Interfacial models in viscoplastic composites materials

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Abstract

The aim of the present work is to extend the concept of interphase and equivalent imperfect interface in the context of viscoplasticity. The interphase takes the form of a thin curved layer of constant thickness, made up of a rigid viscoplastic material located between two other surrounding materials. We aim at representing this interphase by an interface with appropriately devised interface conditions. To reach this objective, a Taylor expansion of the relevant physical fields in the thin region is used. It is shown that, depending of the degree of stiffness of the layer with respect to the neighboring media, this interphase can be replaced by an idealized imperfect interface involving the jump of the velocity field or the traction vector. The first kind of interface model, applicable to soft interphases, is the "spring-type" interface across which the traction are continuous but the velocity field exhibits a discontinuity which is given in term of the traction by a power-law type relation. Moreover, it is shown that the constant of the model can be expressed in terms of the material parameters of the interphase. When the interphase is stiffer than the two surrounding media, one obtain a "stress-type" interface across which the velocity is continuous and a jump condition for the traction is given by a generalization of the so-called Young Laplace model to viscoplastic solids.

Key words: Imperfect Interface, Interphase, Viscoplasticity, Plasticity, Composites.

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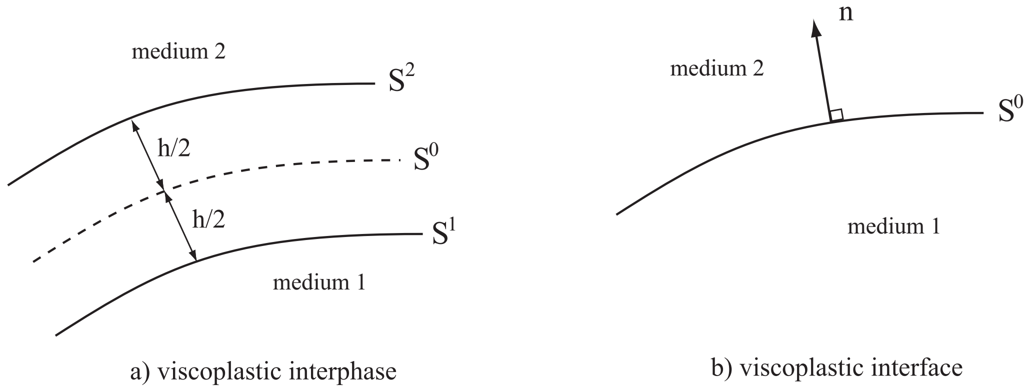
1 Introduction

The properties of a solid can be significantly affected at or near an interface, as it has been shown from atomistic simulations [16], [7,8]. In the last decades, many works deal about the incorporation of these interfacial effects into the overall properties of elastic composites (see for instance Hashin [12], Chen et al. [6], Duan et al. [10]).
Two kinds of imperfect interface models are commonly used:
- the "spring interface" model for which the displacement field is discontinuous and given in terms of the traction vector.
- the "stress interface" model [11,18], in which the jump of the traction vector is given by a generalized Young Laplace equation.
A rigorous methodology of construction of imperfect interfaces models has been firstly proposed by Sanchez-Palencia [20], Pham Huy and Sanchez-Palencia [17] for thermal conduction problem. It consists in replacing a thin interphase by an idealized imperfect interface. This approach has been later generalized in the case of linear elasticity by Karlbring and Mochvan [14,15], Bövik [5], Hashin [13] and Benveniste et al. [2,3] (see [2,19] for a comprehensive list of publications on the subject). In particular, it has been found that the "spring" or "stress" type interfaces appears as the two limit cases of stiff or soft interphases.
Recently, some works dealt about the consideration of imperfect interface in the context of non linear composites materials [21–23], [9]. For instance, Dormieux et al. [9] postulate a plastic interface model to investigate the effect of interfacial stress on the yield strength of nanoporous materials.
In the present study, we derive interface models for viscoplastic materials. To reach this objective, we propose to generalize the Taylor expansion approach initiated by Bövik [5] and formalized in Benveniste [3]. A thin viscoplastic three dimensional curved layer between two surrounding media is considered. This interphase is replaced by an idealized interface involving the jump of both velocity field and traction vector. The two particular cases of a stiff or soft interphase are investigated and allow us to derive a generalization of the "spring" and "stress" interface models in the non linear context. Finally, it is shown that the interface model postulated in [9] appears as a special case of our approach once proper restrictions are applied on the magnitude of the material properties of the interphase.

2 Statement of the problem

Consider a three dimensional medium constituted of two media separated by a thin layer, called interphase or coating, whose thickness is denoted by h. The thin interphase is delimited by two parallel perfectly bonded interfaces,

denoted by S_1 and S_2, the "middle" surface, at half distance between S_1 and S_2, being denoted by S_0 (see figure 1).
All points within the interphase are represented in the curvilinear coordinates system (x_1, x_2, x_3). The iso-x_3 surfaces define surfaces which are parallel to S_0. The surface $x_3 = h/2$ defines the interface between coating and medium "2", namely S_2 whereas the surface $x_3 = -h/2$ defines the interface S_1. The interphase is assumed to be made up of a homogeneous rigid-viscoplastic material whose properties differ from the ones of the two surrounding media.



The equations of the model are summarized below:

$$
\begin{aligned}
&\operatorname{div}(\boldsymbol{\sigma}) = 0 \\
&\boldsymbol{d} = \frac{1}{2}\left[\operatorname{grad}(\underline{v}) + \operatorname{grad}^T(\underline{v})\right] \\
&\boldsymbol{d} = \frac{\partial \psi}{\partial \boldsymbol{\sigma}}, \quad \psi(\boldsymbol{\sigma}) = \frac{\sigma_0 \dot{\varepsilon}_0}{1+n}\left(\frac{\sigma_{eq}}{\sigma_0}\right)^{1+n}, \quad \operatorname{tr}(\boldsymbol{d}) = 0
\end{aligned}
\tag{1}
$$

or, dually:

$$
\boldsymbol{\sigma} = p\boldsymbol{i} + \frac{2\sigma_0}{3\dot{\varepsilon}_0}\left(\frac{d_{eq}}{\dot{\varepsilon}_0}\right)^{m-1}\boldsymbol{d} \tag{2}
$$

In the above expressions, $\underline{v}$ is the velocity field, $\boldsymbol{d}$ and $\boldsymbol{\sigma}$ are the local strain rate and stress tensors. By p, we denote the local pressure, $p = \operatorname{tr}(\boldsymbol{\sigma})/3$, $\boldsymbol{i}$ represent the second order identity tensor. σ_{eq} and d_{eq} are the equivalent "von Mises" stress and strain rate and σ_0 and $\dot{\varepsilon}_0$ are two material parameters. The two exponents n and m are classically related by $n = 1/m$ and are defined such that: $0 \leq m \leq 1$ and $1 \leq n \leq +\infty$. The particular case of an incompressible linear viscous material corresponds to $n = m = 1$ whereas the limited case $n = +\infty$ (and then $m = 0$) corresponds to a rigid ideally-plastic material.
We aim now at replacing this interphase by an idealized surface and at deriving the constitutive equations giving the jump of the velocity and of the traction

across S_0. As previously mentioned, the approach uses the Taylor expansion and is detailed in section 4.

3 Preliminaries

Let us introduce the unit normal vector $\underline{n}$ taken on each iso-x_3 surfaces and the associated two orthogonal projectors $\boldsymbol{\pi}$ and $\boldsymbol{\pi}^\perp$ defined by:

$$\boldsymbol{\pi} = \boldsymbol{i} - \underline{n} \otimes \underline{n}, \qquad \boldsymbol{\pi}^\perp = \underline{n} \otimes \underline{n} \tag{3}$$

where $\boldsymbol{i}$ is the second order identity tensor. The strain rate is decomposed into:

$$\boldsymbol{d} = \boldsymbol{d}_s + \underline{g}_s \otimes \underline{n} + \underline{n} \otimes \underline{g}_s + g_n \boldsymbol{\pi}^\perp \tag{4}$$

where: $\boldsymbol{d}_s = \boldsymbol{\pi}.\boldsymbol{d}.\boldsymbol{\pi}$, $\underline{g}_s = \boldsymbol{\pi}.\boldsymbol{d}.\underline{n}$, $g_n = \boldsymbol{d} : \boldsymbol{\pi}^\perp$
Due to the incompressibility, one has $g_n = -\operatorname{tr}(\boldsymbol{d}_s)$, consequently the strain rate can also be written:

$$\boldsymbol{d} = \boldsymbol{d}_s - \operatorname{tr}(\boldsymbol{d}_s)\boldsymbol{\pi}^\perp + \underline{g}_s \otimes \underline{n} + \underline{n} \otimes \underline{g}_s \tag{5}$$

The equivalent strain rate can be put into the form:

$$d_{eq} = (D_s^2 + G_s^2)^{1/2} \tag{6}$$

with:

$$D_s = \left[\frac{2}{3}\left(\boldsymbol{d}_s : \boldsymbol{d}_s + \operatorname{tr}(\boldsymbol{d}_s)^2\right)\right]^{1/2}, \quad G_s = \left[\frac{4}{3}\underline{g}_s : \underline{g}_s\right]^{1/2} \tag{7}$$

Similarly, the stress field $\boldsymbol{\sigma}$ is decomposed into:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_s + \underline{t}_s \otimes \underline{n} + \underline{n} \otimes \underline{t}_s + t_n \boldsymbol{\pi}^\perp \tag{8}$$

with: $\boldsymbol{\sigma}_s = \boldsymbol{\pi}.\boldsymbol{\sigma}.\boldsymbol{\pi}$, $\underline{t}_s = \boldsymbol{\pi}.\boldsymbol{\sigma}.\underline{n}$, $t_n = \boldsymbol{\sigma} : \boldsymbol{\pi}^\perp$
There are two kinds of quantities which enter into the expression of the strain rate (5) and of the stress (8):
- quantities which are continuous across a perfectly bonded interface: $\boldsymbol{d}_s$, $\underline{t}_s$

and t_n,
- quantities which are discontinuous across a perfect interface: $\boldsymbol{\sigma}_s$ and $\underline{g}_s$.
We aim at expressing $\boldsymbol{\sigma}_s$ and $\underline{g}_s$ as function of $\boldsymbol{d}_s$, $\underline{t}_s$ and t_n. To do that, we decompose the stress-strain relation as follows:

$$\left\{\begin{aligned} \boldsymbol{\sigma}_s &= \frac{2\sigma_0}{3\dot{\varepsilon}_0}\left(\frac{d_{eq}}{\dot{\varepsilon}_0}\right)^{m-1}\boldsymbol{d}_s + p\boldsymbol{\pi} \\ \underline{t}_s &= \frac{2\sigma_0}{3\dot{\varepsilon}_0}\left(\frac{d_{eq}}{\dot{\varepsilon}_0}\right)^{m-1}\underline{g}_s \\ t_n &= -\frac{2\sigma_0}{3\dot{\varepsilon}_0}\left(\frac{d_{eq}}{\dot{\varepsilon}_0}\right)^{m-1}\mathrm{tr}(\boldsymbol{d}_s) + p \end{aligned}\right. \tag{9}$$

From the second relation in (9), one has:

$$\underline{t}_s.\underline{t}_s = \frac{\sigma_0^2}{3\dot{\varepsilon}_0^2}\left(\frac{d_{eq}}{\dot{\varepsilon}_0}\right)^{2m-2} G_s^2 \tag{10}$$

Introducing into the above equation

$$T_s = \sqrt{3\underline{t}_s.\underline{t}_s} \tag{11}$$

it can be observed that G_s is solution of the following non linear equation in z:

$$(z^2 + D_s^2)^{m-1} - \left(\frac{\dot{\varepsilon}_0^m}{\sigma_0}\right)^2 \frac{T_s^2}{z^2} = 0 \tag{12}$$

A closed form solution of the above equation can be found in the special case of a linear viscous material ($m = 1$) and of a perfectly plastic material ($m = 0$) (see sections 5 and 7) or in the case of specific rational values of $m = \frac{1}{2}$ and $m = \frac{1}{3}$. In any case $0 < m < 1$, we formally denote the solution [1] by $z = \mathcal{Z}$ where $\mathcal{Z}$ depends on D_s, T_s and on the material parameters $\dot{\varepsilon}_0$, σ_0.
The equivalent strain reads: $d_{eq} = (D_s^2 + \mathcal{Z}^2)^{1/2}$, and from the second relation in (9), one has:

$$\underline{g}_s = \frac{3\dot{\varepsilon}_0}{2\sigma_0}\left(\frac{(D_s^2 + \mathcal{Z}^2)^{1/2}}{\dot{\varepsilon}_0}\right)^{1-m} \underline{t}_s \tag{13}$$

[1] Equation (12) gives a positive and a negative real solution but also complex solutions, here the solution $\mathcal{Z}$ makes reference to the positives real solution.

The pressure p, can be eliminated from the first and the last relation in (9). It leads to:

$$\boldsymbol{\sigma}_s = \frac{2\sigma_0}{3\dot{\varepsilon}_0} \left(\frac{(D_s^2 + \mathcal{Z}^2)^{1/2}}{\dot{\varepsilon}_0} \right)^{m-1} (\boldsymbol{d}_s + \mathrm{tr}(\boldsymbol{d}_s)\boldsymbol{\pi}) + t_n \boldsymbol{\pi} \tag{14}$$

In (13) and (14), the quantities $\underline{g}_s$ and $\boldsymbol{\sigma}_s$ are only expressed in terms of quantities $\boldsymbol{d}_s$, $\underline{t}_s$ and t_n.

4 The Taylor expansion approach

As explained in the introduction of the paper, an imperfect interface can be seen as the limit case of a thin stiff or soft interface. The connection between the properties of the interphase and of the equivalent interface model has been amply studied by Benveniste [2] in the case of a two dimensional elastic curved surface and has been later generalized in the three dimensional context [3].
In order to establish the connection between the properties of the thin interphase and of the imperfect interface, two methods have been applied in past studies. The first one uses the asymptotic expansion. This method has been initiated by Sanchez-Palencia [20] and Pham Huy and Sanchez-Palencia [17] for conductivity, and applied by Klarbring et al. [14,15], Avila-Pozos et al. [1] and Benveniste [2] in the context of linear elasticity. The second method has been initiated by Bövik [5] and formalized by Hashin [13] and Benveniste [3]. It uses the Taylor expansion and adequate surface differential operators which make possible to derive the general expressions of the jumps of the physical fields. In the special cases of a plane layer [15,1] and of a two dimensional curved layer [2], the results obtained by the asymptotic expansion method are consistent with the one derived form the Taylor expansion approach [3]. Although it has not yet been proven, it is likely that this equivalence between the two methods remains true in the general case of an anisotropic three dimensional curved layer.
Of course, the use of the Taylor expansion based method suppose that the physical fields which are expanded are continuously differentiable, defined and regular everywhere in the thin layer. Such conditions are not needed when the asymptotic expansion method is use, it constitutes the main difference between the two methods. However, the advantage of the Taylor expansion approach over the asymptotic one is that it leads to a compact representation in which the particular cases of "spring interface" and of "stress interface" appear as limit cases.
In the prensent paper we extend the Taylor expansion method to the context of viscoplasticity. The first step consist in expanding the velocity field, $\underline{v}$, and the traction vector $\underline{t}$ at the mid surface, S_0, about the lower interface S_1 or

upper interface S_2:

$$\begin{aligned} \underline{v}|_{S_0} &= \underline{v}|_{S_1} + \frac{h}{2}\,\mathrm{grad}_n(\underline{v})|_{S_1} + o(h^2) \\ \underline{v}|_{S_0} &= \underline{v}|_{S_2} - \frac{h}{2}\,\mathrm{grad}_n(\underline{v})|_{S_2} + o(h^2) \end{aligned} \tag{15}$$

where $\mathrm{grad}_n(\underline{v}) = \mathrm{grad}(\underline{v}).\underline{n}$ are the derivatives of $\underline{v}$ along the x_3 direction. In (15) the series is truncated at the first order, higher terms are neglected in the present work. It supposes that the thickness of the layer is small compared to the characteristic length of the microstructure, as for example the size of the inclusions for a composite material. A combination of the two equations in (15) gives:

$$\underline{v}|_{S_2} - \underline{v}|_{S_1} = \frac{h}{2}\Big[\mathrm{grad}_n(\underline{v})|_{S_1} + \mathrm{grad}_n(\underline{v})|_{S_2}\Big] + o(h^2) \tag{16}$$

Similarly, an equivalent equation can be written for the traction vector $\underline{t}$:

$$\underline{t}|_{S_2} - \underline{t}|_{S_1} = \frac{h}{2}\Big[\mathrm{grad}_n(\underline{t})|_{S_1} + \mathrm{grad}_n(\underline{t})|_{S_2}\Big] + o(h^2) \tag{17}$$

In (15) and (17), we are dealing with continuously differentiable fields, defined and regular everywhere in the thin layer. Following the lines of Benveniste [3] we aim at expressing the differential operators, $\mathrm{grad}_n(\underline{v})$ and $\mathrm{grad}_n(\underline{t})$ in (16) and (17) as functions of the surface derivatives of the velocity, $\mathrm{grad}_s(\underline{v}) = \mathrm{grad}(\underline{v}).\boldsymbol{\pi}$ and of the traction, $\underline{t}$. Once those operators are determined, the derivation of the interface equations will be easier.
Firstly, the normal and tangential components of $\mathrm{grad}_n(\underline{v})$ are given by:

$$\begin{aligned} \mathrm{grad}_n(\underline{v}).\underline{n} &= g_n = -\,\mathrm{tr}(\boldsymbol{d}_s) \\ \mathrm{grad}_n(\underline{v}).\boldsymbol{\pi} &= 2\underline{g}_s - \underline{n}.\,\mathrm{grad}_s(\underline{v}) \end{aligned} \tag{18}$$

where $\underline{g}_s$ is given by (13). It follows that:

$$\mathrm{grad}_n(\underline{v}) = \frac{3\dot{\varepsilon}_0^m}{\sigma_0}(D_s^2 + \mathcal{Z}^2)^{(1-m)/2}\underline{t}_s - \underline{n}.\,\mathrm{grad}_s(\underline{v}) - \mathrm{tr}(\boldsymbol{d}_s)\underline{n} \tag{19}$$

Using now the equilibrium equation, one has:

$$\mathrm{grad}_n(\underline{t}) = \mathrm{grad}(\boldsymbol{\sigma}) : \boldsymbol{\pi}^{\perp} = -\,\mathrm{grad}(\boldsymbol{\sigma}) : \boldsymbol{\pi} \tag{20}$$

In the above expression, the decomposition (8) is used, in which $\boldsymbol{\sigma}_s$ is replaced by (14). It follows that:

$$
\begin{aligned}
\mathrm{grad}_n(\underline{t}) = -\,\mathrm{grad}\Bigg\{ & \frac{2\sigma_0}{3\dot{\varepsilon}_0^m}\left(D_s^2+\mathcal{Z}^2\right)^{(m-1)/2}(\boldsymbol{d}_s+\mathrm{tr}(\boldsymbol{d}_s)\boldsymbol{\pi}) \\
& +\underline{t}_s\otimes\underline{n}+\underline{n}\otimes\underline{t}_s+t_n\boldsymbol{i}\Bigg\}:\boldsymbol{\pi}
\end{aligned}
\tag{21}
$$

In (19) and (21) the "normal" gradient of $\underline{v}$ and $\underline{t}$ are given in term of $\underline{v}$ and $\underline{t}$ and their surface derivatives which are equal to their corresponding quantities in the surrounding media "1" and "2".
In the configuration of figure 1, the interphase has been eliminated and replaced by an idealized interface S_0. Equations (16), (17) with (19) and (21) provide the general expressions for the jump of the velocity field and traction across the equivalent imperfect interface S_0. We aim now at examining the special cases of a soft or stiff interphase for which explicit expressions of the interface model can be derived.

5 The linear case: $m = 1$

Consider the particular case of a linear viscous material, $n = m = 1$. Expressions (19) and (21) become:

$$
\begin{aligned}
&\mathrm{grad}_n(\underline{v}) = \frac{1}{\mu}\underline{t}_s - \underline{n}.\,\mathrm{grad}_s(\underline{v}) - \mathrm{tr}(\boldsymbol{d}_s)\underline{n} \\
&\mathrm{grad}(\underline{t}).\underline{n} = -\,\mathrm{grad}\left\{2\mu(\boldsymbol{d}_s+\mathrm{tr}(\boldsymbol{d}_s)\boldsymbol{\pi})+\underline{t}_s\otimes\underline{n}+\underline{n}\otimes\underline{t}_s+t_n\boldsymbol{i}\right\}:\boldsymbol{\pi}
\end{aligned}
\tag{22}
$$

in which we have introduced $\mu = \sigma_0/(3\dot{\varepsilon}_0)$.
The traction vector and the gradient of the velocity are assumed to takes finite values at the interfaces S_1 and S_2. Consequently, if μ remains finite when the limit $h \to 0$ is taken, the terms at the right of the equality in (16) and (17) vanishes when $h \to 0$. The term containing $h/2$ in both expressions (16) and (17) does not vanish when $h \to 0$ int the following special cases:
- If μ is assumed to be of the same magnitude of the width h, the interphase is soft and can be replaced by a "spring interface" model which involves a jump of the velocity:

$$
[\underline{v}]_{S_0} = \frac{1}{\mu}\underline{t}_s, \quad [\underline{t}]_{S_0} = 0 \tag{23}
$$

where $\bar{\mu} = h\mu$. It can be observed that only the tangential components of the velocity field are discontinuous across the interface whereas the normal components remain continuous, this is a consequence of the material incompressibility. The above results are consistent with the one provided in [3] when a linear incompressible interphase is considered.
- When μ is of the order of magnitude of $1/h$, the interphase is stiff and can be replaced by "stress interface" model which involves a jump of the traction:

$$[\underline{v}]_{S_0} = 0, \quad [\underline{t}]_{S_0} = -\operatorname{grad}(\boldsymbol{\tau}) : \boldsymbol{\pi}, \quad \boldsymbol{\tau} = 2\mu(\boldsymbol{d}_s + \operatorname{tr}(\boldsymbol{d}_s)\boldsymbol{\pi}) \tag{24}$$

with $\bar{\mu} = \mu/h$ and $\boldsymbol{\tau}$ is called interfacial stress. The jump of the traction takes the form of a generalized Young Laplace Law. Again, The above results are consistent with the one provided in [3] for an elastic incompressible interphase.

6 The viscoplastic case: $0 < m < 1$

The value of the exponent m is now arbitrary taken in the interval $]0, 1[$. We first consider the case for which $\dot{\varepsilon}_0$ is proportional to $1/h$, and we put $\dot{\varepsilon}_0 = \bar{\varepsilon}_0/h$. σ_0 and $\bar{\varepsilon}_0$ are assumed to remains finite when $h \to 0$. The non linear equation (12) become:

$$(z^2 + D_s^2)^{m-1} - \frac{1}{h^{2m-2}}\left(\frac{\bar{\varepsilon}_0^m}{\sigma_0}\right)^2 \frac{T_s^2}{z^2} = 0 \tag{25}$$

When low values of h are considered in equation (12), the solution for z is obtained by using an asymptotic expansion. The solution z is taken as: $z = z_{-1}/h + z_0 + z_1 h + ...$ The computation of z_{-1}, z_0... gives:

$$z_{-1} = \bar{\varepsilon}_0\left(\frac{T_s}{\sigma_0}\right)^n, \quad z_0 = 0, \quad z_1 = \frac{1}{2}(1-n)\frac{D_s^2}{\bar{\varepsilon}_0}\left(\frac{T_s}{\sigma_0}\right)^{-n}, \quad etc. \tag{26}$$

Only the higher order term, z_{-1} is kept in the expansion of z, as a consequence:

$$\mathcal{Z} = \frac{\bar{\varepsilon}_0}{h}\left(\frac{T_s}{\sigma_0}\right)^n + o(h) \tag{27}$$

Replacing this expression in (19) and (21), and taking in (16) and (17) the limit $h \to 0$ leads to:

$$[\underline{v}]_{S_0} = \frac{3\bar{\varepsilon}_0}{\sigma_0}\left(\frac{T_s}{\sigma_0}\right)^{n-1} \underline{t}_s, \quad [\underline{t}]_{S_0} = 0 \tag{28}$$

Dually, we suppose that the reference stress σ_0 of the coating is proportional to $1/h$ and we put $\sigma_0 = \bar{\sigma}_0/h$, ($\bar{\sigma}_0$ and $\dot{\varepsilon}_0$ are assumed to remains finite when $h \to 0$). When low values of h are considered in equation (12), the solution z can be taken into the form: $z = z_1 h + z_2 h^2 + ...$ Only the term z_1 is kept in the expansion of z, and solution for $\mathcal{Z}$ can be read:

$$\mathcal{Z} = h\dot{\varepsilon}_0^m D_s^{1-m}\frac{T_s}{\overline{\sigma}_0} + o(1) \tag{29}$$

Taking in (16) and (17) the limit $h \to 0$ leads to:

$$\begin{aligned} &[\underline{v}]_{S_0} = 0, \quad [\underline{t}]_{S_0} = -\operatorname{grad}(\boldsymbol{\tau}) : \boldsymbol{\pi} \\ &\text{with: } \boldsymbol{\tau} = \frac{2\overline{\sigma}_0}{3\dot{\varepsilon}_0}\left(\frac{D_s}{\dot{\varepsilon}_0}\right)^{m-1}(\boldsymbol{d}_s + \operatorname{tr}(\boldsymbol{d}_s)\boldsymbol{\pi}) \end{aligned} \tag{30}$$

As in the linear case, the jump of the traction takes the form of a Young Laplace law. However, the relation giving the interfacial stress, $\boldsymbol{\tau}$ as a function of the surface strain rate is given by a surface viscoplastic law under plane stress conditions.

7 The plastic case: $m = 0$

We now look for the particular case of a rigid perfectly-plastic case corresponding to $n = +\infty$ or $m = 0$. Equation (12) can be explicitly solved and gives:

$$\mathcal{Z} = \frac{T_s D_s}{\sqrt{\sigma_0^2 - T_s^2}} \tag{31}$$

Replacing (31) in (19) and (21) leads to:

$$\begin{aligned} &\operatorname{grad}_n(\underline{v}) = \frac{3D_s}{\sqrt{\sigma_0^2 - T_s^2}}\underline{t}_s - \underline{n}.\operatorname{grad}_s(\underline{v}) - \operatorname{tr}(\boldsymbol{d}_s)\underline{n} \\ &\operatorname{grad}(\underline{t}).\underline{n} = -\operatorname{grad}\left\{\frac{2}{3}\frac{\sqrt{\sigma_0^2 - T_s^2}}{D_s}(\boldsymbol{d}_s + \operatorname{tr}(\boldsymbol{d}_s)\boldsymbol{\pi}) \right. \\ &\qquad\qquad\qquad \left. + \underline{t}_s \otimes \underline{n} + \underline{n} \otimes \underline{t}_s + t_n \boldsymbol{i}\right\} : \boldsymbol{\pi} \end{aligned} \tag{32}$$

Since the material parameter $\dot{\varepsilon}_0$ vanishes in the case of a plastic material, the conditions $[\underline{v}]_{S_0} \neq 0$ and $[\underline{t}]_{S_0} = 0$ cannot be obtained as the limit case of a thin plastic layer. However, when the reference stress is chosen as $\bar{\sigma}_0 = \sigma_0/h$, the above equations gives:

$$\begin{aligned} &\mathrm{grad}_n(\underline{v}) = o(1) \\ &\mathrm{grad}(\underline{t}).\underline{n} = -\frac{1}{h}\,\mathrm{grad}\left\{\frac{2\sigma_0}{3D_s}(\boldsymbol{d}_s + \mathrm{tr}(\boldsymbol{d}_s)\boldsymbol{\pi})\right\} : \boldsymbol{\pi} + o(1) \end{aligned} \tag{33}$$

The limit $h \to 0$ is taken in (16) and (17), giving:

$$[\underline{v}]_{S_0} = 0, \quad [\underline{t}]_{S_0} = -\,\mathrm{div}_s(\boldsymbol{\tau}), \quad \boldsymbol{\tau} = \frac{2\overline{\sigma}_0}{3D_s}(\boldsymbol{d}_s + \mathrm{tr}(\boldsymbol{d}_s)\boldsymbol{\pi}) \tag{34}$$

The dissipation reads $\boldsymbol{\tau} : \boldsymbol{d}_s = \overline{\sigma}_0 D_s$. The surface strain rate, $\boldsymbol{d}_s$, can be eliminated in the relation giving $\boldsymbol{\tau}$ as a function of $\boldsymbol{d}_s$:

$$\frac{3}{2}\left(\boldsymbol{\tau} : \boldsymbol{\tau} - \frac{1}{3}\,\mathrm{tr}(\boldsymbol{\tau})^2\right) = \overline{\sigma}_0^2 \tag{35}$$

The obtained model coincides with the one postulated in [9].

8 Conclusion

Interface models, commonly used in elasticity, have been generalized, in this paper, to the context of viscoplasticity. The methodology, applied to derive the constitutive equations of the interface, uses the concept of equivalent interphase model and Taylor expansions. It has been shown that, when the thickness of the coating, h, is very small compared with the characteristic length of the microstructure, the coating can be replaced by an idealized interface involving the jump of the velocity field and of the traction vector.
Two particular cases has been investigated:
- When the interphase is soft compared with the surrounding media, the obtained interface model involves a jump of the velocity field whereas the traction vector remains continuous. The jump of the velocity field across the interface is given in terms of the surface component of the traction vector and takes the form of a power-law type relation.
- When a stiff interphase is considered, the equivalent interface model involves a jump of the traction vector whereas the velocity field remains continuous. The relation giving the jump across the interface takes the form of the Young-Laplace equation. However, the interfacial stress-strain relation is given by a

viscoplastic law under plane stress conditions.
These interface models can be applied for taking into account the surface effects in viscoplastic composites.

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