Random uncertainties modeling in dynamical systems

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ABSTRACT: In the introduction of this paper, we summarize the usual parametric probabilistic models for random uncertainties modeling in dynamic systems and we introduce a new nonparametric probabilistic approach. In a first part, a random matrix theory, recently developed by the author, is introduced allowing the nonparametric probabilistic approach to be constructed. This theory is compared with the usual GOE from the random matrix theory. In a second part of the paper, we present applications of this new theory for linear structural vibrations with dynamical substructuring techniques and for nonlinear transient elastodynamics.

1 INTRODUCTION

This paper deals with random uncertainties modeling for linear and nonlinear dynamic systems, using a new probabilistic approach, recently proposed by the author, called "nonparametric model of random uncertainties".

Usual parametric probabilistic model of random uncertainties in dynamic systems

In dynamic systems, random uncertainties are usually modeled using parametric models. The uncertain parameters of the boundary value problem can be geometric parameters, boundary conditions, mass density, mechanical parameters of constitutive equations, structural complexity, interface and junction modeling, etc, and are modeled by real- and vector-valued random variables, stochastic processes and stochastic fields. Concerning details related to such a parametric approach,

(1) for general developments and applications, we refer the reader to Collins & Thomson (1969), Shinozuka & Astill (1972), Soong (1973), Chen & Soroka (1973), Prasthofer & Beadle (1975), Haug, Choi & Komkov (1986), Ibrahim (1987), Kotulski & Sobczyk (1987), Shinozuka (1987), Jensen & Iwan (1992), Iwan & Jensen (1993), Lee & Singh (1994), Spanos & Zeldin (1994), Papadimitriou, Katafygiotis *et al* (1995), Lin & Cai (1995), Micaletti *et al* (1998);

(2) for aspects related to stochastic finite elements, we refer the reader to Vanmarcke & Grigoriu (1983), Liu *et al* (1986), Shinozuka & Deodatis (1988), Spanos & Ghanem (1989), Ghanem and Spanos (1991), Kleiber *et al* (1992), Ditlevsen & Tarp-Johansen (1998);

(3) for other aspects related to parametric models in the context of stochastic dynamics and parametric stochastic excitations, we refer the reader to Lin (1967), Kree & Soize (1986), Bergman & Spencer (1987), Roberts & Spanos (1990), Sobczyk (1991), Soong & Grigoriu (1993), Soize (1994), Lin & Cai (1995), Sarkani & Lutes (1996), Schueller (1997).

A new nonparametric probabilistic model of random uncertainties in dynamic systems

In this paper, we present an overview of a set of results recently obtained by the author, concerning a new nonparametric probabilistic model of random uncertainties in linear and nonlinear structural dynamics. In addition, we explain the differences between this probability model and probability models derived from the use of the Gaussian orthogonal ensemble (GOE). The fundamentals of this nonparametric approach and the first development to frequency response calculations in linear structural dynamics with random uncertainties, can be found in Soize (1999 & 2000). The algebraic closure of this nonparametric model, the convergence analysis as dimension goes to infinity and the development to time response calculations in transient linear elastodynamics with random uncertainties, can be found in Soize (2001a). The application of this complete nonparametric model to time response calculations in transient linear structural dynamics can be found in Soize (2001b & 2001c). The development of a random uncertainties model to frequency response calculations in linear structural dynamics using a dynamic substructuring method, can be found in Chebli & Soize (2001) and Soize & Chebli (2002). The use of the dynamic substructuring method allows the nonparametric model of random uncertainties to be used with non-homogeneous uncertainties through the structure. Finally, the introduction of this nonparametric model for transient response calculation in nonlinear structural dynamics with random uncertainties can be found in Soize (2001d).

Linear structural dynamics. The two main objectives introduced to construct such a nonparametric model of random uncertainties in linear structural dynamics are:

(1) the non use of the local parameters of the finite element model of the structure, but the use of generalized quantities directly related to dynamics (non parametric approach);

(2) the use of the available information for constructing the probability model, knowing that, in practice, the main available information related to dynamics is constituted of an adapted mean reduced model derived from the mean finite element model. Usually, for modal range, this mean reduced model is constructed using the generalized coordinates of the mode-superposition method associated with the structural modes corresponding to the lowest eigenfrequencies of the structure.

To satisfy these two objectives, the nonparametric probabilistic model of random uncertainties in linear structural dynamics is constructed by replacing the generalized mass, damping and stiffness matrices of the mean reduced model by full random matrices with values in the set of all the positive-definite symmetric real matrices. The probability model of these random matrices is constructed using the entropy optimization principle from information theory (for the entropy optimization principle, see Shannon (1948), Jaynes (1957) and Kapur & Kesavan (1992)) whose available information is constituted of the following three constraints:

(C1) each full random matrix has to be symmetric positive definite;

(C2) the mean value of each full random matrix is known and is equal to the corresponding generalized matrix of the mean reduced model;

(C3) the second-order moment of the Frobenius norm of the inverse of each random matrix has to exist.

It is natural to introduce constraint (C2). Constraint (C1) has to be taken into account in order to obtain a mechanical system with random uncertainties, which models a dynamic system. For instance, if there are uncertainties in the generalized mass matrix, the probability distribution has to be such that this random generalized mass matrix be positive definite. If not, the probability model would be wrong because the generalized mass matrix of any dynamic system has to be positive definite. Below, we will recall the reason why constraint (C3) has to be introduced.

It should be noted that such a nonparametric model of random uncertainties

(1) allows the uncertainties for the parameters of the finite element model to be taken into account (similarly to the parametric approaches, but using a global approach),

(2) but also allows the model uncertainties to be taken into account, that is to say, modeling the errors which cannot be reached through the model parameters (by definition, any parametric approach cannot model the kind of uncertainties which correspond to non existing parameters in the mean finite element model under consideration); for instance, the use of thick plate finite elements instead of three-dimensional elements, the use of a mean finite element model for which the number of degrees of freedom is fixed and is not considered as a parameter (error induced by the finite element discretization of the boundary value problem), etc.

Nonlinear structural dynamics. The above nonparametric model of random uncertainties in linear structural dynamics can directly be used for modeling random uncertainties existing in the linear part of a nonlinear dynamic system. Consequently, such an approach is very efficient for nonlinear dynamic systems exhibiting a usual linear part and a nonlinear part due to damping and restoring nonlinear forces corresponding to localized nonlinearities (for instance, nonlinear restoring forces induced by stops which limit the vibration amplitudes at a given point on the structure).

Analyzing the role played by the constraints defining the available information

In order to explain the role played by the introduction of constraints (C2) and (C3) as available information for the construction of the nonparametric model, we consider the mean one-DOF linear static problem $\underline{K}\,\underline{x} = f$ in which the mean stiffness is $\underline{K} = 1$, the force f is prescribed and $\underline{x}$ is the unknown displacement. If there are stiffness uncertainties, the stiffness is modeled by a random variable K such that $E\{K\} = \underline{K}$ where E is the mathematical expectation, and the displacement is the random variable X such that $K\,X = f$.

(1) Let K_g be the Gaussian random variable with unit mean value and standard deviation $\delta = 0.5$. Consequently, the probability density function of K_g with respect to dk is written as

$$p_{K_g}(k) = \frac{1}{\sqrt{2\pi}\,\delta} \exp\left\{ \frac{(k-1)^2}{2\delta^2} \right\} \quad . \tag{1}$$

The graph of function $k \mapsto p_{K_g}(k)$ is shown in Figure 1. It is clear that random variable K cannot be modeled by Gaussian random variable K_g because K has to be a positive-valued random variable and this is not the case for K_g. Such a probabilistic model for K is wrong.

(2) If a random variable K_e is constructed using the maximum entropy principle for which the available information is defined by constraints (C1) and (C2), that is to say, K_e is a positive-valued random variable and its mean value $E\{K_e\}$ is equal to 1, then the probability density function of K_e with respect to dk is the exponential distribution written as

$$p_{K_e}(k) = \mathbb{1}_{]0,+\infty[}(k)\,\exp(-k) \quad , \tag{2}$$

in which $\mathbb{1}_{]0,+\infty[}(k) = 0$ if $k \leq 0$ and $\mathbb{1}_{]0,+\infty[}(k) = 1$ if $k > 0$. The standard deviation of K_e is equal to 1 and the graph of function $k \mapsto p_{K_e}(k)$ is shown in Figure 1. Random variable K cannot be modeled by random variable K_e for the following reason. Since K_e is a random variable with values in $]0,+\infty[$, therefore K_e is almost surely invertible and then the random displacement is given by $X = K_e^{-1} f$ almost surely. In this case, X is a random variable, but the second-order moment of X does not exist, that is to say, we have $E\{X^2\} = E\{|K_e^{-1}|^2\}\, f^2 = +\infty$. This is clearly not admissible for such a mechanical problem. Consequently, the constraint $E\{|K^{-1}|^2\} < +\infty$ has to be added in the construction of the probability model of stiffness K and corresponds to constraint (C3) introduced above.

(3) If a random variable K_s is now constructed using the maximum entropy principle for which the available information is such that K_s is a positive-valued random variable, $E\{K_s\} = 1$ and $E\{\ln(K_s)\} = v$ with $|v| < +\infty$, then the probability density function of K_s with respect to dk can be written as

$$p_{K_s}(k) = \mathbb{1}_{]0,+\infty[}(k)\,\frac{\delta^{-2\delta^{-2}}}{\Gamma(\delta^{-2})}\,k^{(\delta^{-2}-1)}\,\exp\left(-\frac{k}{\delta^2}\right) \quad , \tag{3}$$

in which $\delta < 1/\sqrt{2}$ is the standard deviation of K_s. The graph of function $k \mapsto p_{K_s}(k)$ is shown in Figure 1. It is clear that, if condition $E\{\ln(K_s)\} = v$ with $|v| < +\infty$ holds, then constraint (C3) introduced above is satisfied. In this case, we have $E\{|K_s^{-1}|^2\} < +\infty$ and therefore, $X = K_s^{-1} f$ is a second-order random variable, that is to say, $E\{X^2\} < +\infty$. Consequently, random variable K can be modeled by random variable K_s.

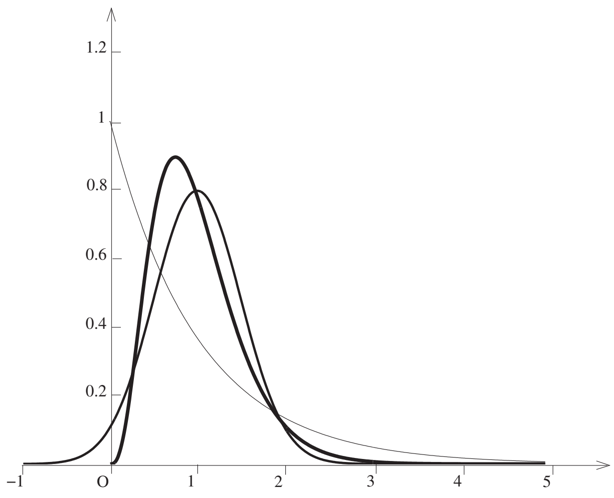

Figure 1. Graphs of the probability density functions of random variables K_e (thin solid line), K_g (med solid line) and K_s (thick solid line).

2 A PROBABILITY MODEL FOR SYMMETRIC POSITIVE-DEFINITE REAL RANDOM MATRICES

Let $\mathbb{M}_n(\mathbb{R})$ be the set of all the $(n \times n)$ real matrices, $\mathbb{M}_n^S(\mathbb{R})$ be the set of all the $(n \times n)$ real symmetric matrices and $\mathbb{M}_n^+(\mathbb{R})$ be the set of all the $(n \times n)$ real symmetric positive-definite matrices.

In this section, we summarize the theory developed in Soize (1999, 2000 & 2001a) concerning the construction of a probability model for a symmetric positive-definite real random matrix $[\mathbf{A}_n]$, defined on a probability space $(\mathcal{A}, \mathcal{T}, P)$, using the entropy optimization principle for which the constraints which define the available information are the following.

(1) Matrix $[\mathbf{A}_n]$ is a symmetric positive-definite real random matrix, that is to say,

$$[\mathbf{A}_n] \in \mathbb{M}_n^+(\mathbb{R}) \quad \text{a.s} \quad . \tag{4}$$

(2) The mean value $[\underline{A}_n]$ of random matrix $[\mathbf{A}_n]$ is a given matrix in $\mathbb{M}_n^+(\mathbb{R})$ such that

$$E\{[\mathbf{A}_n]\} = [\underline{A}_n] \in \mathbb{M}_n^+(\mathbb{R}) \quad . \tag{5}$$

(3) The second-order moment of the Frobenius norm of the inverse $[\mathbf{A}_n]^{-1}$ has to exist,

$$E\{\|[\mathbf{A}_n]^{-1}\|_F^2\} < +\infty \quad , \tag{6}$$

in which $\|[A]\|_F = (\text{tr}\{[A]\,[A]^T\})^{1/2}$ is the Frobenius norm of matrix $[A]$.

Concerning the nonparametric probabilistic model of random uncertainties in dynamic systems, such a random matrix $[\mathbf{A}_n]$ will represent the random generalized mass, damping or stiffness matrix.

2.1 Normalization and dispersion parameter

Since $[\underline{A}_n]$ is a positive-definite real matrix, there is an upper triangular matrix $[\underline{L}_{A_n}]$ in $\mathbb{M}_n(\mathbb{R})$ such that

$$[\underline{A}_n] = [\underline{L}_{A_n}]^T\,[\underline{L}_{A_n}] \quad , \tag{7}$$

which corresponds to the Cholesky factorization of matrix $[\underline{A}_n]$. Considering Eq. (7), random matrix $[\mathbf{A}_n]$ can be written as

$$[\mathbf{A}_n] = [\underline{L}_{A_n}]^T\,[\mathbf{G}_{A_n}]\,[\underline{L}_{A_n}] \quad , \tag{8}$$

in which matrix $[\mathbf{G}_{A_n}]$ is a random variable with values in $\mathbb{M}_n^+(\mathbb{R})$. From Eqs. (5) and (8), we deduce that the mean value $[\underline{G}_{A_n}]$ of random matrix $[\mathbf{G}_{A_n}]$ is such that

$$[\underline{G}_{A_n}] = E\{[\mathbf{G}_{A_n}]\} = [\,I_n] \quad , \tag{9}$$

in which $[\,I_n]$ is the $(n \times n)$ identity matrix. Let $\delta_A > 0$ be the real parameter defined by

$$\delta_A = \left\{\frac{E\{\|\,[\mathbf{G}_{A_n}] - [\underline{G}_{A_n}]\,\|_F^2\}}{\|\,[\underline{G}_{A_n}]\,\|_F^2}\right\}^{1/2} \quad , \tag{10}$$

that allows the dispersion of the probability model of random matrix $[\mathbf{A}_n]$ to be fixed. Let $n_0 \geq 1$ be a fixed integer. Then, the dispersion of the probability model is fixed by giving parameter δ_A, independent of n, a value such that

$$0 < \delta_A < \sqrt{(n_0+1)(n_0+5)^{-1}} \quad . \tag{11}$$

2.2 Probability distribution

The probability distribution $P_{[\mathbf{G}_{A_n}]}$ of random matrix $[\mathbf{G}_{A_n}]$ is defined by a probability density function $[G_n] \mapsto p_{[\mathbf{G}_{A_n}]}([G_n])$ from $\mathbb{M}_n^+(\mathbb{R})$ into $\mathbb{R}^+ = [0\,,+\infty[$, with respect to the measure (volume element) $\widetilde{d}G_n$ on the set $\mathbb{M}_n^S(\mathbb{R})$ such that

$$\widetilde{d}G_n = 2^{n(n-1)/4}\, \Pi_{1\leq i\leq j\leq n}\, d[G_n]_{ij} \quad . \tag{12}$$

We then have

$$P_{[\mathbf{G}_{A_n}]} = p_{[\mathbf{G}_{A_n}]}([G_n])\; \widetilde{d}G_n \quad , \tag{13}$$

with the normalization condition

$$\int_{\mathbb{M}_n^+(\mathbb{R})} p_{[\mathbf{G}_{A_n}]}([G_n])\; \widetilde{d}G_n = 1 \quad . \tag{14}$$

Probability density function $p_{[\mathbf{G}_{A_n}]}([G_n])$ is then written as

$$p_{[\mathbf{G}_{A_n}]}([G_n]) = \mathbb{1}_{\mathbb{M}_n^+(\mathbb{R})}([G_n]) \times C_{\mathbf{G}_{A_n}} \times \left(\det[G_n]\right)^{(n+1)\frac{(1-\delta_A^2)}{2\delta_A^2}} \times \exp\left\{-\frac{(n+1)}{2\delta_A^2}\,\mathrm{tr}\,[G_n]\right\}, \tag{15}$$

in which det is the determinant, $\mathbb{1}_{\mathbb{M}_n^+(\mathbb{R})}([G_n])$ is equal to 1 if $[G_n] \in \mathbb{M}_n^+(\mathbb{R})$ and is equal to zero if $[G_n] \notin \mathbb{M}_n^+(\mathbb{R})$ and where positive constant $C_{\mathbf{G}_{A_n}}$ is such that

$$C_{\mathbf{G}_{A_n}} = \frac{(2\pi)^{-n(n-1)/4}\left(\frac{n+1}{2\delta_A^2}\right)^{n(n+1)(2\delta_A^2)^{-1}}}{\left\{\Pi_{j=1}^n \Gamma\left(\frac{n+1}{2\delta_A^2} + \frac{1-j}{2}\right)\right\}} \quad , \tag{16}$$

where $\Gamma(z)$ is the gamma function defined for $z > 0$ by $\Gamma(z) = \int_0^{+\infty} t^{z-1}\, e^{-t}\, dt$.

2.3 Characteristic function

For all $[\Theta_n]$ in $\mathbb{M}_n^S(\mathbb{R})$, the characteristic function of random matrix $[\mathbf{G}_{A_n}]$ with values in $\mathbb{M}_n^+(\mathbb{R}) \subset \mathbb{M}_n^S(\mathbb{R})$ is defined by $\Phi_{[\mathbf{G}_{A_n}]}([\Theta_n]) = E\{\exp(i \ll [\Theta_n]\,,[\mathbf{G}_{A_n}] \gg)\}$ in which $\ll [\Theta_n],[\mathbf{G}_{A_n}] \gg = \mathrm{tr}\{[\Theta_n][\mathbf{G}_{A_n}]^T\} = \mathrm{tr}\{[\Theta_n][\mathbf{G}_{A_n}]\}$. We then have

$$\Phi_{[\mathbf{G}_{A_n}]}([\Theta_n]) = \int_{\mathbb{M}_n^+(\mathbb{R})} \exp(\,i\,\mathrm{tr}\{[\Theta_n]\,[G_n]\})\; p_{[\mathbf{G}_{A_n}]}([G_n])\; \widetilde{d}G_n\,, \tag{17}$$

which yields

$$\Phi_{[\mathbf{G}_{A_n}]}([\Theta_n]) = \left\{\det\left([\,I_n] - i\frac{2\delta_A^2}{n+1}\,[\Theta_n]\right)\right\}^{-(n+1)(2\delta_A^2)^{-1}}. \tag{18}$$

2.4 Second-order moments

Since $[\underline{G}_{A_n}]_{jk} = \delta_{jk}$ in which $\delta_{jk} = 0$ if $j \neq k$ and $\delta_{jj} = 1$, the covariance $C^{G_n}_{jk,j'k'}$ of random variables $[\mathbf{G}_{A_n}]_{jk}$ and $[\mathbf{G}_{A_n}]_{j'k'}$, defined by

$$C^{G_n}_{jk,j'k'} = E\{([\mathbf{G}_{A_n}]_{jk} - \delta_{jk})([\mathbf{G}_{A_n}]_{j'k'} - \delta_{j'k'})\} \quad , \tag{19}$$

is written as

$$C^{G_n}_{jk,j'k'} = \frac{\delta_A^2}{n+1}\{\delta_{j'k}\,\delta_{jk'} + \delta_{jj'}\,\delta_{kk'}\} \quad . \tag{20}$$

2.5 Invariance under real orthogonal transformations

Let $[\Phi_n]$ be any real orthogonal matrix belonging to $\mathbb{M}_n(\mathbb{R})$ such that $[\Phi_n]^T[\Phi_n] = [\Phi_n]\,[\Phi_n]^T = [\,I_n]$. Let $[\mathbf{G}'_{A_n}]$ be the random matrix with values in $\mathbb{M}_n^+(\mathbb{R})$ defined by $[\mathbf{G}'_{A_n}] = [\Phi_n]^T[\mathbf{G}_{A_n}]\,[\Phi_n]$. We then have

$$[\mathbf{G}_{A_n}] = [\Phi_n]\,[\mathbf{G}'_{A_n}]\,[\Phi_n]^T \quad . \tag{21}$$

The probability density function $p_{[\mathbf{G}'_{A_n}]}([G'_n])$ of random matrix $[\mathbf{G}'_{A_n}]$, with respect to the volume element $\widetilde{d}G'_n$ (see Eq. (12)), is such that

$$p_{[\mathbf{G}'_{A_n}]}([G'_n])\,\widetilde{d}G'_n = p_{[\mathbf{G}_{A_n}]}([G_n])\,\widetilde{d}G_n \quad , \tag{22}$$

in which $p_{[\mathbf{G}_{A_n}]}([G_n])$ is defined by Eq. (15). Let $[G_n]$ and $[G'_n]$ be such that $[G_n] = [\Phi_n]\,[G'_n]\,[\Phi_n]^T$. Since $[\Phi_n]$ is a real orthogonal matrix, we deduce that $\widetilde{d}G_n = \widetilde{d}G'_n$, $\det[G_n] = \det[G'_n]$ and $\mathrm{tr}\,[G_n] = \mathrm{tr}\,[G'_n]$. From Eq. (15), we deduce that

$$p_{[\mathbf{G}_{A_n}]}([G_n])\,\widetilde{d}G_n = p_{[\mathbf{G}_{A_n}]}([G'_n])\,\widetilde{d}G'_n \quad . \tag{23}$$

From Eqs. (22) and (23), we deduce that

$$p_{[\mathbf{G}'_{A_n}]}([G'_n])\,\widetilde{d}G'_n = p_{[\mathbf{G}_{A_n}]}([G'_n])\,\widetilde{d}G'_n \quad , \tag{24}$$

which proves the invariance of random matrix $[\mathbf{G}_{A_n}]$ under real orthogonal transformations. Let $[\mathbf{A}_n]$ be the random matrix defined by Eq. (8). Using the characteristic function (see Eq. (17)), it can easily be proved that the probability distribution $P_{[\mathbf{A}'_n]}$ of the random matrix $[\mathbf{A}'_n] = [\Phi_n]^T[\mathbf{A}_n]\,[\Phi_n]$ with values in $\mathbb{M}_n^+(\mathbb{R})$ and with the mean value $[\underline{A}'_n] = [\Phi_n]^T[\underline{A}_n]\,[\Phi_n]$, is equal to the probability distribution $P_{[\mathbf{A}_n]}$ of random matrix $[\mathbf{A}_n]$ in which mean value $[\underline{A}_n]$ has to be replaced by mean value $[\underline{A}'_n]$.

2.6 Convergence property when dimension goes to infinity

For θ fixed in $\mathcal{A}$, the norm of matrix $[\mathbf{G}_{A_n}(\theta)]^{-1}$ induced by the Euclidean norm of $\mathbb{R}^n$ is defined by

$$\|[\mathbf{G}_{A_n}(\theta)]^{-1}\| = \sup_{\mathbf{q}\in\mathbb{R}^n,\,\|\mathbf{q}\|=1} \|[\mathbf{G}_{A_n}(\theta)]^{-1}\mathbf{q}\| \quad . \tag{25}$$

It should be noted that

$$\|[\mathbf{G}_{A_n}(\theta)]^{-1}\| \leq \|[\mathbf{G}_{A_n}(\theta)]^{-1}\|_F \leq \sqrt{n}\,\|[\mathbf{G}_{A_n}(\theta)]^{-1}\|\,. \tag{26}$$

We then have the following inequality,

$$\forall n \geq n_0 \quad , \quad E\{\|[\mathbf{G}_{A_n}]^{-1}\|^2\} \leq C_{\delta_A} < +\infty \quad , \tag{27}$$

in which C_{δ_A} is a positive finite constant that is independent of n but that depends on δ_A. Equation (27) means that $n \mapsto E\{\|[\mathbf{G}_{A_n}]^{-1}\|^2\}$ is a bounded function from $\{n \geq n_0\}$ into $\mathbb{R}^+$. Figure 2 shows the graph of function $n \mapsto E\{\|[\mathbf{G}_{A_n}]^{-1}\|^2\}$ for $n \geq n_0 = 2$, $\delta_A = 0.1$, 0.3 and 0.5, obtained by a Monte Carlo numerical simulation.

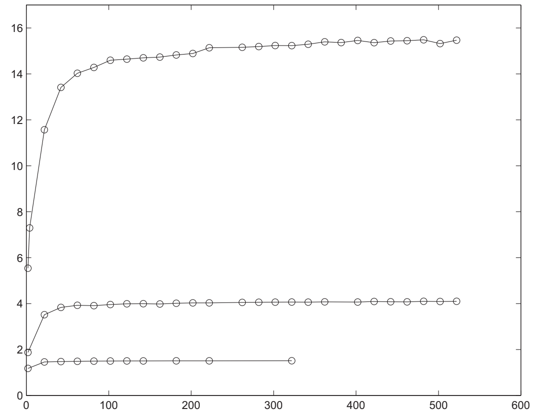

Figure 2. Graph of function $n \mapsto E\{\|[\mathbf{G}_{A_n}]^{-1}\|^2\}$ for $\delta_A = 0.1$ (lower line), 0.3 (mid line) and 0.5 (upper line), in which the horizontal axis is dimension n of the reduced model.

2.7 Algebraic representation of the random matrix

The following algebraic representation of positive-definite real random matrix $[\mathbf{G}_{A_n}]$ allows a procedure for the Monte Carlo numerical simulation of random matrix $[\mathbf{G}_{A_n}]$ to be defined. With this procedure, the numerical cost induced by the simulation is a constant that depends on dimension n but that is independent of the values of parameter δ_A. Random matrix $[\mathbf{G}_{A_n}]$ can be written as

$$[\mathbf{G}_{A_n}] = [\mathbf{L}_{A_n}]^T \, [\mathbf{L}_{A_n}] \quad , \tag{28}$$

in which $[\mathbf{L}_{A_n}]$ is an upper triangular random matrix with values in $\mathbb{M}_n(\mathbb{R})$ such that:

(1) random variables $\{[\mathbf{L}_{A_n}]_{jj'}, j \leq j'\}$ are independent;

(2) for $j < j'$, real-valued random variable $[\mathbf{L}_{A_n}]_{jj'}$ can be written as $[\mathbf{L}_{A_n}]_{jj'} = \sigma_n U_{jj'}$ in which $\sigma_n = \delta_A(n+1)^{-1/2}$ and where $U_{jj'}$ is a real-valued Gaussian random variable with zero mean and variance equal to 1;

(3) for $j = j'$, positive-valued random variable $[\mathbf{L}_{A_n}]_{jj}$ can be written as $[\mathbf{L}_{A_n}]_{jj} = \sigma_n \sqrt{2V_j}$ in which σ_n is defined above and where V_j is a positive-valued gamma random variable whose probability density function $p_{V_j}(v)$ with respect to dv is written as

$$p_{V_j}(v) = \mathbb{1}_{\mathbb{R}^+}(v) \frac{1}{\Gamma\left(\frac{n+1}{2\delta_A^2} + \frac{1-j}{2}\right)} \, v^{\frac{n+1}{2\delta_A^2} - \frac{1+j}{2}} \, e^{-v} \quad . \tag{29}$$

2.8 Probability model of a set of positive-definite symmetric real random matrices

Let us consider ν random matrices $[\mathbf{A}_n^1], \ldots, [\mathbf{A}_n^\nu]$ with values in $\mathbb{M}_n^+(\mathbb{R})$ such that for each j in $\{1, \ldots, \nu\}$, the probability density function of random matrix $[\mathbf{A}_n^j]$ satisfies Eqs. (4), (5) and (6). This means that only the mean values of the random matrices are known. Applying the maximum entropy principle, it can be proved that the probability density function $([A_n^1], \ldots, [A_n^\nu]) \mapsto p_{[\mathbf{A}_n^1],\ldots,[\mathbf{A}_n^\nu]}([A_n^1], \ldots, [A_n^\nu])$ from $\mathbb{M}_n^+(\mathbb{R}) \times \ldots \times \mathbb{M}_n^+(\mathbb{R})$ into $\mathbb{R}^+$ with respect to the measure (volume element) $\widetilde{d}A_n^1 \times \ldots \times \widetilde{d}A_n^\nu$ on $\mathbb{M}_n^S(\mathbb{R}) \times \ldots \times \mathbb{M}_n^S(\mathbb{R})$ is written as

$$p_{[\mathbf{A}_n^1],\ldots,[\mathbf{A}_n^\nu]}([A_n^1], \ldots, [A_n^\nu]) = p_{[\mathbf{A}_n^1]}([A_n^1]) \times \ldots \times p_{[\mathbf{A}_n^\nu]}([A_n^\nu]) \quad , \tag{29}$$

which means that $[\mathbf{A}_n^1], \ldots, [\mathbf{A}_n^\nu]$ are independent random matrices.

3 COMPARISONS BETWEEN THE PROPOSED THEORY AND THE GAUSSIAN ORTHOGONAL ENSEMBLE (GOE)

The theory summarized in Section 2 for symmetric positive-definite real random matrices will be called the "positive-definite" ensemble and differs from the usual Gaussian orthogonal ensemble (GOE) of random matrices which have been extensively studied in the literature (see Dyson (1962), Dyson & Mehta (1963), Mehta (1967 & 1991), Nagao & Slevin (1993), Moorthy (1995) and Katz & Sarnak (1999)) and for which an excellent synthesis is given in Mehta's book (1991). The random matrix theory has already been used for analyzing high-frequency spectral statistics of certain undamped linear vibration problems, i.e. studying the random generalized eigenvalue problem

$$[\mathbf{K}_n] \, \boldsymbol{\varphi} = \Lambda \, [\, \underline{M}_n] \, \boldsymbol{\varphi} \quad , \tag{30}$$

in which $[\,\underline{M}_n]$ is the generalized mass matrix (presently assumed to be deterministic to simplify the developments), $[\mathbf{K}_n]$ is the random generalized stiffness matrix, $\boldsymbol{\varphi}$ is the random eigenvector associated with the random eigenvalue Λ. It has been proved that, if random matrix $[\mathbf{K}_n]$ is taken in the Gaussian orthogonal ensemble (GOE), then a good estimation of the high-frequency spectral statistics can be obtained (see Bohigas *et al* (1984), Weaver (1989), Bohigas *et al* (1991), Schmit (1991), Legrand *et al* (1992)). If $[\mathbf{K}_n]$ belongs to the GOE, then the eigenvalues can be negative with a non zero probability. However, it is well known that, for an undamped linear vibration problem, the generalized mass and stiffness matrices have to be positive definite. Consequently, the eigenvalues have to be positive and cannot be negative. From a theoretical point of view, it is not coherent to use the GOE for modeling random matrix $[\mathbf{K}_n]$ for such an undamped linear vibration problem. Nevertheless, if the GOE is used in the high-frequency range and if the mean values of the considered eigenvalues are large with respect to their standard deviations, then the probability that a random eigenvalue located in the high-frequency range be negative is very small (not significant). This is the reason why the GOE can give good results for certain high-frequency undamped linear vibrations problems. This is very different when the frequency response or the transient response of the linear dynamic system considered is mainly due to the contributions of the first eigenvalues, that is to say for the low-frequency range. In addition, we have to consider damped dynamic system. For this case considered in Soize (1999, 2000, 2001a,b,c,d), it is absolutely necessary to take into account the algebraic property related to the positiveness of these random matrices (that is not the case for a random matrix belonging to the GOE). In addition, as explained in Soize (2000 & 2001a), it is not sufficient to take into account this algebraic property and the second-order moment of the Frobenius norm of their inverse has to exist (of course, such a property does not hold for the GOE).

3.1 Probability density functions of the random eigenvalues

In order to compare the two ensembles, we introduce the two following random eigenvalue problems.

A. "Positive-definite" ensemble. The first eigenvalue problem is written as

$$[\mathbf{G}_{A_n}]\,\boldsymbol{\varphi} = \Lambda\,\boldsymbol{\varphi} \quad , \tag{31}$$

in which random matrix $[\mathbf{G}_{A_n}]$ is the symmetric positive-definite $(n \times n)$ real random matrix defined in Section 2. Therefore, its probability density function is given by Eq. (15) and for all j and k in $\{1,\ldots,n\}$, the mean value and the variance of the random variable $[\mathbf{G}_{A_n}]_{jk}$ are written (see Eqs. (9) and (20)) as

$$E\{[\mathbf{G}_{A_n}]_{jk}\} = \delta_{jk} \quad , \quad V_{jk}^{G_{A_n}} = \frac{\delta_A^2}{(n+1)}(1+\delta_{jk}) \quad . \tag{32}$$

The joint probability density function of the n random eigenvalues is explicitely constructed in Soize (2000). Presently, we are interested in the probability density function of each random egenvalue for the order statistics. Let $\Lambda_1 \leq \Lambda_2 \leq \ldots \leq \Lambda_n$ be the order statistics of the random eigenvalues Λ_j for $j = 1,\ldots n$ of the random eigenvalue problem defined by Eq. (31). Let $p_{\Lambda_j}(\lambda_j)$ be the probability density function with respect to $d\lambda_j$ of random variable Λ_j. Figure 3 shows the graphs of probability density functions p_{Λ_j} for $j = 1,\ldots n$ with $\delta_A = 0.5$, $n = 30$ and estimated by the Monte Carlo numerical method with $10\,000$ samples.

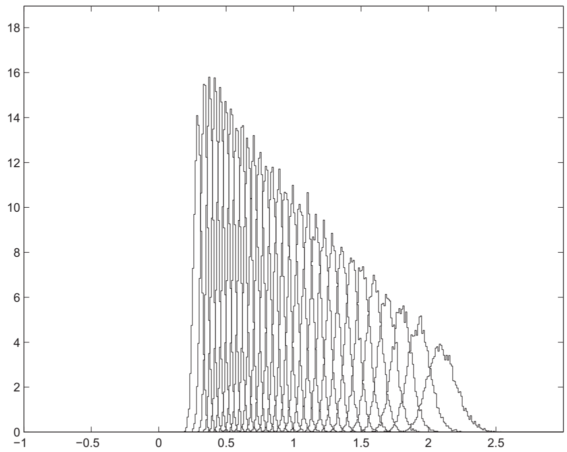

Figure 3. Graphs of probability density functions p_{Λ_j} for $j = 1,\ldots n$ corresponding to the random eigenvalues of symmetric positive-definite real random matrix $[\mathbf{G}_{A_n}]$.

B. Gaussian orthogonal ensemble. The second eigenvalue problem is written as

$$[\mathbf{H}_n]\,\boldsymbol{\psi} = \Xi\,\boldsymbol{\psi} \quad , \tag{33}$$

in which random matrix $[\mathbf{H}_n]$ is a symmetric $(n \times n)$ real random matrix which is defined by

$$[\mathbf{H}_n] = [\,I_n] + [\mathbf{H}_n^{\mathrm{GOE}}] \quad , \tag{34}$$

in which $[\mathbf{H}_n^{\mathrm{GOE}}]$ is a symmetric $(n{\times}n)$ real random matrix belonging to the Gaussian orthogonal ensemble such that its probability density function with respect to the volume element $\widetilde{d}H_n$ (given by Eq. (12)) is written as

$$p_{[\mathbf{H}_n^{\mathrm{GOE}}]}([H_n]) = C_n \times \exp\left\{-\frac{(n+1)}{4\delta_A^2}\,\mathrm{tr}\{[H_n]^2\}\right\} \quad , \tag{35}$$

in which C_n is the constant of normalization. For all j and k in $\{1,\ldots,n\}$, the mean value and the variance of random variable $[\mathbf{H}_n]_{jk}$ are written as

$$E\{[\mathbf{H}_n]_{jk}\} = \delta_{jk} \quad , \quad V_{jk}^{H_n} = \frac{\delta_A^2}{(n+1)}(1+\delta_{jk}) \quad . \tag{36}$$

From Eqs. (32) and (36), we deduce that random matrices $[\mathbf{G}_{A_n}]$ and $[\mathbf{H}_n]$ have the same mean value and the same covariance matrix. The joint probability density function of the n random eigenvalues is explicitely constructed in Mehta (1991). As above, we are interested in the probability density function of each random egenvalue for the order statistics. Let $\Xi_1 \leq \Xi_2 \leq \ldots \leq \Xi_n$ be the order statistics of the random eigenvalues Ξ_j for $j = 1,\ldots n$ of the random eigenvalue problem defined by Eq. (33). Let $p_{\Xi_j}(\xi_j)$ be the probability density function with respect to $d\xi_j$ of random variable Ξ_j. Figure 4 shows the graphs of probability density functions p_{Ξ_j} for $j = 1,\ldots n$ with $\delta_A = 0.5$, $n = 30$ and estimated by the Monte Carlo numerical simulation with $10\,000$ samples.

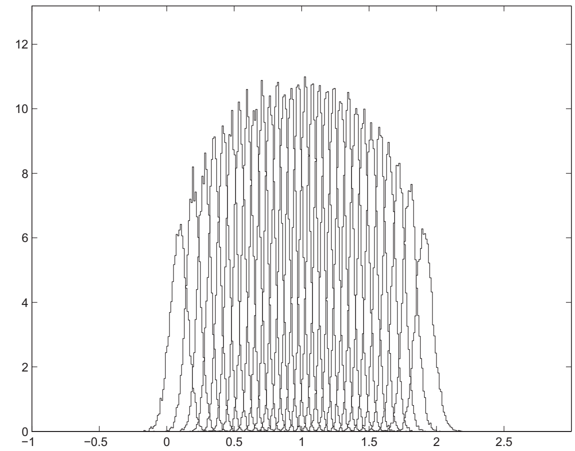

Figure 4. Graphs of probability density functions p_{Ξ_j} for $j = 1,\ldots n$ corresponding to the random eigenvalues of symmetric real random matrix $[\mathbf{H}_n]$.

3.2 Comparisons of the two ensembles for the mean values, standard deviations and variation indexes

Figures 5 to 7 correspond to $n = 30$, $\delta_A = 0.5$ and $10\,000$ samples. Figure 5 shows the graph $j \mapsto m_{\Lambda_j}$ of the mean value $m_{\Lambda_j} = E\{\Lambda_j\}$ of random eigenvalues Λ_j of random matrix $[\mathbf{G}_{A_n}]$ and the graph $j \mapsto m_{\Xi_j}$ of the mean value $m_{\Xi_j} = E\{\Xi_j\}$ of random eigenvalues Ξ_j of random matrix $[\mathbf{H}_n]$. Figure 6 shows the graph $j \mapsto \sigma_{\Lambda_j}$ of the standard deviation of random eigenvalues Λ_j of random matrix $[\mathbf{G}_{A_n}]$ and the graph $j \mapsto \sigma_{\Xi_j}$

of the standard deviation of random eigenvalues Ξ_j of random matrix $[\mathbf{H}_n]$. Finally, Figure 7 shows the graph $j \mapsto \sigma_{\Lambda_j}/m_{\Lambda_j}$ of the variation index of random eigenvalues Λ_j of random matrix $[\mathbf{G}_{A_n}]$ and the graph $j \mapsto \sigma_{\Xi_j}/m_{\Xi_j}$ of the variation index of random eigenvalues Ξ_j of random matrix $[\mathbf{H}_n]$.

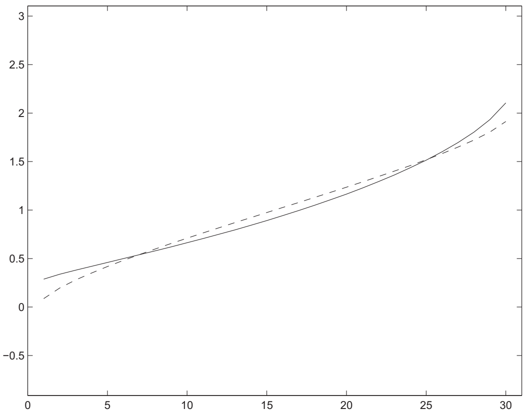

Figure 5. Mean values of the random eigenvalues for the two ensembles. Graphs $j \mapsto m_{\Lambda_j}$ (solid line) and $j \mapsto m_{\Xi_j}$ (dashed line).

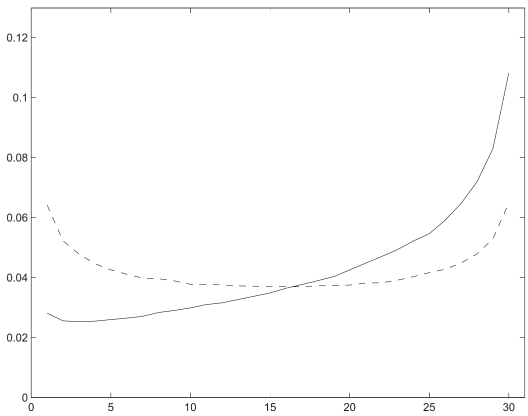

Figure 6. Standard deviation of the random eigenvalues for the two ensembles. Graphs $j \mapsto \sigma_{\Lambda_j}$ (solid line) and $j \mapsto \sigma_{\Xi_j}$ (dashed line).

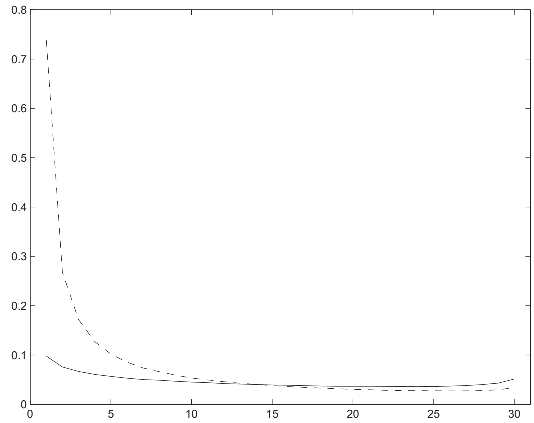

Figure 7. Variation index of the random eigenvalues for the two ensembles. Graphs $j \mapsto \sigma_{\Lambda_j}/m_{\Lambda_j}$ (solid line) and $j \mapsto \sigma_{\Xi_j}/m_{\Xi_j}$ (dashed line).

3.3 Analyzing the differences between the two ensembles

(1) Figure 4 shows that the probability density functions of the first eigenvalues of random matrix $[\mathbf{H}_n]$ are not zero for negative values and consequently, all the random eigenvalues are not positive almost surely. Such a probabilistic model is then wrong because the corresponding dynamic system is not stable almost surely. Random matrix $[\mathbf{K}_n]$ being defined on probability space $(\mathcal{A}, \mathcal{T}, P)$, this means that there exists a subset $\mathcal{A}_0 \subset \mathcal{A}$ with $P(\mathcal{A}_0) \neq 0$, such that for all $\theta \in \mathcal{A}_0$, the solution $\{\mathbf{Q}^n(t,\theta), t \geq 0\}$ of the Cauchy problem $[\underline{M}_n]\,\ddot{\mathbf{Q}}^n(t,\theta) + [\mathbf{K}_n(\theta)]\,\mathbf{Q}^n(t,\theta) = 0$ for $t \geq 0$ with the initial conditions $\mathbf{Q}^n(0,\theta) = \mathbf{q}_0 \neq 0$ and $\dot{\mathbf{Q}}^n(0,\theta) = 0$, is not a bounded function. In opposite, as shown in Figure 3, all the eigenvalues of random matrix $[\mathbf{G}_{A_n}]$ are positive almost surely and consequently, the above dynamic system is stable almost surely.

(2) The second important difference is related to the standard deviation of the random eigenvalues. In structural dynamics, it is known that random uncertainties increase with frequency and consequenly, the standard deviation has to increase with frequency. In addition, this kind of quantities has to be a constant as a function of $\Delta f/f$ in which f is the frequency and Δf the bandwith. Figure 7 which is related to variation indexes $j \mapsto \sigma_{\Lambda_j}/m_{\Lambda_j}$ and $j \mapsto \sigma_{\Xi_j}/m_{\Xi_j}$ shows that this fundamental property holds for random matrix $[\mathbf{G}_{A_n}]$ in all the frequency range, but is not true for random matrix $[\mathbf{H}_n]$ (the GOE) in the low- and med-frequency ranges.

4 NONPARAMETRIC MODEL OF RANDOM UNCERTAINTIES FOR LINEAR AND NONLINEAR DYNAMIC SYSTEMS

4.1 Introduction of the mean finite element model

We consider a nonlinear dynamic system constituted of a three-dimensional damped fixed structure around a static equilibrium configuration considered as a natural state without prestresses and subjected to an external load. The basic finite element model of this nonlinear dynamic system is called the "mean finite element model" (the underlined quantities refer to this "mean finite element model") and leads to the following nonlinear differential equation,

$$[\underline{\mathbb{M}}]\,\ddot{\underline{\mathbf{y}}}(t) + [\underline{\mathbb{D}}]\,\dot{\underline{\mathbf{y}}}(t) + [\underline{\mathbb{K}}]\,\underline{\mathbf{y}}(t) + \mathbf{f}_{\mathrm{NL}}(\underline{\mathbf{y}}(t), \dot{\underline{\mathbf{y}}}(t)) = \mathbf{f}(t)\,, \quad (37)$$

in which $\underline{\mathbf{y}} = (\underline{y}_1, \ldots, \underline{y}_m)$ is the unknown time response vector of the m DOFs (displacements and/or rotations); $\dot{\underline{\mathbf{y}}}$ and $\ddot{\underline{\mathbf{y}}}$ are the velocity and acceleration vectors respectively; $\mathbf{f}(t) = (f_1(t), \ldots, f_m(t))$ is the known external load vector of the m inputs (forces and/or moments); $[\underline{\mathbb{M}}]$, $[\underline{\mathbb{D}}]$ and $[\underline{\mathbb{K}}]$ are the mass, damping and stiffness matrices of the linear part of the model, which are positive-definite symmetric $(m \times m)$ real matrices; $(\mathbf{y}, \mathbf{z}) \mapsto \mathbf{f}_{\mathrm{NL}}(\mathbf{y}, \mathbf{z})$ is a nonlinear mapping from $\mathbb{R}^m \times \mathbb{R}^m$ into $\mathbb{R}^m$ modeling additional nonlinear damping and restoring forces such that $\mathbf{f}_{\mathrm{NL}}(0,0) = 0$. The linear case can be derived from Eq. (37) in taking $\mathbf{f}_{\rm NL} = 0$.

4.2 Introduction of the mean reduced models

The generalized eigenvalue problem associated with the mean mass and stiffness matrices of the mean finite element model is written as $[\mathbb{K}]\underline{\varphi} = \underline{\lambda}[\mathbb{M}]\underline{\varphi}$. Since $[\mathbb{K}]$ is a positive-definite matrix, we have $0 < \underline{\lambda}_1 \leq \underline{\lambda}_2 \leq \ldots \leq \underline{\lambda}_m$ and the associated elastic modes $\{\underline{\varphi}_1, \underline{\varphi}_2, \ldots\}$ are such that $< [\mathbb{M}]\underline{\varphi}_\alpha, \underline{\varphi}_\beta >= \underline{\mu}_\alpha\, \delta_{\alpha\beta}$ and $< [\mathbb{K}]\underline{\varphi}_\alpha, \underline{\varphi}_\beta >= \underline{\mu}_\alpha\, \underline{\omega}_\alpha^2\, \delta_{\alpha\beta}$ in which $\underline{\omega}_\alpha = \sqrt{\underline{\lambda}_\alpha}$ is the eigenfrequency of elastic mode $\underline{\varphi}_\alpha$ whose normalization is defined by the generalized mass $\underline{\mu}_\alpha$. The mean reduced models of the dynamic systems whose mean finite element models are defined by Eq. (37) is obtained by constructing the projection of the mean finite element models on the subspace V_n of $\mathbb{R}^m$ spanned by $\{\underline{\varphi}_1, \ldots, \underline{\varphi}_n\}$ with $n \ll m$. Let $[\underline{\Phi}_n]$ be the $(m \times n)$ real matrix whose columns are vectors $\{\underline{\varphi}_1, \ldots, \underline{\varphi}_n\}$. The generalized force $\mathbf{F}^n(t)$ is an $\mathbb{R}^n$-vector such that $\mathbf{F}^n(t) = [\underline{\Phi}_n]^T\, \mathbf{f}(t)$. The generalized mass, damping and stiffness matrices $[\underline{M}_n]$, $[\underline{D}_n]$ and $[\underline{K}_n]$ are positive-definite symmetric $(n\times n)$ real matrices such that $[\underline{M}_n]_{\alpha\beta} = \underline{\mu}_\alpha\, \delta_{\alpha\beta}$, $[\underline{D}_n]_{\alpha\beta} =< [\mathbb{D}]\,\underline{\varphi}_\beta, \underline{\varphi}_\alpha >$ and $[\underline{K}_n]_{\alpha\beta} = \underline{\mu}_\alpha\, \underline{\omega}_\alpha^2\, \delta_{\alpha\beta}$, in which, generally, $[\underline{D}_n]$ is a full matrix. Consequently, the mean reduced model of the nonlinear dynamic system is written as the projection $\underline{\mathbf{y}}^n$ of $\underline{\mathbf{y}}$ on V_n can be written as $\underline{\mathbf{y}}^n(t) = [\underline{\Phi}_n]\, \underline{\mathbf{q}}^n(t)$ in which the vector $\underline{\mathbf{q}}^n(t) \in \mathbb{R}^n$ of the generalized coordinates verifies the mean nonlinear differential equation,

$$[\underline{M}_n]\, \ddot{\underline{\mathbf{q}}}^n(t) + [\underline{D}_n]\, \dot{\underline{\mathbf{q}}}^n(t) + [\underline{K}_n]\, \underline{\mathbf{q}}^n(t) + \mathbf{F}_{\rm NL}^n(\underline{\mathbf{q}}^n(t), \dot{\underline{\mathbf{q}}}^n(t)) = \mathbf{F}^n(t)\ ,\ \forall t \geq 0\ , \qquad (38)$$

where, for all $\mathbf{q}$ and $\mathbf{p}$ in $\mathbb{R}^n$,

$$\mathbf{F}_{\rm NL}^n(\mathbf{q}, \mathbf{p}) = [\underline{\Phi}_n]^T\, \mathbf{f}_{\rm NL}([\underline{\Phi}_n]\, \mathbf{q}, [\underline{\Phi}_n]\, \mathbf{p}) \quad . \qquad (39)$$

4.3 Nonparametric model of random uncertainties

The principle of construction of the nonparametric model of random uncertainties for the linear and nonlinear dynamic systems whose mean finite element model is defined by Eq. (37), is given in Section 1. It consists in modeling the generalized mass, damping and stiffness matrices of the reduced model (see Eq. (38)) by random matrices $[\mathbf{M}_n]$, $[\mathbf{D}_n]$ and $[\mathbf{K}_n]$. If the nonlinear forces are uncertain, a usual parametric model can be used for these nonlinear forces. In this case, a nonparametric-parametric mixed formulation can easily be constructed.

The construction of the probability model of random matrices $[\mathbf{M}_n]$, $[\mathbf{D}_n]$ and $[\mathbf{K}_n]$ is based on the available information defined by constraints (C1), (C2) and (C3) introduced in Section 1, that is to say by Eqs. (4), (5) and (6)). Consequently, random matrices $[\mathbf{M}_n]$, $[\mathbf{D}_n]$ and $[\mathbf{K}_n]$ are defined on probability space $(\mathcal{A}, \mathcal{T}, \mathcal{P})$, with values in $\mathbb{M}_n^+(\mathbb{R})$, whose mean values are such that $E\{[\mathbf{M}_n]\} = [\underline{M}_n]$, $E\{[\mathbf{D}_n]\} = [\underline{D}_n]$ and $E\{[\mathbf{K}_n]\} = [\underline{K}_n]$ and such that $E\{\|[\mathbf{M}_n]^{-1}\|_F^2\} < +\infty$, $E\{\|[\mathbf{D}_n]^{-1}\|_F^2\} < +\infty$ and $E\{\|[\mathbf{K}_n]^{-1}\|_F^2\} < +\infty$. From Section 2.8, we deduce that random matrices $[\mathbf{M}_n]$, $[\mathbf{D}_n]$ and $[\mathbf{K}_n]$ are independent, each one being a random matrix for which the probability model in given in Section 2. Consequently, we have $[\mathbf{M}_n] = [\underline{L}_{M_n}]^T\, [\mathbf{G}_{M_n}]\, [\underline{L}_{M_n}]$, $[\mathbf{D}_n] = [\underline{L}_{D_n}]^T\, [\mathbf{G}_{D_n}]\, [\underline{L}_{D_n}]$ and $[\mathbf{K}_n] = [\underline{L}_{K_n}]^T\, [\mathbf{G}_{K_n}]\, [\underline{L}_{K_n}]$. The parameters δ_M, δ_D and δ_K allowing the dispersion of random matrices $[\mathbf{M}_n]$, $[\mathbf{D}_n]$ and $[\mathbf{K}_n]$ to be controlled are defined by Eq. (10). The probability distribution of each random matrix $[\mathbf{G}_{M_n}]$, $[\mathbf{G}_{D_n}]$ or $[\mathbf{G}_{K_n}]$ is defined in Sections 2.2 and 2.3.

CASE 1: The stochastic frequency response of the linear dynamic system with a nonparametric probabilistic model of random uncertainties, whose mean reduced model is defined by Eq. (37) with $\mathbf{f}_{\rm NL} = 0$, is the stochastic process $\widehat{\mathbf{Y}}_{\rm LF}^n(\omega)$ indexed by $\mathbb{R}$, with valued in $\mathbb{C}^m$ such that $\widehat{\mathbf{Y}}_{\rm LF}^n(\omega) = [\underline{\Phi}_n]\, \widehat{\mathbf{Q}}_{\rm LF}^n(\omega)$ in which, for all ω fixed in $\mathbb{R}$, the random variable $\widehat{\mathbf{Q}}_{\rm LF}^n(\omega)$ with values in $\mathbb{C}^n$ is such that

$$(-\omega^2[\mathbf{M}_n] + i\omega\, [\mathbf{D}_n] + [\mathbf{K}_n])\, \widehat{\mathbf{Q}}_{\rm LF}^n(\omega) = \widehat{\mathbf{F}}^n(\omega)\,. \qquad (40)$$

CASE 2: The stochastic transient response of the nonlinear dynamic system with a nonparametric probabilistic model of random uncertainties, whose mean reduced model is defined by Eq. (37), is the stochastic process $\mathbf{Y}^n(t)$, indexed by $\mathbb{R}^+$, with values $\mathbb{R}^m$, such that $\mathbf{Y}^n(t) = [\underline{\Phi}_n]\, \mathbf{Q}^n(t)$ in which the stochastic process $\mathbf{Q}^n(t)$, indexed by $\mathbb{R}^+$, with values $\mathbb{R}^n$, is such that

$$[\mathbf{M}_n]\, \ddot{\mathbf{Q}}^n(t) + [\mathbf{D}_n]\, \dot{\mathbf{Q}}^n(t) + [\mathbf{K}_n]\, \mathbf{Q}^n(t) + \mathbf{F}_{\rm NL}^n(\mathbf{Q}^n(t), \dot{\mathbf{Q}}^n(t)) = \mathbf{F}^n(t)\ ,\ \forall t \geq 0\ , \qquad (41)$$

with the initial conditions, $\mathbf{Q}^n(0) = 0$ and $\dot{\mathbf{Q}}^n(0) = 0$.

4.4 Stochastic solution as a second-order stochastic process

CASE 1 of Section 4.3. Using the fundamental property defined by Eq. (27) for each random matrix $[\mathbf{G}_{M_n}]$, $[\mathbf{G}_{D_n}]$, $[\mathbf{G}_{K_n}]$ and a development similar to the proof given in Soize (2001a), it can be proved that, if $\int_{\mathbb{R}} \|\mathbf{f}(t)\|^2 dt = \frac{1}{2\pi} \int_{\mathbb{R}} \|\widehat{\mathbf{f}}(\omega)\|^2 d\omega < +\infty$, then, we have

$$E\{\int_{\mathbb{R}} \|i\omega \widehat{\mathbf{Y}}_{\rm LF}^n(\omega)\|^2 d\omega\} \leq C_0 \int_{\mathbb{R}} \|\widehat{\mathbf{f}}(\omega)\|^2 d\omega < +\infty \quad , \qquad (42)$$

in which C_0 is a positive constant independent of n and ω, and $\|\,.\,\|$ denotes the Euclidean or the Hermitian norm in $\mathbb{R}^n$ or $\mathbb{C}^n$.

CASE 2 of Section 4.3. For any $T > 0$, it is proved (see Soize (2001d)) that, under reasonable assumptions concerning the nonlinear damping and restoring forces and if $\int_0^T \|\mathbf{f}(t)\|^2 dt < +\infty$, then for all t in $[0\,,T]$, we have

$$E\{\|\mathbf{Y}^n(t)\|^2\} \leq C_3 < +\infty\,,\ E\{\|\dot{\mathbf{Y}}^n(t)\|^2\} \leq C_4 < +\infty\,. \qquad (43)$$

in which C_3 and C_4 are positive constants that are independent of n and t.

4.5 Construction of the stochastic solution

The stochastic solution of Eq. (40) or (41) can easily be constructed using the Monte Carlo numerical simulation, the samples of random matrix $[\mathbf{A}_n]$, in which $[\mathbf{A}_n]$ represents random matrices $[\mathbf{M}_n]$, $[\mathbf{D}_n]$ or $[\mathbf{K}_n]$, being constructed by using Eqs. (8) and (28). It should be noted that the numerical cost is low with such a method because Eq. (40) or (41) correspond to a stochastic reduced model with $n \ll m$.

5 EXAMPLES

5.1 CASE 1. Stochastic frequency response of a linear dynamic system

We consider the stochastic frequency response of the linear dynamic system studied in (Chebli & Soize 2001; Soize and Chebli 2002) and corresponding to Eq. (40), for which the Craig-Bampton (1968) dynamic substructuring method is used. Such an approach allows the nonparametric model to be used when random uncertainties are not homogeneous in the structure. The system under consideration is a nonhomogeneous structure constituted of a rectangular, homogeneous, isotropic thin plate in bending mode, simply supported with 2 point masses and 3 springs (see Figure 8). We consider the cross-frequency response function whose input is the point force and the output is the normal displacement defined in Figure 8.

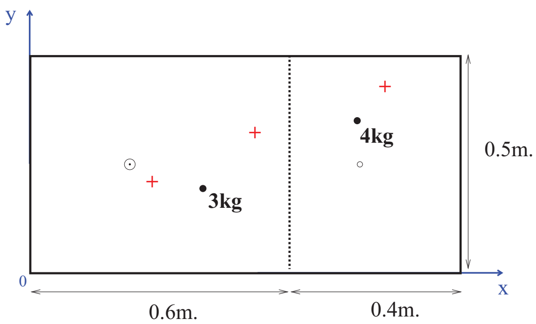

Figure 8. Nonhomogeneous structure: rectangular plate in bending mode with 2 point masses (•) and 3 springs (+). Excitation: point force (⊙). Response: normal displacement (○).

Two substructures are considered (see Figure 8). The mean finite element model is constituted of four-node square plate finite elements leading, respectively, to 8840 degrees of freedom (DOFs) and 5860 DOFs for the two substructures, and to 149 DOFs for the coupling interface. The eigenfrequencies of the complete structure, calculated with the mean finite element model are such that $\underline{\nu}_1 = 2.6\,Hz$ and $\underline{\nu}_{35} = 106.4\,Hz$. The dispersion parameters for the random generalized mass, damping and stiffness matrices are $\delta_M = \delta_D = \delta_K = 0.1$. The Monte Carlo numerical simulation is carried out with 500 samples. Figure 9 shows the mean-square convergence of the stochastic displacement field with respect to the dimension n of the reduced model of each substructure (in this example, the dimension of the reduced model is the same for the two substructures). It can be seen that convergence is reached for $n = 25$.

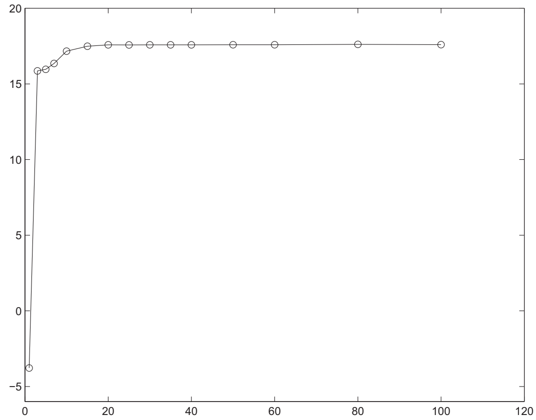

Figure 9. Mean-square convergence of the stochastic displacement field (vertical axis) with respect to the dimension of the reduced models (horizontal axis).

Figure 10 is related to the modulus in dB of the stochastic cross-frequency response function defined above. The thin dashed line represents the mean reponse. The upper and lower thick solid lines represent the confidence region for a probability level of 0.95. The upper and lower thin solid lines represent the extreme value statistics. It should be noted that the role plays by the random uncertainties increases with the frequency.

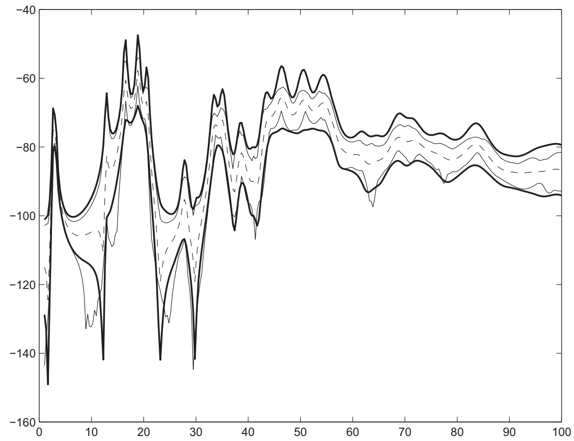

Figure 10. Modulus in dB of the stochastic cross-frequency response function in frequency band $[1\,,100]$ Hz (horizontal axis). Mean response (thin dashed line), confidence region corresponding to a probabilty level of 0.95 (thick solid lines), extreme value statistics (thin solid lines).

5.2 CASE 2. Stochastic transient response of a nonlinear dynamic system

In this section, we consider the stochastic transient response of the nonlinear dynamic system studied in Soize (2001d) and corresponding to Eq. (41). The nonlinear dynamic system is composed of a linear thin plane in bending mode with a nonlinearity due to a nonlinear restoring force induced by two stops modeled by high stiffness symmetric barriers which limit the vibration amplitudes of the plate. The plate is rectangular, homogeneous, isotropic, in bending mode, with constant thickness $4\times10^{-4}\,m$, width $0.40\,m$, length $0.50\,m$, mass density $7800\,kg/m^3$, Young's modulus $2.1\times10^{11}\,N/m^2$ and Poisson ratio 0.29 . This plate is simply supported on 3 edges and free on the fourth edge corresponding to $x_2 = 0$ (see Figure 11). To this plate are attached one point mass having a mass of $4\,kg$ and one spring having a stiffness coefficient $k = 2.388 \times 10^7\,N/m$. Consequently, the dynamic system is not homogeneous. The two stops are located at the free edge. The plate is free between the stops $[-0.002\,m\; 0.002\,m]$ and the stiffness of the two symmetric barriers is $25000\,N/m$.

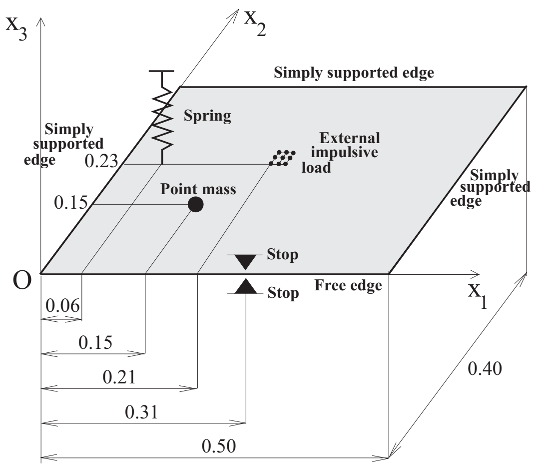

Figure 11. Geometry of the structure

The mean finite element model of the plate is composed of 2000 four-node square plate finite elements and there are $m = 6009$ degrees of freedom. The eigenfrequencies calculated with the mean finite element model of the linear plate without the stops are such that $\underline{\nu}_1 = 1.94$, $\underline{\nu}_2 = 10.28$, $\underline{\nu}_3 = 15.47$, $\ldots$, $\underline{\nu}_8 = 53.5$, $\underline{\nu}_9 = 66.1$, $\underline{\nu}_{10} = 68.9$, $\ldots$, $\underline{\nu}_{30} = 198.3$, $\underline{\nu}_{31} = 206.0$, $\underline{\nu}_{32} = 208.9$, $\ldots$, $\underline{\nu}_{50} = 330.9$, $\underline{\nu}_{51} = 336.3$, $\ldots$, $\underline{\nu}_{100} = 670.8$, $\underline{\nu}_{120} = 817.6$ Hz.

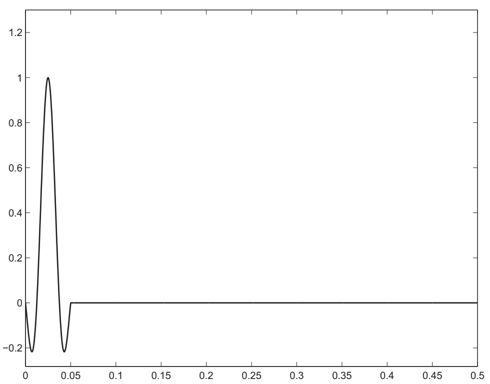

Figure 12. Graph of wave impulse function as a function of time

The excitation is an impulsive load defined in Figure 11 whose impulse function is defined in Figure 12. The main part of the energy of this impulse function is distributed over the $[0\,,60]$ Hz frequency band in which there are 8 elastic modes of the mean linearized dynamic system. The damping matrix $[\underline{\mathbb{D}}]$ of the mean finite element model is written as $[\underline{\mathbb{D}}] = a\,[\underline{\mathbb{M}}] + b\,[\underline{\mathbb{K}}]$ in which a and b are defined by $a = 2\,\underline{\xi}\,\Omega_{\text{max}}\,\Omega_{\text{min}}/(\Omega_{\text{max}} + \Omega_{\text{min}})$, $b = 2\,\underline{\xi}/(\Omega_{\text{max}} + \Omega_{\text{min}})$ in which $\underline{\xi} = 0.04$, $\Omega_{\text{min}} = 2\pi \times 2\,rad/s$ and $\Omega_{\text{max}} = 2\pi \times 100\,rad/s$. Below, we are interested in the normalized response spectrum with respect to $g = 9.81\,m/s^2$, over the $[1\,,200]\,Hz$ frequency band and for the DOF corresponding to the normal displacement of the mesh node located at coordinates $x_1 = 0.37$, $x_2 = 0.15$. The dispersion parameters for the random generalized mass, damping and stiffness matrices are $\delta_M = \delta_D = \delta_K = 0.2$.

The nonlinear transient response of the structure with random uncertainties is calculated by using the Monte Carlo numerical simulation method with a maximum of 500 samples. For given generalized mass, damping and stiffness matrices , the nonlinear evolution problem defined by Eq. (41) is solved by using the Newmark implicit step-by-step integration scheme and an additional numerical iteration procedure for solving the nonlinear algebraic equations at each time step. The value of the time-step size is $\Delta t = 1/2000\,s$ and the number of time steps is 8000. Convergence with respect to the dimension n of the reduced model and the number n_s of samples used in the Monte Carlo numerical method, is studied by constructing the following function,

$$\text{Conv}(n_s, n) = \left\{ \frac{1}{n_s} \sum_{k=1}^{n_s} \int_0^T \|\mathbf{Q}^n(t, \theta_k)\|^2\, dt \right\}^{1/2} \quad .$$

Figure 13 displays the graphs of functions $n_s \mapsto \log_{10}\{\text{Conv}(n_s, n)\}$ from $\{1, 2, \ldots, 500\}$ into $\mathbb{R}$, for $n = 10, 20, 30, 50, 100$. Convergence with respect to n and n_s is obtained for $n = 50$ and $n_s = 300$.

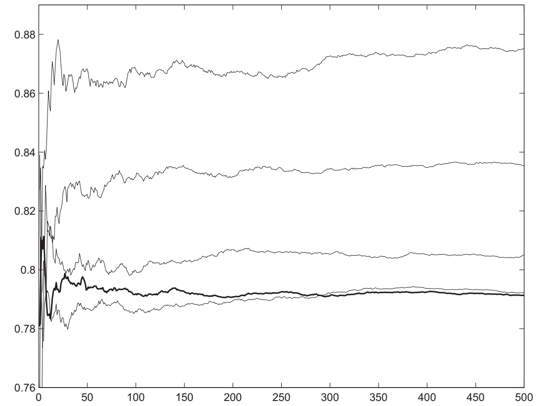

Figure 13. Convergence in the nonlinear case. Graphs of functions $n_s \mapsto \log_{10}\{\text{Conv}(n_s, n)\}$ for $n = 10, 20$ and 30 (three upper thin solid lines, for $n = 50$ (lower thin solid line) and for $n = 100$ (lower thick solid line).

Figure 14 corresponds to the transient response of the nonlinear dynamic system with random uncertainties. This figure corresponds to the base 10 logarithm of the random normalized response spectrum for the observation defined above (vertical axis) as a function of the base 10 logarithm of the frequency in Hertz for the $[1\,,200]\,Hz$ frequency band. The mid irregular thin solid line represents the deterministic response of the mean finite element model. The mid smoothed thin solid line represents the mean value of the model with random uncertainties. The lower and upper thick solid lines represent the lower and upper envelopes of the confidence region corresponding to a probability level of 0.95; this confidence region is estimated by using the Chebychev inequality. The lower and upper thin solid lines correspond to the extreme value statistics. It can be seen that the confidence region gives a good estimation of the extreme value statistics. Figure 15 is similar to Figure 14 but for the transient response of the linearized dynamic system (without the stops). The comparison of Figure 14 with Figure 15 allows the following conclusions to be obtained. For the linearized dynamic system (Figure 15), since the energy of the impulse input is concentrated in the $[0, 60]Hz$ frequency band and since there are 8 eigenfrequencies in this frequency band, then the dynamics of the transient output is modal type and is concentrated in the same $[0,60]Hz$ frequency band. For this linear case, the size of the confidence region does not increase in the $[60,200]Hz$ frequency band when frequency is increasing. In opposite, for the nonlinear dynamic system, the energy of the impulse input, which is always concentrated in the $[0,60]Hz$ frequency band, is spread out over the $[0,200]Hz$ broad frequency band due to the nonlinearity in the dynamic system. This energy is sufficient for exciting the eigenmodes whose eigenfrequencies belong to the $[60,200]Hz$ frequency band. These modes are sensitive to random uncertainties and it should be noted that the size of the confidence region increases in the $[60,200]Hz$ frequency band when frequency is increasing. This means that, for the nonlinear dynamic system studied in the example presented, the role plays by random uncertainties increases in the upper part of the frequency band which is not directly excited by the impulse input.

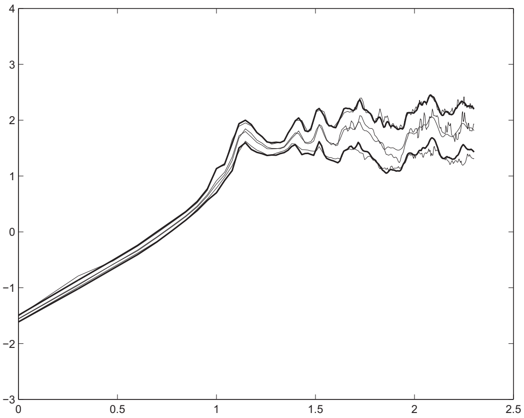

Figure 14. Nonlinear case: $\log 10$ of the random normalized response spectrum (vertical axis) as a function of $\log 10$ of the frequency for the $[1\,,200]\,Hz$ frequency band. Deterministic response of the mean model (mid irregular thin solid line). Mean value of the model with random uncertainties (mid smoothed thin solid line). Lower and upper envelopes of the confidence region (lower and upper thick solid lines). Extreme value statistics (lower and upper thin solid lines).

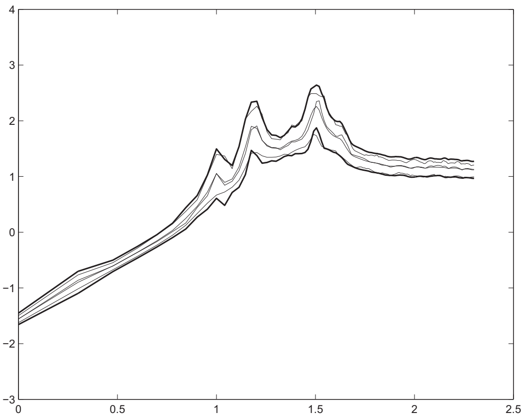

Figure 15. Linear case: similar to Figure 14 for the linear dynamic system (without the stops).

6 CONCLUSIONS

The nonparametric probabilistic model of random uncertainties recently introduced by the author allows linear and nonlinear dynamic systems to be analyzed. Nonhomogeneous uncertainties can be modeled with such a nonparametric approach using dynamic substructuring methods. For this nonparametric approach, the information used does not require the description of the local parameters of the mechanical model. The parametric approaches existing in literature are very useful when the number of uncertain parameters is small and when the probabilistic model can be constructed for the set of parameters considered. The nonparametric approach proposed is useful when the number of uncertain parameters is high or when the probabilistic model is difficult to construct for the set of parameters considered. In addition, the parametric approaches do not allow the model uncertainties to be taken into account (because a parametric approach is associated with a fixed model exhibiting some parameters), whereas the nonparametric approach proposed allows the model uncertainties to be taken into account. For nonlinear dynamic systems, the nonparametric model is only applied to the linear part of the reduced model. This nonparametric model of random uncertainties can simultaneously be used with the usual parametric model of random uncertainties which allows random uncertainties on nonlinear damping and restoring forces to be taken into account. In this case, a nonparametric-parametric model of random uncertianties has to be considered. Concerning this nonparametric model, the probability distribution of each random generalized matrix of the linear part of the random reduced model depends only on two parameters: the mean generalized matrix associated with the mean finite element model and corresponding to the design model, and a scalar parameter δ whose values has to be fixed by the designer in the interval $[0\,,1[$ in order to give the dispersion level attached to the random generalized matrix. It seems clear that parameter δ is a global parameter resulting from expertise or identification. Concerning the construction of the confidence region by using the Chebychev inequality, this method seems to give a good approximation for the extreme value statistics. This result, which is not theoretically proved, is interesting because the convergence speed of the variance estimator is faster than the convergence speed of the extreme value statistics (with respect to the number of samples required in the Monte Carlo numerical simulation).

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