Exact solution of a plastic hollow sphere with a Mises-Schleicher matrix

Vincent Monchiet, D. Kondo

Exact limit-analysis solution of a plastic hollow sphere with a Mises-Schleicher matrix

Vincent MONCHIET [a], Djimedo KONDO [b]

[a] *Université Paris-Est, Modélisation et Simulation Multi Echelle, MSME UMR 8208 CNRS,*
5 bd Descartes, 77454 Marne-la-Vallée, France

[b] *Université Pierre et Marie Curie, Institut D'Alembert, UMR 7190 CNRS,*
4, Place Jussieu, 75252 Paris Cedex 05, France

Abstract

In this paper we establish the exact solution for a hollow sphere with a rigid-plastic pressure-sensitive matrix and subjected to hydrostatic traction or compression. The matrix is assumed to obey to a parabolic Mises-Schleicher criterion. The closed-form expressions of the velocity field and of the stress field are provided. These exact solutions, expressed by means of the Lambert W function, allow to assess and discuss existing results.

Key words: Gurson Model, Limit Analysis, Hollow Sphere, Pressure-Sensitive Plasticity, Mises-Schleicher, Parabolic Type Criterion.

1 Introduction

The study by Gurson [7] is known to provide an upper bound for the macroscopic yield function of porous plastic materials having von Mises matrix. The approach is based on limit analysis of a hollow sphere made up of a rigid ideal plastic von Mises material and subjected to an arbitrary loading in the context of homogeneous strain rate boundary conditions. The macroscopic yield function has been derived by considering two velocity fields: the first one corresponds to a homogeneous strain rate field, while the second one, which is heterogeneous, accounts for the cavity expansion. For the latter, use has been made of the exact solution of the hollow sphere subjected to a hydrostatic

Email addresses: vincent.monchiet@univ-mlv.fr (Vincent MONCHIET), djimedo.kondo@upmc.fr (Djimedo KONDO).

external loading. As a consequence, the obtained macroscopic yield function provides the exact solution for this particular loading case. Improvements of the Gurson criterion have been recently proposed by Monchiet al. [12], [13] by making use of Eshelby-like trial velocity field.
Despite its great interest, the Gurson criterion appears to be inappropriate for a large class of ductile porous materials (for instance some polymers or some geomaterials) having a pressure-dependent matrix. To circumvent this limitation, several studies have been further proposed in order to extend the Gurson approach to porous materials with a Drucker-Prager type plastic matrix [6] (see also [8,9]). Still, these authors used a two-field trial velocity which contains the exact solution for the Drucker-Prager hollow sphere subjected to a purely hydrostatic loading (see [19,14]). It must be emphasized that the macroscopic yield function derived by [6] allows to recover the Gurson criterion in the limit case of von Mises matrix.
Unfortunately, the above extensions of the Gurson model do not apply to porous media with a pressure-dependent matrix obeying to a parabolic type criterion (instead of the Drucker-Prager linear one), the so-called Mises-Schleicher criterion [11] [1] (see also [15,16]) which exhibits also an asymmetry between tension and compression [2]. It must be mentioned that this type of matrix has been considered by Lee and Oung [10] (see also [5]) to model the ductile failure behavior of glassy polymers. To this end, these authors used a trial velocity field inspired from the one already considered by Gurson and accounting for the matrix pressure sensitivity. A crucial observation, emphasized by the authors themselves, is that their macroscopic yield function does not recover the Gurson criterion in the limit of the von-Mises matrix. In particular, the exact solution established by [7] for a hydrostatic loading is not retrieved. Based on these observations, the authors proposed a heuristical modification of the original criterion which allows to retrieve the Gurson exact solution. In the case of the pressure-dependent matrix, this modification still misses the exact solution for a hydrostatic external loading.
In the present study, we derive the exact stress and velocity fields for the hollow sphere (with a matrix which obeys to a Mises-Schleicher criterion) subjected to an external hydrostatic traction or compression. The closed-form expressions of the velocity field is shown to be expressed in term of the Lambert W function (see for instance [3]). These new results allows to assess the relevance of the Lee and Oung (2000)'s criterion. Their use to derive an upper bound of the macroscopic yield function for arbitrary loadings is addressed.

[1] *The original publication of this criterion can be found in [17]*
[2] *Note that recent investigations on ductile porous metallic materials with matrix having asymmetry between tension and compression has been carried out by [2].*

2 Statement of the problem

2.1 The local constitutive behavior

We consider a homogeneous plastic material obeying to the following pressure sensitive criterion:

$$f(\boldsymbol{\sigma}) = \sigma_{eq}^2 + 3\alpha\sigma_0\sigma_m - \sigma_0^2 = 0 \tag{1}$$

known as the Mises-Schleicher criterion [17] (see also [11] (page 28) [15,16]) in which α and σ_0 are related to the tensile yield stress, T, and absolute yield stress in compression, C, by:

$$\sigma_0 = \sqrt{CT}, \quad \alpha = \frac{C - T}{\sqrt{CT}} \tag{2}$$

Note that the Mises-Schleicher criterion predicts asymmetry between tension and compression.
In (1), σ_{eq} and σ_m denote the equivalent von-Mises stress and the mean stress respectively. σ_0 and α are two material parameters with the assumption that $\alpha \geq 0$ which physically means that the yield stress in traction is lower than in compression. The strain rate is derived from the normality flow rule:

$$\boldsymbol{d} = \dot{\lambda}\frac{\partial f}{\partial \boldsymbol{\sigma}}(\boldsymbol{\sigma}) = \dot{\lambda}\Big(3\overline{\boldsymbol{\sigma}} + \alpha\sigma_0\boldsymbol{I}\Big) \tag{3}$$

where $\overline{\boldsymbol{\sigma}}$ is the deviatoric part of the local stress, $\overline{\boldsymbol{\sigma}} = \boldsymbol{\sigma} - \sigma_m\boldsymbol{I}$ and $\boldsymbol{I}$ is the second order identity tensor. In (3), $\dot{\lambda}$ is the plastic multiplier. The computation of the von-Mises equivalent strain rate, d_{eq}, and the mean part of the local strain rate, d_m, leads to:

$$d_{eq} = 2\dot{\lambda}\sigma_{eq}, \qquad d_m = \dot{\lambda}\alpha\sigma_0 \tag{4}$$

The plastic multiplier being positive, the means strain rate is then positive, $d_m \geq 0$.
The local dissipation reads:

$$\pi(\boldsymbol{d}) = \boldsymbol{\sigma} : \boldsymbol{d} = \sigma_{eq}d_{eq} + 3\sigma_m d_m = (2\sigma_{eq}^2 + 3\alpha\sigma_0\sigma_m)\dot{\lambda} = \dot{\lambda}(\sigma_0^2 + \sigma_{eq}^2) \tag{5}$$

Eliminating $\dot{\lambda}$ between the two equations in (4), one has:

$$\sigma_{eq} = \frac{\alpha\sigma_0}{2}\frac{d_{eq}}{d_m} \tag{6}$$

Replacing now the equivalent stress in (5) by its expression (6) and the plastic multiplier by $\dot{\lambda} = d_m/(\alpha\sigma_0)$ (see equation (4)) one obtains:

$$\pi(\boldsymbol{d}) = \frac{\sigma_0 d_m}{\alpha} + \frac{\alpha\sigma_0}{4}\frac{d_{eq}^2}{d_m} \tag{7}$$

2.2 Application of the Gurson limit analysis approach

Let us now consider a hollow sphere, with the internal and external radii a and b respectively, subjected at its outer boundary to homogeneous strain rate conditions:

$$\underline{v}(r = b) = \boldsymbol{D}.\underline{x} \tag{8}$$

For this homogeneous strain rate boundary conditions, the following inequality holds for all macroscopic stress $\boldsymbol{\Sigma}$ and macroscopic strain rate $\boldsymbol{D}$ [18], [4]:

$$\boldsymbol{\Sigma} : \boldsymbol{D} \leq \Pi(\boldsymbol{D}) = \inf_{\underline{v}\ K.A}\left[\frac{1}{|\Omega|}\int_{\Omega-\omega}\pi(\boldsymbol{d})dV\right] \tag{9}$$

$\Pi(\boldsymbol{D})$ represents the macroscopic dissipation, Ω denotes the volume of the unit cell, $|\Omega| = 4\pi b^3/3$, whereas ω denotes the volume of the void, $|\omega| = 4\pi a^3/3$. In (9), the local dissipation $\pi(\boldsymbol{d})$ is given by (7). The infimum is taken over all kinematically admissible (K.A) velocity fields, $\underline{v}$, complying then with (8). As classically, the limit stress states at the macroscopic scale are shown to be of the form:

$$\boldsymbol{\Sigma} = \frac{\partial\Pi}{\partial\boldsymbol{D}} \tag{10}$$

2.3 Lee and Oung's macroscopic yield surface [10]

An upper bound for the macroscopic yield function has been derived by Lee and Oung [10]. These authors considered a trial velocity field inspired of the one used by Gurson [7] in the case of a von-Mises matrix:

$$\underline{v} = (\overline{\boldsymbol{D}} + a_0 \boldsymbol{I}).\underline{x} + (D_m - a_0)\frac{b^3}{r^2}\underline{e}_r \tag{11}$$

This field is compatible with boundary conditions (8) and coincide with Gurson's trial velocity field when $a_0 = 0$. The constant a_0 is introduced in order to takes into account the compressibility of the matrix. Due to the presence of this constant, it must be emphasized that the velocity field given by (11) is not entirely determined by the boundary conditions. Its determination has required the minimization of the macroscopic dissipation (9) according to a_0 which is then given by (see [10]):

$$a_0 = \frac{\alpha}{2}\left(\frac{4D_m^2 + f^2 D_{eq}^2}{f + \alpha^2}\right)^{1/2} \tag{12}$$

The macroscopic yield function, derived from (10) reads:

$$\Sigma_{eq}^2 + \frac{9f}{4}\Sigma_m^2 + 3(1-f)\alpha\sigma_0\Sigma_m - (1-f)^2\sigma_0^2 = 0 \tag{13}$$

Note that the above criterion constitutes an upper bound of the exact yield surface. However, the most important criticism which can be formulated about (13) is that it does not allow to recover the Gurson criterion [7] when the limit $\alpha \rightarrow 0$ is taken [3]. Moreover, under purely hydrostatic loadings (traction or compression) and for $\alpha = 0$, it provides:

$$|\Sigma_m| = \frac{2}{3}\sigma_0\frac{1-f}{f} \tag{15}$$

Which differs from the well-known exact solution:

$$|\Sigma_m| = -\frac{2}{3}\sigma_0 \ln(f) \tag{16}$$

Since the trial velocity field (11) contains the exact solution for purely hydrostatic loading and $\alpha = 0$ (then $a_0 = 0$), this observation is questionable. The explanation may be found in the fact that, although the velocity field (11) coincides with the one used by Gurson for $\alpha = 0$, it may not be valid for the

[3] *To correct this shortcoming of the upper bound* (13)*, Lee and Oung [10] has proposed a heuristical modification of their result in the form:*

$$\Sigma_{eq}^2 + 2f\sigma_0^2\cosh\left(\frac{3\Sigma_m}{2\sigma_0}\right) + 3(1-f)\alpha\sigma_0\Sigma_m - (1-f)^2\sigma_0^2 = 0 \tag{14}$$

dissipation (7). Indeed, when the hollow sphere is subjected to an hydrostatic traction or compression ($\overline{\boldsymbol{D}} = 0$ and $\boldsymbol{D} = D_m \boldsymbol{I}$), the exact expression of the local dissipation is:

$$\pi(\boldsymbol{d}) = 2\sigma_0 |D_m| \frac{b^3}{r^3} \tag{17}$$

whereas with the trial velocity field (11), $\pi(\boldsymbol{d})$ takes the following form when $\alpha \to 0$:

$$\pi(\boldsymbol{d}) = \sigma_0 \frac{|D_m|}{\sqrt{f}} \left(1 + f \frac{b^6}{r^6}\right) \tag{18}$$

3 Derivation of the local stress fields and the yield strength

3.1 The local stress field

Let us first introduce the spherical basis $(\underline{e}_r, \underline{e}_\theta, \underline{e}_\varphi)$ and the associated coordinates system (r, θ, φ), with $\theta \in [0, 2\pi]$ and $\varphi \in [0, \pi]$. The macroscopic loading is assumed to be isotropic (the hollow sphere is subjected at its external boundary to hydrostatic traction or compression). Since the problem is invariant by any rotation with an angle φ or θ, the stress tensor has the form:

$$\sigma_{ij} = \begin{pmatrix} \sigma_{rr} & 0 & 0 \\ 0 & \sigma_{\theta\theta} & 0 \\ 0 & 0 & \sigma_{\varphi\varphi} \end{pmatrix}_{(\underline{e}_r, \underline{e}_\theta, \underline{e}_\varphi)} \tag{19}$$

where σ_{rr} and $\sigma_{\theta\theta} = \sigma_{\varphi\varphi}$ are both functions of the coordinate r. The equivalent stress as well as the mean stress, computed from (19), read then:

$$\sigma_{eq} = \epsilon(\sigma_{\theta\theta} - \sigma_{rr}), \quad \sigma_m = \frac{1}{3}(\sigma_{rr} + 2\sigma_{\theta\theta}) \tag{20}$$

in which $\epsilon = \text{sign}(\sigma_{\theta\theta} - \sigma_{rr})$. Let us introduce the positive function of the radial coordinate, $G(r)$, such that

$$\sigma_{eq} = \sigma_0 G(r) \tag{21}$$

Due to the condition $f(\boldsymbol{\sigma}) = 0$, the mean stress reads, for $\alpha \neq 0$:

$$\sigma_m = \frac{\sigma_0}{3\alpha}(1 - G^2(r)) \tag{22}$$

The components of the stress tensor are:

$$\begin{aligned} \sigma_{rr} &= \sigma_m - \frac{2\epsilon}{3}\sigma_{eq} = \frac{\sigma_0}{3\alpha}\Big[1 - G^2(r)\Big] - \frac{2\epsilon}{3}\sigma_0 G(r) \\ \sigma_{\theta\theta} &= \sigma_m + \frac{\epsilon}{3}\sigma_{eq} = \frac{\sigma_0}{3\alpha}\Big[1 - G^2(r)\Big] + \frac{\epsilon}{3}\sigma_0 G(r) \end{aligned} \tag{23}$$

As classically, the local equilibrium is expressed as:

$$\frac{d\sigma_{rr}}{dr} + \frac{2}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = 0 \tag{24}$$

Introducing (23) into (24) leads to the following non linear differential equation for $G(r)$:

$$G(r)G'(r) + \epsilon\alpha G'(r) + \frac{3\epsilon\alpha}{r}G(r) = 0 \tag{25}$$

The integration of the above equation allows to obtain the solution in the following implicit form:

$$\ln(G(r)) + \frac{G(r)}{\epsilon\alpha} + \ln(r^3) = K \tag{26}$$

Where K is a constant. The above solution can be put into the equivalent form:

$$\frac{G(r)}{\epsilon\alpha}\exp\left(\frac{G(r)}{\epsilon\alpha}\right) = p\frac{a^3}{r^3} \tag{27}$$

where the notation $p = \exp(K)/(\epsilon\alpha a^3)$ has been adopted. The explicit solution reads then:

$$G(r) = \epsilon\alpha W\left(p\frac{a^3}{r^3}\right) \tag{28}$$

where W denotes the "Lambert W" function which is defined to be the inverse of the function $x \mapsto x\exp(x)$. Function W is not injective and is double-valued on the interval $[-1/e, 0]$. Indeed, if $x > 0$ the equation $x\exp(x) = y$ has only one solution for x but has two solutions when $-1/e \leq x > 0$ (see figure 1). For this reason, two branches of function W are distinguished. The upper branch, corresponding to $W(x) \geq -1$, is denoted $W_{-1}(x)$ whereas $W_0(x)$ denotes the lower branch, corresponding to $W(x) \leq -1$. On figure 1 are represented the two branches of the Lambert W function.

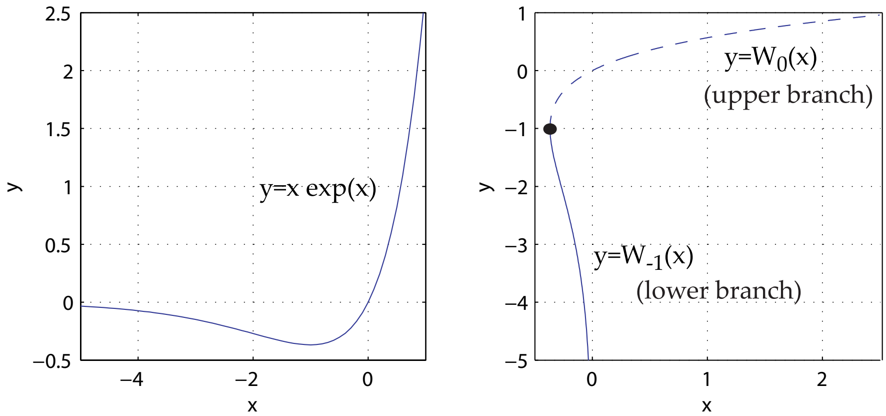

Fig. 1. At the left: function $y = x\exp(x)$. At the right: the two branches of the W Lambert function, $y = W_0(x)$ and $y = W_{-1}(x)$.

Function G must be positive since $G(r) = \sigma_{eq}/\sigma_0$. However $W(x)$ has the sign of x; it follows that, in (28), $\epsilon = \text{sign}(p)$. Finally, reporting (28) into (23) gives:

$$\begin{aligned} \sigma_{rr} &= \frac{\sigma_0}{3\alpha}\left[1 - 2\alpha^2 W\left(p\frac{a^3}{r^3}\right) - \alpha^2 W^2\left(p\frac{a^3}{r^3}\right)\right] \\ \sigma_{\theta\theta} &= \frac{\sigma_0}{3\alpha}\left[1 + \alpha^2 W\left(p\frac{a^3}{r^3}\right) - \alpha^2 W^2\left(p\frac{a^3}{r^3}\right)\right] \end{aligned} \tag{29}$$

Coefficient p is determined with condition $\sigma_{rr}(r = a) = 0$ (traction free condition on the void boundary). This identification leads to :

$$\begin{cases} p = p_+ = z_+\exp(z_+), \quad z_+ = \dfrac{-\alpha + \sqrt{\alpha^2+1}}{\alpha} \\ or \\ p = p_- = z_-\exp(z_-), \quad z_- = \dfrac{-\alpha - \sqrt{\alpha^2+1}}{\alpha} \end{cases} \tag{30}$$

The variations of p_+ and p_- according to α are represented on figure 2. It can be observed that coefficient p_+ is positive and tends to zero when $\alpha \to +\infty$ whereas p_- is negative and tends to the finite value $z_- = -2\exp(-2)$ when $\alpha \to +\infty$.

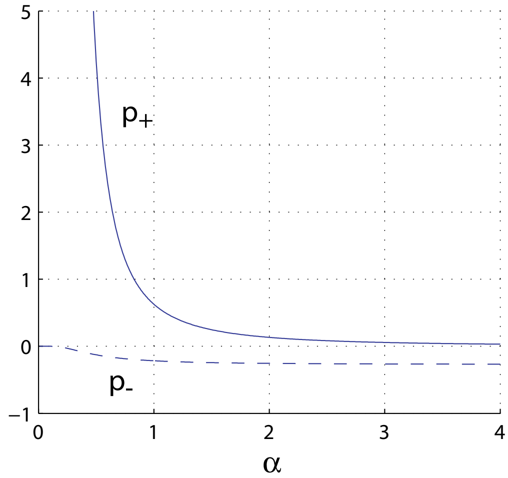

Fig. 2. Representation of the coefficients p_+ and p_- as function α

3.2 The yield strength

Under uniform strain rate boundary conditions, the macroscopic stress is defined as the average of the local stress field:

$$\boldsymbol{\Sigma} =< \boldsymbol{\sigma} >_\Omega \tag{31}$$

Here, the macroscopic strain rate is assumed to be isotropic, $\boldsymbol{D} = D_m \boldsymbol{I}$. Therefore, the macroscopic stress is isotropic and takes the form $\boldsymbol{\Sigma} = \Sigma_m \boldsymbol{I}$ with:

$$\Sigma_m =< \sigma_m >_\Omega = \frac{b}{3|\Omega|} \int_{\partial\Omega} \sigma_{rr}(r = b) dS \tag{32}$$

for which use has been done of the divergence theorem. Since σ_{rr} is only function of the radial coordinate, r, one has:

$$\Sigma_m = \sigma_{rr}(r = b) \tag{33}$$

(28) together with (30) lead to four expressions for each component of the stress field, depending on whether one considers the upper or lower branch of the Lambert W function but also whether one considers expressions p_+ or p_- for coefficient p. Note that $W_{-1}(x)$ takes complex values for $x < 0$ and consequently the solution $W_{-1}(p_- a^3/r^3)$ must be discarded. The solution which uses the branch W_0 with coefficient p_- can be also discarded since it

leads to the solution $\Sigma_m = 0$ when $\alpha = 0$. The solution for traction is given by (29) in which the upper branch W_0 together with p_+ are used:

$$\Sigma_m = \frac{\sigma_0}{3\alpha}\left[1 - \alpha^2 W_0^2(fp_+) - 2\alpha^2 W_0(fp_+)\right] \tag{34}$$

where $f = a^3/b^3$ is the porosity.

Remark 1: The Taylor expansion of $W_0(fp_+)$ at $\alpha = 0$ gives: $W_0(fp_+) = \frac{1}{\alpha} + \ln(f) - 1 + o(\alpha)$. It can then be shown that the expression of Σ_m, given by (34), takes the finite limit $\Sigma_m = -\frac{2\sigma_0}{3}\ln(f)$ when $\alpha \to 0$ which coincides with the result of Gurson (equation (16)).

The solution for compression use the lower branch W_{-1} together with p_-:

$$\Sigma_m = \frac{\sigma_0}{3\alpha}\left[1 - \alpha^2 W_{-1}^2(p_- f) - 2\alpha^2 W_{-1}(p_- f)\right] \tag{35}$$

Remark 2: The Taylor expansion of $W_{-1}(fp_-)$ at $\alpha = 0$ is: $W_{-1}(fp_-) = -\frac{1}{\alpha} + \ln(f) - 1 + o(\alpha)$. It can then be shown that the expression of Σ_m, given by (35), has a finite limit when $\alpha \to 0$. This limit is $\Sigma_m = \frac{2\sigma_0}{3}\ln(f)$ which coincides also with the result of Gurson (equation (16)).

Note that in [10] the authors derived the macroscopic flow stress for purely hydrostatic loading by a direct method, i.e not requiring the determination of the local stress field. This solution can be recovered from (34) and (35) by reexpressing $W(fp)$ as function of Σ_m and taking thereafter the exponential. Doing that, the two solutions (34) and (35) can be reexpressed in the following implicit form:

- if $\Sigma_m \geq 0$

$$\ln\left[\frac{-\alpha + \sqrt{1 + \alpha^2 - 3\alpha\Sigma_m/\sigma_0}}{-\alpha + \sqrt{1+\alpha^2}}\right] + \frac{1}{\alpha}\sqrt{1 + \alpha^2 - 3\alpha\Sigma_m/\sigma_0} - \frac{1}{\alpha}\sqrt{1+\alpha^2} = \ln(f) \tag{36}$$

- if $\Sigma_m < 0$

$$\ln\left[\frac{\alpha + \sqrt{1 + \alpha^2 - 3\alpha\Sigma_m/\sigma_0}}{\alpha + \sqrt{1+\alpha^2}}\right] - \frac{1}{\alpha}\sqrt{1 + \alpha^2 - 3\alpha\Sigma_m/\sigma_0} + \frac{1}{\alpha}\sqrt{1+\alpha^2} = \ln(f) \tag{37}$$

On figure 3 is represented the yield stress Σ_m/σ_0 as function of the parameter α for a porosity $f = 0.1$. The upper bound (13) and the modified criterion (14) both proposed by Lee and Oung [10] are compared to the exact solution (34) for a macroscopic hydrostatic tension. It is observed that the upper bound (13) differs from the exact solution (34) when low values of the coefficient α are considered. However for larger values of α the upper bound (13) coincide with (34). The case of a hydrostatic compression is illustrated on figure 4 in which the upper bound (13) and the modified criterion (14) are compared to the exact solution (35). It appears that the upper bound differs from the exact solution, whatever the value of α. The modified criterion (14) allows to retrieve the exact solution provided by Gurson criterion ($\alpha = 0$). However, for strictly positive values of α, the approximate solution gives an estimation of the yield stress which is different from the exact solution.

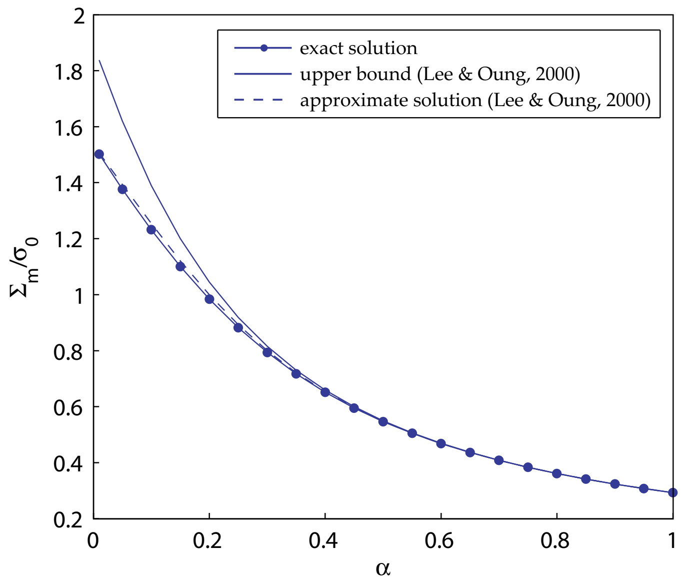

Fig. 3. Normalized yield stress Σ_m/σ_0 as function of α for a porosity $f = 0.1$. Case of purely hydrostatic traction.

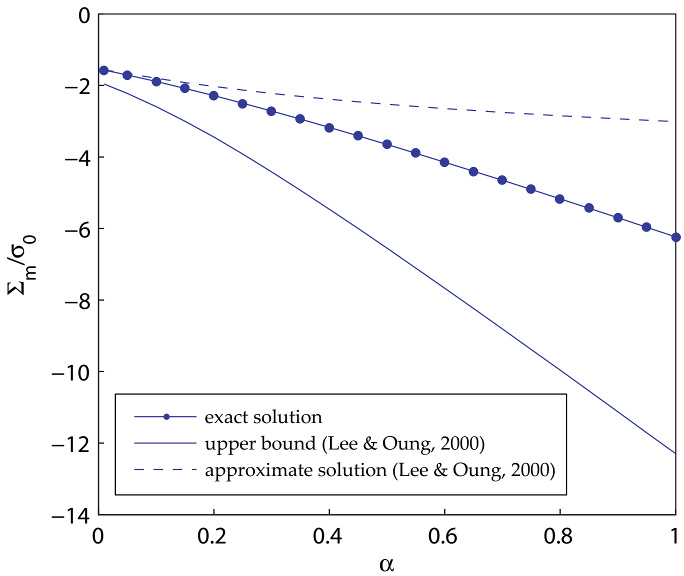

Fig. 4. Normalized yield stress Σ_m/σ_0 as function of α for a porosity $f = 0.1$. Case of purely hydrostatic compression.

4 Derivation of the velocity field

We now propose to derive the expression of the velocity field, solution of the rigid-plastic hollow sphere obeying to the Mises-Schleicher criterion. When the sphere is subjected to a hydrostatic loading, the velocity field is in the form:

$$\underline{v} = F(r)\underline{e}_r \tag{38}$$

and the local strain rate field reads:

$$d_{ij} = \begin{pmatrix} d_{rr} & 0 & 0 \\ 0 & d_{\theta\theta} & 0 \\ 0 & 0 & d_{\varphi\varphi} \end{pmatrix}_{(\underline{e}_r,\underline{e}_\theta,\underline{e}_\varphi)} \tag{39}$$

with:

$$d_{rr} = F'(r), \quad d_{\theta\theta} = d_{\varphi\varphi} = \frac{1}{r}F(r) \tag{40}$$

From relation (3), one has $\epsilon = \text{sign}(\sigma_{\theta\theta} - \sigma_{rr}) = \text{sign}(d_{\theta\theta} - d_{rr})$. Consequently the equivalent and mean strain rate read:

$$
\begin{aligned}
d_{eq} &= \frac{2}{3}|d_{\theta\theta} - d_{rr}| = \frac{2\epsilon}{3}\left(\frac{1}{r}F(r) - F'(r)\right) \\
d_m &= \frac{1}{3}(d_{rr} + 2d_{\theta\theta}) = \frac{1}{3}\left(F'(r) + \frac{2}{r}F(r)\right)
\end{aligned}
\tag{41}
$$

Recalling that $\sigma_{eq} = \sigma_0 G(r)$ (see (21)) where the expression of $G(r)$ is given in (28), and taking into account relation (6) with (41), it comes:

$$
W\left(p\frac{a^3}{r^3}\right) = \frac{F(r) - rF'(r)}{2F(r) + rF'(r)} \tag{42}
$$

Introducing in the above equation $F(r) = \exp(X(u))$, with the change of variable $u = a^3/r^3$, one obtains:

$$
X'(u) = \frac{W(pu)}{u(1 + W(pu))} - \frac{1}{3u} \tag{43}
$$

Due to the property:

$$
\frac{dW(x)}{dx} = \frac{W(x)}{x(1 + W(x))} \tag{44}
$$

the integration of $X(u)$ from (43) gives:

$$
X(u) = \frac{1}{3}W(pu) - \frac{1}{3}\ln(u) + K' \tag{45}
$$

It follows that:

$$
F(r) = \frac{C}{r^2 W(pa^3/r^3)} \tag{46}
$$

Where $C = a^2 \exp(K')$ is a constant which has to be determined from boundary conditions.
The hollow sphere being subjected to the uniform strain boundary conditions $\underline{v}(r = b) = \boldsymbol{D}.\underline{x}$ with $\boldsymbol{D} = D_m \boldsymbol{I}$, $F(r)$ must satisfy $F(r = b) = bD_m$. It comes, for the coefficient C:

$$
C = b^3 D_m W(pf) \tag{47}
$$

for which it is recalled that the porosity reads $f = a^3/b^3$. The velocity field is then given by:

$$v_r = D_m \frac{b^3}{r^2} \frac{W(pf)}{W(pa^3/r^3)} \tag{48}$$

In the case of a hydrostatic tension, the above solution is used with the upper branch W_0 and coefficient p_+:

$$v_r = D_m \frac{b^3}{r^2} \frac{W_0(p_+ f)}{W_0(p_+ a^3/r^3)} \tag{49}$$

whereas in the case of compression:

$$v_r = D_m \frac{b^3}{r^2} \frac{W_{-1}(p_- f)}{W_{-1}(p_- a^3/r^3)} \tag{50}$$

Figures 5 and 6 show the variations of $v_r/(bD_m)$ with r/a for traction and compression, respectively. The porosity value is $f = 0.01$ and different values of α are considered. On figures 5 and 6 are also reported the variations of the trial velocity field (in fact the unique radial component) used by Lee and Oung given by (11) together with (12) in which has been put $\overline{\boldsymbol{D}} = 0$. As already mentioned, this velocity field coincides with the exact solution for $\alpha = 0$. For completeness we propose to compare the values of d_{eq} and d_m, which enters into the definition of the local dissipation, for low values of the coefficient α. The Taylor expansions of d_{eq} and d_m at $\alpha = 0$ reads, for the exact solution:

$$d_{eq} = 2|D_m| \frac{b^3}{r^3} + o(\alpha); \qquad d_m = |D_m| \frac{b^3}{r^3} \alpha + o(\alpha^3) \tag{51}$$

whereas, from the trial velocity field (11), we get:

$$d_{eq} = 2|D_m| \frac{b^3}{r^3} + o(\alpha); \qquad d_m = \frac{|D_m|}{\sqrt{f}} \alpha + o(\alpha^3) \tag{52}$$

It is observed that the terms of order 0 in the series of d_{eq} and d_m are the same for the exact and approximate solutions. However, the terms of order 1 in the series of d_m differs.

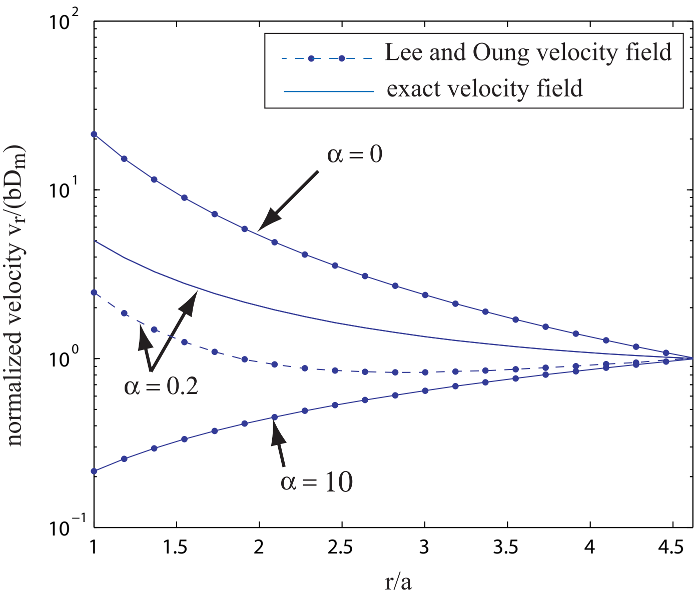

Fig. 5. Normalized radial component of the velocity field, $v_r/(bD_m)$ as function of r/a for $f = 0.01$ and various values of coefficient α in the case of traction.

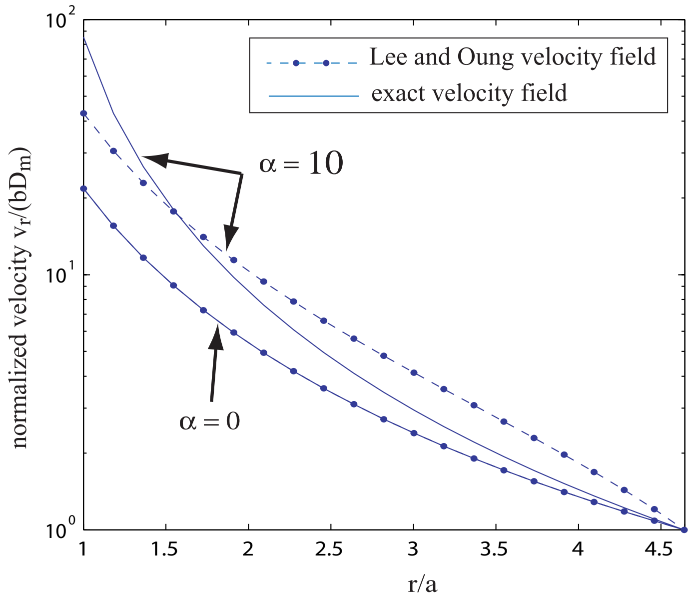

Fig. 6. Normalized radial component of the velocity field, $v_r/(bD_m)$ as function of r/a for $f = 0.01$ and various values of coefficient α in the case of compression.

We aim now at comparing the exact and approximate expressions of the local dissipation. The sphere being still submitted to an external hydrostatic strain rate, the exact expression of the local dissipation is then obtained by replacing in (7) the expressions of d_{eq} and d_m obtained with (38) with (48). The resulting expression reads:

$$\pi(\boldsymbol{d}) = \sigma_0|D_m|\frac{b^3}{r^3}\frac{W(pf)[1+\alpha^2 W(pa^3/r^3)]}{\alpha W(pr^3/a^3)[1+W(pr^3/a^3)]} \tag{53}$$

whereas the approximate expression of the local dissipation, obtained by Lee and Oung [10], is:

$$\pi(\boldsymbol{d}) = \sigma_0|D_m|\frac{r^6/b^6 + f + 2\alpha^2 - 2\alpha\sqrt{f+\alpha^2}}{r^6/b^6\sqrt{f+\alpha^2}} \tag{54}$$

On figure 7 are compared the two expressions (53) and (54) of the local dissipation for $\alpha = 0$ and a porosity $f = 0.01$. It is observed that, although the approximate velocity field coincides with the exact solution for $\alpha = 0$ (see figure 5), the local dissipation significantly differs. This is due to the fact that in the two solutions (see (51) and (52)) the ratio d_m/α which directly enters in the expression of the local dissipation does not deliver the same limit when $\alpha \to 0$.

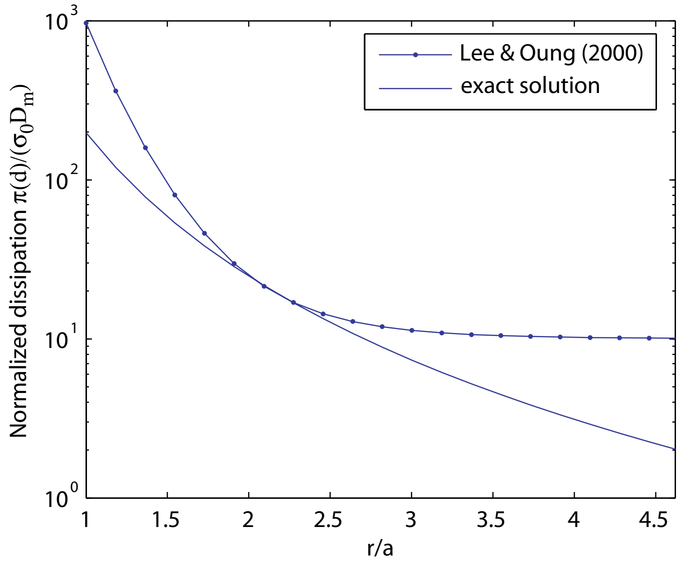

Fig. 7. Normalized dissipation, $\pi(\boldsymbol{d})/(\sigma_0 D_m)$ as function of r/a for $f = 0.01$ and $\alpha = 0$ in the case of a hydrostatic traction.

Figure 8 corresponds to the comparison for $\alpha = 0.2$.

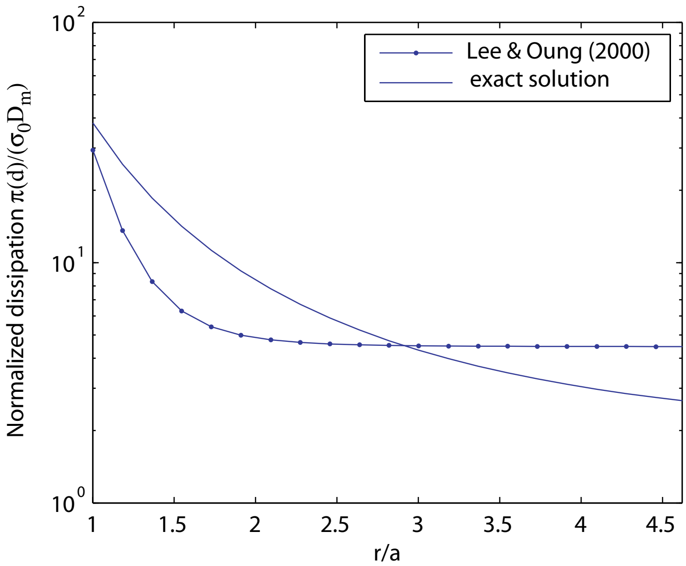

Fig. 8. Normalized dissipation, $\pi(\boldsymbol{d})/(\sigma_0 D_m)$ as function of r/a for $f = 0.01$ and $\alpha = 0.2$ in the case of a hydrostatic traction.

5 Conclusion

In this paper we derive the exact solution for a pressure-sensitive plastic hollow sphere subjected to a purely hydrostatic traction and compression. The hollow sphere is made up of a rigid-plastic material which obeys to the Mises-Schleicher criterion [17]. The closed-form expressions for the velocity and strain field, as well as for the stress field are shown to be expressed in term of the Lambert W function. These exact solutions are then used to assess the accuracy of approximate expressions in tension and compression established by [10]. The exact velocity field found in the present study can be used to construct a trial velocity field for the limit analysis of the plastic hollow sphere under arbitrary loadings.

References

[1] **J.-F. Barthélémy, L. Dormieux**.
Détermination du critère de rupture macroscopique d'un milieu poreux par homognisation non linaire. *C. R. Méc., Acad. Sci. Paris.* **331**:271-276, 2003.

[2] **O. Cazacu and J. B. Stewart**.
Plastic potentials for porous aggregates with the matrix exhibiting tension-compression asymmetry. *Journal of Mechanics and Physics of Solids.* **57**:325-341, 2009.

[3] **R.M. Corless, G.H. Gonnet, D.E.G Hare, D.J. Jeffrey, D.E. Knuth.** On the Lambert W function. *Adv. Comput. Math.* **5**:329-359, 1996.

[4] **P. De Buhan**. A fundamental approach to the yield design of reinforced soil structures - Chap. 2: yield design homogenization theory for periodic media. *Thèse de doctorat d'état, Université Pierre et Marie Curie, Paris VI*, 1986.

[5] **D. Durban, T. Cohen, Y. Hollander**. Plastic response of porous solids with pressure sensitive matrix. *Mech. Res. Comm.* **37**:636-641, 2010.

[6] **T.F. Guo, J. Faleskog, C.F. Shih**. Continuum modeling of a porous solid with pressure sensitive dilatant matrix. *J. Mech. Phys. Solids.* **56**:21882212, 2008.

[7] **A.L. Gurson**. Continuum theory of ductile rupture by void nucleation and growth: Part I. - Yield criterion and flow rules for porous ductile media. *J. Eng. Mat. Tech.* **99**:2-15, 1977.

[8] **H.-Y. Jeong, J. Pan**. A macroscopic constitutive law for porous solids with pressure-sensitive matrices and its implications to plastic flow localization. *Int. J. Sol. Struct.* **32(24)**: 3669–3691, 1995.

[9] **H.-Y. Jeong**. A new yield function and a hydrostatic stress-controlled model for porous solids with pressure-sensitive matrices. *Int. J. Sol. Struct.* **39**:1385-1403, 2002.

[10] **J.H. Lee, J. Oung**. Yield functions and flow rules for porous pressure-dependent strain-hardening polymeric materials. *J. Appl. Mech.* **67**:288-297, 2000.

[11] **J. Lubliner.** Plasticity Theory. Macmillan, New York, 1990.

[12] **V. Monchiet, E. Charkaluk, D. Kondo**. An improvement of Gurson-type models of porous materials by using Eshelby-like trial velocity fields. *Comptes Rendus Mcanique.* **335(1)**:32-41, 2007.

[13] **V. Monchiet, E. Charkaluk, D. Kondo**. A Micromechanics-based modification of the Gurson criterion by using Eshelby-like velocity fields. *European Journal of Mechanics A/solids.* **30**:940-949, 2011.

[14] **J. Pastor, P. Thoré, F. Pastor.** Limit analysis and numerical modeling of spherically porous solids with Coulomb and Drucker-Prager matrices. *J. Comp. Appl. Mathematics.* **234**:2162-2174, 2010.

[15] **R.S. Raghava, R.M. Caddell, G.S.Y. Yeh.** The macroscopic yield behavior of polymers. *J. Mater. Sci.* **9**:225-232, 1973.

[16] **R.S. Raghava, R.M. Caddell, G.S.Y. Yeh.**
A Macroscopic Yield Criterion for Crystalline Polymers. *Int. J. Mech. Sci.* **15**:967-974, 1973.

[17] **F. Schleicher.**
Z. angew. Math. Mech. **6**:199, 1926.

[18] **P. Suquet**.
Plasticité et homogénéisation. Thèse de doctorat d'état, Université Pierre et Marie Curie, Paris VI, 1982.

[19] **P. Thoré, F. Pastor, J. Pastor, D. Kondo.**
Closed-form solutions for the hollow sphere model with Coulomb and Drucker-Prager materials under isotropic loadings. *C. R. Méc., Acad. Sci. Paris.* **337**:260-267, 2009.