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Abstract

Coupling between initial and damage-induced anisotropies in 3D elastic damaged materials has been so far addressed by homogenization techniques only for particular microcracks configurations. The main difficulty in developing a general 3D micromechanical model lies in the lack of a closed-form solution of the Eshelby tensor corresponding to cracks in an elastic anisotropic medium. In this study, we begin to present closed form expressions of the Eshelby tensor $\mathbb{S}$ that we derived for a cylindrical crack embedded in an orthotropic material. The 3D obtained expressions reduce to existing results in the 2D case or in particular 3D cracks configurations. The effective compliance of an orthotropic medium containing arbitrarily oriented cracks is then derived by using the new Eshelby tensor. Finally, a damage model is formulated by combining the above results with classical thermodynamics approach. The ability of the model to capture coupling between initial orthotropy and damage induced anisotropy is demonstrated through a comparison with experimental data available for a ceramic matrix composite (unidirectional SiC-SiC).

Key words: Homogenization; Eshelby tensor; Anisotropy; Damage; Brittle materials; Ceramic Matrix Composites; Micromechanics; Cracks.

1 Introduction

In the last few decades the study of inelastic behavior and failure characteristics of man-made or natural brittle materials under mechanical loading has been the object of numerous studies. Both phenomenological (i.e. continuum damage mechanics - CDM) or micromechanical based approaches have been considered (see for instance the reference works of [25], [39, 8] where some of the main contributions are reviewed). Since the mechanical behavior and some other microstructure-sensitive properties of heterogeneous media are strongly influenced by damage mechanisms occurring at different scales, the micromechanical approach of damage modeling appears to be very powerful. In particular, for quasi brittle materials considered in the present study, the inelastic deformation or other various complex effects observed at the macroscopic level by means of standard mechanical tests (e.g. deterioration of elastic properties, damage-induced anisotropy, volumetric dilatancy, unilateral effects, etc) are commonly attributed to matrix microcracking.

The present study addresses the problem of estimating the effective elastic properties of anisotropic bodies permeated by cylindrical cracks in the framework of Eshelby-like methods and the formulation of a micro-macro model able to account for the anisotropy of crack-induced damage. Despite the progresses concerning the mathematical formulation of constitutive laws and the evaluation of overall properties of engineering materials ([29, 9, 10, 48, 19, 51, 34, 41, 21, 45] and many others, especially in the 2D context), the 3D modelling of interaction between initial and damage-induced anisotropies remains a difficult task, even in the context of phenomenological models (see, for example, [26, 44, 16, 5]). In this paper, this coupling is addressed by means of Eshelby-type homogenization techniques. The general case of arbitrarily oriented cylindrical cracks with respect to the symmetry axis of an orthotropic medium is considered.

It is worth emphasizing that the analysis of a plastic matrix containing microcracks is beyond the scope of the present work. Such an analysis can be conducted following the classic approach of [15]. The reader is referred to yielding of plastic microcracked materials (see [12, 13], the recent studies of [36, 35] and the recent review by [3]).

In the context of upscaling techniques, the determination of the overall elastic properties of a cracked material requires the computation of the Eshelby tensor $\mathbb{S}$ associated to any crack. Many studies have addressed the case of

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a linearly elastic isotropic matrix containing penny-shaped and/or elliptical cracks (see for instance [18, 38, 39, 20]) for which the Eshelby tensor is well known. In the case of materials with matrix exhibiting structural (primary) anisotropy, few analytical results exist in literature. In a three dimensional case, only particular crack configurations have been considered, i.e. when the crack lies in one of the symmetry planes of the solid matrix (see [27, 28, 38] and the more recent work of [23]). For an arbitrarily oriented crack in an anisotropic matrix, the results existing in literature are restricted to planar elasticity. The principal difficulty in conducting for a 3D analysis is that the Green function associated to anisotropic solids (from which is classically determined the analytical expression of the Eshelby tensor $\mathbb{S}$) is generally not known in closed form (see, for example, [45], [32], [22]). In this paper, we seek to address these limitations. The objectives of these study are:

(i) to obtain the closed-form expression of the effective compliance of an orthotropic matrix containing cylindrical cracks. To this end we will recall and use analytical expressions of the Eshelby tensor $\mathbb{S}$ that we briefly exposed in [14] [1] based on the general methodology introduced by [24] and [11] (see also the works of [46, 43]).

(ii) to propose an anisotropic damage model developed by the implementation of these new results in an homogenization scheme, namely the Mori-Tanaka model, in order to determine the macroscopic properties of the orthotropic material weakened by an arbitrarily oriented system of cracks. The anisotropic damage model is further applied for the prediction of the inelastic behavior of a ceramic matrix composite.

The paper is organized as follows. We begin with a brief description of the Mori-Tanaka homogenization scheme, that will be further used to obtain the effective compliance (2). The focus of section 3 is the derivation of an analytic expression of the Eshelby tensor for 3D microcracks systems. in particular, we present the basic methodology in subsection 3.1 while section 3.2 is devoted to the validation of the results and to the presentation of their connection with existing results in planar elasticity. The determination of the effective compliance and the illustration of combined effects of microcracks-induced anisotropic damage and initial anisotropy are performed in section 4. These results are then used in section 5 where is presented a micromechanical damage model able to describe the inelastic behavior of orthotropic composites. The proposed model is applied and tested on a ceramic matrix composite (SiC-SiC), in order to illustrate it's ability to account for the interaction ef-

1. *It is convenient to note that the developed approach has been recently followed and extended on some aspects by [33] and by [2] who proposed semi analytical techniques for determining the effective compliance for higher anisotropic matrix containing inclusions.*

fects between the initial anisotropy and cracks orientation.

Notations: Standard tensorial notations will be used throughout. Lower underlined case letters will describe vectors, bold script capital letters will be associated to second-order tensors and mathematical double-struck capital letters will denote fourth-order tensors respectively. The following vector and tensor products are exemplified: $(\mathbf{A}.\underline{b})_i = A_{ij}b_j$, $(\mathbf{A}.\mathbf{B})_{ij} = A_{ik}B_{kj}$, $(\mathbb{A} : \mathbf{B})_{ij} = A_{ijkl}B_{kl}$, $(\mathbb{A} : \mathbb{B})_{ijkl} = A_{ijpq}B_{pqkl}$. Einstein summation convention, applied for the repeated indices and Cartesian coordinates are used. As usual, small respectively large characters refer to microscopic (resp. macroscopic) quantities. The reference configuration corresponds to the natural state (undamaged material); $\mathbf{1}$ and $\mathbb{I}$ are, respectively, the second and fourth order identity tensors, the former represents the Kronecker symbol (δ_{ij}) and the latter is defined as: $I_{ijkl} = (1/2)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$.

2 Mori-Tanaka scheme for orthotropic matrix weakened by arbitrary oriented cracks

As already mentioned, in contrast to the continuum macroscopic damage approaches, the micromechanically based approaches to modeling microcraks-induced anisotropy present the advantage of incorporating damage mechanisms occurring at microscopic scales. In addition to having a physical basis, the resulting damage models provide an explicit way of modeling the coupling between the structural and microcracks-induced anisotropies. This section begins with a short description of the general concepts of the homogenization of disordered media using the classical Eshelby's equivalent inclusion method and how the Eshelby results can be further used in the Mori-Tanaka scheme to derive the effective properties of the cracked media.

2.1 Basic principles of Eshelby-based homogenization techniques

Consider a representative volume element (RVE) Ω of a material consisting of an orthotropic solid matrix Ω^s (of stiffness tensor $\mathbb{C}^s$ and volume fraction φ^s) and a system of microcracks occupying a domain Ω^f. The orientation of each family r of microcracks ($r = 1$ to $\mathcal{N}$) is characterized by a normal unit vector denoted by $\underline{n}$. Uniform strain conditions are prescribed on the outer boundary $\partial\Omega$ of the RVE (see Figure 1). The crack is considered as an elastic material of stiffness tensor $\mathbb{C}^f$.

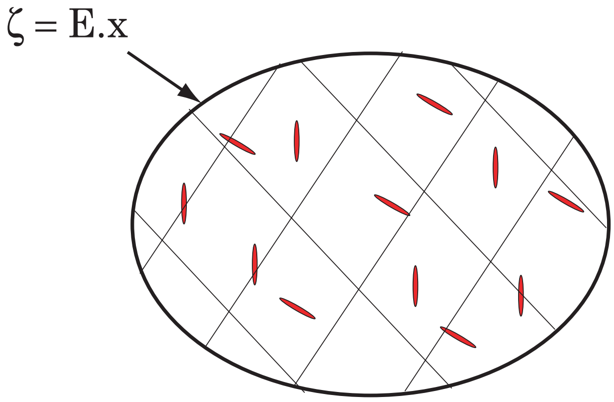


Figure 1. The representative volume element (RVE) of the orthotropic cracked solid.

The microscopic stress and strain fields are respectively denoted by $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ whereas $\underline{x}$ and $\underline{\zeta}$ denote respectively the position vector in the RVE and the microscopic displacement.
A crucial step of the homogenization techniques consists in finding the fourth-order localization tensor $\mathbb{A}(\underline{x})$ which relates the microscopic strain field to the macroscopic one, i.e. $\boldsymbol{\varepsilon}(\underline{x}) = \mathbb{A}(\underline{x}) : \mathbf{E}$. It follows that the homogenized stiffness tensor $\mathbb{C}^{hom}$ is (see, for instance, [52]):

$$\mathbb{C}^{hom} = \langle \mathbb{C}(\underline{x}) : \mathbb{A}(\underline{x}) \rangle_{\Omega} = \mathbb{C}^s + \sum_{r=1}^{\mathcal{N}} \varphi_r^f \left(\mathbb{C}_r^f - \mathbb{C}^s \right) : \mathbb{A}_r^f \tag{1}$$

where φ_r^f is the volume fraction of the r^{th} family of microcracks ($r = 1$ to $\mathcal{N}$), and the identity $\langle \mathbb{A}(\underline{x}) \rangle_{\Omega} = \mathbb{I}$ (which results from the strain average rule) is used.
The localization problem is solved by using the fundamental result provided by [9]: for an ellipsoidal crack (defining the r^{th} family) embedded in a linear elastic solid matrix. In the dilute homogenization scheme, the localization tensor reads:

$$\mathbb{A}_r^f = (\mathbb{I} + \mathbb{P}_r : \delta\mathbb{C}) = (\mathbb{I} - \mathbb{S}_r)^{-1} \tag{2}$$

In (2), $\mathbb{S}_r$ and $\mathbb{P}_r$ denote the Eshelby and the Hill tensors respectively ($\mathbb{S}_r = \mathbb{P}_r : \mathbb{C}^s$). Since only opened cracks are considered in the present study, $\mathbb{C}^f = 0$ and $\delta\mathbb{C} = \mathbb{C}_r^f - \mathbb{C}^s$ ($= -\mathbb{C}^s$ for opened cracks). Let us recall that the dilute scheme is restricted to infinitesimal concentrations of the inhomogeneous inclusions.

2.2 The Mori-Tanaka scheme applied to cracked media

The Mori-Tanaka homogenization scheme (see [37]) accounts for interaction effects. It is classically formulated by adapting the Eshelby result to this situation. The resulting localization tensor for the r^{th} family of microcracks ($r = 1$ to $\mathcal{N}$) takes then the form (see for instance [6] or [7]):

$$\mathbb{A}_r^f = (\mathbb{I} - \mathbb{S}_r)^{-1} : \left[\varphi^s \mathbb{I} + \sum_{j=1}^{\mathcal{N}} \varphi_j^f (\mathbb{I} - \mathbb{S}_j)^{-1} \right]^{-1} \tag{3}$$

which, when substituted in (1), leads to:

$$\mathbb{C}^{hom} = \mathbb{C}^s - \sum_{r=1}^{\mathcal{N}} \varphi_r^f \mathbb{C}^s : (\mathbb{I} - \mathbb{S}_r)^{-1} : \left[\varphi^s \mathbb{I} + \sum_{j=1}^{\mathcal{N}} \varphi_j^f : (\mathbb{I} - \mathbb{S}_j)^{-1} \right]^{-1} \tag{4}$$

Note that the determination of the homogenized stiffness (or compliance) tensor reduces to the computation of $\mathbb{S}_r$. For orthotropic media, this is not a straightforward task and is the focus of the next subsection.

3 Closed-form expression of Eshelby tensor $\mathbb{S}_r$ for an arbitrarily oriented cylindrical crack in an orthotropic medium

3.1 Methodology and results

Consider first that the solid matrix is orthotropic (of stiffness tensor $\mathbb{C}^s$) and is weakened by a set of parallel microcracks. Let (z_1, z_2,z_3) be the rectangular cartesian coordinates system and ($\underline{k}_1, \underline{k}_2, \underline{k}_3$) the local frame associated to the microcracks. Let also define by (x_1, x_2, x_3) the global coordinate system and by ($\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$) the global frame associated to the axes of symmetry (orthotropy) of the matrix. We model the crack as an infinite cylinder aligned along the direction $\underline{e}_3 \equiv \underline{k}_3$ and elliptical cross-section of low aspect ratio $X = b/a$ (see Figure 2) geometrically prescribed by:

$$\frac{z_1^2}{a^2} + \frac{z_2^2}{b^2} = 1, \qquad -\infty < z_3 < \infty \tag{5}$$

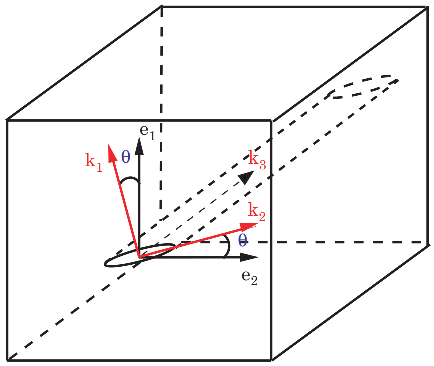


Figure 2. Crack modelled as an infinite cylinder with elliptical section; the aspect ratio is $X = b/a$

Let us also recall that the components of the solid matrix stiffness tensor in the local frame (the angle θ gives the orientation of the crack with respect to the axes of orthotropy of the matrix), are obtained using classical tensor transformation rules as: $C_{ijkl} = U_{ip}U_{jq}U_{kr}U_{ls}C^s_{pqrs}$. The second-order orthogonal transformation tensor $\mathbf{U}$ used for the local frame coordinate system associated to the crack (z_1, z_2, z_3) is:

$$U = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6}$$

Note that, for an arbitrary orientation of the cracks plane, θ, the stiffness tensor in the local frame displays a matrix of 13 non-zero components. The determination of the interaction tensor $\mathbb{P}$, which is based on the work of [24], (see also [11], [27] or [50]), consists in evaluating the integral:

$$P_{ijkl} = \frac{ab}{\pi} \int_0^{2\pi} \frac{N_{ijkl}(\xi_1, \xi_2)}{(a^2\xi_1^2 + b^2\xi_2^2)} d\psi \tag{7}$$

The integration is performed on the unit circle centered at the origin, in the plane (ξ_1, ξ_2): $\|\underline{\xi}\| = 1$ (e.g. $\underline{\xi} = \cos\psi \underline{e}_1 + \sin\psi \underline{e}_2$). The components of the fourth order tensor $\mathbb{N}$ read:

$$N_{ijkl}(\xi_1, \xi_2) = D_{ijkl}(\xi_1, \xi_2, 0) \qquad \text{with:} \tag{8}$$

$$D_{ijkl} = \frac{1}{4}\left(\xi_i K^{-1}_{jk}\xi_l + \xi_j K^{-1}_{ik}\xi_l + \xi_i K^{-1}_{jl}\xi_k + \xi_j K^{-1}_{il}\xi_k\right) \tag{9}$$

In equation(9), $\mathbf{K} = \underline{\xi}.\mathbb{C}.\underline{\xi}$ represents the acoustic tensor associated to $\mathbb{C}$ and the vector $\underline{\xi}$. It can be noted that the anisotropy of the elastic solid matrix affects $\mathbb{P}$ through the acoustic tensor $\mathbf{K}$. Since the above expressions (equation 7, together with 8 and 9) imply computations in the local frame of the crack, $\mathbb{P}$ depends a priori on the orientation of the latter. Equivalently, the components of the tensor $\mathbb{P}$ can be expressed as:

$$P_{ijkl} = \frac{1}{4}(M_{ijkl} + M_{jikl} + M_{ijlk} + M_{jilk}) \tag{10}$$

$$\text{where:} \qquad M_{ijkl} = \frac{ab}{\pi}\int_0^{2\pi} \frac{\xi_i K^{-1}_{jk}\xi_l}{(a^2\xi_1^2 + b^2\xi_2^2)} d\psi \tag{11}$$

As mentioned before, note that the analytical evaluation of the interaction tensor $\mathbb{P}$ is not an easy task. An interesting and elegant method for estimating this integral which is based on the residue theorem was used by [14] (see appendix A). The analytical expressions of the components of the tensor $\mathbb{P}$ are given in the appendix B.

It is worth noting that only the orientation and the geometry of the elliptical inhomogeneity (i.e. the angle θ and the aspect ratio $X = b/a$) as well as the initial anisotropy of the solid matrix (the stiffness tensor $\mathbb{C}^s$) affects the components of the interaction tensor $\mathbb{P}$. Thus, the influence of the initial anisotropy on the overall elastic properties of the material (which are now obtainable by incorporating the previous results in various homogenization schemes) and the coupling between initial and crack induced anisotropy is described. To the best of our knowledge (especially for the validation of the obtained results) the closed form expressions of the interaction tensor $\mathbb{P}$ of ellipsoidal inhomogeneities in orthotropic media are known only for cylindrical cracks normal to one of the material symmetry axis (see [27, 11]). The only 3D analytical expressions were obtained by [28] for penny-shaped cracks normal to the symmetry axis, i.e. lying in the isotropy plane of the transversely isotropic matrix. Thus, for validation purposes, we can only consider these particular cases and show that the new results reduce to the corresponding existing ones.

3.2 Connection with existing results

In the present section are presented several validations of the previous results implying directly either the $\mathbb{P}$ or equivalently $\mathbb{Q}$ tensor (introduced in section 3.2.2).

3.2.1 Aligned cracks normal to a symmetry axis of the orthotropic matrix

A first validation of the new results is done by considering the particular case of a system of aligned cracks normal to the symmetry axis $\underline{e}_1$ of the solid matrix. This problem has been studied by [27], who obtained the following non-zero components for the fourth-order interaction tensor $\mathbb{P}$:

$$
\begin{aligned}
P_{1111} &= \frac{C^s_{2222} + C^s_{1212}\sqrt{\alpha\beta}}{C^s_{1111}C^s_{1212}\ \sqrt{\alpha\beta}(\sqrt{\alpha}+\sqrt{\beta})}X \\
P_{2222} &= \frac{1}{C^s_{2222}} + \frac{C^s_{2222} - C^s_{1212}(\alpha+\beta+\sqrt{\alpha\beta})}{C^s_{2222}C^s_{1212}\ \sqrt{\alpha\beta}(\sqrt{\alpha}+\sqrt{\beta})}X \\
P_{1122} &= -\frac{C^s_{1122} + C^s_{1212}}{C^s_{1111}C^s_{1212}\ \sqrt{\alpha\beta}(\sqrt{\alpha}+\sqrt{\beta})}X \\
P_{2323} &= \frac{1}{4C^s_{2323}} - \frac{\sqrt{C^s_{1313}}}{4{C^s_{2323}}^{3/2}}X; \qquad P_{1313} = \frac{1}{4\sqrt{C^s_{2323}C^s_{1313}}}X \\
P_{1212} &= \frac{1}{4C^s_{1212}} - \frac{C^s_{1111}C^s_{2222} - {C^s_{1122}}^2}{2C^s_{1111}{C^s_{1212}}^2\ \sqrt{\alpha\beta}(\sqrt{\alpha}+\sqrt{\beta})}X
\end{aligned}
\tag{12}
$$

where α and β are the complex conjugate roots of the characteristic equation for the orthotropic solid matrix:

$$
C^s_{1111}C^s_{1212}x^2 - \Big[C^s_{1111}C^s_{2222} - C^s_{1122}(C^s_{1122} + 2C^s_{1212})\Big]x + C^s_{2222}C^s_{1212} = 0 \tag{13}
$$

It can be easily verified that for $\theta = 0$, the expressions (B.1) that we established coincide with the expressions (12).

3.2.2 Validation through comparison with classic results obtained from the study in planar elasticity of orthotropic cracked media

Rigorous results for planar elasticity problems [2] involving elliptical inclusions (of aspect ratio $X = b/a$) embedded in an anisotropic matrix were available long time ago by using the complex potentials approach (see [29]). Interestingly, they are recently applied by [47] for the homogenization of cracked materials with orthotropic matrix. At the difference of the complex potentials method, the Eshelby-based results obtained in the present study allows to investigate 3D cracks configurations. The purpose of this second level of validation is to show that the results derived in the present study reduce to

2. We recall that, in the present study, the analytical expressions of the components of $\mathbb{P}$ tensor were obtained in a tridimensional context.

existing ones when 2D cracks are considered. The starting point is the relationship already established by [47] between the strain induced by the cavity, (ε^c), and the macroscopic stress field $\mathbf{\Sigma}$ applied to the RVE:

$$\varepsilon^c = \mathbb{H} : \mathbf{\Sigma} \tag{14}$$

in which, following [27], the $\mathbb{H}$ tensor is related to the Hill tensor by:

$$\mathbb{H} = dX\mathbb{Q}^{-1} \quad \text{with} \quad \mathbb{Q} = \mathbb{C}^s - \mathbb{C}^s : \mathbb{P} : \mathbb{C}^s \tag{15}$$

where d is the crack density parameter ($d = \mathcal{M}a^2$, $\mathcal{M}$ being the number of cracks per unit volume) introduced by [4] and $\mathbb{Q}$ the second Hill tensor.
As a validation, it is convenient to compute tensor $\mathbb{Q}$ (corresponding to the arbitrarily oriented elliptical crack in the orthotropic matrix) from (B.1) and then its inverse $\mathbb{Q}^{-1}$. It follows that:

$$\begin{aligned}
XQ^{-1}_{1111} &= t\,\sqrt{\alpha\beta}(\sqrt{\alpha}+\sqrt{\beta})\left[X^2\cos^2(\theta)+\sin^2(\theta)+\frac{C^s_{1122}X}{C^s_{1111}\sqrt{\alpha\beta}(\sqrt{\alpha}+\sqrt{\beta})}\right]\\
XQ^{-1}_{1112} &= XQ^{-1}_{1211} = t\,\sqrt{\alpha\beta}(\sqrt{\alpha}+\sqrt{\beta})\cos\theta\sin\theta(X^2-1)\\
XQ^{-1}_{1122} &= XQ^{-1}_{2211} = -t\,\sqrt{\alpha\beta}X\\
XQ^{-1}_{1212} &= \frac{t(\sqrt{\alpha}+\sqrt{\beta})}{4}\cdot\\
&\left\{X^2\left[\cos^2(\theta)+\sqrt{\alpha\beta}\sin^2(\theta)\right]+\left[\sin^2(\theta)+\sqrt{\alpha\beta}\cos^2(\theta)\right]+X(\sqrt{\alpha}+\sqrt{\beta})\right\}\\
XQ^{-1}_{2212} &= XQ^{-1}_{1222} = \frac{t(\sqrt{\alpha}+\sqrt{\beta})}{2}(X^2-1)\cos\theta\sin\theta\\
XQ^{-1}_{2222} &= t(\sqrt{\alpha}+\sqrt{\beta})\left\{X^2\sin^2(\theta)+\cos^2(\theta)+\frac{X}{\sqrt{\alpha}+\sqrt{\beta}}\right\}
\end{aligned} \tag{16}$$

where $t = C^s_{1111}/(C^s_{111}C^s_{2222} - {C^s_{1122}}^2)$. The final verification requires to compare the components of $X\mathbb{Q}^{-1}$ given by (16) with that of $\mathbb{H}$ given by [47] and reported in appendix (D) (in which the definition $d = \pi a^2/A$ must be adopted, A being the reference cell area).

4 The Mori-Tanaka estimate of the effective compliance: combined effects of initial and damage-induced anisotropies

4.1 Determination of the effective compliance

We aim at deriving the macroscopic elastic properties of an orthotropic solid matrix containing randomly oriented microcracks. To this end, a Mori-Tanaka estimate of the homogenized stiffness tensor of the microcracked medium is considered (4). Following [4], let us recall the 2D cracks density parameter $d_r = \mathcal{M}{a_r}^2$, where $\mathcal{M}$ denotes the number of cracks per unit area of the considered cracks system. Recalling the notation φ_r^f for the volume fraction of the r^{th} family of microcracks ($r = 1$ to $\mathcal{N}$), one has $\varphi_r^f = \pi d_r X_r$. Taking then the limit for low aspect ratio ($\lim_{X_r \to 0}$), it can be shown that the homogenized stiffness tensor, given by (4), takes the form (as in the isotropic case):

$$\mathbb{C}^{hom} = \mathbb{C}^s : \left(\mathbb{I} + \sum_{r=1}^{r=\mathcal{N}} \pi d_r \mathbb{T}_r \right)^{-1} \quad \text{where} \quad \mathbb{T}_r = \lim_{X_r \to 0} X_r \left[\mathbb{I} - \mathbb{S}_r \right]^{-1} \tag{17}$$

The components of tensor $\mathbb{T}_r$ are given in Appendix D. Note that they depend on the cracks orientation defined by the θ angle (see figure 2).
Moreover, the material orthotropy is only preserved when cracks are aligned along a material symmetry axis ($\theta = 0$ or $\theta = \pi/2$). This corresponds to the cancelation of some components of $\mathbb{T}_r$.
The homogenized compliance tensor is obtained by inverting (17):

$$\mathbb{S}^{hom} = \left(\mathbb{I} + \sum_{r=1}^{r=\mathcal{N}} \pi d_r \mathbb{T}_r \right) : \mathbb{S}^s \tag{18}$$

where $\mathbb{S}^s = (\mathbb{C}^s)^{-1}$ represents the compliance tensor of the solid matrix. The components of $\mathbb{S}^{hom}$ are given by:

$$
\begin{aligned}
S^{hom}_{1111} &= S^s_{1111} + d\pi \frac{(S^s_{2222}S^s_{3333} - {S^s_{2233}}^2)(\sqrt{\alpha}+\sqrt{\beta})\sqrt{\alpha\beta}(\sin\theta)^2}{S^s_{3333}} \\
S^{hom}_{1122} &= S^s_{1122} \qquad S^{hom}_{1133} = S^s_{1133} \qquad S^{hom}_{2233} = S^s_{2233} \qquad S^{hom}_{3333} = S^s_{3333} \\
S^{hom}_{2222} &= S^s_{2222} + d\pi \frac{(S^s_{2222}S^s_{3333} - {S^s_{2233}}^2)(\sqrt{\alpha}+\sqrt{\beta})(\cos\theta)^2}{S^s_{3333}} \\
S^{hom}_{1112} &= S^s_{1112} + d\pi \frac{({S^s_{2233}}^2 - S^s_{2222}S^s_{3333})(\sqrt{\alpha}+\sqrt{\beta})\sqrt{\alpha\beta}\sin\theta\cos\theta}{2S^s_{3333}} \\
S^{hom}_{2212} &= S^s_{2212} + d\pi \frac{({S^s_{2233}}^2 - S^s_{2222}S^s_{3333})(\sqrt{\alpha}+\sqrt{\beta})\sin\theta\cos\theta}{2S^s_{3333}} \\
S^{hom}_{2323} &= S^s_{2323} + d\pi(\sin\theta)^2 \frac{\sqrt{S^s_{2323}S^s_{1313}}}{2} \\
S^{hom}_{1313} &= S^s_{1313} + d\pi(\cos\theta)^2 \frac{\sqrt{S^s_{2323}S^s_{1313}}}{2} \\
S^{hom}_{1323} &= S^s_{1323} + d\pi\sin\theta\cos\theta \frac{\sqrt{S^s_{2323}S^s_{1313}}}{2} \\
S^{hom}_{1212} &= S^s_{1212} + d\pi \frac{({S^s_{2233}}^2 - S^s_{2222}S^s_{3333})(\sqrt{\alpha}+\sqrt{\beta})\left[(\sin\theta)^2 + \sqrt{\alpha\beta}(\cos\theta)^2\right]}{S^s_{3333}}
\end{aligned}
\tag{19}
$$

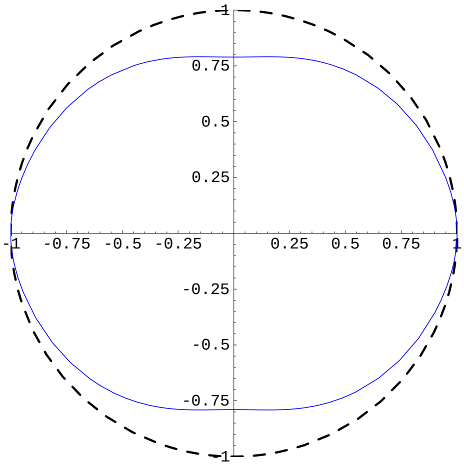
Note that the components of the homogenized compliance tensor depends on the elastic properties of the solid matrix and strongly on the crack orientation (the θ angle). Such dependence accounts for the interaction between the initial anisotropy and the cracks orientation. The deviation from material orthotropy can be found in components $S^{hom}_{1112} = S^{hom}_{1211}$, $S^{hom}_{2212} = S^{hom}_{1222}$ and $S^{hom}_{1323} = S^{hom}_{2313}$.

4.2 Effects of interaction between the initial and micro-cracks induced anisotropy on the elastic moduli

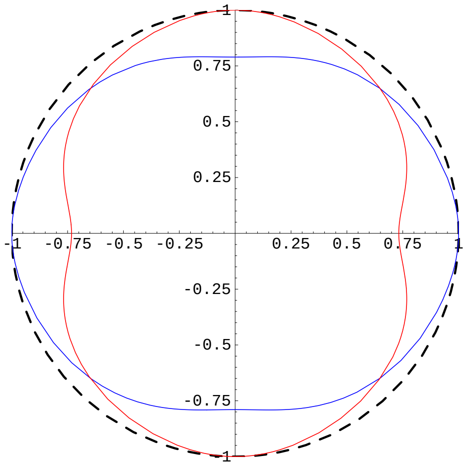
We propose to illustrate here the effect of open cracks systems on the homogenized elastic properties, the solid matrix being orthotropic. Following [49], one can derive from $\mathbb{S}^{hom}$ the expressions of the generalized elastic moduli: the Young modulus, $E(\underline{m})$, corresponding to an arbitrary direction defined by a unit vector $\underline{m}$, the Poisson ratio $\nu(\underline{m},\underline{p})$, and the shear modulus $\mu(\underline{m},\underline{p})$, associated to two orthogonal directions of respective unit vectors $\underline{m}$ and $\underline{p}$.

These moduli are respectively defined as (see also [17]):

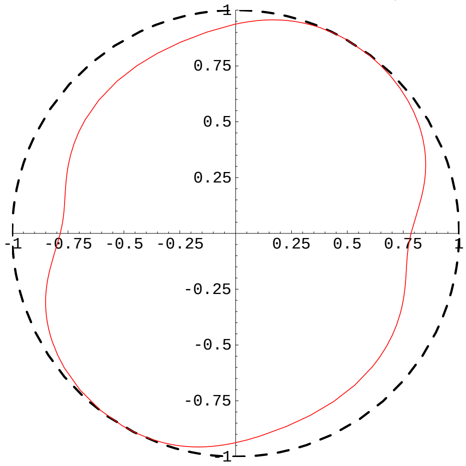
$$\begin{cases} E(\underline{m}) = [(\underline{m} \otimes \underline{m}) : \mathbb{S}^{hom} : (\underline{m} \otimes \underline{m})]^{-1} \\ \nu(\underline{m}, \underline{p}) = -\dfrac{(\underline{p} \otimes \underline{p}) : \mathbb{S}^{hom} : (\underline{m} \otimes \underline{m})}{(\underline{m} \otimes \underline{m}) : \mathbb{S}^{hom} : (\underline{m} \otimes \underline{m})} \\ \mu(\underline{m}, \underline{p}) = [4(\underline{m} \otimes \underline{p}) : \mathbb{S}^{hom} : (\underline{m} \otimes \underline{p})]^{-1} \end{cases} \tag{20}$$



a) $\theta = 0$



b) $\theta = 0$, $\theta = \pi/2$



c) $\theta = \pi/3$

Figure 3. Variation of the generalized Young's modulus produced by opened cracks systems. Three distinct families are respectively considered: a) $\theta = 0$; b) comparison for orientations $\theta = 0$ and $\theta = \pi/2$; c) $\theta = \pi/3$. The results are normalized with the initial values of the Young moduli, which are represented by the unit circle

As an example, the effect of cracks on the generalized Young modulus $E(\underline{m})$ will be only presented here. To this end, we consider a set of parallel cracks whose unit normal is $\underline{n}$, e.g. associated to the angle θ with respect to the symmetry axis $\underline{e}_2$ of the matrix (see Figure 2). For each space direction given

by a vector $\underline{m}$, taken in the plane $(\underline{e}_1, \underline{e}_2)$, the value of the elastic moduli is normalized by the corresponding value of the solid matrix (plotted as the unit circle in dashed line).
The analysis is done for the case of open cracks. The elastic moduli of the orthotropic solid matrix, which are assumed to correspond to the composite studied in section are: $E_1^s = 320000$ MPa, $E_2^s = 170000$ MPa for the Young moduli, $G_{12}^s = 90000$ MPa for the shear modulus and $\nu_{12}^s = 0.18$ for the Poisson ratio.
The results (see for instance Figure 3a corresponding to $\theta = 0$) show a significant degradation of the Young modulus in the normal direction to the crack. The same observation can be done for a cracks family oriented with an angle $\theta = \pi/2$ (see Figure 3b). However, it must be emphasized that this second curve is not obtained by a simple rotation of the previous one ($\theta = 0$), as it would be the case for an isotropic solid matrix. This is clearly a consequence of the orthotropy of the solid matrix. The effect of the matrix anisotropy is even more pronounced for $\theta = \pi/3$ (see Figure 3c which shows a notable distortion).

5 A micromechanical damage model for initially orthotropic materials

In this section we propose a micromechanical damage model for initially orthotropic materials. The model is based on the results presented in the preceding section and on the choice of a damage criterion which will allow to describe the propagation of distributed microcracks. The predictive capabilities of the micromechanical model are demonstrated by application to a brittle matrix composite. Comparison with experimental data obtained by [1] on a SiC-SiC ceramic composite shows the ability of the proposed model to describe the overall stress-response of this material when submitted to off axis loadings.

5.1 Free enthalpy. Damage propagation by cracks growth

A stress-based formulation of the damage model is proposed. The corresponding macroscopic free enthalpy for an arbitrarily oriented cracks system depends on the macroscopic stress and on the damage state:

$$W^* = \frac{1}{2}\mathbf{\Sigma} : \mathbb{S}^{hom} : \mathbf{\Sigma} \qquad (21)$$

$\mathbb{S}^{hom}$ depends on the cracks density parameters d_r (representing the damage variables) and is given by (18) when a Mori-Tanaka scheme is considered.

The first state law provides the macroscopic strain tensor $\mathbf{E}$ as a function of the macroscopic stress $\mathbf{\Sigma}$:

$$\mathbf{E} = \frac{\partial W^*}{\partial \mathbf{\Sigma}} = \mathbb{S}^{hom} : \mathbf{\Sigma} \tag{22}$$

To derive the complete formulation of the model, one needs to propose a damage criterion and a damage evolution law. Let us recall that the quantities d_r (for $r = 1$ to $\mathcal{N}$) constitute the set of internal damage variables. The damage criterion being based on the intrinsic dissipation due to damage, the latter can be obtained from the expression:

$$\mathcal{D} = \sum_{r=1}^{r=\mathcal{N}} \frac{\partial W^*}{\partial d_r} \dot{d}_r = \sum_{r=1}^{r=\mathcal{N}} F^{d_r} \dot{d}_r \geq 0 \tag{23}$$

in which, F^{d_r} is the thermodynamic force (energy release rate associated to the r^{th} cracks family) associated to damage variable d_r:

$$F^{d_r} = \frac{\partial W^*}{\partial d_r} \qquad r = 1 \text{ to } \mathcal{N} \tag{24}$$

The previous expression constitutes the second state law which was obtained by adopting as damage variable the crack density parameter d_r corresponding to a crack family denoted i. Based on the above thermodynamic arguments, the damage criterion for the considered cracks family can be put in the form (see also [31]):

$$f_r(F^{d_r}, d_r) = F^{d_r} - \mathcal{R}(d_r) \qquad r = 1 \text{ to } \mathcal{N} \tag{25}$$

In (25) the function $\mathcal{R}(d_r)$ describes the crack resistance to damage propagation. In principle, the damage activation criterion can be determined from experimental investigations (see [30] and [40]). It's dependency on d_r confers to the latter the role of a hardening variables. For simplicity, the following form is chosen for $\mathcal{R}(d_r)$:

$$\mathcal{R}(d_r) = k(1 + \eta d_r) \qquad r = 1 \text{ to } \mathcal{N} \tag{26}$$

where k is a parameter which describes the damage threshold and η accounts for the hardening effect of damage. These two parameters can be identified from standard tensile tests.
The damage evolution is obtained by assuming the normality rule:

$$\left| \begin{array}{l} \dot{d}_r = \dot{\Lambda}_{d_r} \dfrac{\partial f_r(F^{d_r}, d_r)}{\partial F^{d_r}} = \dot{\Lambda}_{d_r} \quad ; \quad \dot{\Lambda}_{d_r} \geq 0 \\ \text{so } \ \dot{d}_r = \begin{cases} 0 & \text{if } \ f_r < 0 \ \text{ or } \ f_r = 0, \ \dot{f}_r < 0 \\ \dot{\Lambda}_{d_r} & \text{if } \ f_r = 0 \ \text{ and } \ \dot{f}_r = 0 \end{cases} \end{array} \right. \tag{27}$$

The damage multiplier $\dot{\Lambda}_{d_r}$ is derived from the classical Kuhn-Tucker consistency condition: $\dot{f}_r = 0$. From a physical point of view, this implies that the representing point of loading is placed on the loading surface, so:

$$\dot{\Lambda}_{d_r} = \frac{1}{\mathcal{R}'(d_r)} (\mathbf{B}_r : \dot{\mathbf{\Sigma}})^+ \tag{28}$$

with $\mathbf{B}_r = \partial F^{d_r} / \partial \mathbf{\Sigma}$. It should be stated that the adopted damage evolution law supposes implicitly that cracks are propagating in their own plane. The adopted framework (of thermodynamic irreversible processes) indicates also that the propagation speed $\dot{d}^r$ of each variable is dependent of the macroscopic stress $\mathbf{\Sigma}$ only via the associated thermodynamic force. This framework is similar to the one adopted by [42] for the micro-macro transition, by using the parameters describing the microstructural arrangement of the material as internal variables.

5.2 Rate formulation of the stress-based model

For a damage activation state unchanged by the application of a stress increment, the tangent compliance tensor is determined by the use of the damage evolution law (relations (27-28)). The rate form of the damage model can be then deduced as:

$$\dot{\mathbf{E}} = \mathbb{S}_t^{hom} : \dot{\mathbf{\Sigma}} \tag{29}$$

$$\text{with } \; \mathbb{S}_t^{hom} = \mathbb{S}^{hom} - \sum_{r=1}^{\mathcal{N}} \varpi_r \frac{1}{\mathcal{R}'(d_r)} \mathcal{G}_r \mathbf{B}_r \otimes \mathbf{B}_r \tag{30}$$

$$\text{and } \; \mathcal{G}^r = \begin{cases} 0 \text{ if } \; f_r < 0 \;\text{ or }\; f_r = 0, \; \dot{f}_r < 0 \\ 1 \text{ if } \; f_r = 0 \;\text{ and }\; \dot{f}_r = 0 \end{cases} \tag{31}$$

ϖ_r denotes the associated weight of the r^{th} integration point. Due to the symmetry of the second order tensor $\mathbf{B}_r$, the homogenized tangent compliance $\mathbb{S}_t^{hom}$ has the minor and major symmetries.

5.3 Numerical local integration of the constitutive damage law

Algorithmic aspects related with the numerical integration of the proposed model are presented in the following. The formulation is strain based.
During a time step (t_j, t_{j+1}) an elastic trial state is computed:

$$f_r(F^{d_r j+1}, d_r^j) \leq 0 \tag{32}$$

then the trial state is the final state (i.e. the solution to the problem):

$$\begin{cases} \mathbf{E}^{j+1} = \mathbf{E}^j + \Delta\mathbf{E} \\ d_r^{j+1} = d_r^j \quad (r = 1 \text{ to } \mathcal{N}) \end{cases} \tag{33}$$

If the condition (32) is not satisfied, a linear correction should be imposed:

$$\begin{cases} \Delta d_r = \dfrac{1}{k\eta} f_r(F^{d_r j+1}, d_r^j) \\ d_r^{j+1} = d_r^j + \Delta d_r \end{cases} \tag{34}$$

5.4 Application of the model to an unidirectional SiC-SiC composite

In this section, we apply the proposed model to study the response of a brittle matrix SiC-SiC ceramic composite subjected to uniaxial off-axis tensile loading in different directions φ with respect to the axis of symmetry of the material (see Figure 4). The experimental data were reported by [1]. The elastic moduli of the orthotropic solid matrix, which are assumed to correspond to the initial elastic moduli of the composite are: $E_1 = 320000$ MPa, $E_2 = 170000$ MPa for the Young moduli, $G_{12} = 90000$ MPa for the shear modulus and $\nu_{12} = 0.18$ for the Poisson ratio.
The model involves two material parameters k and η that appear in the expression of the damage criterion. The identification procedure consists in calibrating the model parameters k and η from the tensile test corresponding to $\varphi = 0^\circ$. The obtained numerical values are: $k = 3.75 J.m^{-2}$ and $\eta = 140$. No experimental data were available on the initial cracks density parameter. In all the simulations, we set this parameter value to 0.01. We also assumed that there are 60 distinct families of cracks randomly oriented in the solid matrix. The validation of the model is done by simulating off-axis tests (corresponding to $\varphi = 20^0$ and $\varphi = 45^0$). Figure 5 shows the overall stress-strain response in tensile tests for $\varphi = 0^0, \varphi = 20^0$ and $\varphi = 45^0$. The agreement between the simulated response at $\varphi = 0^0$ confirms the relevance of the parameters calibration.

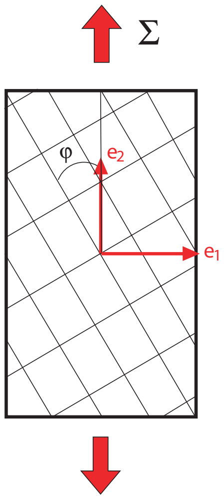


Figure 4. Representation of the tensile test applied with the φ angle

The results predicted for the off-axis loading tests show also a good agreement with experimental data; this indicates the good predictive capabilities of the proposed micromechanical damage model. In particular, the relative positions of the three curves manifest the combined effect of the initial anisotropy and the evolving damage.

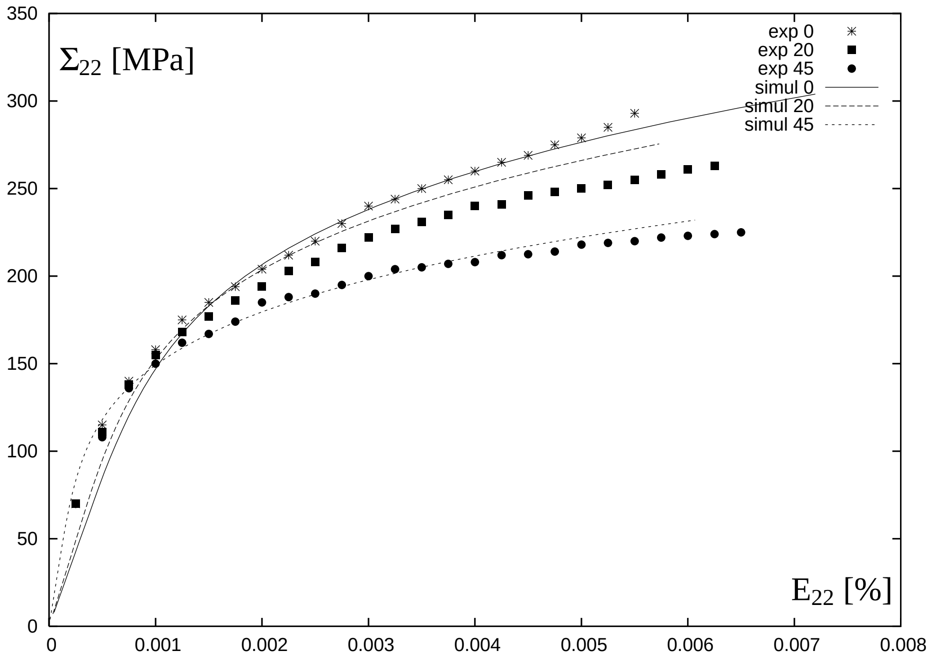


Figure 5. Comparison of the tensile tests of unidirectional SiC-SiC composite

6 Conclusions

In the present study we investigate and evaluate the effective properties of an orthotropic matrix weakened by arbitrarily oriented microcracks. The determination of the macroscopic elastic properties of this class of materials has been performed in the framework of Eshelby's equivalent inclusion problems. To this end, we have derived original closed-form expressions of the Eshelby tensor (or equivalently, of the Hill tensor). The macroscopic elastic properties (compliance), deduced from these results, shows a coupling between the initial orthotropy of the solid matrix and the cracks-induced anisotropy. As a consequence, the deviation from elastic orthotropy can be characterized. A new damage model for initially orthotropic materials is then proposed and implemented by adopting a Mori-Tanaka homogenization scheme and a damage criterion based on the derived energy release rate. The numerical predictions of the damage model are in good agreement with experimental data reported by [1] on a ceramic matrix composite (SiC-SiC), particularly for off-axis loadings experiments.

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A Proposed methodology for the determination of $\mathbb{P}$ tensor

The starting point here is the recent study of [43] which follows the procedure described by [45] [3]. Let us consider the two global unit orthogonal vectors $\underline{e}_1$ and $\underline{e}_2$ in the plane $\xi_3 = 0$; any unit vector in this plane reads then $\underline{\xi} = \cos\psi\underline{e}_1 + \sin\psi\underline{e}_2$. It follows that the acoustic tensor $\mathbf{K} = \underline{\xi}.\mathbb{C}.\underline{\xi}$ takes the form:

$$\mathbf{K} = (\cos\psi)^2\mathbf{Q} + \cos\psi\sin\psi(\mathbf{R} + \mathbf{R}^T) + (\sin\psi)^2\mathbf{T} \tag{A.1}$$

Substituting $y = \cot\psi$ in this expression yields:

$$\mathbf{K}(\psi) = (\sin\psi)^2\left[\mathbf{Q}y^2 + y(\mathbf{R} + \mathbf{R}^T) + \mathbf{T}\right] = (\sin\psi)^2\mathbf{K}(y), \qquad \text{with:} \tag{A.2}$$

$$\mathbf{K}(y) = y^2\mathbf{Q} + y(\mathbf{R} + \mathbf{R}^T) + \mathbf{T}. \tag{A.3}$$

The second order tensors $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{T}$ are defined, respectively as:

$$\mathbf{Q} = \underline{e}_1.\mathbb{C}.\underline{e}_1 \quad \mathbf{R} = \underline{e}_1.\mathbb{C}.\underline{e}_2 \quad \mathbf{T} = \underline{e}_2.\mathbb{C}.\underline{e}_2 \tag{A.4}$$

3. These authors have not studied the case of arbitrarily oriented inclusions in an orthotropic solid matrix

For the computation of $\mathbb{P}$ (equations (7-9)), one needs to invert $\mathbf{K}(y)$. By denoting $\mid \mathbf{K}(y) \mid$ the determinant of $\mathbf{K}(y)$ and by $\tilde{\mathbf{K}}(y)$ it's adjoint, it follows that: $\mathbf{K}(y).\tilde{\mathbf{K}}(y) = \mid \mathbf{K}(y) \mid \mathbf{1}$, and respectively:

$$\mathbf{K}^{-1}(y) = \frac{\tilde{\mathbf{K}}(y)}{\mid \mathbf{K}(y) \mid} \tag{A.5}$$

For the second-order tensor $\boldsymbol{\Delta} = \underline{\xi} \otimes \underline{\xi}$ which, as for $\mathbf{K}^{-1}$, appears in the definition (9) of the tensor $\mathbb{N}$, it can be shown that: $\boldsymbol{\Delta}(\psi) = (\sin\psi)^2 \boldsymbol{\Delta}(y)$ with $\boldsymbol{\Delta}(y) = y^2 \underline{e}_1 \otimes \underline{e}_1 + y(\underline{e}_1 \otimes \underline{e}_2 + \underline{e}_2 \otimes \underline{e}_1) + \underline{e}_2 \otimes \underline{e}_2$. Note also that $a^2\xi_1^2 + b^2\xi_2^2 = (\sin\psi)^2(a^2y^2 + b^2)$. Using then equation (A.2) and the change of variable $y = \cot\psi$, the expression (11) becomes:

$$M_{ijkl} = \frac{ab}{\pi} \int_{-\infty}^{\infty} \frac{\tilde{K}_{jk}(y)\Delta_{il}(y)dy}{(a^2y^2 + b^2) \mid \mathbf{K}(y) \mid} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{X\tilde{K}_{jk}(y)\Delta_{il}(y)dy}{(y^2 + X^2) \mid \mathbf{Q} \mid f(y)} \tag{A.6}$$

where $X = b/a$ denotes the microcrack aspect ratio. In general, $f(y)$ is a polynomial of degree 6:

$$f(y) = (y - y_1)(y - \bar{y}_1)(y - y_2)(y - \bar{y}_2)(y - y_3)(y - \bar{y}_3). \tag{A.7}$$

where y_p $(p = 1, 2, 3)$ are the roots with positive imaginary part, while $\bar{y}_p$ their complex conjugate, respectively. The $\mathbb{M}$ tensor depends only on the shape of the inclusion and on the material's stiffness. Note also that the representation (A.6) allows the evaluation of M_{ijkl} by the residues theorem. The first order approximation of $\mathbb{M}$ at $X = 0$ (X being the microcrack aspect ratio) gives:

$$\begin{aligned} M_{ijkl} = {} & \frac{\tilde{K}_{jk}(0)\Delta_{il}(0)}{f(0)\mid \mathbf{Q} \mid} + \frac{2iX}{\mid \mathbf{Q} \mid} \sum_{i=1}^{3} \frac{\tilde{K}_{jk}(y_i)\Delta_{il}(y_i)}{f'(y_i)y_i^2} + \frac{iX}{f(0) \mid \mathbf{Q} \mid} \cdot \\ & \left[\tilde{K}'_{jk}(0)\Delta_{il}(0) + \tilde{K}_{jk}(0)\Delta'_{il}(0) - \tilde{K}_{jk}(0)\Delta_{il}(0)\frac{f'(0)}{f(0)} \right] \end{aligned} \tag{A.8}$$

where $\tilde{K}_{jk}$ are the components of the adjoint of $\mathbf{K}$. The quantity $\mid \mathbf{Q} \mid$, the determinant of $\mathbf{Q}$ (see (A.4)) is expressed as:

$$\begin{aligned} \mid \mathbf{Q} \mid = {} & C^s_{1111}C^s_{1212}(\cos^2(\theta) + \alpha\sin^2(\theta))(\cos^2(\theta) + \beta\sin^2(\theta)) \\ & (C^s_{3232}\sin^2(\theta) + C^s_{3131}\cos^2(\theta)) \end{aligned} \tag{A.9}$$

with α and β are given by (13). Thus,

$$\alpha + \beta = \frac{C^s_{1111}C^s_{2222} - {C^s_{1122}}^2 - 2C^s_{1122}C^s_{1212}}{C^s_{1111}C^s_{1212}} \qquad \alpha\beta = \frac{C^s_{2222}}{C^s_{1111}} \tag{A.10}$$

It can be shown that the imaginary part, $\Im(\mathbb{M})$, of $\mathbb{M}$ is null:

$$\Im(M_{ijkl}) = \frac{2X}{\mid \mathbf{Q} \mid} \Re \sum_{i=1}^{3} \frac{\tilde{K}_{jk}(y_i)\Delta_{il}(y_i)}{f'(y_i)y_i^2} + \frac{X}{f(0) \mid \mathbf{Q} \mid} \cdot \\ \left[\tilde{K}'_{jk}(0)\Delta_{il}(0) + \tilde{K}_{jk}(0)\Delta'_{il}(0) - \tilde{K}_{jk}(0)\Delta_{il}(0)\frac{f'(0)}{f(0)} \right] = 0. \tag{A.11}$$

Hence its real part, $\Re(\mathbb{M})$, is:

$$M_{ijkl} = \Re(M_{ijkl}) = \frac{\tilde{K}_{jk}(0)\Delta_{il}(0)}{f(0) \mid \mathbf{Q} \mid} - \frac{2X}{\mid \mathbf{Q} \mid} \Im \sum_{i=1}^{3} \frac{\tilde{K}_{jk}(y_i)\Delta_{il}(y_i)}{f'(y_i)y_i^2} \tag{A.12}$$

In summary, we have obtained the expression of $\mathbb{M}$ in the local frame of the crack. Its expression in the global frame is given by a simple change of frame.

B Expression of the components of Hill's tensor $\mathbb{P}$

In the global frame, the components of Hill's tensor $\mathbb{P}$ are:

$$
\begin{aligned}
P_{1111} &= \frac{(C_{2222}u^2 + C_{1212}v^2)v^2}{C_{1111}C_{1212}(\alpha u^2 + v^2)(\beta u^2 + v^2)} + \frac{X}{C_{1111}C_{1212}(\alpha - \beta)} \cdot \\
&\quad \left\{ \frac{\sqrt{\alpha}(\alpha u^2 - v^2)(\alpha C_{1212} - C_{2222})}{(\alpha u^2 + v^2)^2} - \frac{\sqrt{\beta}(\beta u^2 - v^2)(\beta C_{1212} - C_{2222})}{(\beta u^2 + v^2)^2} \right\} \\
P_{1122} &= -\frac{(C_{1122} + C_{1212})u^2 v^2}{C_{1111}C_{1212}(\alpha u^2 + v^2)(\beta u^2 + v^2)} \\
&\quad + \frac{X(C_{1122} + C_{1212})}{C_{1111}C_{1212}(\alpha - \beta)} \left\{ \frac{\sqrt{\alpha}(\alpha u^2 - v^2)}{(\alpha u^2 + v^2)^2} - \frac{\sqrt{\beta}(\beta u^2 - v^2)}{(\beta u^2 + v^2)^2} \right\} \\
P_{2222} &= \frac{(C_{1111}v^2 + C_{1212}u^2)u^2}{C_{1111}C_{1212}(\alpha u^2 + v^2)(\beta u^2 + v^2)} + \frac{X}{C_{1111}C_{1212}(\alpha - \beta)} \cdot \\
&\quad \left\{ \frac{(\alpha u^2 - v^2)(\alpha C_{1111} - C_{1212})}{\sqrt{\alpha}(\alpha u^2 + v^2)^2} - \frac{(\beta u^2 - v^2)(\beta C_{1111} - C_{1212})}{\sqrt{\beta}(\beta u^2 + v^2)^2} \right\} \\
P_{1112} &= \frac{(C_{2222}u^2 - C_{1122}v^2)uv}{2C_{1111}C_{1212}(\alpha u^2 + v^2)(\beta u^2 + v^2)} + \frac{Xuv}{C_{1111}C_{1212}(\alpha - \beta)} \cdot \\
&\quad \left\{ \frac{\sqrt{\alpha}(\alpha C_{1122} + C_{2222})}{(\alpha u^2 + v^2)^2} - \frac{\sqrt{\beta}(\beta C_{1122} + C_{2222})}{(\beta u^2 + v^2)^2} \right\} \qquad \text{(B.1)} \\
P_{2212} &= \frac{(C_{1111}v^2 - C_{1122}u^2)uv}{2C_{1111}C_{1212}(\alpha u^2 + v^2)(\beta u^2 + v^2)} - \frac{Xuv}{C_{1111}C_{1212}(\alpha - \beta)} \cdot \\
&\quad \left\{ \frac{\sqrt{\alpha}(\alpha C_{1111} + C_{1122})}{(\alpha u^2 + v^2)^2} - \frac{\sqrt{\beta}(\beta C_{1111} + C_{1122})}{(\beta u^2 + v^2)^2} \right\} \\
P_{1212} &= \frac{C_{1111}v^4 + C_{2222}u^4 - 2C_{1122}v^2u^2}{4C_{1111}C_{1212}(\alpha u^2 + v^2)(\beta u^2 + v^2)} + \frac{X(C_{1111}C_{2222} - C_{1122}^2)}{4C_{1111}C_{1212}^2(\alpha - \beta)} \cdot \\
&\quad \left\{ \frac{\sqrt{\alpha}(\alpha u^2 - v^2)}{(\alpha u^2 + v^2)^2} - \frac{\sqrt{\beta}(\beta u^2 - v^2)}{(\beta u^2 + v^2)^2} \right\} \\
P_{1313} &= \frac{v^2}{4(C_{2323}u^2 + C_{1313}v^2)} + \frac{X}{4}\sqrt{\frac{C_{2323}}{C_{1313}}} \frac{(C_{2323}u^2 - C_{1313}v^2)}{(C_{3232}u^2 + C_{3131}v^2)^2} \\
P_{2323} &= \frac{u^2}{4(C_{3232}u^2 + C_{1313}v^2)} - \frac{X}{4}\sqrt{\frac{C_{1313}}{C_{2323}}} \frac{C_{2323}u^2 - C_{1313}v^2}{(C_{3232}u^2 + C_{1313}v^2)^2} \\
P_{1323} &= \frac{uv}{4(C_{2323}u^2 + C_{3131}v^2)} - \frac{X\sqrt{C_{1313}C_{2323}}uv}{2(C_{2323}u^2 + C_{1313}v^2)^2}
\end{aligned}
$$

with $u = \cos(\theta)$ and $v = \sin(\theta)$.

C Components of the $\mathbb{T}$ tensor

Componentwise $\mathbb{T}_{ijkl}$ terms deduced in the section 4.1 are as follows (using the notation $t = C^s_{1111}/(C^s_{111}C^s_{2222} - {C^s_{1122}}^2)$):

$$
\begin{aligned}
T_{1111} &= t\ C^s_{1111}(\sqrt{\alpha}+\sqrt{\beta})\sqrt{\alpha\beta}\sin^2(\theta)\\
T_{1122} &= t\ C^s_{1122}(\sqrt{\alpha}+\sqrt{\beta})\sqrt{\alpha\beta}\sin^2(\theta)\\
T_{2211} &= t\ C^s_{1122}(\sqrt{\alpha}+\sqrt{\beta})\cos^2(\theta)\\
T_{2222} &= t\ C^s_{2222}(\sqrt{\alpha}+\sqrt{\beta})\cos^2(\theta)\\
T_{1133} &= t\ C^s_{1133}(\sqrt{\alpha}+\sqrt{\beta})\sqrt{\alpha\beta}\sin^2(\theta)\\
T_{2233} &= t\ C^s_{2233}(\sqrt{\alpha}+\sqrt{\beta})\cos^2(\theta)\\
T_{1112} &= -t\ C^s_{1212}(\sqrt{\alpha}+\sqrt{\beta})\sqrt{\alpha\beta}\sin\theta\cos\theta\\
T_{2212} &= -t\ C^s_{1212}(\sqrt{\alpha}+\sqrt{\beta})\sin\theta\cos\theta\\
T_{1211} &= -\frac{t}{2}(C^s_{1122}+C^s_{1111}\sqrt{\alpha\beta})(\sqrt{\alpha}+\sqrt{\beta})\sin\theta\cos\theta\\
T_{2212} &= -\frac{t}{2}(C^s_{2222}+C^s_{1122}\sqrt{\alpha\beta})(\sqrt{\alpha}+\sqrt{\beta})\sin\theta\cos\theta\\
T_{1212} &= \frac{t}{2}C^s_{1212}\left[\sin^2(\theta)+\sqrt{\alpha\beta}\cos^2(\theta)\right](\sqrt{\alpha}+\sqrt{\beta})\\
T_{1233} &= -\frac{t}{2}(C^s_{2233}+C^s_{1133}\sqrt{\alpha\beta})(\sqrt{\alpha}+\sqrt{\beta})\sin\theta\cos\theta;\\
T_{3232} &= \frac{1}{2}\sqrt{\frac{C^s_{3232}}{C^s_{3131}}}\cos^2(\theta) \qquad T_{3132} = \frac{1}{2}\sqrt{\frac{C^s_{3231}}{C^s_{3131}}}\cos\theta\sin\theta\\
T_{3231} &= \frac{1}{2}\sqrt{\frac{C^s_{3131}}{C^s_{3232}}}\cos\theta\sin\theta \qquad T_{3131} = \frac{1}{2}\sqrt{\frac{C^s_{3131}}{C^s_{3232}}}\sin^2(\theta)
\end{aligned}
\tag{C.1}
$$

D Components of the $\mathbb{H}$ tensor

The components of the $\mathbb{H}$ tensor obtained by [47] in planar elasticity are as follows:

$$
\begin{aligned}
H_{1111} &= \frac{\pi L}{A\sqrt{E_1^s}}\left[(b^2-a^2)\cos^2(\theta)+a^2+\frac{ab}{L\sqrt{E_1^s}}\right]\\
H_{1122} &= -\frac{\pi ab}{A\sqrt{E_1^s E_2^s}}\\
H_{1112} &= \frac{\pi(b^2-a^2)L}{2A\sqrt{E_1^s}}\sin(\theta)\cos(\theta)\\
H_{2222} &= \frac{\pi L}{A\sqrt{E_2^s}}\left[(a^2-b^2)\cos^2(\theta)+b^2+\frac{ab}{L\sqrt{E_2^s}}\right]\\
H_{2212} &= \frac{\pi(b^2-a^2)L}{2A\sqrt{E_2^s}}\sin(\theta)\cos(\theta)\\
H_{1212} &= \frac{\pi L}{4A\sqrt{E_1^s E_2^s}}\\
&\left[(a^2-b^2)\left(\sqrt{E_2^s}-\sqrt{E_1^s}\right)\cos^2(\theta)+a^2\sqrt{E_1^s}+abL\sqrt{E_1^s E_2^s}+b^2\sqrt{E_2^s}\right]
\end{aligned}
\tag{D.1}
$$

in which A is the representative area of the considered cell. The constant L is related to the elastic moduli of the orthotropic 2D matrix by:

$$
L=\sqrt{\frac{1}{G_{12}^s}+\frac{2}{\sqrt{E_1^s\ E_2^s}}-\frac{2\nu_{12}^s}{E_1^s}}=(\sqrt{\alpha}+\sqrt{\beta})\sqrt{\frac{C_{1111}^s}{C_{1111}^s C_{2222}^s-{C_{1122}^s}^2}} \tag{D.2}
$$

α, β are the roots of the characteristic equation of the bidimensional orthotropic medium.