property. Let Ω be a bounded open domain of $\mathbb{R}^d$ and let $\overline{\Omega} = \Omega \cup \partial\Omega$ be its closure in which $\partial\Omega$ is the boundary of Ω. We then have

$$E\big\{(\sup_{\mathbf{x}\in\overline{\Omega}} \| [\mathbf{G}_n(\mathbf{x})]^{-1}\|)^2\big\} = c_G^2 \; < \; +\infty \quad , \tag{3}$$

in which sup is the supremum and where $0 < c_G < +\infty$ is a finite positive constant.

Remark concerning the proof of Eq. (3). Let us consider the case $d = 1$ with $\overline{\Omega}$ be a compact interval of $\mathbb{R}$. Since the stochastic process $\{\| G_n(\mathbf{x})^{-1}\| \,, \mathbf{x} \in \overline{\Omega} \subset \mathbb{R}\}$ is not a continuous local martingal with respect to an increasing family of σ-fields, the following fundamental Doob maximal inequality (Doob, 1953) $E\big\{\sup_{\mathbf{x}\in\overline{\Omega}} \| [\mathbf{G}_n(\mathbf{x})]^{-1}\|^2\big\} \le 4\,E\big\{\| [\mathbf{G}_n(\mathbf{x})]^{-1}\|^2\big\}$ cannot be used. In addition, we have to consider the non Gaussian random field case $d \ge 2$. Consequently, there is no known result allowing a direct proof of Eq. (3) to be obtained and a complete proof of this fundamental result is given in (Soize, 2004).

3. Construction and properties of the ensemble SFE$^+$ of non Gaussian positive-definite matrix-valued random fields

3.1. Definition of the ensemble SFE$^+$

Let $d \ge 1$ and $n \ge 2$ be two fixed integers. Let Ω be an open (or closed) bounded (or not) domain of $\mathbb{R}^d$ (we can have $\Omega = \mathbb{R}^d$). Let $\mathbf{x} \mapsto [\underline{a}_n(\mathbf{x})]$ be a matrix-valued field from Ω into $\mathbb{M}_n^+(\mathbb{R})$. Then, for all $\mathbf{x}$ fixed in Ω, there is an upper triangular invertible matrix $[\underline{L}_n(\mathbf{x})]$ in $\mathbb{M}_n(\mathbb{R})$ such that $[\underline{a}_n(\mathbf{x})] = [\underline{L}_n(\mathbf{x})]^T\,[\underline{L}_n(\mathbf{x})]$. It is assumed that: (i) there is a real positive constant $0 < \underline{c}_0 < +\infty$ independent of $\mathbf{x}$ such that, for all $\mathbf{x}$ in Ω and for all $\mathbf{y} \in \mathbb{R}^n$, $<[\underline{a}_n(\mathbf{x})]\,\mathbf{y}\,,\mathbf{y}> \;\ge\; \underline{c}_0\,\|\mathbf{y}\|^2$; (ii) there is a real positive constant $0 < \underline{c}_1 < +\infty$ independent of $\mathbf{x}$ such that, for all $\mathbf{x}$ in Ω, we have $\|[\underline{L}_n(\mathbf{x})]\| \le \sqrt{\underline{c}_1}$ which yields $<[\underline{a}_n(\mathbf{x})]\,\mathbf{y}\,,\mathbf{y}> \;\le\; \underline{c}_1\,\|\mathbf{y}\|^2$, for all $\mathbf{y}$ in $\mathbb{R}^n$ and for all $\mathbf{x}$ in Ω. Consequently, for all $\mathbf{x}$ in Ω, we have $\|[\underline{a}_n(\mathbf{x})]\| \le \underline{c}_1$ and $\|[\underline{a}_n(\mathbf{x})]\|_F \le \sqrt{n}\,\underline{c}_1$. The ensemble SFE$^+$ is then defined as the set of all the random fields $\mathbf{x} \mapsto [\mathbf{A}_n(\mathbf{x})]$, defined on probability space $(\Theta, \mathcal{T}, P)$, indexed by Ω, with values in $\mathbb{M}_n^+(\mathbb{R})$, such that

$$\forall\,\mathbf{x} \in \Omega \quad , \quad [\mathbf{A}_n(\mathbf{x})] = [\underline{L}_n(\mathbf{x})]^T\,[\mathbf{G}_n(\mathbf{x})]\,[\underline{L}_n(\mathbf{x})] \quad , \tag{4}$$

in which $\mathbf{x} \mapsto [\mathbf{G}_n(\mathbf{x})]$ is the random field in SFG$^+$, defined on $(\Theta, \mathcal{T}, P)$, indexed by $\mathbb{R}^d$ and with values in $\mathbb{M}_n^+(\mathbb{R})$ (see Section 2.2).

3.2. Properties of the random field $\mathbf{x} \mapsto [\mathbf{A}_n(\mathbf{x})]$

Basic properties. For all $\mathbf{x}$ in Ω, $[\mathbf{A}_n(\mathbf{x})]$ is a random matrix with values in $\mathbb{M}_n^+(\mathbb{R})$, the mean function is such that $\mathbf{x} \mapsto E\{[\mathbf{A}_n(\mathbf{x})]\} = [\underline{a}_n(\mathbf{x})] \in \mathbb{M}_n^+(\mathbb{R})$ and $E\{\|[\mathbf{A}_n(\mathbf{x})]\|^2\} \leq E\{\|[\mathbf{A}_n(\mathbf{x})]\|_F^2\} \leq n\,\underline{c}_1^2\, E\{\|[\mathbf{G}_n(\mathbf{x})]\|^2\} \leq n\,\underline{c}_1^2\, E\{\|[\mathbf{G}_n(\mathbf{x})]\|_F^2\} < +\infty$ which proves that $\mathbf{x} \mapsto [\mathbf{A}_n(\mathbf{x})]$ is a second-order random field on Ω. In general, since $[\underline{a}_n(\mathbf{x})]$ depends on $\mathbf{x}$, then the random field $\{[\mathbf{A}_n(\mathbf{x})]\,, \mathbf{x} \in \Omega\}$ is non homogeneous. Nevertheless, if $[\underline{a}_n(\mathbf{x})] = [\underline{a}_n]$ is independent of $\mathbf{x}$, then the random field $\{[\mathbf{A}_n(\mathbf{x})] = [\underline{L}_n]^T\,[\mathbf{G}_n(\mathbf{x})]\,[\underline{L}_n]\,, \mathbf{x} \in \Omega\}$ can be viewed as the restriction to Ω of a homogeneous random field indexed by $\mathbb{R}^d$. We have $E\{\|[\mathbf{A}_n(\mathbf{x})] - [\underline{a}_n(\mathbf{x})]\|_F^2\} = \{\delta^2/(n+1)\}\{\|[\underline{a}_n(\mathbf{x})]\|_F^2 + (\text{tr}\,[\underline{a}_n(\mathbf{x})])^2\}$. The dispersion parameter, defined by $\delta_{A_n}(\mathbf{x}) = \{E\{\|[\mathbf{A}_n(\mathbf{x})] - [\underline{a}_n(\mathbf{x})]\|_F^2\}/\|[\underline{a}_n(\mathbf{x})]\|_F^2\}^{1/2}$, is such that $\delta_{A_n}(\mathbf{x}) = (\delta/\sqrt{n+1})\{1 + (\text{tr}\,[\underline{a}_n(\mathbf{x})])^2/\text{tr}\{[\underline{a}_n(\mathbf{x})]^2\}\}^{1/2}$.

Spatial correlation lengths for the homogeneous case. Then (see above), $\delta_{A_n}(\mathbf{x}) = \delta_{A_n}$ is independent of $\mathbf{x}$. Let $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_d) \mapsto r^{A_n}(\boldsymbol{\eta})$ be the function defined from $\mathbb{R}^d$ into $\mathbb{R}$ by $r^{A_n}(\boldsymbol{\eta}) = \text{tr}\, E\{([\mathbf{A}_n(\mathbf{x}+\boldsymbol{\eta})] - [\underline{a}_n])\,([\mathbf{A}_n(\mathbf{x})] - [\underline{a}_n])\}/E\{\|[\mathbf{A}_n(\mathbf{x})] - [\underline{a}_n]\|_F^2\}$. We have $r^{A_n}(0) = 1$ and $r^{A_n}(-\boldsymbol{\eta}) = r^{A_n}(\boldsymbol{\eta})$. For all $j = 1, \ldots, d$, the spatial correlation length $L_j^{A_n}$ of the homogeneous random field $\mathbf{x} \mapsto [\mathbf{A}_n(\mathbf{x})]$ indexed by $\mathbb{R}^d$, relative to coordinate x_j, can then be defined by $L_j^{A_n} = \int_0^{+\infty} |r^{A_n}(0, \ldots, 0, \eta_j, 0, \ldots, 0)|\, d\eta_j$.

4. Elliptic stochastic partial differential operator

The presentation is limited to the second-order stochastic differential operator defined by Eq. (2) on an open bounded domain Ω of $\mathbb{R}^3$ whose boundary $\partial\Omega$ is written as $\Gamma_0 \cup \Gamma$. On Γ_0, there is a zero Dirichlet boundary condition. We introduce the real Hilbert spaces $H = (L^2(\Omega))^3$ and $V = \{\mathbf{u} \in (H^1(\Omega))^3,\ \mathbf{u} = 0 \text{ on } \Gamma_0\}$ whose inner products are denoted by $< \mathbf{u},\mathbf{w} >_H$ and $< \mathbf{u},\mathbf{w} >_V$ respectively, and where the associated norms are denoted by $\|\mathbf{u}\|_H$ and $\|\mathbf{u}\|_V$ respectively. Let $\mathbb{H} = L^2(\Theta, H)$ and $\mathbb{V} = L^2(\Theta, V))$ be the real Hilbert spaces of all the second-order random variables $\theta \mapsto \{\mathbf{x} \mapsto \mathbf{U}(\mathbf{x}, \theta)\}$ defined on probability space $(\Theta, \mathcal{T}, P)$, with values in H and V respectively, equipped with the inner products $\ll \mathbf{U},\mathbf{W} \gg_{\mathbb{H}} = E\{< \mathbf{U},\mathbf{W} >_H\}$ and $\ll \mathbf{U},\mathbf{W} \gg_{\mathbb{V}} = E\{< \mathbf{U},\mathbf{W} >_V\}$ respectively, and where the associated norms are denoted by $\|\mathbf{U}\|_{\mathbb{H}}$ and $\|\mathbf{U}\|_{\mathbb{V}}$ respectively.

4.1. Weak formulation of the elliptic stochastic partial differential operator.

Let $n = 6$ and let us introduce the new indices I and J belonging to $\{1, \ldots, 6\}$ such that $I = (i, j)$ and $J = (k, h)$ with the following correspondence: $1 = (1, 1), 2 = (2, 2), 3 = (3, 3), 4 = (1, 2), 5 = (2, 3)$ and $6 = (3, 1)$. Thus, for all $\mathbf{x}$ in Ω, we introduce the matrix $[\underline{a}_n(\mathbf{x})]$ in $\mathbb{M}_n^+(\mathbb{R})$ such that $[\underline{a}_n(\mathbf{x})]_{IJ} = \underline{c}_{ijkh}(\mathbf{x})$ and the random $(n \times n)$ real matrix $[\mathbf{A}_n(\mathbf{x})]$ such that $[\mathbf{A}_n(\mathbf{x})]_{IJ} = C_{ijkh}(\mathbf{x})$. A nonparametric probabilistic model of the random fourth-order elasticity tensor $C_{ijkh}(\mathbf{x})$ consists in choosing the random field $\mathbf{x} \mapsto [\mathbf{A}_n(\mathbf{x})]$ in SFE$^+$ with the mean value $[\underline{a}_n(\mathbf{x})] = E\{[\mathbf{A}_n(\mathbf{x})]\}$. The weak formulation of the stochastic partial differential operator defined by Eq. (2) leads the random bilinear form $(\mathbf{U}, \mathbf{W}) \mapsto \mathbf{K}(\mathbf{U}, \mathbf{W})$ on $\mathbb{V} \times \mathbb{V}$ to be introduced, such that

$$\mathbf{K}(\mathbf{U}, \mathbf{W}) = \int_\Omega < [\mathbf{A}_n(\mathbf{x})]\, \mathbf{e}(\mathbf{U}(\mathbf{x}))\,, \mathbf{e}(\mathbf{W}(\mathbf{x})) > d\mathbf{x} \quad , \tag{5}$$

in which $\mathbf{e}(\mathbf{u}) = (\varepsilon_{11}(\mathbf{u}), \varepsilon_{22}(\mathbf{u}), \varepsilon_{33}(\mathbf{u}), 2\,\varepsilon_{12}(\mathbf{u}), 2\,\varepsilon_{23}(\mathbf{u}), 2\,\varepsilon_{31}(\mathbf{u}))$.

4.2. Ellipticity of the random bilinear form

Let $(\mathbf{U},\mathbf{W}) \mapsto \mathcal{K}(\mathbf{U},\mathbf{W})$ be the bilinear form on $\mathbb{V} \times \mathbb{V}$ defined by $\mathcal{K}(\mathbf{U},\mathbf{W}) = E\{\mathbf{K}(\mathbf{U},\mathbf{W})\}$. If the following property was introduced: for all $\mathbf{x} \in \Omega$ and for all $\mathbb{R}^n$-valued random variable $\mathbf{Y}$ defined on $(\Theta, \mathcal{T}, P)$, $< [\mathbf{A}_n(\mathbf{x})]\,\mathbf{Y}\,,\mathbf{Y} > \;\geq\; c\,\|\mathbf{Y}\|^2$ a.s in which $0 < c < +\infty$ is independent of $\mathbf{x}$, then the bilinear form $(\mathbf{U},\mathbf{W}) \mapsto \mathcal{K}(\mathbf{U},\mathbf{W})$ on $\mathbb{V}\times\mathbb{V}$ would be coercive in $\mathbb{V}$ (i.e. $\mathbb{V}$-elliptic) because, we would have $\mathcal{K}(\mathbf{U},\mathbf{U}) \;\geq\; cE\big\{\int_\Omega \|\mathbf{e}(\mathbf{U}(\mathbf{x}))\|^2\,d\mathbf{x}\big\} \;\geq\; c_{\mathcal{K}}\|\mathbf{U}\|^2_{\mathbb{V}}$ with $0 \;<\; c_{\mathcal{K}} \;<\; +\infty$. This property, which is generally not coherent with the available information which can be deduced from objective data, does not hold for the random field $\mathbf{x} \mapsto [\mathbf{A}_n(\mathbf{x})]$ belonging to SFE$^+$ and consequently, the usual analysis given above cannot presently be used. Another analysis has to be developed using the fundamental property defined by Eq. (3): it is proved (Soize, 2004) that, for all random field $\{\mathbf{x} \mapsto \mathbf{U}(\mathbf{x})\}$ in $\mathbb{V}$, we have

$$\sqrt{E\{\mathbf{K}(\mathbf{U},\mathbf{U})^2\}} \;\geq\; c_K\,\|\mathbf{U}\|^2_{\mathbb{V}} \quad , \tag{6}$$

in which c_K is a positive finite real constant. It should be noted that Eq. (6) differs from equation $E\{\mathbf{K}(\mathbf{U},\mathbf{U})\} \;\geq\; c_{\mathcal{K}}\,\|\mathbf{U}\|^2_{\mathbb{V}}$ due to the fact that the two positive-valued random variables $\sup_{\mathbf{x}\in\overline{\Omega}} \|\,[\mathbf{G}_n(\mathbf{x})]^{-1}\|$ and $\mathbf{K}(\mathbf{U},\mathbf{U})$ are dependent.

4.3. Existence and uniqueness of a weak second-order stochastic solution for a stochastic BVP

Let $\mathbf{w} \mapsto f(\mathbf{w})$ be a given continuous linear form on V, that is to say such that $|f(\mathbf{w})| \leq c_f\,\|\mathbf{w}\|_V$ with $0 < c_f < +\infty$. Then, the following random problem: find a random field $\{\mathbf{x} \mapsto \mathbf{U}(\mathbf{x})\}$ in $\mathbb{V}$ such that, for all $\mathbf{W} \in \mathbb{V}$, $\mathbf{K}(\mathbf{U},\!\mathbf{W}) = f(\mathbf{W})$ a.s , has a unique stochastic solution $\{\mathbf{x} \mapsto \mathbf{U}(\mathbf{x})\}$ in $\mathbb{V}$.

The proof can easily be constructed. From equations $\mathbf{K}(\mathbf{U},\!\mathbf{W}) = f(\mathbf{W})$ and $|f(\mathbf{w})| \leq c_f\,\|\mathbf{w}\|_V$, we deduce that $\mathbf{K}(\mathbf{U},\mathbf{U}) \;\leq\; c_f\,\|\mathbf{U}\|_V$ and consequently, $E\{\mathbf{K}(\mathbf{U},\mathbf{U})^2\} \;\leq\; c_f^2\,E\{\|\mathbf{U}\|^2_V\}$. Using Eq. (6) yields $c_K^2\,\|\mathbf{U}\|^4_{\mathbb{V}} \;\leq\; c_f^2\,\|\mathbf{U}\|^2_{\mathbb{V}}$ which can be rewritten as $\|\mathbf{U}\|_{\mathbb{V}} \leq c_U < +\infty$ with $c_U = c_f/c_K$ and yields the existence. Finally, the proof of the uniqueness is straightforward because, if $\mathbf{U}$ and $\mathbf{U}'$ are two solutions in $\mathbb{V}$, for all $\mathbf{W}$ in $\mathbb{V}$, we have $\mathbf{K}(\mathbf{U}-\mathbf{U}',\!\mathbf{W}) = 0$ a.s and thus $E\{\mathbf{K}(\mathbf{U}-\mathbf{U}',\!\mathbf{W})^2\} = 0$. Taking $\mathbf{W} = \mathbf{U}-\mathbf{U}'$ and from Eq. (6) yield $\|\mathbf{U}-\mathbf{U}'\|^2_{\mathbb{V}} = 0$, i.e., $\mathbf{U} = \mathbf{U}'$ in $\mathbb{V}$.

5. Conclusions

We have presented the mathematical construction of a non Gaussian positive-definite $(n\times n)$ real matrix-valued random field, indexed by any domain of $\mathbb{R}^d$, depending only on its mean function and on a smaller number of scalar parameters constituted of a dispersion parameter and d spatial correlation lengths. Such a random field is adapted to the inverse problem relative to the experimental identification. A fundamental mathematical property is proved and allows the ellipticity of stochastic partial differential operators to be obtained.

References

Doob, J.L. (1953), *Stochastic Processes*, John Wiley and Sons, New York (Wiley Classics Library Edition Published 1990).

Ghanem,R. and P.D. Spanos (2003), *Stochastic Finite Elements: A spectral Approach*, revised edition, Dover Publications, New York.

Kleiber, M., D.H. Tran and T.D. Hien (1992), *The Stochastic Finite Element Method*, John Wiley and Sons, New York.

Soize, C. (2001), "Maximum entropy approach for modeling random uncertainties in transient elastodynamics", *J. Acoust. Soc. Amer.*, **109**(5),1979-1996.

Soize, C. (2004), "Non Gaussian positive-definite matrix-valued random fields for elliptic stochastic partial differential operators", *Computer Methods in Applied Mechanics and Engineering*, submitted in March 2004.