Maximum entropy approach for modeling random uncertainties in transient elastodynamics

Christian Soize

MAXIMUM ENTROPY APPROACH FOR MODELING RANDOM UNCERTAINTIES IN TRANSIENT ELASTODYNAMICS

Christian Soize

Structural Dynamics and Coupled Systems Department, ONERA, BP 72, 92322 Chatillon Cedex, France

ABSTRACT

A new approach is presented for analyzing random uncertainties in dynamical systems. This approach consists in modeling random uncertainties by a nonparametric model allowing transient responses of mechanical systems submitted to impulsive loads to be predicted in the context of linear structural dynamics. The information used does not require the description of the local parameters of the mechanical model. The probability model is deduced from the use of the entropy optimization principle whose available information is constituted of the algebraic properties related to the generalized mass, damping and stiffness matrices which have to be positive-definite symmetric matrices, and the knowledge of these matrices for the mean reduced matrix model. An explicit construction and representation of the probability model have been obtained and are very well suited to algebraic calculus and to Monte Carlo numerical simulation in order to compute the transient responses of structures submitted to impulsive loads. The fundamental properties related to the convergence of the stochastic solution with respect to the dimension of the random reduced matrix model is analyzed. Finally, an example is presented.

PACS numbers: 43.40

Keywords: Random uncertainties; dynamical systems, structural dynamics; structural acoustics; transient response; impulsive load; entropy optimization principle

INTRODUCTION

This paper deals with predicting the transient responses of structures submitted to impulsive loads in linear structural dynamics. The theory presented below can be extended without any difficulties to structural-acoustic problems such as a structure coupled with an internal acoustic cavity. In general, this kind of prediction is relatively difficult because the structural models have to be adapted to large, medium and small vibrational wavelengths which correspond to the low-, medium- and high-frequency ranges.

Here, we are interested in the case where the impulsive load under consideration has an energy which is almost entirely distributed over a broad low-frequency band and for which prediction of the impulsive load response can be obtained with a reduced matrix model constructed using the generalized coordinates of the mode-superposition method associated with the structural modes corresponding to the n lowest eigenfrequencies of the structure. It should be noted that, for a complex structure, only a numerical approximation of the first structural modes can be calculated using a large finite element model of the structure. The low-frequency case considered in this paper is important for many applications, and details concerning such a case can be found in the literature on structural dynamics and vibrations (see Refs. 1 to 8).

Under the above assumptions and for a complex structure, dimension n of the reduced matrix model generally has to be high (several dozen or hundred structural modes may be necessary to predict transient responses). However, it is known that the higher the eigenfrequency of a structural mode, the lower its accuracy because the uncertainties in the model increase (in linear structural dynamics and vibrations, the effects of uncertainties on the model increase with the frequency and it should be kept in mind that the mechanical model and the finite element model of a complex structure tend to be less reliable in predicting the higher structural modes). This is why random uncertainties in the mechanical model have to be taken into account. This is a fundamental problem in structural dynamics and in structural acoustics when the mechanical model has to be adapted to predict a transient response for which not only the low-frequency band is mainly concerned, but also the upper part of this low-frequency band and maybe the medium-frequency-band have to be taken into account.

Random uncertainties in linear structural dynamics and structural acoustics are usually modeled using parametric models. This means that 1) the uncertain parameters (scalars, vectors or fields) occurring in the boundary value problem (geometrical parameters; boundary conditions; mass density; mechanical parameters of constitutive equations; structural complexity, interface and junction modeling, etc.) have to be identified; 2) appropriate probabilistic models of these uncertain parameters have to be constructed, and 3) functions mapping the domains of uncertain parameters into the mass, damping and stiffness operators have to be constructed. Concerning details related to such a parametric approach, we refer the reader to Refs. 9 to 15 for general developments, to

Refs. 16 to 21 for general aspects related to stochastic finite elements and to Refs. 22 to 27 for other aspects related to this kind of parametric models of random uncertainties in the context of developments written in stochastic dynamics and parametric stochastic excitations.

In this paper we present a new approach, that we will call a nonparametric approach, for constructing a model of random uncertainties in linear structural dynamics in order to predict the transient response of complex structures submitted to impulsive loads (as indicated above, this approach can be directly extended to structural-acoustic problems). This nonparametric model of random uncertainties does not require identifying the uncertain parameters in the boundary value problem as described above for the parametric approach but is based on the use of recent research (see Refs. 28 and 29) in which the construction of a probability model for symmetric positive-definite real random matrices using the entropy optimization principle has been introduced and developed. These results will allow the direct construction of a probabilistic model of the reduced matrix model deduced from the variational formulation of the boundary value problem to be obtained, for which the only information used in this construction is the available information constituted of the mean reduced matrix model, the existence of second-order moments of inverses of the random matrices and some algebraic properties relative to the positive-definiteness of these random matrices. It should be noted that these properties have to be taken into account in order to obtain a mechanical system with random uncertainties, which models a dynamical system. For instance if there are uncertainties on the generalized mass matrix, the probability distribution has to be such that this random matrix be positive definite. If not, the probability model would be wrong because the generalized mass matrix of any dynamical system has to be positive definite.

In Refs. 28 and 29, we presented the calculation of the matrix-valued frequency response functions for discretized linear dynamical systems with random uncertainties. Unfortunately, convergence results were not obtained yet and consequently, a parameter of the probability model were not clearly defined for a designer. In this paper, an explicit construction of the probabilistic reduced matrix model of finite dimension n is given and its convergence is studied as n approaches infinity. In such a probabilistic theory, it seems absolutely fundamental to prove the convergence. It is not self-evident that convergence properties exist in such a construction. In addition, it should be noted that Eqs. (65)-(68) have been deduced from the convergence analysis carried out. Thanks to this new analysis presented in this paper, we have obtained a new consistent and coherent theory in which all the parameters are clearly defined. In Section I, the mean boundary value problem is introduced and its variational formulation is given in order to construct the mean reduced matrix model, which is carried out in Section II, using the mode-superposition method. Section III is devoted to construction of the nonparametric model of random uncertainties for the reduced matrix model. In this section, we introduce the available information which is directly used for constructing the probabilistic model of random uncertainties. In Section IV, we give a summary of the main results established in Refs. 28 and 29 concerning the probability model for symmetric positive-definite real random matrices and we complete this construction in order to obtain a consistent probabilistic model useful for studying convergence as dimension n approaches infinity. The nonparametric model of random uncertainties for the reduced matrix model constructed using Sections III and IV, is presented in Section V. The convergence properties of this nonparametric model of random uncertainties as dimension n approaches infinity are given in Section VI. The convergence properties prove that the construction proposed is consistent. Finally, an example is presented in Section VII.

I. MEAN BOUNDARY VALUE PROBLEM FOR MEAN TRANSIENT RESPONSE AND ITS VARIATIONAL FORMULATION

A. Definition of the mean boundary value problem

We consider the linear transient response of a three-dimensional damped fixed structure around a static equilibrium configuration considered as a natural state without prestresses, submitted to an impulsive load. The mean mechanical model is described by the following mean boundary value problem. Let Ω be the bounded open domain of $\mathbb{R}^3$ occupied by the mean structure at static equilibrium and made of viscoelastic material without memory. Let $\partial\Omega = \Gamma_0 \cup \Gamma$ be the boundary such that $\Gamma_0 \cap \Gamma = \emptyset$ and let $\mathbf{n}$ be its outward unit normal. Let $\mathbf{u} = (u_1, u_2, u_3)$ be the displacement field at each point $\mathbf{x} = (x_1, x_2, x_3)$ in Cartesian coordinates. On part Γ_0 of the boundary, the structure is fixed ($\mathbf{u} = \mathbf{0}$) while on part Γ it is free. There are external prescribed impulsive volumetric and surface force fields applied to Ω and Γ, written as $\{\mathbf{g}_{vol}(\mathbf{x}, t), t \geq 0\}$ and $\{\mathbf{g}_{surf}(\mathbf{x}, t), t \geq 0\}$ respectively. Let T be a positive real number. The mean transient response $\{\mathbf{u}(\mathbf{x}, t), \mathbf{x} \in \Omega, t \in [0, T]\}$ is the solution of the following mean boundary value problem:

$$\rho\, \ddot{u}_i - \frac{\partial \sigma_{ij}}{\partial x_j} = g_{vol,i} \quad \text{in} \quad \Omega \quad , \quad t \in [0, T] \quad , \tag{1}$$

$$\sigma_{ij} n_j = g_{surf,i} \quad \text{on} \quad \Gamma \quad , \quad t \in [0, T] \quad , \tag{2}$$

$$u_i = 0 \quad \text{on} \quad \Gamma_0 \quad , \quad t \in [0, T] \quad , \tag{3}$$

for $i = 1, 2, 3$, with zero initial conditions at time $t = 0$,

$$\mathbf{u}(\mathbf{x}, 0) = 0 \quad , \quad \dot{\mathbf{u}}(\mathbf{x}, 0) = 0 \quad , \quad \forall \mathbf{x} \in \Omega \quad , \tag{4}$$

in which $\dot{\mathbf{u}}$ and $\ddot{\mathbf{u}}$ mean $\partial \mathbf{u}/\partial t$ and $\partial^2 \mathbf{u}/\partial t^2$ respectively, where the index summation convention is used and where $\rho(\mathbf{x}) > 0$ is the mass density of the mean model (which is assumed to be a bounded function on Ω). For a viscoelastic material without memory, stress tensor σ_{ij} of the mean model is written as

$$\sigma_{ij} = a_{ijkh}(\mathbf{x})\, \varepsilon_{kh}(\mathbf{u}) + b_{ijkh}(\mathbf{x})\, \varepsilon_{kh}(\dot{\mathbf{u}}) \quad , \tag{5}$$

in which $\varepsilon_{kh}(\mathbf{u}) = (\partial u_k/\partial x_h + \partial u_h/\partial x_k)/2$ is the linearized strain tensor. The mechanical coefficients of the mean model $a_{ijkh}(\mathbf{x})$ and $b_{ijkh}(\mathbf{x})$ are real, depend on $\mathbf{x}$ and verify the usual properties of symmetry and positiveness[30−32,8].

B. Variational formulation of the mean boundary value problem

The variational formulation of the mean boundary value problem is absolutely necessary to construct the mean reduced matrix model in the general case. In addition, in order to prove the convergence properties of the stochastic transient response of the dynamical system with random uncertainties as the dimension of the reduction approaches infinity, we need to introduce the set $\mathbb{V}$ of admissible displacement fields. For the mathematical notations used in this section, we refer the reader to Refs. 33-35, and for the general methodology for constructing a variational formulation of a boundary value problem, we refer the reader to Refs. 8,33,36-37.

Set of admissible displacement fields

We introduce the real Hilbert space $\mathbb{H} = \{\, \mathbf{u} = (u_1, u_2, u_3)\,,\, u_j \in L^2(\Omega) \,\}$ equipped with the inner product

$$(\mathbf{u}\,,\mathbf{v})_{\mathbb{H}} = \int_{\Omega} \mathbf{u}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x})\, d\mathbf{x} \quad , \tag{6}$$

and the associated norm

$$\|\mathbf{u}\|_{\mathbb{H}} = (\mathbf{u}\,,\mathbf{u})_{\mathbb{H}}^{1/2} \quad , \tag{7}$$

in which $\mathbf{u} \cdot \mathbf{v} = \sum_{j=1}^{3} u_j v_j$ and where $L^2(\Omega)$ denotes the set of all the square integrable functions from Ω into $\mathbb{R}$. Let $\mathbb{V} \subset \mathbb{H}$ be the Hilbert space representing the set of admissible displacement fields with values in $\mathbb{R}^3$ such that

$$\mathbb{V} = \{\mathbf{u} \in \mathbb{H} \quad , \quad \partial \mathbf{u}/\partial x_j \in \mathbb{H} \quad , \quad \mathbf{u} = \mathbf{0} \quad \text{on} \quad \Gamma_0\} \quad , \tag{8}$$

equipped with the inner product

$$(\mathbf{u}\,,\mathbf{v})_{\mathbb{V}} = (\mathbf{u}\,,\mathbf{v})_{\mathbb{H}} + \sum_{j=1}^{3} \left(\frac{\partial \mathbf{u}}{\partial x_j}\,, \frac{\partial \mathbf{v}}{\partial x_j} \right)_{\mathbb{H}} \quad , \tag{9}$$

and the associated norm

$$\|\mathbf{u}\|_{\mathbb{V}} = (\mathbf{u}\,,\mathbf{u})_{\mathbb{V}}^{1/2} \quad . \tag{10}$$

Linear form representing the prescribed external forces

For all fixed t, it is assumed that prescribed external forces $\mathbf{x} \mapsto \mathbf{g}_{vol}(\mathbf{x}, t)$ and $\mathbf{x} \mapsto \mathbf{g}_{surf}(\mathbf{x}, t)$ are square integrable functions on Ω and Γ respectively. For all fixed t, we introduce the linear form $\mathbf{v} \mapsto g(\mathbf{v}\,; t)$ on $\mathbb{V}$ representing the prescribed external forces and defined by

$$g(\mathbf{v}\,; t) = \int_{\Gamma} \mathbf{g}_{surf}(\mathbf{x}, t) \cdot \mathbf{v}(\mathbf{x})\, ds(\mathbf{x}) + \int_{\Omega} \mathbf{g}_{vol}(\mathbf{x}, t) \cdot \mathbf{v}(\mathbf{x})\, d\mathbf{x} \quad , \tag{11}$$

in which ds is the surface element.

Variational formulation

Below, for t fixed in $[0, T]$, the mapping $\mathbf{x} \mapsto \mathbf{u}(\mathbf{x}, t)$is denoted $\mathbf{u}(t)$. The variational formulation of the mean boundary value problem defined by Eqs. (1)-(3) consists in finding a function $t \mapsto \mathbf{u}(t)$ with values in $\mathbb{V}$ such that

$$\underline{m}(\ddot{\mathbf{u}}, \mathbf{v}) + \underline{d}(\dot{\mathbf{u}}, \mathbf{v}) + \underline{k}(\mathbf{u}, \mathbf{v}) = g(\mathbf{v}\,; t) \quad , \quad \forall \mathbf{v} \in \mathbb{V} \quad , \quad \forall t \in [0, T] \quad , \tag{12}$$

with the initial conditions defined by Eq. (4). Bilinear form $\underline{m}(\mathbf{u}, \mathbf{v})$ (mass term) is defined by

$$\underline{m}(\mathbf{u}, \mathbf{v}) = \int_{\Omega} \rho(\mathbf{x})\, \mathbf{u}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x})\, d\mathbf{x} \quad , \tag{13}$$

and is symmetric, positive definite, continuous on $\mathbb{H} \times \mathbb{H}$ and such that

$$\underline{m}(\mathbf{u}, \mathbf{u}) \geq c_{\underline{m}} \|\mathbf{u}\|_{\mathbb{H}}^2 \quad , \tag{14}$$

in which $c_{\underline{m}}$ is a positive real constant. The properties of mechanical coefficients $b_{ijkh}(\mathbf{x})$ and $a_{ijkh}(\mathbf{x})$ are such that bilinear forms $\underline{d}(\mathbf{u}, \mathbf{v})$ (damping term) and $\underline{k}(\mathbf{u}, \mathbf{v})$ (stiffness term) which are defined by

$$\underline{d}(\mathbf{u}, \mathbf{v}) = \int_{\Omega} b_{ijkh}(\mathbf{x})\, \varepsilon_{kh}(\mathbf{u})\, \varepsilon_{ij}(\mathbf{v})\, d\mathbf{x} \quad , \tag{15}$$

$$\underline{k}(\mathbf{u},\mathbf{v}) = \int_\Omega a_{ijkh}(\mathbf{x})\,\varepsilon_{kh}(\mathbf{u})\,\varepsilon_{ij}(\mathbf{v})\,d\mathbf{x} \quad , \tag{16}$$

are symmetric, positive definite, continuous on $\mathbb{V}\times\mathbb{V}$ and are such that

$$\underline{d}(\mathbf{u},\mathbf{u}) \geq c_{\underline{d}}\|\mathbf{u}\|^2_{\mathbb{V}} \quad , \quad \underline{k}(\mathbf{u},\mathbf{u}) \geq c_{\underline{k}}\|\mathbf{u}\|^2_{\mathbb{V}} \quad , \tag{17}$$

in which $c_{\underline{d}}$ and $c_{\underline{k}}$ are positive real constants. Below, Eqs. (14) and (17) will be used to prove the convergence properties of the stochastic transient response. From the usual reference books (see for instance Ref. 33), we deduce that the problem defined by Eq. (12) with Eq. (4) has a unique solution $t \mapsto \mathbf{u}(t)$ with values in $\mathbb{V}$, that we refer to as the transient response of the mean model of the structure submitted to impulsive loads. Note that underlined quantities refer to the mean mechanical model.

Response ratio and dynamic magnification factor of the mean model

The complete history of the transient response and its Fourier transform are of most interest to analyze structural-dynamic and structural-acoustic systems subjected to impulsive loads. Nevertheless, the main objective of this paper being to present a new theory for modeling random uncertainties in such systems, the presentation is limited to the calculation of the maximum response produced by the impulsive load rather than the complete history. Consequently, we introduce the usual dynamic magnification factor which is also of interest to the designer and which allows the random uncertainties modeling to be easily analyzed. It should be noted that this choice for presenting the results of the example does not suppress the generality of the theory presented.

The elastic energy at time t, associated with the transient response $\mathbf{u}(t)$ of the mean model is equal to $\frac{1}{2}\underline{k}(\mathbf{u}(t),\mathbf{u}(t))$. The quasi-static response $\{\mathbf{x} \mapsto \mathbf{u}_{stat}(\mathbf{x},t)\}$ at time t of the mean model, also denoted $\mathbf{u}_{stat}(t)$ and associated with Eq. (12), is the unique solution of the following quasi-static problem

$$\underline{k}(\mathbf{u}_{stat}(t),\mathbf{v}) = g(\mathbf{v}\,;t) \quad , \quad \forall\mathbf{v}\in\mathbb{V} \quad , \quad \forall t\in[0,T] \quad , \tag{18}$$

and its maximum $\mathbf{u}_S(\mathbf{x})$ is such that

$$\mathbf{u}_S(\mathbf{x}) = \max_{t\geq 0}\,\mathbf{u}_{stat}(\mathbf{x},t) \quad . \tag{19}$$

The maximum of the elastic energy associated with the maximum of the quasi-static response of the mean model is then equal to $\frac{1}{2}\underline{k}(\mathbf{u}_S,\mathbf{u}_S)$. Finally, the response ratio $\underline{r}(t)$ at time t, associated with the elastic energy of transient response $\mathbf{u}(t)$ of the mean model, is defined by

$$\underline{r}(t) = \sqrt{\frac{\frac{1}{2}\underline{k}(\mathbf{u}(t), \mathbf{u}(t))}{\frac{1}{2}\underline{k}(\mathbf{u}_S, \mathbf{u}_S)}} \quad , \tag{20}$$

and the dynamic magnification factor $\underline{b}$ of the mean model (associated with the elastic energy) can then be defined by

$$\underline{b} = \max_{t \in [0,T]} \underline{r}(t) \quad . \tag{21}$$

II. MEAN REDUCED MATRIX MODEL

A. Spectral problem associated with the mean model

We consider the following spectral problem associated with Eq. (12), corresponding to the construction of the structural modes of the associated conservative mean structural model and consisting in finding $\underline{\lambda}$ and $\underline{\boldsymbol{\varphi}}$ in $\mathbb{V}$ such that

$$\underline{k}(\underline{\boldsymbol{\varphi}}, \mathbf{v}) = \underline{\lambda}\, \underline{m}(\underline{\boldsymbol{\varphi}}, \mathbf{v}) \quad , \quad \forall \mathbf{v} \in \mathbb{V} \quad . \tag{22}$$

Taking into account the properties of bilinear forms $\underline{m}$ and $\underline{k}$, there exists an increasing sequence of positive eigenvalues with finite multiplicity $0 < \underline{\lambda}_1 \leq \underline{\lambda}_2 \ldots \leq \underline{\lambda}_\alpha \leq \ldots$. Let $\underline{\boldsymbol{\varphi}}_\alpha$ be the eigenfunction associated with eigenvalue $\underline{\lambda}_\alpha$. Then, eigenfunctions $\{\underline{\boldsymbol{\varphi}}_\alpha, \alpha \geq 1\}$ form a complete set in $\mathbb{V}$ which means that an arbitrary function $\mathbf{u}$ belonging to $\mathbb{V}$ can be expanded as $\mathbf{u} = \sum_{\alpha=1}^{+\infty} q_\alpha \underline{\boldsymbol{\varphi}}_\alpha$, in which $\{q_\alpha\}_\alpha$ is a sequence of real numbers. The eigenfunctions satisfy the following orthogonality conditions

$$\underline{m}(\underline{\boldsymbol{\varphi}}_\alpha, \underline{\boldsymbol{\varphi}}_\beta) = \underline{\mu}_\alpha \delta_{\alpha\beta} \quad , \quad \underline{k}(\underline{\boldsymbol{\varphi}}_\alpha, \underline{\boldsymbol{\varphi}}_\beta) = \underline{\mu}_\alpha \underline{\omega}_\alpha^2 \delta_{\alpha\beta} \quad , \tag{23}$$

in which $\underline{\omega}_\alpha = \sqrt{\underline{\lambda}_\alpha}$ is the eigenfrequency of the mean model associated with structural mode $\underline{\boldsymbol{\varphi}}_\alpha$ whose normalization $\underline{\mu}_\alpha > 0$ is the mean generalized mass and where $\delta_{\alpha\alpha} = 1$ and $\delta_{\alpha\beta} = 0$ for $\alpha \neq \beta$. Consequently, the eigenfrequencies of the associated conservative mean structural model are such that

$$0 < \underline{\omega}_1 \leq \underline{\omega}_2 \ldots \leq \underline{\omega}_\alpha \leq \ldots \quad . \tag{24}$$

B. Mean reduced matrix model

The mean reduced matrix model is obtained using the Ritz-Galerkin projection of the variational formulation of the mean boundary value problem on the subspace $\mathbb{V}_n$ of $\mathbb{V}$ spanned by the structural modes $\{\underline{\boldsymbol{\varphi}}_1, \ldots, \underline{\boldsymbol{\varphi}}_n\}$ of the mean structural model, which correspond to the n lowest eigenfrequencies $\{\underline{\omega}_1, \ldots, \underline{\omega}_n\}$. Let $t \mapsto \mathbf{u}(t)$ from $\mathbb{R}^+$ into $\mathbb{V}$ be the unique solution of Eq. (12) with the initial conditions defined by Eq. (4) and let $\mathbf{u}_n(t)$ be the projection of $\mathbf{u}(t)$ on $\mathbb{V}_n$ (in structural dynamics, this corresponds to the usual mode-superposition method). Consequently, the mean reduced matrix model related to structural modes $\{\underline{\boldsymbol{\varphi}}_1, \ldots, \underline{\boldsymbol{\varphi}}_n\}$ of the mean structural model is then written as

$$\mathbf{u}_n(\mathbf{x}, t) = \sum_{\alpha=1}^{n} q_\alpha^n(t)\, \underline{\boldsymbol{\varphi}}_\alpha(\mathbf{x}) \quad , \tag{25}$$

in which the $\mathbb{R}^n$-valued vector $\mathbf{q}^n(t) = (q_1^n(t), \ldots, q_n^n(t))$ of the generalized coordinates is the solution of the time reduced problem,

$$[\,\underline{M}_n]\, \ddot{\mathbf{q}}^n(t) + [\,\underline{D}_n]\, \dot{\mathbf{q}}^n(t) + [\,\underline{K}_n]\, \mathbf{q}^n(t) = \mathbf{F}^n(t) \quad , \quad t \in [0, T] \quad , \tag{26}$$

with the initial conditions,

$$\mathbf{q}^n(0) = 0 \quad , \quad \dot{\mathbf{q}}^n(0) = 0 \quad , \tag{27}$$

in which the generalized force $\mathbf{F}^n(t)$ is the n real vector $(F_1(t), \ldots, F_n(t))$ such that

$$F_\alpha^n(t) = g(\underline{\boldsymbol{\varphi}}_\alpha\, ; t) \quad , \quad \alpha = 1, \ldots, n \quad . \tag{28}$$

The mean generalized mass, damping and stiffness matrices $[\,\underline{M}_n]$, $[\,\underline{D}_n]$ and $[\,\underline{K}_n]$ are positive-definite symmetric $(n \times n)$ real matrices such that

$$[\,\underline{M}_n]_{\alpha\beta} = \underline{\mu}_\alpha\, \delta_{\alpha\beta} \quad , \quad [\,\underline{D}_n]_{\alpha\beta} = \underline{d}(\underline{\boldsymbol{\varphi}}_\beta, \underline{\boldsymbol{\varphi}}_\alpha) \quad , \quad [\,\underline{K}_n]_{\alpha\beta} = \underline{\mu}_\alpha\, \underline{\omega}_\alpha^2\, \delta_{\alpha\beta} \quad . \tag{29}$$

C. Response ratio and dynamic magnification factor of the mean reduced matrix model

From Eqs. (23), (25) and (29), we deduce that the elastic energy $\frac{1}{2}\underline{k}(\mathbf{u}_n(t), \mathbf{u}_n(t))$ at time t, associated with transient response $\mathbf{u}_n(t)$ of the mean model, is equal to $\frac{1}{2} < [\,\underline{K}_n]\mathbf{q}^n(t), \mathbf{q}^n(t) >$ in which $<\mathbf{y}\,, \mathbf{x}>= y_1\, x_1 + \ldots + y_n\, x_n$ is the Euclidean inner product of $\mathbb{R}^n$. Consequently, the response ratio $\underline{r}_n(t)$ at time t, associated with the elastic energy of transient response $\mathbf{u}_n(t)$ (with values in $\mathbb{V}_n$) of the mean model and defined by Eq. (20), is such that

$$\underline{r}_n(t) = \sqrt{\frac{\frac{1}{2} < [\,\underline{K}_n]\mathbf{q}^n(t), \mathbf{q}^n(t) >}{\frac{1}{2}\underline{k}(\mathbf{u}_S, \mathbf{u}_S)}} \quad , \tag{30}$$

and the dynamic magnification factor $\underline{b}_n$ of the mean model (associated with the elastic energy) and defined by Eq. (21), is such that

$$\underline{b}_n = \max_{t\in[0,T]} \underline{r}_n(t) \quad . \tag{31}$$

We have $\lim_{n\to+\infty} \underline{r}_n(t) = \underline{r}(t)$ and $\lim_{n\to+\infty} \underline{b}_n = \underline{b}$ in which $\underline{r}(t)$ and $\underline{b}$ are defined by Eqs. (20) and (21) respectively.

III. PRINCIPLE OF CONSTRUCTION OF A NONPARAMETRIC MODEL OF RANDOM UNCERTAINTIES FOR THE REDUCED MATRIX MODEL

In this section, we introduce the principle of construction of a nonparametric model of random uncertainties and we define the available information for the construction of the nonparametric model.

A. Principle of construction

It should be noted that if the boundary value problem defined by Eqs. (1)-(3) corresponds to an exact mechanical model of the structure under consideration, there are no uncertainties in the model which is then sure. However, in continuum mechanics, the exact boundary value problem cannot be written for a given complex structure due to uncertainties in the data (geometry, boundary conditions, constitutive equation, structural complexity, etc.) and the most advanced deterministic model which it is possible to construct can only be considered as a mean mechanical model leading to the notion of the mean boundary value problem introduced above. In addition, Eq. (12), which corresponds to the variational formulation of this mean boundary value problem, does not constitute available information for predicting the transient response $\{\mathbf{u}(\mathbf{x},t), \mathbf{x} \in \Omega, t \in [0,T]\}$ of the mean structure subjected to any impulsive loads for the following reason. For a complex structure, such a mean boundary value problem defined by Eqs. (1)-(3) is not able to predict the transient response due to impulsive loads whose energy is distributed over a very broad frequency band, i.e. over the low-, medium- and high-frequency ranges (for instance, if there is energy in the medium-frequency range, more advanced probabilistic mechanical models such as the fuzzy structure theory have to be used to take into account the role played by the structural complexity[8]); the most that this kind of deterministic mean boundary value problem is able to predict is the transient response due to impulsive loads whose energy is mainly distributed over a broad low-frequency range for which the mean reduced matrix model defined by Eqs. (25)-(27) is suitable and allows the transient response to be predicted with good accuracy. This means that the variational formulation of the mean boundary value problem does not constitute available information for constructing the nonparametric model of random uncertainties. However, the mean reduced matrix model defined by Eqs. (25)-(27) (with n not too large) does constitute the available information for constructing the transient response of the mean model, then constructing the probability model of random uncertainties. This probabilistic model is a nonparametric model of random uncertainties because the sources of random uncertainties in the mechanical model which are due to uncertain mechanical parameters such as geometrical parameters, boundary conditions, junction stiffness, mass density, Young's modulus, etc., are not directly modeled by random variables or stochastic fields. These random uncertain geometrical and mechanical parameters mean that the generalized mass, damping and stiffness matrices of the reduced matrix model are random matrices. In addition, it should be noted that such a nonparametric approach also allows the model uncertainties to be taken into account whereas parametric approaches do not allow it.

Consequently, the main idea for this nonparametric construction is that not only all the parametric random uncertainties of the mechanical model are taken into account but also the model uncertainties, which lead to a random set of coupled generalized linear oscillators in the space of generalized coordinates, represented by the system of random generalized matrices of the random reduced matrix model. The problem is then to construct the probability distribution of this set of generalized oscillators, i.e. the probability distribution of the random generalized matrices of the random reduced matrix model, using only the available information. The available information is constituted of the mean generalized matrices of the mean reduced matrix model, the positive-definiteness of the random generalized matrices and the existence of second-order moments of inverses of these random generalized matrices. The nonparametric model of random uncertainties which is proposed consists in introducing a direct construction of a probabilistic model of these random generalized matrices.

B. Random reduced matrix model

In this paper, $\mathbb{M}_n(\mathbb{R})$, $\mathbb{M}_n^S(\mathbb{R})$ and $\mathbb{M}_n^+(\mathbb{R})$ are the set of all the $(n \times n)$ real matrices, the set of all the symmetric $(n \times n)$ real matrices and the set of all the positive-definite symmetric $(n \times n)$ real matrices, respectively; we have $\mathbb{M}_n^+(\mathbb{R}) \subset \mathbb{M}_n^S(\mathbb{R}) \subset \mathbb{M}_n(\mathbb{R})$; all the random variables are defined on a probability space $(\mathcal{A}, \mathcal{T}, P)$.

Using the construction principle presented in Section III.A, the random reduced matrix model associated with the mean reduced matrix model introduced in Section II.B is written as

$$\mathbf{U}_n(\mathbf{x},t) = \sum_{\alpha=1}^{n} Q_\alpha^n(t)\, \underline{\boldsymbol{\varphi}}_\alpha(\mathbf{x}) \quad , \tag{32}$$

in which $\mathbf{Q}^n(t) = (Q_1^n(t), \ldots, Q_n^n(t))$ is the $\mathbb{R}^n$-valued random variable such that

$$[\mathbf{M}_n]\, \ddot{\mathbf{Q}}^n(t) + [\mathbf{D}_n]\, \dot{\mathbf{Q}}^n(t) + [\mathbf{K}_n]\, \mathbf{Q}^n(t) = \mathbf{F}^n(t) \quad , \quad t \geq 0 \quad , \tag{33}$$

with the initial conditions:

$$\mathbf{Q}^n(0) = 0 \quad , \quad \dot{\mathbf{Q}}^n(0) = 0 \quad \text{a.s.} \quad , \tag{34}$$

in which generalized force $\mathbf{F}^n(t)$ is the $\mathbb{R}^n$-valued vector defined by Eq. (28) and where $[\mathbf{M}_n]$, $[\mathbf{D}_n]$ and $[\mathbf{K}_n]$ are the random generalized mass, damping and stiffness matrices with values in $\mathbb{M}_n^+(\mathbb{R})$. It should be noted that the mathematical property related to the positiveness of the random matrices is absolutely fundamental and is required so that the second-order differential equation in time corresponds effectively to a dynamical system.

C. Random response ratio and random dynamic magnification factor

For the elastic energy of the model with random uncertainties and due to Eqs. (30)-(32), the random response ratio $R_n(t)$ at time t, associated with stochastic transient response $\mathbf{U}_n(t)$ with values in $\mathbb{V}_n$, is written as

$$R_n(t) = \sqrt{\frac{\frac{1}{2} < [\underline{K}_n]\mathbf{Q}^n(t), \mathbf{Q}^n(t) >}{\frac{1}{2}\underline{k}(\mathbf{u}_S, \mathbf{u}_S)}} \quad . \tag{35}$$

The random dynamic magnification factor B_n is then defined by

$$B_n = \max_{t \in [0,T]} R_n(t) \quad . \tag{36}$$

D. Available information for the construction of the nonparametric model

We have to define the available information which is useful for constructing the probabilistic model. The basic available informations are the mean reduced matrix model, the positive-definiteness of the random generalized matrices and the existence of second-order moments of inverses of these random generalized matrices. The mean reduced matrix model is constituted of mean generalized mass, damping and stiffness matrices $[\underline{M}_n]$, $[\underline{D}_n]$ and $[\underline{K}_n]$ defined in Section II.B and which belong to $\mathbb{M}_n^+(\mathbb{R})$. Random generalized mass, damping and stiffness matrices $[\mathbf{M}_n]$, $[\mathbf{D}_n]$ and

$[\mathbf{K}_n]$ are second-order random variables with values in $\mathbb{M}_n^+(\mathbb{R})$ (the fundamental algebraic property relative to the positive-definiteness of the random generalized matrices) such that

$$E\{[\mathbf{M}_n]\} = [\,\underline{M}_n] \quad , \quad E\{[\mathbf{D}_n]\} = [\,\underline{D}_n] \quad , \quad E\{[\mathbf{K}_n]\} = [\,\underline{K}_n] \quad . \tag{37}$$

In addition, in order to obtain a consistent probabilistic model and in particular, to obtain convergence properties of stochastic transient response $\{\mathbf{U}_n(\mathbf{x},t), \mathbf{x} \in \Omega, t \in [0,T]\}$ when dimension n approaches infinity, we need to introduce information relative to the existence of moments of random variables $[\mathbf{M}_n]^{-1}$, $[\mathbf{D}_n]^{-1}$ and $[\mathbf{K}_n]^{-1}$ (such as second-order moments). It should be noted that since random matrices $[\mathbf{M}_n]$, $[\mathbf{D}_n]$ and $[\mathbf{K}_n]$ are almost surely positive definite, the inverse matrices exist almost surely, but the existence of moments does not follow. We therefore introduce the following constraints,

$$E\big\{\|[\mathbf{M}_n]^{-1}\|_F^\gamma\big\} < +\infty \quad , \quad E\big\{\|[\mathbf{D}_n]^{-1}\|_F^\gamma\big\} < +\infty \quad , \quad E\big\{\|[\mathbf{K}_n]^{-1}\|_F^\gamma\big\} < +\infty \quad , \tag{38}$$

in which $\gamma \geq 1$ is a positive integer and

$$\|\,[A\,]\,\|_F = \big(\mathrm{tr}\{[A\,]\,[A\,]^T\}\big)^{1/2} \quad , \tag{39}$$

is the Frobenius norm of matrix $[A\,] \in \mathbb{M}_n(\mathbb{R})$ where tr is the trace of the matrices and $[A\,]^T$ is the transpose of matrix $[A\,]$. We then have to construct a probability model for symmetric positive-definite real random matrices $[\mathbf{M}_n]$, $[\mathbf{D}_n]$ and $[\mathbf{K}_n]$ with the available information defined by Eqs. (37) and (38). This construction is presented in Section IV.

IV. PROBABILITY MODEL FOR SYMMETRIC POSITIVE-DEFINITE REAL RANDOM MATRICES

In a part of this section, we recall the main results established in Refs. 28 and 29 concerning the construction of a probability model for random matrices with values in $\mathbb{M}_n^+(\mathbb{R})$ using the entropy optimization principle which allows only the available information to be used. It should be noted that the results obtained and presented below differ from the known results concerning Gaussian and circular ensembles for random matrices (orthogonal, symplectic, unitary and antisymmetric Hermitian ensembles) which have been extensively studied in the literature (see for instance Refs 38 to 43). In another part of this section, we complete the construction given in Refs. 28 and 29 in order to obtain a consistent probabilistic model which allows the convergence properties to be studied when dimension n approaches infinity. In particular, we give an explicit expression of parameter λ_{A_n} as a function of scalar parameter δ_A which is independent of dimension n of random matrix $[\mathbf{A}_n]$ and which allows the dispersion of random matrix $[\mathbf{A}_n]$ to be given.

A. Probability density function on the space of positive-definite symmetric real matrices and characteristic function

Let $[\mathbf{A}_n]$ be a random matrix with values in $\mathbb{M}_n^+(\mathbb{R})$, defined on probability space $(\mathcal{A}, \mathcal{T}, P)$, whose probability distribution

$$P_{[\mathbf{A}_n]} = p_{[\mathbf{A}_n]}([A_n])\, \widetilde{d}A_n \tag{40}$$

is defined by a probability density function $[A_n] \mapsto p_{[\mathbf{A}_n]}([A_n])$ from $\mathbb{M}_n^+(\mathbb{R})$ into $\mathbb{R}^+$ with respect to the measure (volume element) $\widetilde{d}A_n$ on $\mathbb{M}_n^S(\mathbb{R})$ defined by[28,29]

$$\widetilde{d}A_n = 2^{n(n-1)/4}\, \Pi_{1\leq i \leq j \leq n}\, d[A_n]_{ij} \quad . \tag{41}$$

This probability density function is such that

$$\int_{\mathbb{M}_n^+(\mathbb{R})} p_{[\mathbf{A}_n]}([A_n])\, \widetilde{d}A_n = 1 \quad . \tag{42}$$

For all $[\Theta_n]$ in $\mathbb{M}_n^S(\mathbb{R})$, the characteristic function of random matrix $[\mathbf{A}_n]$ with values in $\mathbb{M}_n^+(\mathbb{R}) \subset \mathbb{M}_n^S(\mathbb{R})$ is defined by $\Phi_{[\mathbf{A}_n]}([\Theta_n]) = E\big\{\exp(i \ll [\Theta_n]\,, [\mathbf{A}_n] \gg)\big\}$ in which $\ll [\Theta_n], [\mathbf{A}_n] \gg = \text{tr}\{[\Theta_n]\, [\mathbf{A}_n]^T\} = \text{tr}\{[\Theta_n]\, [\mathbf{A}_n]\}$. We then have

$$\Phi_{[\mathbf{A}_n]}([\Theta_n]) = \int_{\mathbb{M}_n^+(\mathbb{R})} \exp(\, i\, \text{tr}\{[\Theta_n]\, [A_n]\})\, p_{[\mathbf{A}_n]}([A_n])\, \widetilde{d}A_n \quad . \tag{43}$$

B. Available information for construction of the probability model

We are interested in constructing the probability distribution of a second-order random variable $[\mathbf{A}_n]$ with values in $\mathbb{M}_n^+(\mathbb{R})$ for which the available information is the mean value $[\underline{A}_n]$ of random matrix $[\mathbf{A}_n]$,

$$E\{[\mathbf{A}_n]\} = \int_{\mathbb{M}_n^+(\mathbb{R})} [A_n]\, p_{[\mathbf{A}_n]}([A_n])\, \widetilde{d}A_n = [\underline{A}_n] \quad , \tag{44}$$

in which E denotes the mathematical expectation and where mean value $[\underline{A}_n]$ is given in $\mathbb{M}_n^+(\mathbb{R})$. In addition, we assume that random matrix $[\mathbf{A}_n]$ is such that

$$E\{\ln(\det[\mathbf{A}_n])\} = v \quad \text{with} \quad |v| < +\infty \quad . \tag{45}$$

Below, we prove that the constraint defined by Eq. (45) allows us to demonstrate the existence of moments related to the inverse random matrix $[\mathbf{A}_n]^{-1}$,

$$E\{\|[\mathbf{A}_n]^{-1}\|_F^\gamma\} < +\infty \quad , \tag{46}$$

in which $\gamma \geq 1$ is a positive integer. Consequently, from Eqs. (42), (44) and (45), we deduce that the constraints imposed for the construction of the probability model of random matrix $[\mathbf{A}_n]$ with values in $\mathbb{M}_n^+(\mathbb{R})$ are

$$\int_{\mathbb{M}_n^+(\mathbb{R})} p_{[\mathbf{A}_n]}([A_n]) \, \widetilde{d}A_n = 1 \quad , \tag{47}$$

$$\int_{\mathbb{M}_n^+(\mathbb{R})} [A_n] \, p_{[\mathbf{A}_n]}([A_n]) \, \widetilde{d}A_n = [\underline{A}_n] \in \mathbb{M}_n^+(\mathbb{R}) \quad , \tag{48}$$

$$\int_{\mathbb{M}_n^+(\mathbb{R})} \ln(\det[A_n]) \, p_{[\mathbf{A}_n]}([A_n]) \, \widetilde{d}A_n = v \quad \text{with} \quad |v| < +\infty \quad . \tag{49}$$

C. Probability model using the maximum entropy principle

By introducing the measure of entropy[44] (uncertainty) and the maximum entropy principle[45,46] to construct the probability model of random matrix $[\mathbf{A}_n]$ with values in $\mathbb{M}_n^+(\mathbb{R})$ based only on the use of the available information defined by Eqs. (47)-(49), we proved that, for $\lambda_{A_n} > 0$ and $[\Theta_n] \in \mathbb{M}_n^S(\mathbb{R})$, probability density function $p_{[\mathbf{A}_n]}([A_n])$ and characteristic function $\Phi_{[\mathbf{A}_n]}([\Theta_n])$ of positive-definite random matrix $[\mathbf{A}_n]$ are written as[28,29]

$$p_{[\mathbf{A}_n]}([A_n]) = \mathbb{1}_{\mathbb{M}_n^+(\mathbb{R})}([A_n]) \times c_{A_n} \times \big(\det[A_n]\big)^{\lambda_{A_n}-1} \times \exp\big(-\tfrac{(n-1+2\lambda_{A_n})}{2}\mathrm{tr}\{[\underline{A}_n]^{-1}[A_n]\}\big) \quad , \tag{50}$$

$$\Phi_{[\mathbf{A}_n]}([\Theta_n]) = \Big\{\det\Big([\,I_n] - \frac{2i}{(n-1+2\lambda_{A_n})}\,[\underline{A}_n]\,[\Theta_n]\Big)\Big\}^{-(n-1+2\lambda_{A_n})/2} \quad , \tag{51}$$

in which det is the determinant of the matrices, $[\,I_n]$ is the $(n \times n)$ identity matrix and where $\mathbb{1}_{\mathbb{M}_n^+(\mathbb{R})}([A_n])$ is equal to 1 if $[A_n] \in \mathbb{M}_n^+(\mathbb{R})$ and is equal to zero if $[A_n] \notin \mathbb{M}_n^+(\mathbb{R})$. When λ_{A_n} is an integer, the probability distribution defined by Eq. (50) or (51) is a Wishart distribution[47,48]. If λ_{A_n} is not an integer, the probability distribution defined by Eq. (50) or (51) is not a Wishart distribution. In Eq. (50), positive constant c_{A_n} is written as

$$c_{A_n} = \frac{(2\pi)^{-n(n-1)/4}\left(\frac{n-1+2\lambda_{A_n}}{2}\right)^{n(n-1+2\lambda_{A_n})/2}}{\big\{\Pi_{j=1}^n \Gamma\big(\frac{n-j+2\lambda_{A_n}}{2}\big)\big\}(\det[\underline{A}_n])^{(n-1+2\lambda_{A_n})/2}} \quad , \tag{52}$$

where $\Gamma(z)$ is the gamma function defined for $\Re e\, z > 0$ by $\Gamma(z) = \int_0^{+\infty} t^{z-1}\, e^{-t}\, dt$. The range of parameter λ_{A_n} satisfying Eq. (46) in which $\gamma \geq 1$ is a positive integer has to be determined. It is proved (see Appendix A) that

$$\lambda_{A_n} > \gamma + 1 \implies E\big\{\|[\mathbf{A}_n]^{-1}\|_F^{\gamma}\big\} < +\infty \quad , \quad \gamma \geq 1 \quad . \tag{53}$$

In addition, it can easily be proved that

$$\lambda_{A_n} > 0 \implies E\big\{\|[\mathbf{A}_n]\|_F^{\eta}\big\} < +\infty \quad , \quad \eta > 0 \quad . \tag{54}$$

Equation (54) means that for $\lambda_{A_n} > 0$, all the moments of random matrix $[\mathbf{A}_n]$ exist (η is any positive integer). The covariance $C^{A_n}_{jk,j'k'} = E\big\{([\mathbf{A}_n]_{jk} - [\underline{A}_n]_{jk})([\mathbf{A}_n]_{j'k'} - [\underline{A}_n]_{j'k'})\big\}$ of random variables $[\mathbf{A}_n]_{jk}$ and $[\mathbf{A}_n]_{j'k'}$ is written as

$$C^{A_n}_{jk,j'k'} = (n - 1 + 2\lambda_{A_n})^{-1}\big\{[\underline{A}_n]_{j'k}[\underline{A}_n]_{jk'} + [\underline{A}_n]_{jj'}[\underline{A}_n]_{kk'}\big\} \quad , \tag{55}$$

and the variance $V^{A_n}_{jk} = C^{A_n}_{jk,jk}$ of random variable $[\mathbf{A}_n]_{jk}$ is such that $V^{A_n}_{jk} = (n - 1 + 2\lambda_{A_n})^{-1}$ $\times\big\{[\underline{A}_n]^2_{jk} + [\underline{A}_n]_{jj}[\underline{A}_n]_{kk}\big\}$.

D. Dispersion parameter δ_A of random matrix $[\mathbf{A}_n]$

Since $[\underline{A}_n]$ is a positive-definite real matrix, there is an upper triangular matrix $[\underline{L}_{A_n}]$ in $\mathbb{M}_n(\mathbb{R})$ such that

$$[\underline{A}_n] = [\underline{L}_{A_n}]^T\, [\underline{L}_{A_n}] \quad , \tag{56}$$

which corresponds to the Cholesky factorization of matrix $[\underline{A}_n]$. Considering Eq. (56), random matrix $[\mathbf{A}_n]$ can be written as

$$[\mathbf{A}_n] = [\underline{L}_{A_n}]^T\, [\mathbf{G}_{A_n}]\, [\underline{L}_{A_n}] \quad , \tag{57}$$

in which matrix $[\mathbf{G}_{A_n}]$ is a random variable with values in $\mathbb{M}^+_n(\mathbb{R})$. From Eqs. (44) and (57), we deduce that the mean value $[\underline{G}_{A_n}]$ of random matrix $[\mathbf{G}_{A_n}]$ is such that

$$[\underline{G}_{A_n}] = E\{[\mathbf{G}_{A_n}]\} = [\, I_n] \quad . \tag{58}$$

The probability density function $p_{[\mathbf{G}_{A_n}]}([G_n])$ with respect to measure $\widetilde{d}G_n$ on $\mathbb{M}^S_n(\mathbb{R})$ of random matrix $[\mathbf{G}_{A_n}]$ with values in $\mathbb{M}^+_n(\mathbb{R})$ is given by Eqs. (50) and (52) in which $[\underline{A}_n]$ has to be replaced by $[\, I_n]$. We then have

$$p_{[\mathbf{G}_{A_n}]}([G_n]) = \mathbb{1}_{\mathbb{M}^+_n(\mathbb{R})}([G_n]) \times C_{G_{A_n}} \times \big(\det[G_n]\big)^{\lambda_{A_n}-1} \times \exp\left(-\frac{(n-1+2\lambda_{A_n})}{2}\, \mathrm{tr}[G_n]\right) \quad , \tag{59}$$

in which positive constant $C_{G_{A_n}}$ is such that

$$C_{G_{A_n}} = \frac{(2\pi)^{-n(n-1)/4}\left(\frac{n-1+2\lambda_{A_n}}{2}\right)^{n(n-1+2\lambda_{A_n})/2}}{\left\{\Pi_{j=1}^{n}\Gamma\left(\frac{n-j+2\lambda_{A_n}}{2}\right)\right\}} \quad . \tag{60}$$

From Eqs. (55) and (57), we deduce that the covariance $C^{G_n}_{jk,j'k'}$ of random variables $[\mathbf{G}_{A_n}]_{jk}$ and $[\mathbf{G}_{A_n}]_{j'k'}$, defined by $C^{G_n}_{jk,j'k'} = E\{([\mathbf{G}_{A_n}]_{jk} - [\underline{G}_{A_n}]_{jk})([\mathbf{G}_{A_n}]_{j'k'} - [\underline{G}_{A_n}]_{j'k'})\}$, is written as

$$C^{G_n}_{jk,j'k'} = (n-1+2\lambda_{A_n})^{-1}\left\{[\underline{G}_{A_n}]_{j'k}[\underline{G}_{A_n}]_{jk'} + [\underline{G}_{A_n}]_{jj'}[\underline{G}_{A_n}]_{kk'}\right\} \quad . \tag{61}$$

Since $[\underline{G}_{A_n}] = [\,I_n]$, the variance $V^{G_n}_{jk} = C^{G_n}_{jk,jk}$ of random variable $[\mathbf{G}_{A_n}]_{jk}$ is such that

$$V^{G_n}_{jk} = (n-1+2\lambda_{A_n})^{-1}(1+\delta_{jk}) \quad , \tag{62}$$

in which $\delta_{jk} = 0$ if $j \neq k$ and $\delta_{jj} = 1$. Let $\delta_A > 0$ be defined by

$$\delta_A = \left\{\frac{E\{\|\,[\mathbf{G}_{A_n}] - [\underline{G}_{A_n}]\,\|_F^2\}}{\|\,[\underline{G}_{A_n}]\,\|_F^2}\right\}^{1/2} \quad . \tag{63}$$

Equation (62) yields $E\{\|\,[\mathbf{G}_{A_n}] - [\underline{G}_{A_n}]\,\|_F^2\} = \sum_j \sum_k V^{G_n}_{jk} = n(n+1)(n-1+2\lambda_{A_n})^{-1}$ and since $\|\,[\underline{G}_{A_n}]\,\|_F^2 = \|\,[\,I_n]\,\|_F^2 = n$, we deduce that

$$\delta_A = \left\{\frac{n+1}{n-1+2\lambda_{A_n}}\right\}^{1/2} \quad , \tag{64}$$

and consequently,

$$\lambda_{A_n} = \ell_A(n) \quad , \tag{65}$$

in which $n \mapsto \ell_A(n)$ is the mapping defined on the set $\mathbb{N}^*$ of all positive integers such that

$$\ell_A(n) = \frac{1-\delta_A^2}{2\delta_A^2}\, n + \frac{1+\delta_A^2}{2\delta_A^2} \quad . \tag{66}$$

From Eqs. (53) and (64), we deduce that parameter δ_A has to be such that

$$0 < \delta_A < \sqrt{\frac{n+1}{n+1+2\gamma}} < 1 \quad , \quad \gamma \geq 1 \quad , \quad \forall n \geq 1 \quad . \tag{67}$$

Equation (67) shows that γ has to be chosen as small as possible in order to increase the domain of possible values for δ_A. From convergence considerations when $n \to +\infty$ (see Section VI) and from Eq. (67), we deduce that $\gamma = 2$ is the optimal value. Let $n_0 \geq 1$ be a fixed integer. Taking the value $\gamma = 2$, we then deduce that, if parameter δ_A satisfies

$$0 < \delta_A < \sqrt{\frac{n_0 + 1}{n_0 + 5}} \quad , \tag{68}$$

then, $\forall\, n \geq n_0$, we have $\lambda_{A_n} = \ell_A(n) > \gamma + 1 = 3$ and consequently, Eq. (53) holds. These equations will be used as follows. The lower bound n_0 of positive integer n is fixed. Then, the dispersion of the probability model is fixed by giving parameter δ_A, independent of n, a value such that Eq. (68) is satisfied. For each value of integer $n \geq n_0$, parameter $\lambda_{A_n} = \ell_A(n)$ is then calculated by using Eq. (66). Consequently, $\lambda_{A_n} = \ell_A(n)$ appears as a function of n.

E. Algebraic representation of random matrix $[\mathbf{A}_n]$ when λ_{A_n} is an integer

When $\lambda_{A_n} = \ell_A(n)$ is a positive integer, we introduce the positive integer m_A such that

$$m_A(n) = n - 1 + 2\ell_A(n) \quad . \tag{69}$$

Substituting Eq. (66) in the right-hand side of Eq. (69) yields

$$m_A(n) = (n+1)/\delta_A^2 \quad . \tag{70}$$

Since $m_A(n)$ is a positive integer, it can be verified that the probability distribution defined by Eq. (50) or (51) is a Wishart distribution[47,48] and that random matrix $[\mathbf{A}_n]$ can be written as[28,29]

$$[\mathbf{A}_n] = \frac{1}{m_A(n)} \sum_{j=1}^{m_A(n)} \left([\underline{L}_{A_n}]^T \mathbf{X}_j\right) \left([\underline{L}_{A_n}]^T \mathbf{X}_j\right)^T \quad , \tag{71}$$

in which $[\underline{L}_{A_n}]$ is the upper triangular matrix defined by Eq. (56) and where $\mathbf{X}_1, \ldots, \mathbf{X}_{m_A(n)}$ are independent random vectors, each vector $\mathbf{X}_j$ being an $\mathbb{R}^n$-valued second-order Gaussian random variable, centered and whose covariance matrix is $[C_{\mathbf{X}_j}] = E\{\mathbf{X}_j \mathbf{X}_j^T\} = [\,I_n]$. Consequently, Eq. (71) gives an efficient procedure for algebraic calculations and the Monte Carlo numerical simulation of random matrix $[\mathbf{A}_n]$.

F. Algebraic representation of random matrix $[\mathbf{A}_n]$ when λ_{A_n} is not an integer

Let us now assume that $\lambda_{A_n} = \ell_A(n)$, given by Eq. (65), is a positive real number (the particular case for which λ_{A_n} is a positive integer is presented above in Section IV.E). Since $[\mathbf{G}_{A_n}]$ defined by Eq. (57), is a random matrix with values in $\mathbb{M}_n^+(\mathbb{R})$, the Cholesky factorization allows us to write

$$[\mathbf{G}_{A_n}] = [\mathbf{L}_{A_n}]^T\, [\mathbf{L}_{A_n}] \quad \text{a.s.} \quad , \tag{72}$$

in which $[\mathbf{L}_{A_n}]$ is an upper triangular random matrix with values in $\mathbb{M}_n(\mathbb{R})$. The following results, which allow a procedure for the Monte Carlo simulation of random matrix $[\mathbf{A}_n]$ to be defined, are proved[28,29]:

(1) Random variables $\{[\mathbf{L}_{A_n}]_{jj'}, j \leq j'\}$ are independent.

(2) For $j < j'$, real-valued random variable $[\mathbf{L}_{A_n}]_{jj'}$ can be written as $[\mathbf{L}_{A_n}]_{jj'} = 2^{-1/2}\,[\widetilde{\mathbf{L}}_{A_n}]_{jj'}$ in which $[\widetilde{\mathbf{L}}_{A_n}]_{jj'}$ is a real-valued Gaussian random variable with zero mean and variance given by

$$v = 2\,(n - 1 + 2\ell_A(n))^{-1} \quad . \tag{73}$$

(3) For $j = j'$, positive-valued random variable $[\mathbf{L}_{A_n}]_{jj}$ can be written as $[\mathbf{L}_{A_n}]_{jj} = \sqrt{v\mathbf{Y}_j}$ in which v is given by Eq. (73) and where $\mathbf{Y}_j$ is a positive-valued gamma random variable whose probability density function with respect to dy is given by

$$\Gamma_j(y) = \frac{\mathbb{1}_{[0,+\infty[}(y)}{\Gamma\left(\frac{n-j+2\ell_A(n)}{2}\right)}\; y^{\frac{(n-j+2\ell_A(n))}{2}-1}\; e^{-y} \quad . \tag{74}$$

(4) We have $[\mathbf{G}_{A_n}] = [\mathbf{L}_{A_n}]^T\,[\mathbf{L}_{A_n}]$ and $[\mathbf{A}_n] = [\underline{L}_{A_n}]^T\,[\mathbf{G}_{A_n}]\,[\underline{L}_{A_n}]$.

G. Probability model of a set of positive-definite symmetric real random matrices

Let us consider ν random matrices $[\mathbf{A}_n^1], \ldots, [\mathbf{A}_n^\nu]$ with values in $\mathbb{M}_n^+(\mathbb{R})$ such that for each j in $\{1, \ldots, \nu\}$, the probability density function of random matrix $[\mathbf{A}_n^j]$ satisfies Eqs. (47)-(49). This means that only the mean values of the random matrices are known. Applying the maximum entropy principle, it can be proved that the probability density function $([A_n^1], \ldots, [A_n^\nu]) \mapsto p_{[\mathbf{A}_n^1],\ldots,[\mathbf{A}_n^\nu]}([A_n^1], \ldots, [A_n^\nu])$ from $\mathbb{M}_n^+(\mathbb{R}) \times \ldots \times \mathbb{M}_n^+(\mathbb{R})$ into $\mathbb{R}^+$ with respect to the measure (volume element) $\widetilde{d}A_n^1 \times \ldots \times \widetilde{d}A_n^\nu$ on $\mathbb{M}_n^S(\mathbb{R}) \times \ldots \times \mathbb{M}_n^S(\mathbb{R})$ is written as

$$p_{[\mathbf{A}_n^1],\ldots,[\mathbf{A}_n^\nu]}([A_n^1], \ldots, [A_n^\nu]) = p_{[\mathbf{A}_n^1]}([A_n^1]) \times \ldots \times p_{[\mathbf{A}_n^\nu]}([A_n^\nu]) \quad , \tag{75}$$

which means that $[\mathbf{A}_n^1], \ldots, [\mathbf{A}_n^\nu]$ are independent random matrices.

V. NONPARAMETRIC MODEL OF RANDOM UNCERTAINTIES

In this section we complete the construction of the probability model introduced in Section III using the developments of Section IV.

A. Probability model of the reduced matrix model

Let $n_0 \geq 1$ be a fixed integer and $n \geq n_0$. We apply the results of Section IV to the set of positive-definite symmetric real random matrices $\{[\mathbf{M}_n], [\mathbf{D}_n], [\mathbf{K}_n]\}$ defined in Section III, for which the available information is described by Eqs. (37)-(38). As indicated in Section IV.D, we take $\gamma = 2$ and the levels of dispersion of random matrices $[\mathbf{M}_n]$, $[\mathbf{D}_n]$ and $[\mathbf{K}_n]$ are controlled by parameters δ_M, δ_D and δ_K respectively, which are independent of n and are chosen such that (see Eq. (68)),

$$0 < \delta_M \, , \, \delta_D \, , \, \delta_K < \sqrt{\frac{n_0+1}{n_0+5}} \quad . \tag{76}$$

Parameters λ_M, λ_D and λ_K are defined by Eq. (65),

$$\lambda_M = \ell_M(n) \ , \ \lambda_D = \ell_D(n) \ , \ \lambda_K = \ell_K(n) \quad , \tag{77}$$

in which $\ell_M(n)$, $\ell_D(n)$ and $\ell_K(n)$ are given by Eq. (66),

$$\ell_M(n) = \frac{1-\delta_M^2}{2\delta_M^2}\, n + \frac{1+\delta_M^2}{2\delta_M^2} \quad , \tag{78}$$

$$\ell_D(n) = \frac{1-\delta_D^2}{2\delta_D^2}\, n + \frac{1+\delta_D^2}{2\delta_D^2} \quad , \tag{79}$$

$$\ell_K(n) = \frac{1-\delta_K^2}{2\delta_K^2}\, n + \frac{1+\delta_K^2}{2\delta_K^2} \quad . \tag{80}$$

From Section IV.G, we deduce that random matrices $[\mathbf{M}_n]$, $[\mathbf{D}_n]$ and $[\mathbf{K}_n]$ are independent random variables with values in $\mathbb{M}_n^+(\mathbb{R})$ and their probability density functions $p_{[\mathbf{M}_n]}([M_n])$, $p_{[\mathbf{D}_n]}([D_n])$ and $p_{[\mathbf{K}_n]}([K_n])$ with respect to the measures (volume elements) $\widetilde{d}M_n$, $\widetilde{d}D_n$ and $\widetilde{d}K_n$ on $\mathbb{M}_n^S(\mathbb{R})$ are given by Eqs. (50) and (52), and their characteristic functions by Eq. (51).

B. Construction of the stochastic transient response

For fixed positive integer $n \geq n_0$, we have to construct stochastic processes $\{\mathbf{U}_n(\mathbf{x}, t), \mathbf{x} \in \Omega, t \in [0, T]\}$ defined by Eqs. (32)-(34) and $\{R_n(t), t \geq 0\}$ defined by Eq. (35), and random variable B_n defined by Eq. (36). Below, we present a formulation which is adapted to Monte Carlo numerical simulation.

For given matrices $[M_n], [D_n], [K_n]$ in $\mathbb{M}_n^+(\mathbb{R})$, let $t \mapsto \mathbf{q}_\mathbf{F}^n(t; [M_n], [D_n], [K_n])$ be the solution from $\mathbb{R}^+$ into $\mathbb{R}^n$ of the deterministic second-order differential equation

$$[M_n]\, \ddot{\mathbf{q}}_\mathbf{F}^n(t) + [D_n]\, \dot{\mathbf{q}}_\mathbf{F}^n(t) + [K_n]\, \mathbf{q}_\mathbf{F}^n(t) = \mathbf{F}^n(t) \quad , \quad t \geq 0 \quad , \tag{81}$$

with the initial conditions

$$\mathbf{q}_\mathbf{F}^n(0) = 0 \quad , \quad \dot{\mathbf{q}}_\mathbf{F}^n(0) = 0 \quad . \tag{82}$$

We deduce that stochastic process $\{\mathbf{Q}^n(t), t \geq 0\}$ which is the solution of the stochastic dynamical problem defined by Eqs. (33)-(34), can be written as

$$\mathbf{Q}^n(t) = \mathbf{q}_\mathbf{F}^n(t; [\mathbf{M}_n], [\mathbf{D}_n], [\mathbf{K}_n]) \quad . \tag{83}$$

It should be noted that $\mathbf{Q}^n(t)$ can be usually written as

$$\mathbf{Q}^n(t) = \int_0^t [h_n(t-\tau)]\, \mathbf{F}^n(\tau)\, d\tau \quad , \tag{84}$$

in which $t \mapsto [h_n(t)]$ is the matrix-valued impulse response function of the linear filter associated with second-order differential Eq. (81). If $\mathbf{q} \mapsto [\theta_n(\mathbf{q})]$ is a mapping from $\mathbb{R}^n$ into the set $\mathbb{M}_{\nu_1,\nu_2}(\mathbb{R})$ of all the $(\nu_1 \times \nu_2)$ real matrices, we have

$$E\{[\theta_n(\mathbf{Q}^n(t))]\} = \int_{\mathbb{M}_n^+(\mathbb{R})} \int_{\mathbb{M}_n^+(\mathbb{R})} \int_{\mathbb{M}_n^+(\mathbb{R})} [\theta_n(\mathbf{q}_\mathbf{F}^n(t; [M_n], [D_n], [K_n]))] \\ \times p_{[\mathbf{M}_n]}([M_n]) \times p_{[\mathbf{D}_n]}([D_n]) \times p_{[\mathbf{K}_n]}([K_n])\, \widetilde{d}M_n\, \widetilde{d}D_n\, \widetilde{d}K_n \quad . \tag{85}$$

For instance, $R_n(t)$ defined by Eq. (35) can be written as $R_n(t) = [\theta_n(\mathbf{Q}^n(t))]$ with $\nu_1 = \nu_2 = 1$. Calculation of the stochastic transient response of the dynamical system with random uncertainties requires the numerical construction of mapping $t \mapsto \mathbf{q}_\mathbf{F}^n(t; [M_n], [D_n], [K_n])$ as the solution of the deterministic Eqs. (81)-(82). Since matrices $[M_n]$, $[D_n]$ and $[K_n]$ are full matrices (not diagonal) as samplings of random matrices $[\mathbf{M}_n]$, $[\mathbf{D}_n]$ and $[\mathbf{K}_n]$, Eq. (84) is not used but second-order differential Eq. (81) is solved directly using an unconditionally stable implicit step-by-step integration method (such as the Newmark integration scheme[2]) with initial conditions defined by Eq. (82). In addition, we have to calculate multiple integrals in a higher dimension (see Eq. (85)) for which a well suited method consists in using a Monte Carlo calculation with or without variance reduction procedures[49−55]. This method is very efficient if there is a Monte Carlo simulation procedure for random matrices $[\mathbf{M}_n]$, $[\mathbf{D}_n]$ and $[\mathbf{K}_n]$ which is the case of the method presented in Sections IV.E and IV.F. It should be noted that for many applications, integer n is sufficiently high that λ_M, λ_D and λ_K can be considered as positive integers without introducing any significant limitation in the model. Applying Eqs. (70)-(71) to random matrices $[\mathbf{M}_n]$, $[\mathbf{D}_n]$ and $[\mathbf{K}_n]$ yields

$$[\mathbf{M}_n] = \frac{1}{m_M(n)} \sum_{j=1}^{m_M(n)} \left([\underline{L}_{M_n}]^T \mathbf{X}_j\right) \left([\underline{L}_{M_n}]^T \mathbf{X}_j\right)^T \quad , \tag{86}$$

$$[\mathbf{D}_n] = \frac{1}{m_D(n)} \sum_{j=1}^{m_D(n)} \left([\underline{L}_{D_n}]^T\, \mathbf{Y}_j\right) \left([\underline{L}_{D_n}]^T\, \mathbf{Y}_j\right)^T \quad , \tag{87}$$

$$[\mathbf{K}_n] = \frac{1}{m_K(n)} \sum_{j=1}^{m_K(n)} \left([\underline{L}_{K_n}]^T\, \mathbf{Z}_j\right) \left([\underline{L}_{K_n}]^T\, \mathbf{Z}_j\right)^T \quad , \tag{88}$$

in which

$$m_M(n) = \text{Fix}\left(\frac{n+1}{\delta_M^2}\right) \quad , \quad m_D(n) = \text{Fix}\left(\frac{n+1}{\delta_D^2}\right) \quad , \quad m_K(n) = \text{Fix}\left(\frac{n+1}{\delta_K^2}\right) \quad , \tag{89}$$

where $\text{Fix}(x)$ is equal to x when x is an integer and $\text{Fix}(x)$ rounds down $x+1$ to the nearest integer when x is not an integer. In Eqs. (86)-(88), $[\underline{L}_{M_n}]$, $[\underline{L}_{D_n}]$ and $[\underline{L}_{K_n}]$ are upper triangular matrices in $\mathbb{M}_n(\mathbb{R})$ corresponding to the Cholesky factorization of symmetric positive-definite matrices $[\underline{M}_n]$, $[\underline{D}_n]$ and $[\underline{K}_n]$:

$$[\underline{M}_n] = [\underline{L}_{M_n}]^T\, [\underline{L}_{M_n}] \quad , \quad [\underline{D}_n] = [\underline{L}_{D_n}]^T\, [\underline{L}_{D_n}] \quad , \quad [\underline{K}_n] = [\underline{L}_{K_n}]^T\, [\underline{L}_{K_n}] \quad . \tag{90}$$

The set of all the components of vectors $\mathbf{X}_1, \ldots, \mathbf{X}_{m_M(n)}$, $\mathbf{Y}_1, \ldots, \mathbf{Y}_{m_D(n)}$ and $\mathbf{Z}_1, \ldots, \mathbf{Z}_{m_K(n)}$ with values in $\mathbb{R}^n$ is constituted of $m_M(n)\times n + m_D(n)\times n + m_K(n)\times n$ independent random variables, each of which is a real-valued second-order normalized Gaussian random variable (zero mean value and unit variance).

VI. CONVERGENCE PROPERTIES AS THE DIMENSION APPROACHES INFINITY

For each $n \geq n_0$ fixed, stochastic transient response $\{\mathbf{U}_n(\mathbf{x},t),\ \mathbf{x}\in\Omega,\ t\in[0,T]\}$ of the dynamical system with random uncertainties can be constructed using Sections III to V. A major problem concerns the convergence properties of stochastic transient response $\{\mathbf{U}_n(\mathbf{x},t),\ \mathbf{x}\in\Omega,\ t\in[0,T]\}$ and related quantities as $n \to +\infty$ for the nonparametric probabilistic model proposed in Sections III to V. This problem is studied below.

A. Introduction of norms useful for the convergence properties

As above, all the random variables are defined on probability space $(\mathcal{A}, \mathcal{T}, P)$. Let $\mathbf{Q} = (Q_1, \ldots, Q_n)$ be an $\mathbb{R}^n$-valued random variable. The norm $|||\mathbf{Q}|||$ of $\mathbf{Q}$ is defined by

$$|||\mathbf{Q}||| = \sqrt{E\{\|\mathbf{Q}\|^2\}} \quad , \tag{91}$$

in which $\|\mathbf{Q}\| = (\sum_{\alpha=1}^n Q_\alpha^2)^{1/2}$ is the Euclidean norm of $\mathbb{R}^n$ and where E is the mathematical expectation. It should be noted that vector $\mathbf{Q}$ is a second-order random variable if $|||\mathbf{Q}||| < +\infty$. Let

$\mathbf{U}$ be an $\mathbb{H}$-valued random variable (see Section I.B). Its norm in $\mathbb{H}$ is $\|\mathbf{U}\|_{\mathbb{H}}$ and is a positive-valued random variable defined (see Eq. (6)) by

$$\|\mathbf{U}\|_{\mathbb{H}} = \left(\int_{\Omega} \mathbf{U}(\mathbf{x})\cdot\mathbf{U}(\mathbf{x})\, d\mathbf{x}\right)^{1/2} \quad .$$

The norm $|||\mathbf{U}|||_{\mathbb{H}}$ of $\mathbf{U}$ is defined by

$$|||\mathbf{U}|||_{\mathbb{H}} = \sqrt{E\{\|\mathbf{U}\|^2_{\mathbb{H}}\}} \quad . \tag{92}$$

Similarly, if $\mathbf{U}$ is a $\mathbb{V}$-valued random variable (see Section I.B), its norm $\|\mathbf{U}\|_{\mathbb{V}}$ is the positive-valued random variable defined (see Eq. (7)) by

$$\|\mathbf{U}\|_{\mathbb{V}} = \left(\|\mathbf{U}\|^2_{\mathbb{H}} + \sum_{j=1}^{3} \left\|\frac{\partial \mathbf{U}}{\partial x_j}\right\|^2_{\mathbb{H}}\right)^{1/2} \quad ,$$

and the norm $|||\mathbf{U}|||_{\mathbb{V}}$ of $\mathbf{U}$ is defined by

$$|||\mathbf{U}|||_{\mathbb{V}} = \sqrt{E\{\|\mathbf{U}\|^2_{\mathbb{V}}\}} \quad . \tag{93}$$

B. Prerequisite to the construction of basic inequalities

Below, the $\mathbb{M}_n^+(\mathbb{R})$-valued random matrix $[\mathbf{A}_n]$ denotes $\mathbb{M}_n^+(\mathbb{R})$-valued random matrices $[\mathbf{M}_n]$, $[\mathbf{D}_n]$ or $[\mathbf{K}_n]$ defined on probability space $(\mathcal{A}, \mathcal{T}, P)$ and introduced in Sections III and V. Let $[\mathbf{G}_{A_n}]$ be the $\mathbb{M}_n^+(\mathbb{R})$-valued random matrix on probability space $(\mathcal{A}, \mathcal{T}, P)$ defined by Eq. (57), whose probability density function $p_{[\mathbf{G}_{A_n}]}([G_n])$ is given by Eq. (59). For ω fixed in $\mathcal{A}$, the norm of matrix $[\mathbf{G}_{A_n}(\omega)]^{-1}$ induced by the Euclidean norm of $\mathbb{R}^n$ is defined by

$$\|[\mathbf{G}_{A_n}(\omega)]^{-1}\| = \sup_{\mathbf{q}\in\mathbb{R}^n, \|\mathbf{q}\|=1} \|[\mathbf{G}_{A_n}(\omega)]^{-1}\mathbf{q}\| \quad , \tag{94}$$

and can be written as

$$\|[\mathbf{G}_{A_n}(\omega)]^{-1}\| = \frac{1}{\widetilde{\Sigma}_1(\omega)} \quad , \tag{95}$$

in which $\widetilde{\Sigma}_1(\omega) > 0$ is the smallest eigenvalue of matrix $[\mathbf{G}_{A_n}(\omega)] \in \mathbb{M}_n^+(\mathbb{R})$ whose eigenvalues are such that $0 < \widetilde{\Sigma}_1(\omega) \leq \widetilde{\Sigma}_2(\omega) \leq \ldots \leq \widetilde{\Sigma}_n(\omega)$. It should be noted that $\|[\mathbf{G}_{A_n}(\omega)]^{-1}\| \leq \|[\mathbf{G}_{A_n}(\omega)]^{-1}\|_F \leq \sqrt{n}\, \|[\mathbf{G}_{A_n}(\omega)]^{-1}\|$ in which the Frobenius norm is defined by Eq. (39). We then have

$$E\{\|[\mathbf{G}_{A_n}]^{-1}\|^2\} = E\{\widetilde{\Sigma}_1^{-2}\} \quad . \tag{96}$$

In Appendix B, the following inequality is proved:

$$\forall n \geq n_0 \quad , \quad E\{\|[\mathbf{G}_{A_n}]^{-1}\|^2\} \leq C_{\delta_A} < +\infty \quad , \tag{97}$$

in which $n_0 \geq 1$ is the fixed integer defined in Section V.A and where C_{δ_A} is a positive finite constant which is independent of n but which depends on δ_A defined by Eq. (76). Equation (97) means that $n \mapsto E\{\|[\mathbf{G}_{A_n}]^{-1}\|^2\}$ is a bounded function from $\{n \geq n_0\}$ into $\mathbb{R}^+$. We have numerically verified Eq. (97) using a Monte Carlo numerical simulation based on Eqs. (95)-(96), Section IV.E and the usual estimator of the second-order moment of random variable $1/\widetilde{\Sigma}_1$. Figure 1 shows the graph of function $n \mapsto E\{\|[\mathbf{G}_{A_n}]^{-1}\|^2\}$ for $n \geq n_0 = 2$, $\delta_A = 0.1$, 0.3 and 0.5, and corresponds to 100 samples in the Monte Carlo numerical simulation. These numerical results confirm Eq. (97) which is mathematically proved.

C. Basic inequalities derived from the random energy equation

Let us assume that prescribed external forces $(\mathbf{x}, t) \mapsto \mathbf{g}_{vol}(\mathbf{x}, t)$ and $(\mathbf{x}, t) \mapsto \mathbf{g}_{surf}(\mathbf{x}, t)$ are such that

$$\int_0^T \|\mathbf{f}(\tau)\|^2_{\mathbb{V}'}\, d\tau < +\infty \quad , \tag{98}$$

in which $\mathbf{f}(\tau)$ and $\|\mathbf{f}(\tau)\|_{\mathbb{V}'}$ are defined by Eqs. (D.3) and (D.2) in Appendix D. From Appendix E, we deduce that

$$\forall\, n \geq n_0 \quad , \quad \forall\, t \in [0, T] \quad , \quad |||\mathbf{U}_n(t)|||^2_{\mathbb{V}} \quad \leq \quad C_1 \quad < \quad +\infty \quad , \tag{99}$$

$$\forall\, n \geq n_0 \quad , \quad \forall\, t \in [0, T] \quad , \quad |||\dot{\mathbf{U}}_n(t)|||^2_{\mathbb{H}} \quad \leq \quad C_2 \quad < \quad +\infty \quad , \tag{100}$$

in which C_1 and C_2 are positive constants which are independent of n and t but which depend on the prescribed external forces, parameters T, δ_M, δ_D, δ_K (see Eq. (76)), and which are written (see Eqs. (E.22) and (E.21)) as

$$C_1 = \left(\frac{C_{\delta_K}}{c_{\underline{k}}^2} + \frac{C_{\delta_D}}{c_{\underline{d}}^2} \right) \int_0^T \|\mathbf{f}(\tau)\|^2_{\mathbb{V}'}\, d\tau \quad , \tag{101}$$

$$C_2 = \left(\frac{C_{\delta_M}}{c_{\underline{m}}^2} + \frac{C_{\delta_D}}{c_{\underline{d}}^2} \right) \int_0^T \|\mathbf{f}(\tau)\|^2_{\mathbb{V}'}\, d\tau \quad , \tag{102}$$

where positive finite constants $c_{\underline{m}}$, $c_{\underline{d}}$ and $c_{\underline{k}}$ are defined by Eqs. (14) and (17) and where positive finite constants C_{δ_M}, C_{δ_D} and C_{δ_K} are defined by Eq. (97).

Let $\mathbb{X}_{\mathbb{V}}$ and $\mathbb{X}_{\mathbb{H}}$ be the vector spaces of all the second-order random variables defined on probability space $(\mathcal{A}, \mathcal{T}, P)$ with values in $\mathbb{V}$ and $\mathbb{H}$ respectively. If random variables $\mathbf{U} = \{\mathbf{U}(\mathbf{x}), \mathbf{x} \in \Omega\}$ and $\mathbf{W} = \{\mathbf{W}(\mathbf{x}), \mathbf{x} \in \Omega\}$ belong to $\mathbb{X}_{\mathbb{V}}$ and $\mathbb{X}_{\mathbb{H}}$ respectively, then $|||\mathbf{U}|||_{\mathbb{V}}^2 = E\{\|\mathbf{U}\|_{\mathbb{V}}^2\} < +\infty$ and $|||\mathbf{W}|||_{\mathbb{H}}^2 = E\{\|\mathbf{W}\|_{\mathbb{H}}^2\} < +\infty$. Denoting $\{\mathbf{U}_n(\mathbf{x}, t), \mathbf{x} \in \Omega\}$ by $\mathbf{U}_n(t)$ and since $[0, T]$ is a bounded interval, the inequality defined by Eq. (99) means that, for all fixed $n \geq n_0$, function $t \mapsto \mathbf{U}_n(t)$ belongs to the set $L^2([0, T], \mathbb{X}_{\mathbb{V}})$ of all the square integrable functions from $[0, T]$ into $\mathbb{X}_{\mathbb{V}}$ and that the sequence of functions $\{t \mapsto \mathbf{U}_n(t)\}_{n \geq n_0}$ belongs to a bounded set of $L^2([0, T], \mathbb{X}_{\mathbb{V}})$. Similarly, denoting $\{\dot{\mathbf{U}}_n(\mathbf{x}, t), \mathbf{x} \in \Omega\}$ as $\dot{\mathbf{U}}_n(t)$, the inequality defined by Eq. (100) means that for all fixed $n \geq n_0$, function $t \mapsto \dot{\mathbf{U}}_n(t)$ belongs to the set $L^2([0, T], \mathbb{X}_{\mathbb{H}})$ of all the square integrable functions from $[0, T]$ into $\mathbb{X}_{\mathbb{H}}$ and that the sequence of functions $\{t \mapsto \dot{\mathbf{U}}_n(t)\}_{n \geq n_0}$ belongs to a bounded set of $L^2([0, T], \mathbb{X}_{\mathbb{H}})$. It should be noted that the above results hold if spaces $L^2([0, T], \mathbb{X}_{\mathbb{V}})$ and $L^2([0, T], \mathbb{X}_{\mathbb{H}})$ are replaced by the sets of all the bounded functions from $[0, T]$ into $\mathbb{X}_{\mathbb{V}}$ and $\mathbb{X}_{\mathbb{H}}$ respectively.

D. Convergence as dimension n approaches infinity

Since $\{t \mapsto \mathbf{U}_n(t)\}_n$ and $\{t \mapsto \dot{\mathbf{U}}_n(t)\}_n$ are bounded sequences in $L^2([0, T], \mathbb{X}_{\mathbb{V}})$ and $L^2([0, T], \mathbb{X}_{\mathbb{H}})$ respectively (see Section VI.C), from sequences $\{t \mapsto \mathbf{U}_n(t)\}_n$ and $\{t \mapsto \dot{\mathbf{U}}_n(t)\}_n$ can be extracted[33] subsequences $\{t \mapsto \mathbf{U}_{n_k}(t)\}_k$ and $\{t \mapsto \dot{\mathbf{U}}_{n_k}(t)\}_k$ respectively, which are weakly convergent in $L^2([0, T], \mathbb{X}_{\mathbb{V}})$ and $L^2([0, T], \mathbb{X}_{\mathbb{H}})$ respectively, as k approaches infinity.

VII. EXAMPLE

A. Definition of the mean model

The mean structure is constituted of a rectangular homogeneous and isotropic plate located in the plane (Ox_1, Ox_2) of a Cartesian coordinate system $(Ox_1x_2x_3)$, in bending mode (the outplane displacement is x_3), with constant thickness 4×10^{-4} m, width 0.40 m, length 0.50 m, mass density 7800 kg/m^3, Young's modulus 2.1×10^{11} N/m^2 and Poisson ratio 0.29 . This plate is simply supported on 3 edges and free on the fourth edge corresponding to $x_2 = 0$ (see Figure 2). The spectral problem related to the mean reduced matrix model is analyzed using the finite element method. The mean finite element model is constituted of a regular rectangular mesh with a constant step size of 0.01 m in x_1 and x_2 (41 nodes in the width, 51 nodes in the length). Consequently, all the finite elements are the same and each one is a 4-node square plate element. There are 2000 finite elements and $m = 6009$ degrees of freedom (x_3-translations and x_1- and x_2-rotations). The eigenfrequencies $\underline{\omega}_\alpha = 2\pi\underline{\nu}_\alpha$ of the mean model (see Section II.A) are $\underline{\nu}_1 = 1.94$, $\underline{\nu}_2 = 10.28$, $\underline{\nu}_3 = 15.47, \ldots, \underline{\nu}_{23} = 150.89, \underline{\nu}_{24} = 155.82, \underline{\nu}_{25} = 167.44, \ldots, \underline{\nu}_{37} = 247.89, \underline{\nu}_{38} = 253.01, \ldots,$ $\underline{\nu}_{80} = 527.29, \ldots, \underline{\nu}_{120} = 817.65$ Hz. The finite element discretization of the prescribed external forces yields an impulsive load vector denoted $\mathbf{F}(t) \in \mathbb{R}^m$ which is written as $\mathbf{F}(t) = e(t)\,\mathbf{Z}$. Spatial part $\mathbf{Z} = (Z_1, \ldots, Z_m) \in \mathbb{R}^m$ is independent of time t and is such that $Z_j = 0$ for all j in $\{1, \ldots, m\}$ except for the nine DOFs corresponding to the nodes whose (x_1, x_2) coordinates are $(0.30, 0.25)$, $(0.30, 0.26)$, $(0.30, 0.27)$, $(0.31, 0.25)$, $(0.31, 0.26)$, $(0.31, 0.27)$, $(0.32, 0.25)$, $(0.32, 0.26)$ and $(0.32, 0.27)$, for which $Z_j = 1$. Let t_1 be defined by $t_1 = 2\pi/\Delta\omega$ in which $\Delta\omega = 2\pi \times 60$ rad/s and let $\Omega_c = 2\pi \times 200$ rad/s. The impulse function $t \mapsto e(t)$ is a wave-type impulse function such that, for $t < 0$ and $t > t_1$, $e(t) = 0$, and for $0 \leq t \leq t_1$,

$$e(t) = \frac{1}{\Delta\omega\,(t - t_1)}\,\{\sin\{(\Omega_c + \Delta\omega/2)(t - t_1)\} - \sin\{(\Omega_c - \Delta\omega/2)(t - t_1)\}\} \quad . \qquad (103)$$

Figure 3 shows the graph of function $t \mapsto e(t)$ and Figure 4 shows the graph of the modulus of its Fourier transform. It can be seen in Figure 4 that the main part of the energy of impulse function e is distributed over the $[150\,, 250]$ Hz frequency band in which there are 15 structural modes of the mean model (one has $\underline{\nu}_{23} = 150.89$ Hz and $\underline{\nu}_{37} = 247.89$ Hz). It is assumed that the damping rate $\underline{\xi}$ of the mean model is 0.001 for frequencies around 200 Hz. The generalized damping matrix $[\,\underline{D}_n]$ of the mean reduced matrix model, defined by Eq. (29), is written as $[\,\underline{D}_n] = 2\,\underline{\xi}\,\Omega_{\text{ref}}[\,\underline{M}_n]$ in which $[\,\underline{M}_n]$ is the generalized mass matrix of the mean reduced matrix model, defined by Eq. (29), and where $\Omega_{\text{ref}} = 2\,\pi \times 200$ rad/s.

B. Transient response of the mean model

The transient response of the mean model is calculated by solving the evolution problem defined by Eqs. (26)-(27) using an unconditionally stable implicit step-by-step integration method (Newmark integration scheme) with a time step size $\Delta t = 1/4000$ s. This time-step corresponds to 10 time-steps per period for the structural mode of the mean model whose eigenfrequency is $\underline{\nu}_{61} = 402.24$ Hz. The finite element approximation of the maximum $0.5\underline{k}(\mathbf{u}_S, \mathbf{u}_S)$ of the quasi-static response of the mean model (see Section I.B) is equal to 2.5384. Figure 5 shows the convergence of the dynamic magnification factor $\underline{b}_n$ of the mean model, defined by Eq. (31), as dimension n of the mean reduced matrix model increases. From Figure 5, it can be deduced that the transient response of the mean model is reasonably converged when $n \geq 40$. Figure 6 shows the graph of function

$t \mapsto \underline{r}_n(t)$ for $n = 40$ in which $\underline{r}_n(t)$ is the response ratio for the elastic energy calculated with the mean reduced matrix model and defined by Eq. (30). For $n = 40$, the corresponding value of the dynamic magnification factor is $\underline{b}_n = 1.85$.

C. Transient response of the model with random uncertainties

Let us choose $n_0 = 4$. Therefore, for the convergence analysis with respect to dimension n of the reduced matrix model with random uncertainties, we have to consider $n \geq n_0 = 4$. The dispersions of the generalized mass, damping and stiffness random matrices of the reduced matrix model with random uncertainties, are controlled by parameters δ_M, δ_D and δ_K introduced in Section V.A, which have to verify the constraints defined by Eq. (76),

$$0 < \delta_M, \delta_D, \delta_K < 0.7453 \quad . \tag{104}$$

The numerical simulations presented below correspond to the values

$$\delta_M = 0.3 \quad , \quad \delta_D = 0.3 \quad , \quad \delta_K = 0.3 \quad , \tag{105}$$

which verify Eq. (104). We are interested in the random response ratio $R_n(t)$ defined by Eq. (35) and the random dynamic magnification factor $B_n = \max_{t\geq 0} R_n(t)$ defined by Eq. (36). The transient response of the structure with random uncertainties is calculated using the Monte Carlo numerical simulation method. For given generalized mass, damping and stiffness matrices, the evolution problem defined by Eqs. (81)-(82) is solved with the same Newmark integration scheme and with the same time step size $\Delta t = 1/4000$ s.

The Monte Carlo numerical simulation is carried out with n_S samples, denoted $\theta_1, \ldots, \theta_{n_S}$, for which the samples $t \mapsto R_n(t\,;\theta_1), \ldots, t \mapsto R_n(t\,;\theta_{n_S})$ are numerically calculated. For t fixed, the mean value of random variable $R_n(t)$ is estimated by

$$E\{R_n(t)\} \simeq \frac{1}{n_S} \sum_{j=1}^{n_S} R_n(t\,;\theta_j) \quad . \tag{106}$$

The samples of random variable B_n are such that

$$B_n(\theta_j) = \max_{t \geq 0} \; R_n(t\,;\theta_j) \quad . \tag{107}$$

The mean value of random variable B_n is estimated by

$$E\{B_n\} \simeq \frac{1}{n_S} \sum_{j=1}^{n_S} B_n(\theta_j) \quad . \tag{108}$$

Finally, we introduce the function $t \mapsto R_{n,max}(t\,;\boldsymbol{\theta}(n_S))$ and the real number $B_{n,max}(\boldsymbol{\theta}(n_S))$ defined by

$$R_{n,max}(t\,;\boldsymbol{\theta}(n_S)) = \max_{j=1,\ldots,n_S} R_n(t\,;\theta_j) \quad , \tag{109}$$

$$B_{n,max}(\boldsymbol{\theta}(n_S)) = \max_{j=1,\ldots,n_S} B_n(\theta_j) \quad , \tag{110}$$

in which $\boldsymbol{\theta}(n_S) = (\theta_1, \ldots, \theta_{n_S})$. Figure 7 shows the functions $n \mapsto E\{B_n\}$ calculated by Eq. (108) and Figure 8 shows the function $n \mapsto B_{n,max}(\boldsymbol{\theta}(n_S))$ calculated by Eq. (110) for $n_S = 50, 300, 600$ and 900. For n_S sufficiently high ($n_S \geq 300$) the Monte Carlo numerical method is reasonably converged and it can be seen that the nonparametric model proposed is convergent with respect to dimension n of the random reduced matrix model (see Section VI.D). For $n = 120$ and $n_S = 900$, the value of $B_{n,max}(\boldsymbol{\theta}(n_S))$ is 3.45. This value has to be compared to the value for the mean model which is 1.85. Finally, Figure 9 is relative to $n = 120$ and $n_S = 900$ and shows three curves: the lower thin solid line corresponds to the graph of function $t \mapsto \underline{r}_n(t)$, the thick solid line to the graph of function $t \mapsto E\{R_n(t)\}$ calculated by Eq. (106) and the upper thin solid line to the graph of function $t \mapsto R_{n,max}(t\,;\boldsymbol{\theta}(n_S))$ defined by Eq. (109). This figure shows the sensitivity of the maximum transient response due to random uncertainties. The dynamic magnification factor increases when the random uncertainties increase, and is greater than the deterministic dynamic amplification factor of the mean model.

VIII. CONCLUSIONS

We have presented a new approach allowing the random uncertainties to be modeled by a nonparametric model for prediction of transient responses to impulsive loads in linear structural dynamics. This approach has been presented in the context of structural dynamics but can be extended without any difficulty to structural-acoustic problems such as a structure coupled with an internal acoustic cavity. The parametric approaches existing in literature are very useful when the number of uncertain parameters is small and when the probabilistic model can be constructed for the set of parameters considered. The nonparametric approach presented is useful when the number of uncertain parameters is high or when the probabilistic model is difficult to construct for the set of parameters considered. In addition, the parametric approaches do not allow the model uncertainties to be taken into account (because a parametric approach is associated with a fixed model exhibiting some parameters), whereas the nonparametric approach proposed allows to take into account the model uncertainties. For this nonparametric approach, the information used does not require the description of the local parameters of the mechanical model. The probability model is deduced from the use of the entropy optimization principle whose available information is constituted of the fundamental algebraic properties related to the generalized mass, damping and stiffness matrices which have to be positive-definite symmetric matrices for which the second-order moments of their inverse have to exist, and the knowledge of these matrices for the mean reduced matrix model. An explicit construction and representation of the probability model have been obtained and are very well suited to algebraic calculus and to Monte Carlo numerical simulation. The fundamental properties related to the convergence of the stochastic solution with respect to the dimension of the random reduced matrix model has been analyzed. This convergence analysis carried out has allowed the consistency of the theory proposed to be proved and the parameters of the probability distribution of the random generalized matrices to be clearly defined. With this nonparametric model, the probability distribution of each random generalized matrix of the random reduced matrix model depends only on two parameters : the mean generalized matrix associated with the mean mechanical model and corresponding to the design model, and a scalar parameter δ whose values has to be fixed by the designer in the interval $[0\,,1[$ in order to give the dispersion level attached to the random generalized matrix. It seems clear that parameter δ should be a global parameter resulting from expertise, because the model uncertainties which can be taken into account with the nonparametric model, cannot be quantified in terms of correlation between random variables. For instance, if there is no uncertainty for the stiffness model, then $\delta_K = 0$. On the other hand, if it is assumed that the global uncertainty for the stiffness model is 10%, then δ_K has to be 0.1. Nevertheless, experiments are in progress to study the correlation which could exist between the dispersion of the random responses and parameters δ_M, δ_D and δ_K associated with the random generalized matrices.

APPENDIX A: PROOF OF EQ. (53)

In this Appendix, we prove Eq. (53), i.e.,

$$\lambda_{A_n} > \gamma + 1 \implies E\{\|[\mathbf{A}_n]^{-1}\|_F^\gamma\} < +\infty \quad , \quad \gamma \geq 1 \quad , \tag{A.1}$$

in which $\gamma \geq 1$ is a positive integer. Since $[\mathbf{A}_n]$ is a positive-definite random matrix, it can be written as $[\mathbf{A}_n] = [\mathbf{R}_n]\,[\mathbf{\Sigma}_n]\,[\mathbf{R}_n]^T$ in which $[\mathbf{R}_n]$ is an orthogonal random matrix and $[\mathbf{\Sigma}_n]$ is a diagonal positive-definite random matrix whose diagonal elements are the random eigenvalues $\Sigma_1, \ldots, \Sigma_n$.

We then have

$$\| [\mathbf{A}_n]^{-1} \|_F^2 = \frac{1}{\Sigma_1^2} + \ldots + \frac{1}{\Sigma_n^2} \quad . \tag{A.2}$$

Using the probability density function of random vector $\Sigma_1, \ldots, \Sigma_n$ constructed in Ref. 29 and reusing the proof given in Section 3.6 of this reference, it can be proved that $E\{\| [\mathbf{A}_n]^{-1} \|_F^\gamma\} < +\infty$ if and only if

$$\mathcal{I}_\varepsilon = \int_{\|\boldsymbol{\sigma}\|<\varepsilon} (\sigma_1 \times \ldots \times \sigma_n)^{\lambda_A - 1} \left(\frac{1}{\sigma_1^2} + \ldots + \frac{1}{\sigma_n^2} \right)^{\gamma/2} \times \{\Pi_{j<k} |\sigma_k - \sigma_j|\} \, d\boldsymbol{\sigma} < +\infty \quad , \tag{A.3}$$

in which $0 < \varepsilon \ll 1$, $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$ and $d\boldsymbol{\sigma} = (d\sigma_1, \ldots, d\sigma_n)$. We introduce polar coordinates r and $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_{n-1})$ in $\mathbb{R}^n$ such that $\sigma_1 = r \sin\theta_1$, $\sigma_2 = r \cos\theta_1 \sin\theta_2$, $\ldots$, $\sigma_{n-1} = r \cos\theta_1 \cos\theta_2 \ldots \cos\theta_{n-2} \sin\theta_{n-1}$ and $\sigma_n = r \cos\theta_1 \cos\theta_2 \ldots \cos\theta_{n-2} \cos\theta_{n-1}$, in which $-\pi/2 < \theta_j \leq \pi/2$ for $j = 1, \ldots, n-2$ and $-\pi < \theta_{n-1} \leq \pi$. We have $d\boldsymbol{\sigma} = r^{n-1} h_1(\boldsymbol{\theta}) \, dr \, d\boldsymbol{\theta}$ in which $h_1(\boldsymbol{\theta}) = |\cos^{n-2}\theta_1 \cos^{n-3}\theta_3 \ldots \cos\theta_{n-2}|$. If we assume that $\lambda_{A_n} > \gamma + 1$ with $\gamma > 0$, then Eq. (A.3) holds if $\lambda_{A_n} > \gamma/n + (1-n)/2$ and consequently, λ_{A_n} has to be such that $\lambda_{A_n} > \max\{\gamma + 1 , \gamma/n + (1-n)/2\}$. Since $n \geq 1$ and $\gamma > 0$, we deduce Eq. (A.1).

APPENDIX B: PROOF OF EQ. (97)

In this Appendix, we prove Eq. (97). Let $\Sigma_1, \ldots, \Sigma_n$ be the positive-valued random eigenvalues of random matrix $[\mathbf{G}_{A_n}]$ with values in $\mathbb{M}_n^+(\mathbb{R})$. It is proved in Refs. 28 and 29 that the probability density function $p_{\boldsymbol{\Sigma}}(\boldsymbol{\sigma})$ with respect to $d\boldsymbol{\sigma} = \sigma_1 \ldots \sigma_n$ of the random vector $\boldsymbol{\Sigma} = (\Sigma_1, \ldots, \Sigma_n)$ with values in $\mathcal{D}_n = (]0, +\infty[)^n \subset \mathbb{R}^n$ is written as

$$p_{\boldsymbol{\Sigma}}(\boldsymbol{\sigma}) = \mathbb{1}_{\mathcal{D}_n}(\boldsymbol{\sigma}) \times c_\Sigma \times (\sigma_1 \times \ldots \times \sigma_n)^{\lambda_{A_n} - 1} \{\Pi_{\alpha<\beta} |\sigma_\beta - \sigma_\alpha|\} \, e^{-\frac{1}{2}(n-1+2\lambda_{A_n})(\sigma_1 + \ldots + \sigma_n)} \quad , \tag{B.1}$$

in which c_Σ is a constant of normalization defined by the equation $\int_{\mathcal{D}_n} p_{\boldsymbol{\Sigma}}(\boldsymbol{\sigma}) \, d\boldsymbol{\sigma} = 1$ and where $\lambda_{A_n} = \ell_A(n)$ is given by Eq. (66). Let $\widetilde{\boldsymbol{\Sigma}} = (\widetilde{\Sigma}_1, \ldots, \widetilde{\Sigma}_n)$ be the order statistics of $\boldsymbol{\Sigma} = (\Sigma_1, \ldots, \Sigma_n)$ such that $0 \leq \widetilde{\Sigma}_1 \leq \widetilde{\Sigma}_2 \leq \ldots \leq \widetilde{\Sigma}_n$. Let $\mathcal{P}_n$ be the group of all the permutations τ of the first n positive integers $\{1, 2, \ldots, n\}$. Since $p_{\boldsymbol{\Sigma}}$ is a symmetric function in all the variables $\sigma_1, \ldots, \sigma_n$, that is to say, for any permutation τ in $\mathcal{P}_n$,

$$p_{\boldsymbol{\Sigma}}(\sigma_{\tau(1)}, \ldots, \sigma_{\tau(n)}) = p_{\boldsymbol{\Sigma}}(\sigma_1, \ldots, \sigma_n) \quad , \tag{B.2}$$

then the probability density function $p_{\widetilde{\boldsymbol{\Sigma}}}(\boldsymbol{\sigma})$ of order statistics $\widetilde{\boldsymbol{\Sigma}}$ with respect to $d\boldsymbol{\sigma} = d\sigma_1 \ldots d\sigma_n$ is written as[52]

$$p_{\widetilde{\boldsymbol{\Sigma}}}(\boldsymbol{\sigma}) = \frac{\mathbb{1}_{\mathcal{S}_n}(\boldsymbol{\sigma}) \, p_{\boldsymbol{\Sigma}}(\boldsymbol{\sigma})}{\int_{\mathcal{S}_n} p_{\boldsymbol{\Sigma}}(\boldsymbol{\sigma}) \, d\boldsymbol{\sigma}} \quad , \tag{B.3}$$

in which $\mathcal{S}_n$ is the simplex defined by

$$\mathcal{S}_n = \{\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n) \in \mathbb{R}^n \, ; \, 0 < \sigma_1 < \ldots < \sigma_n < +\infty\} \quad . \qquad (B.4)$$

Consequently, using Eqs. (96), (B.3), (B.1) and (65), (66) yields

$$E\{\|[\mathbf{G}_{A_n}]^{-1}\|^2\} = \frac{\int_{\mathcal{S}_n} \sigma_1^{-2} h(\boldsymbol{\sigma}) \, d\boldsymbol{\sigma}}{\int_{\mathcal{S}_n} h(\boldsymbol{\sigma}) \, d\boldsymbol{\sigma}} \quad , \qquad (B.5)$$

in which

$$h(\boldsymbol{\sigma}) = (\sigma_1 \times \ldots \times \sigma_n)^{a(n+1)} \, \{\Pi_{\alpha<\beta} |\sigma_\beta - \sigma_\alpha|\} \, e^{-b(n+1)(\sigma_1 + \ldots + \sigma_n)} \quad , \qquad (B.6)$$

where

$$a = \frac{1 - \delta_A^2}{2\delta_A^2} > 0 \quad , \quad b = \frac{1}{2\delta_A^2} \quad .$$

Let $\varepsilon > 0$ be a positive real number independent of n. Since $h(\boldsymbol{\sigma}) > 0$ for $\boldsymbol{\sigma} \in \mathcal{S}_n$, we have

$$\int_\varepsilon^{+\infty} d\sigma_1 \int_{\sigma_1}^{+\infty} d\sigma_2 \ldots \int_{\sigma_{n-1}}^{+\infty} d\sigma_n \; \sigma_1^{-2} \, h(\boldsymbol{\sigma}) \leq \frac{1}{\varepsilon^2} \int_{\mathcal{S}_n} h(\boldsymbol{\sigma}) \, d\boldsymbol{\sigma} \quad , \qquad (B.7)$$

and consequently, from Eqs. (B.5) and (B.7), we deduce that

$$E\{\|[\mathbf{G}_{A_n}]^{-1}\|^2\} \leq \frac{1}{\varepsilon^2} + \frac{\int_0^\varepsilon d\sigma_1 \int_{\sigma_1}^{+\infty} d\sigma_2 \ldots \int_{\sigma_{n-1}}^{+\infty} d\sigma_n \sigma_1^{-2} \, h(\boldsymbol{\sigma})}{\int_0^{+\infty} d\sigma_1 \int_{\sigma_1}^{+\infty} d\sigma_2 \ldots \int_{\sigma_{n-1}}^{+\infty} d\sigma_n \sigma_1^{-2} \, h(\boldsymbol{\sigma})} \quad . \qquad (B.8)$$

Since function $\boldsymbol{\sigma} \mapsto \sigma_1^{-2} \, h(\boldsymbol{\sigma})$ is symmetric in the variables $\sigma_2, \ldots, \sigma_n$, we have

$$\begin{aligned} \int_0^\varepsilon d\sigma_1 \int_{\sigma_1}^{+\infty} d\sigma_2 \ldots \int_{\sigma_{n-1}}^{+\infty} d\sigma_n \; \sigma_1^{-2} \, h(\boldsymbol{\sigma}) &= \frac{1}{(n-1)!} \int_0^\varepsilon d\sigma_1 \int_{\sigma_1}^{+\infty} d\sigma_2 \ldots \int_{\sigma_1}^{+\infty} d\sigma_n \; \sigma_1^{-2} \, h(\boldsymbol{\sigma}) \\ &\leq \frac{1}{(n-1)!} \int_0^\varepsilon d\sigma_1 \int_0^{+\infty} d\sigma_2 \ldots \int_0^{+\infty} d\sigma_n \; \sigma_1^{-2} \, h(\boldsymbol{\sigma}) \quad , \qquad (B.9) \end{aligned}$$

and since function $\boldsymbol{\sigma} \mapsto h(\boldsymbol{\sigma})$ is symmetric in the variables $\sigma_1, \ldots, \sigma_n$, we have

$$\int_0^{+\infty} d\sigma_1 \int_{\sigma_1}^{+\infty} d\sigma_2 \ldots \int_{\sigma_{n-1}}^{+\infty} d\sigma_n \;\; h(\boldsymbol{\sigma}) = \frac{1}{n!} \int_0^{+\infty} d\sigma_1 \int_0^{+\infty} d\sigma_2 \ldots \int_0^{+\infty} d\sigma_n \;\; h(\boldsymbol{\sigma}) \quad . \qquad (B.10)$$

We then deduce that

$$E\{\|[\mathbf{G}_{A_n}]^{-1}\|^2\} \leq \frac{1}{\varepsilon^2} + H_n(\varepsilon) \quad , \qquad (B.11)$$

with

$$H_n(\varepsilon) = \frac{n \int_0^\varepsilon d\sigma_1 \int_0^{+\infty} d\sigma_2 \ldots \int_0^{+\infty} d\sigma_n \; \sigma_1^{-2} \, h(\boldsymbol{\sigma})}{\int_0^{+\infty} d\sigma_1 \int_0^{+\infty} d\sigma_2 \ldots \int_0^{+\infty} d\sigma_n \;\; h(\boldsymbol{\sigma})} \quad . \qquad (B.12)$$

Using a similar proof to the proof of Lemma 4.4, page 196 of Ref. 56, it can be proved that, for $\varepsilon > 0$ taken sufficiently small and independent of n, we have

$$\lim_{n \to +\infty} H_n(\varepsilon) = 0 \quad . \qquad (B.13)$$

From Eqs. (B.11) and (B.13), we deduce that $n \mapsto E\{\|[\mathbf{G}_{A_n}]^{-1}\|^2\}$ is a bounded function and consequently, Eq. (97) is proved.

APPENDIX C: INEQUALITIES FOR THE RANDOM INSTANTANEOUS KINETIC ENERGY, POTENTIAL ENERGY AND DISSIPATED POWER

In this Appendix, we construct inequalities for the random instantaneous kinetic energy defined by $\frac{1}{2} < [\mathbf{M}_n]\dot{\mathbf{Q}}^n(t), \dot{\mathbf{Q}}^n(t) >$, the random instantaneous potential energy defined by $\frac{1}{2} < [\mathbf{K}_n]\mathbf{Q}^n(t), \mathbf{Q}^n(t) >$ and the random instantaneous dissipated power $< [\mathbf{D}_n]\dot{\mathbf{Q}}^n(\tau), \dot{\mathbf{Q}}^n(\tau) >$.

Let $[\mathbf{A}_n]$ be random matrix $[\mathbf{M}_n]$, $[\mathbf{D}_n]$ or $[\mathbf{K}_n]$ defined on probability space $(\mathcal{A}, \mathcal{T}, P)$ which is written (see Eq. (57)) as $[\mathbf{A}_n] = [\underline{L}_{A_n}]^T\,[\mathbf{G}_{A_n}]\,[\underline{L}_{A_n}]$. Let $\mathbf{Q}^n$ be an $\mathbb{R}^n$-valued random vector defined on the same probability space $(\mathcal{A}, \mathcal{T}, P)$ and which is not independent of random matrix $[\mathbf{G}_{A_n}]$. Let $[\underline{A}_n] = E\{[\mathbf{A}_n]\} = [\underline{L}_{A_n}]^T\,[\underline{L}_{A_n}] \in \mathbb{M}_n^+(\mathbb{R})$. We then have

$$< [\underline{A}_n]\mathbf{Q}^n, \mathbf{Q}^n > \;\leq\; \|\,[\mathbf{G}_{A_n}]^{-1}\|\; < [\mathbf{A}_n]\mathbf{Q}^n, \mathbf{Q}^n > \quad . \tag{C.1}$$

To prove Eq. (C.1), we write $< [\underline{A}_n]\mathbf{Q}^n, \mathbf{Q}^n > = < \mathbf{S}^n, \mathbf{S}^n >$ in which $\mathbf{S}^n = [\underline{L}_{A_n}]\mathbf{Q}^n$. Since $[\mathbf{G}_{A_n}]$ is a random matrix with values in $\mathbb{M}_n^+(\mathbb{R})$, we can write

$$\begin{aligned}
< [\underline{A}_n]\mathbf{Q}^n, \mathbf{Q}^n > &= < [\mathbf{G}_{A_n}]^{-1}[\mathbf{G}_{A_n}]^{1/2}\mathbf{S}^n, [\mathbf{G}_{A_n}]^{1/2}\mathbf{S}^n > \\
&\leq \|\,[\mathbf{G}_{A_n}]^{-1}\| \times \|\,[\mathbf{G}_{A_n}]^{1/2}\mathbf{S}^n\|^2 \\
&= \|\,[\mathbf{G}_{A_n}]^{-1}\|\; < [\mathbf{G}_{A_n}]\mathbf{S}^n, \mathbf{S}^n > \\
&= \|\,[\mathbf{G}_{A_n}]^{-1}\|\; < [\underline{L}_{A_n}]^T\,[\mathbf{G}_{A_n}]\,[\underline{L}_{A_n}]\mathbf{Q}^n, \mathbf{Q}^n > \\
&= \|\,[\mathbf{G}_{A_n}]^{-1}\|\; < [\mathbf{A}_n]\mathbf{Q}^n, \mathbf{Q}^n >
\end{aligned}$$

which is the inequality defined by Eq. (C.1).

Let t and τ be two fixed times in $[0, T]$. Let $\mathbf{U}_n(t)$ be the mapping $\{\mathbf{x} \mapsto \mathbf{U}_n(\mathbf{x}, t)\}$ and $\dot{\mathbf{U}}_n(t) = \partial\mathbf{U}_n(t)/\partial t$. From Eqs. (32),(13),(15),(16) and using Eqs. (23) and (29), we deduce that

$$\underline{m}(\dot{\mathbf{U}}_n(t), \dot{\mathbf{U}}_n(t)) \;=\; < [\underline{M}_n]\dot{\mathbf{Q}}^n(t), \dot{\mathbf{Q}}^n(t) > \quad , \tag{C.2}$$

$$\underline{k}(\mathbf{U}_n(t), \mathbf{U}_n(t)) \;=\; < [\underline{K}_n]\mathbf{Q}^n(t), \mathbf{Q}^n(t) > \quad , \tag{C.3}$$

$$\underline{d}(\dot{\mathbf{U}}_n(\tau), \dot{\mathbf{U}}_n(\tau)) \;=\; < [\underline{D}_n]\dot{\mathbf{Q}}^n(\tau), \dot{\mathbf{Q}}^n(\tau) > \quad . \tag{C.4}$$

Equations (14),(17) and Eqs. (C.2)-(C.4) yield

$$c_{\underline{m}}\|\dot{\mathbf{U}}_n(t)\|^2_{\mathbb{H}} \;\leq\; < [\underline{M}_n]\dot{\mathbf{Q}}^n(t), \dot{\mathbf{Q}}^n(t) > \quad , \tag{C.5}$$

$$c_{\underline{k}}\|\mathbf{U}_n(t)\|^2_{\mathbb{V}} \quad \leq \quad <[\underline{K}_n]\mathbf{Q}^n(t), \mathbf{Q}^n(t)> \quad , \tag{C.6}$$

$$c_{\underline{d}}\|\dot{\mathbf{U}}_n(\tau)\|^2_{\mathbb{V}} \quad \leq \quad <[\underline{D}_n]\dot{\mathbf{Q}}^n(\tau), \dot{\mathbf{Q}}^n(\tau)> \quad , \tag{C.7}$$

in which $c_{\underline{m}}$, $c_{\underline{k}}$ and $c_{\underline{d}}$ are positive finite constants independent of n. Applying Eq. (C.1) to Eqs. (C.5)-(C.7), we deduce the following inequalities of positive-valued random variables,

$$\frac{c_{\underline{m}}}{\|\,[\mathbf{G}_{M_n}]^{-1}\|}\,\|\dot{\mathbf{U}}_n(t)\|^2_{\mathbb{H}} \quad \leq \quad <[\mathbf{M}_n]\dot{\mathbf{Q}}^n(t), \dot{\mathbf{Q}}^n(t)> \quad , \tag{C.8}$$

$$\frac{c_{\underline{k}}}{\|\,[\mathbf{G}_{K_n}]^{-1}\|}\,\|\mathbf{U}_n(t)\|^2_{\mathbb{V}} \quad \leq \quad <[\mathbf{K}_n]\mathbf{Q}^n(t), \mathbf{Q}^n(t)> \quad , \tag{C.9}$$

$$\frac{c_{\underline{d}}}{\|\,[\mathbf{G}_{D_n}]^{-1}\|}\,\|\dot{\mathbf{U}}_n(\tau)\|^2_{\mathbb{V}} \quad \leq \quad <[\mathbf{D}_n]\dot{\mathbf{Q}}^n(\tau), \dot{\mathbf{Q}}^n(\tau)> \quad . \tag{C.10}$$

APPENDIX D: INEQUALITY FOR THE RANDOM INSTANTANEOUS INPUT POWER

In this Appendix, we introduce a mapping $\mathbf{f}(t)$ representing the prescribed external forces and we deduce an inequality for the random instantaneous input power which is absolutely necessary to construct the energy inequality. It should be noted that this kind of inequality cannot be constructed without introducing the continuous dual space of $\mathbb{V}$ and we give the reason below.

Let $\mathbb{V}'$ be the continuous dual space of $\mathbb{V}$ (i.e. the set of all the continuous linear forms on vector space $\mathbb{V}$ defined in Section I.B) and $<\mathbf{f}\,,\mathbf{v}>_{\mathbb{V}',\mathbb{V}}$ be the duality bracket between $\mathbf{f}\in\mathbb{V}'$ and $\mathbf{v}\in\mathbb{V}$ which is linear with respect to $\mathbf{f}$ and $\mathbf{v}$. For all $\mathbf{f}$ in $\mathbb{V}'$ and $\mathbf{v}$ in $\mathbb{V}$, we have

$$|<\mathbf{f}\,,\mathbf{v}>_{\mathbb{V}',\mathbb{V}}| \leq \|\mathbf{f}\|_{\mathbb{V}'}\,\|\mathbf{v}\|_{\mathbb{V}} \quad , \tag{D.1}$$

in which

$$\|\mathbf{f}\|_{\mathbb{V}'} = \sup_{\mathbf{v}\in\mathbb{V},\mathbf{v}\neq 0} \{|<\mathbf{f}\,,\mathbf{v}>_{\mathbb{V}',\mathbb{V}}|\,/\,\|\mathbf{v}\|_{\mathbb{V}}\} \quad , \tag{D.2}$$

is the norm on $\mathbb{V}'$. Since Hilbert space $\mathbb{H}$ (defined in Section I.B) is identified to its continuous dual space $\mathbb{H}'$, we have $\mathbb{V}\subset\mathbb{H}\subset\mathbb{V}'$ and, if $\mathbf{f}$ is in $\mathbb{H}$, we have $<\mathbf{f}\,,\mathbf{v}>_{\mathbb{V}',\mathbb{V}}=(\mathbf{f}\,,\mathbf{v})_{\mathbb{H}}$.
Since $g(\mathbf{v}\,;t)$ defined by Eq. (11) is continuous on $\mathbb{V}$, there exists a unique element $\mathbf{f}(t)$ in $\mathbb{V}'$ such that

$$g(\mathbf{v}\,;t) = <\mathbf{f}(t)\,,\mathbf{v}>_{\mathbb{V}',\mathbb{V}} \quad , \quad \forall\mathbf{v}\in\mathbb{V} \quad , \tag{D.3}$$

and Eq. (D.1) holds. It should be noted that if $\mathbf{g}_{surf}=0$, then $\mathbf{f}$ coincides with $\mathbf{g}_{vol}$ and consequently $\mathbf{f}(t)$ is in $\mathbb{H}$; in general, $\mathbf{g}_{surf}\neq 0$ and then $\mathbf{f}(t)$ is not in $\mathbb{H}$ but is in $\mathbb{V}'$. This is why we need vector space $\mathbb{V}'$ (it is wrong to consider a nonzero prescribed surface force field ($\mathbf{g}_{surf} \neq 0$) and to assume that $\mathbf{f}(t)$ is a square integrable function on domain Ω, i.e. that it belongs to $\mathbb{H}$).

Let $\mathbf{U}_n(t)$ be the mapping $\{\mathbf{x} \mapsto \mathbf{U}_n(\mathbf{x}, t)\}$ and $\dot{\mathbf{U}}_n(t) = \partial \mathbf{U}_n(t)/\partial t$. From Eqs. (D.3), (28) and (32), we deduce the expression of the random instantaneous input power,

$$< \mathbf{F}^n(\tau) \,, \dot{\mathbf{Q}}^n(\tau) > = < \mathbf{f}(\tau) \,, \dot{\mathbf{U}}_n(\tau) >_{\mathbb{V}',\mathbb{V}} \quad . \tag{D.4}$$

Using Eqs. (D.1) and (D.4) yields

$$| < \mathbf{F}^n(\tau) \,, \dot{\mathbf{Q}}^n(\tau) > | \leq \|\mathbf{f}(\tau)\|_{\mathbb{V}'} \, \|\dot{\mathbf{U}}_n(\tau)\|_{\mathbb{V}} \quad . \tag{D.5}$$

APPENDIX E: BASIC INEQUALITIES DERIVED FROM THE RANDOM ENERGY EQUATION

In this Appendix, for any t fixed in $[0, T]$ and for any $n \geq n_0$, using the energy inequality, we prove basic inequalites relative to $|||\dot{\mathbf{U}}_n(t)|||^2_{\mathbb{H}}$ and $|||\mathbf{U}_n(t)|||^2_{\mathbb{V}}$.

Taking the inner product of the two members of Eq. (33) with $\dot{\mathbf{Q}}^n(t)$ yields

$$\begin{aligned} &< [\mathbf{M}_n]\ddot{\mathbf{Q}}^n(t) \,, \dot{\mathbf{Q}}^n(t) > + < [\mathbf{D}_n]\dot{\mathbf{Q}}^n(t) \,, \dot{\mathbf{Q}}^n(t) > \\ &+ < [\mathbf{K}_n]\mathbf{Q}^n(t) \,, \dot{\mathbf{Q}}^n(t) > = < \mathbf{F}^n(t) \,, \dot{\mathbf{Q}}^n(t) > \ . \end{aligned} \tag{E.1}$$

Using the symmetry properties of random matrices $[\mathbf{M}_n]$, $[\mathbf{D}_n]$ and $[\mathbf{K}_n]$, Eq. (E.1) is rewritten as

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big\{ < [\mathbf{M}_n]\dot{\mathbf{Q}}^n(t) \,, \dot{\mathbf{Q}}^n(t) > + < [\mathbf{K}_n]\mathbf{Q}^n(t) \,, \mathbf{Q}^n(t) > \Big\} \\ &\quad + < [\mathbf{D}_n]\dot{\mathbf{Q}}^n(t) \,, \dot{\mathbf{Q}}^n(t) > = < \mathbf{F}^n(t) \,, \dot{\mathbf{Q}}^n(t) > \quad . \end{aligned} \tag{E.2}$$

Integrating the two members of Eq. (E.2) with respect to t over $[0, t]$ with $0 \leq t \leq T$ and taking into account Eq. (34) yields the energy random equation

$$\begin{aligned} &< [\mathbf{M}_n]\dot{\mathbf{Q}}^n(t) \,, \dot{\mathbf{Q}}^n(t) > + < [\mathbf{K}_n]\mathbf{Q}^n(t) \,, \mathbf{Q}^n(t) > \\ &+ 2\int_0^t < [\mathbf{D}_n]\dot{\mathbf{Q}}^n(\tau) \,, \dot{\mathbf{Q}}^n(\tau) > d\tau = 2\int_0^t < \mathbf{F}^n(\tau) \,, \dot{\mathbf{Q}}^n(\tau) > d\tau \quad . \end{aligned} \tag{E.3}$$

which is an equality of random variables. Since $[\mathbf{M}_n]$, $[\mathbf{D}_n]$ and $[\mathbf{K}_n]$ are random matrices with values in $\mathbb{M}_n^+(\mathbb{R})$, for any fixed t in $[0, T]$, we have $< [\mathbf{M}_n]\dot{\mathbf{Q}}^n(t) \,, \dot{\mathbf{Q}}^n(t) > \geq 0$, $< [\mathbf{K}_n]\mathbf{Q}^n(t) \,, \mathbf{Q}^n(t) > \geq 0$ and $\int_0^t < [\mathbf{D}_n]\dot{\mathbf{Q}}^n(\tau) \,, \dot{\mathbf{Q}}^n(\tau) > d\tau \ \geq 0$. From Eq. (E.3), we then deduce that

$$< [\mathbf{M}_n]\dot{\mathbf{Q}}^n(t) \,, \dot{\mathbf{Q}}^n(t) > \quad \leq \quad 2\int_0^t | < \mathbf{F}^n(\tau) \,, \dot{\mathbf{Q}}^n(\tau) > | \, d\tau \quad , \tag{E.4}$$

$$< [\mathbf{K}_n]\mathbf{Q}^n(t)\,,\mathbf{Q}^n(t) > \quad \leq \quad 2\int_0^t | < \mathbf{F}^n(\tau)\,,\dot{\mathbf{Q}}^n(\tau) > |\, d\tau \quad , \tag{E.5}$$

$$\int_0^t < [\mathbf{D}_n]\dot{\mathbf{Q}}^n(\tau)\,,\dot{\mathbf{Q}}^n(\tau) > d\tau \quad \leq \quad \int_0^t | < \mathbf{F}^n(\tau)\,,\dot{\mathbf{Q}}^n(\tau) > |\, d\tau \quad . \tag{E.6}$$

Using the inequalities defined by Eqs. (C.8)-(C.10) and (D.5), Eqs. (E.4)-(E.6) yield

$$c_{\underline{m}} \|\dot{\mathbf{U}}_n(t)\|_{\mathbb{H}}^2 \quad \leq \quad 2\int_0^t \|\mathbf{f}(\tau)\|_{\mathbb{V}'} \, \| \, [\mathbf{G}_{M_n}]^{-1}\| \, \|\dot{\mathbf{U}}_n(\tau)\|_{\mathbb{V}}\, d\tau \quad , \tag{E.7}$$

$$c_{\underline{k}} \|\dot{\mathbf{U}}_n(t)\|_{\mathbb{V}}^2 \quad \leq \quad 2\int_0^t \|\mathbf{f}(\tau)\|_{\mathbb{V}'} \, \| \, [\mathbf{G}_{K_n}]^{-1}\| \, \|\dot{\mathbf{U}}_n(\tau)\|_{\mathbb{V}}\, d\tau \quad , \tag{E.8}$$

$$c_{\underline{d}} \int_0^t \|\dot{\mathbf{U}}_n(\tau)\|_{\mathbb{V}}^2\, d\tau \quad \leq \quad \int_0^t \|\mathbf{f}(\tau)\|_{\mathbb{V}'} \, \| \, [\mathbf{G}_{D_n}]^{-1}\| \, \|\dot{\mathbf{U}}_n(\tau)\|_{\mathbb{V}}\, d\tau \quad . \tag{E.9}$$

Since the left-hand sides and the right-hand sides of Eqs. (E.7)-(E.9) are positive-valued random variables, taking the mathematical expectation of the two members of inequalities (E.7)-(E.9) and using Eqs. (92) and (93) yield

$$c_{\underline{m}} |||\dot{\mathbf{U}}_n(t)|||_{\mathbb{H}}^2 \quad \leq \quad 2\int_0^t \|\mathbf{f}(\tau)\|_{\mathbb{V}'} \, E\{\| \, [\mathbf{G}_{M_n}]^{-1}\| \, \|\dot{\mathbf{U}}_n(\tau)\|_{\mathbb{V}}\}\, d\tau \quad , \tag{E.10}$$

$$c_{\underline{k}} |||\dot{\mathbf{U}}_n(t)|||_{\mathbb{V}}^2 \quad \leq \quad 2\int_0^t \|\mathbf{f}(\tau)\|_{\mathbb{V}'} \, E\{\| \, [\mathbf{G}_{K_n}]^{-1}\| \, \|\dot{\mathbf{U}}_n(\tau)\|_{\mathbb{V}}\}\, d\tau \quad , \tag{E.11}$$

$$c_{\underline{d}} \int_0^t |||\dot{\mathbf{U}}_n(\tau)|||_{\mathbb{V}}^2\, d\tau \quad \leq \quad \int_0^t \|\mathbf{f}(\tau)\|_{\mathbb{V}'} \, E\{\| \, [\mathbf{G}_{D_n}]^{-1}\| \, \|\dot{\mathbf{U}}_n(\tau)\|_{\mathbb{V}}\}\, d\tau \quad . \tag{E.12}$$

Let $[\mathbf{G}_{A_n}]$ be $[\mathbf{G}_{M_n}]$, $[\mathbf{G}_{K_n}]$ or $[\mathbf{G}_{D_n}]$. Using the Holder inequality, we can write

$$E\{\| \, [\mathbf{G}_{A_n}]^{-1}\| \, \|\dot{\mathbf{U}}_n(\tau)\|_{\mathbb{V}}\} \quad \leq \quad \sqrt{E\{\| \, [\mathbf{G}_{A_n}]^{-1}\|^2\}} \; |||\dot{\mathbf{U}}_n(\tau)|||_{\mathbb{V}} \quad . \tag{E.13}$$

For $n \geq n_0$, Eqs. (97) and (E.13) yield

$$E\{\| \, [\mathbf{G}_{A_n}]^{-1}\| \, \|\dot{\mathbf{U}}_n(\tau)\|_{\mathbb{V}}\} \quad \leq \quad C_{\delta_A}^{1/2} \; |||\dot{\mathbf{U}}_n(\tau)|||_{\mathbb{V}} \quad , \tag{E.14}$$

in which C_{δ_A} is a finite real constant which is independent of n. Substituting the left-hand side of Eq. (E.14) into the right-hand side of Eqs. (E.10)-(E.12) yields

$$|||\dot{\mathbf{U}}_n(t)|||_{\mathbb{H}}^2 \quad \leq \quad \frac{2\, C_{\delta_M}^{1/2}}{c_{\underline{m}}} \int_0^t \|\mathbf{f}(\tau)\|_{\mathbb{V}'} \; |||\dot{\mathbf{U}}_n(\tau)|||_{\mathbb{V}}\}\, d\tau \quad , \tag{E.15}$$

$$|||\dot{\mathbf{U}}_n(t)|||_{\mathbb{V}}^2 \quad \leq \quad \frac{2\, C_{\delta_K}^{1/2}}{c_{\underline{k}}} \int_0^t \|\mathbf{f}(\tau)\|_{\mathbb{V}'} \; |||\dot{\mathbf{U}}_n(\tau)|||_{\mathbb{V}}\}\, d\tau \quad , \tag{E.16}$$

$$\int_0^t |||\dot{\mathbf{U}}_n(\tau)|||^2_{\mathbb{V}}\, d\tau \quad \leq \quad \frac{C^{1/2}_{\delta_D}}{c_{\underline{d}}} \int_0^t \|\mathbf{f}(\tau)\|_{\mathbb{V}'}\ |||\dot{\mathbf{U}}_n(\tau)|||_{\mathbb{V}}\}\, d\tau \quad . \tag{E.17}$$

If a, b and η are three positive real numbers, we have

$$a\, b \quad \leq \quad \frac{a^2}{4\eta} + \eta\, b^2 \quad .$$

Applying this inequality to the right-hand sides of Eqs. (E.15)-(E.17) with $\eta = 0.5\, c_{\underline{m}}\, C^{-1/2}_{\delta_M}$, $\eta = 0.5\, c_{\underline{k}}\, C^{-1/2}_{\delta_K}$ and $\eta = 0.5\, c_{\underline{d}}\, C^{-1/2}_{\delta_D}$ respectively, yields

$$|||\dot{\mathbf{U}}_n(t)|||^2_{\mathbb{H}} \quad \leq \quad \int_0^t |||\dot{\mathbf{U}}_n(\tau)|||^2_{\mathbb{V}}\}\, d\tau + \frac{C_{\delta_M}}{c^2_{\underline{m}}} \int_0^T \|\mathbf{f}(\tau)\|^2_{\mathbb{V}'}\, d\tau \quad , \tag{E.18}$$

$$|||\dot{\mathbf{U}}_n(t)|||^2_{\mathbb{V}} \quad \leq \quad \int_0^t |||\dot{\mathbf{U}}_n(\tau)|||^2_{\mathbb{V}}\}\, d\tau + \frac{C_{\delta_K}}{c^2_{\underline{k}}} \int_0^T \|\mathbf{f}(\tau)\|^2_{\mathbb{V}'}\, d\tau \quad , \tag{E.19}$$

$$\int_0^t |||\dot{\mathbf{U}}_n(\tau)|||^2_{\mathbb{V}}\, d\tau \quad \leq \quad \frac{C_{\delta_D}}{c^2_{\underline{d}}} \int_0^T \|\mathbf{f}(\tau)\|^2_{\mathbb{V}'}\, d\tau \quad . \tag{E.20}$$

Adding Eq. (E.18) to Eq. (E.20) yields

$$|||\dot{\mathbf{U}}_n(t)|||^2_{\mathbb{H}} \quad \leq \quad \left(\frac{C_{\delta_M}}{c^2_{\underline{m}}} + \frac{C_{\delta_D}}{c^2_{\underline{d}}} \right) \int_0^T \|\mathbf{f}(\tau)\|^2_{\mathbb{V}'}\, d\tau \quad , \tag{E.21}$$

and adding Eq. (E.19) to Eq. (E.20) yields

$$|||\mathbf{U}_n(t)|||^2_{\mathbb{V}} \quad \leq \quad \left(\frac{C_{\delta_K}}{c^2_{\underline{k}}} + \frac{C_{\delta_D}}{c^2_{\underline{d}}} \right) \int_0^T \|\mathbf{f}(\tau)\|^2_{\mathbb{V}'}\, d\tau \quad . \tag{E.22}$$

[1] R. W. Clough and J. Penzien, *Dynamics of Structures* (McGraw-Hill, New York, 1975).

[2] K. J. Bathe and E. L. Wilson, *Numerical Methods in Finite Element Analysis* (Prentice-Hall, New York, 1976).

[3] L. Meirovitch, *Computational Methods in Structural Dynamics* (Sijthoff and Noordhoff, The Netherlands, 1980).

[4] O. C. Zienkiewicz and R. L. Taylor, *The Finite Element Method* (4th edition) (McGraw-Hill, New York, 1989) (vol. 1, 1989 and vol. 2, 1991).

[5] J. Argyris and H. P. Mlejnek, *Dynamics of Structures* (North-Holland, Amsterdam, 1991).

[6] Y. K. Cheung and A. Y. T. Leung, *Finite Element Methods in Dynamics* (Kluwer Academic Pub, 1992).

[7] M. Géradin and D. Rixen, *Mechanical Vibrations* (Wiley, Chichester, U.K., 1994).

[8] R. Ohayon and C. Soize, *Structural Acoustics and Vibration* (Academic Press, San Diego, London, 1998).

[9] T. T. Soong, *Random Differential Equations in Science and Engineering* (Academic Press, New York, 1973).

[10] E. J. Haug, K. K. Choi and V. Komkov, *Design Sensitivity Analysis of Structural Systems* (Academic Press, 1986).

[11] R. A. Ibrahim, "Structural dynamics with parameter uncertainties", Applied Mechanics Reviews **40**(3), 309-328 (1987).

[12] P. D. Spanos and B. A. Zeldin, "Galerkin sampling method for stochastic mechanics problems", Journal of Engineering Mechanics **120**(5), 1091-1106 (1994).

[13] W. D. Iwan and H. Jensen, "On the dynamical response of continuous systems including model uncertainty", Transactions of ASME **60**, 484-490 (1993).

[14] C. Lee and R. Singh, "Analysis of discrete vibratory systems with parameter uncertainties. Part II: Impulse response", Journal of Sound and Vibration, **174**(3), 395-412 (1994).

[15] Y. K. Lin and G. Q. Cai, *Probabilistic Structural Dynamics* (McGraw-Hill, New York, 1995).

[16] E. Vanmarcke and M. Grigoriu, "Stochastic finite element analysis of simple beams", Journal of Engineering Mechanics **109**(5), 1203-1214 (1983).

[17] W. K. Liu, T. Belytschko and A. Mani, "Random field finite elements", International Journal of Numerical Methods in Engineering **23**, 1832-1845 (1986).

[18] M. Shinozuka and G. Deodatis, "Response variability of stochastic finite element systems", Journal of Engineering Mechanics **114**(3), 499-519 (1988).

[19] P. D. Spanos and R. G. Ghanem, "Stochastic finite element expansion for random media", ASCE Journal of Engineering Mechanics, **115**(5), 1035-1053 (1989).

[20] R. G. Ghanem and P. D. Spanos, *Stochastic Finite Elements: A Spectral Approach* (Springer-Verlag, New York, 1991).

[21] M. Kleiber, D. H. Tran and T. D. Hien, *The Stochastic Finite Element Method* (John Wiley and Sons, 1992).

[22] Y. K. Lin, *Probabilistic Theory of Structural Dynamics* (McGraw-Hill, New York, 1967) (Reprint R.E., Krieger, Melbourne, Fla., 1976).

[23] R. A. Ibrahim, *Parametric Random Vibration* (John Wiley and Sons, New York, 1985).

[24] P. Kree and C. Soize, *Mathematics of Random Phenomena* (Reidel, Dordrecht, 1986).

[25] J. B. Roberts and P. D. Spanos, *Random Vibration and Statistical Linearization* (John Wiley and Sons, New York, 1990).

[26] T. T. Soong and M. Grigoriu, *Random Vibration of Mechanical and Structural Systems* (P T R Prentice Hall, Englewood Cliffs, New Jersey, 1993).

[27] C. Soize, *The Fokker-Planck Equation for Stochastic Dynamical Systems and its Explicit Steady State Solutions* (World Scientific, Singapore, 1994).

[28] C. Soize, "A nonparametric model of random uncertainties in linear structural dynamics", Publication du LMA-CNRS, ISBN 2-909669-16-5, **152**, 109-138 (1999).

[29] C. Soize, "A nonparametric model of random uncertainties for reduced matrix models in structural dynamics", Probabilistic Engineering Mechanics, **15**(3), 277-294 (2000).

[30] Y. C. Fung, *Foundations of Solid Mechanics* (Prentice Hall, Englewood Cliffs, New Jersey, 1968).

[31] C. Truesdell, *Mechanics of Solids, Vol. III, Theory of Viscoelasticity, Plasticity, Elastic Waves and Elastic Stability* (Springer-Verlag, Berlin, 1984).

[32] P. G. Ciarlet, *Mathematical Elasticity, Vol. I: Three-Dimensional Elasticity* (North-Holland, Amsterdam, 1988).

[33] R. Dautray and J.-L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology* (Springer-Verlag, Berlin, 1992).

[34] M. Reed and B. Simon, *Methods of Modern Mathematical Physics. I. Functional Analysis* (Academic Press, New York, 1980).

[35] F. Treves, *Basic Linear Partial Differential Equations* (Academic Press, New York, 1975).

[36] J. E. Marsden and T. J. R. Hughes, *Mathematical Foundations of Elasticity* (Prentice Hall, Englewood Cliffs, New Jersey, 1983).

[37] J. T. Oden and J. N. Reddy, *Variational Methods in Theoretical Mechanics* (Springer-Verlag, New York, 1983).

[38] M. L. Mehta, *Random Matrices* (Revised and Enlarged Second Edition, Academic Press, 1991).

[39] P. A. Deift, *Orthogonal Polynomials and Random Matrices: A Riemann-Hilbert Approach* (Courant Institute, New York University, 1999).

[40] N. M. Katz and P. Sarnak, *Random Matrices, Frobenius Eigenvalues and Monodromy* (American Mathematical Society, Providence, Rhode Island, 1999).

[41] D. Fox and P. B. Kahn, “Higher order spacing distributions for a class of unitary ensembles”,

Physical Review, **134**(5B), 1151-1155 (1964).

[42] M. V. Moorthy, “Inversion of the multi-dimensional Laplace transform-expansion by Laguerre series”, Z angew Math. Phys., **46**, 792-806 (1995).

[43] T. Nagao and K. Slevin, “Laguerre ensembles of random matrices: Nonuniversal correlation functions”, J. Math. Phys. **34**(6), 2317-2330 (1993).

[44] C. E. Shannon, "A mathematical theory of communication", Bell System Tech. J., **27**, 379-423 and 623-659 (1948).

[45] E. T. Jaynes, "Information theory and statistical mechanics", Physical Review **106**(4), 620-630 and **108**(2), 171-190 (1957).

[46] J. N. Kapur and H. K. Kesavan, *Entropy Optimization Principles with Applications* (Academic Press, San Diego, 1992).

[47] C. Fougeaud and A. Fuchs, *Statistique* (Dunod, Paris, 1967) (2nd edition 1972).

[48] T. W. Anderson, *Introduction to Multivariate Statistical Analysis* (John Wiley and Sons, 1958).

[49] J. M. Hammersley and D. C. Handscomb, *Monte Carlo Methods* (Chapman and Hall, 1964).

[50] W. G. Cochran, *Sampling Techniques* (John Wiley and Sons, 1977).

[51] R. Y. Rubinstein, *Simulation and the Monte Carlo Method* (John Wiley and Sons, New York, 1981).

[52] L. Devroye, *Non-Uniform Random Variate Generation* (Springer-Verlag, New York, 1986).

[53] M. H. Kalos and P. A. Whitlock, *Monte Carlo Methods, Volume 1: Basics* (John Wiley and Sons, 1986).

[54] P. Bratley, B. L. Fox and E. L. Schrage, *A Guide to Simulation* (Springer-Verlag, New York, 2nd edition, 1987).

[55] P. K. MacKeown, *Stochastic Simulation in Physics* (Springer-Verlag, Singapore, 1997).

[56] K. Johansson, “On fluctuations of eigenvalues of random Hermitian matrices”, Duke Mathematical Journal, **91**(1), 151-204 (1998).

LEGENDS ACCOMPANYING EACH FIGURE

FIG. 1. Graph of function $n \mapsto E\{\|[\mathbf{G}_{A_n}]^{-1}\|^2\}$ for $\delta_A = 0.1$, 0.3 and 0.5.

FIG. 2. Geometry of the mean structure.

FIG. 3. Graph of wave impulse function $t \mapsto e(t)$.

FIG. 4. Graph of the modulus of the Fourier transform of wave impulse function.

FIG. 5. Graph of the convergence of dynamic magnification factor $\underline{b}_n$ as function of n for the mean reduced matrix model.

FIG. 6. Graph of function $t \mapsto \underline{r}_n(t)$ for $n = 40$ corresponding to the response ratio for the mean reduced matrix model.

FIG. 7. Graph of function $n \mapsto E\{B_n\}$ (mathematical expectation of the random dynamic magnification factor) for $n_S = 50$ (circle symbol), 300 (x-mark symbol), 600 (plus symbol) and 900 (square symbol).

FIG. 8. Graph of function $n \mapsto B_{n,max}(\boldsymbol{\theta}(n_S))$ (maximum of the random dynamic magnification factor) for $n_S = 50$ (circle symbol), 300 (x-mark symbol), 600 (plus symbol) and 900 (square symbol).

FIG. 9. Transient responses $t \mapsto \underline{r}_n(t)$ (lower thin solid line), $t \mapsto E\{R_n(t)\}$ (thick solid line) and $t \mapsto R_{n,max}(t\,;\boldsymbol{\theta}(n_S))$ (upper thin solid line) for $n = 120$ and $n_S = 900$.

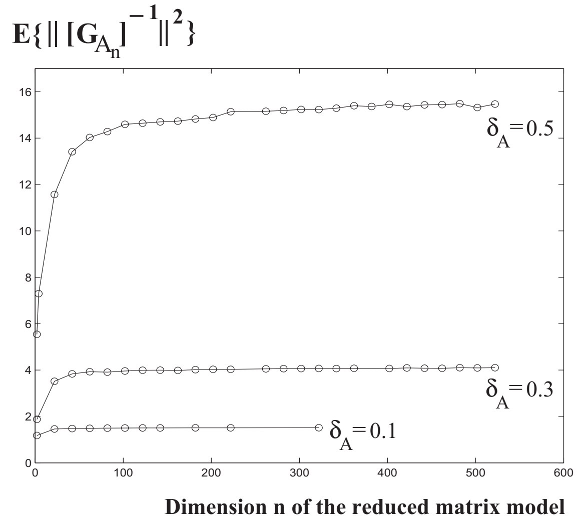

Fig. 1, Christian Soize, J. Acoust. Soc. Am.

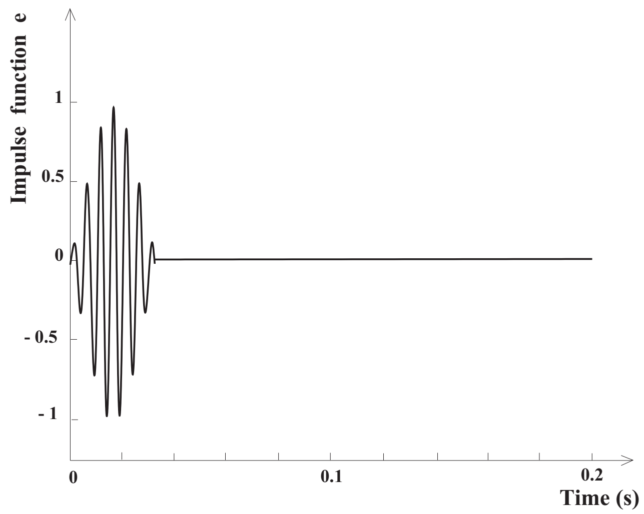

Fig. 3, Christian Soize, J. Acoust. Soc. Am.

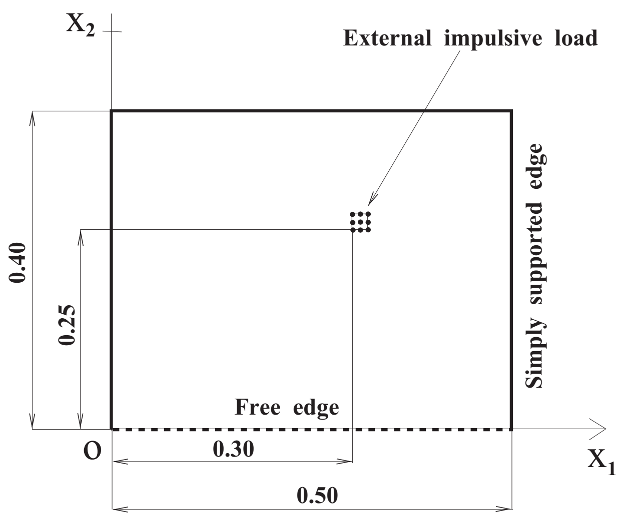

Fig. 2, Christian Soize, J. Acoust. Soc. Am.

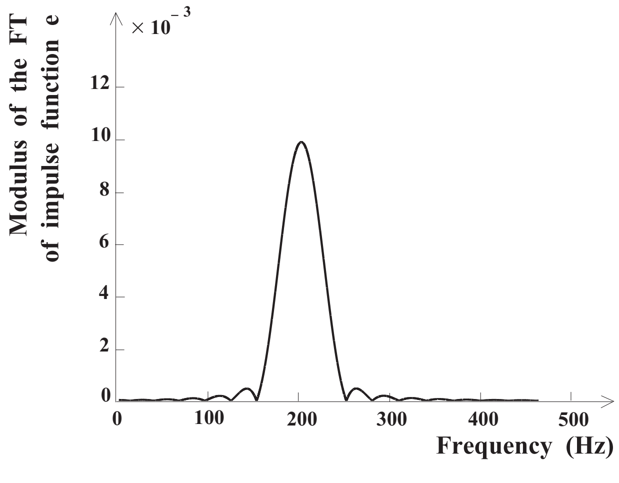

Fig. 4, Christian Soize, J. Acoust. Soc. Am.

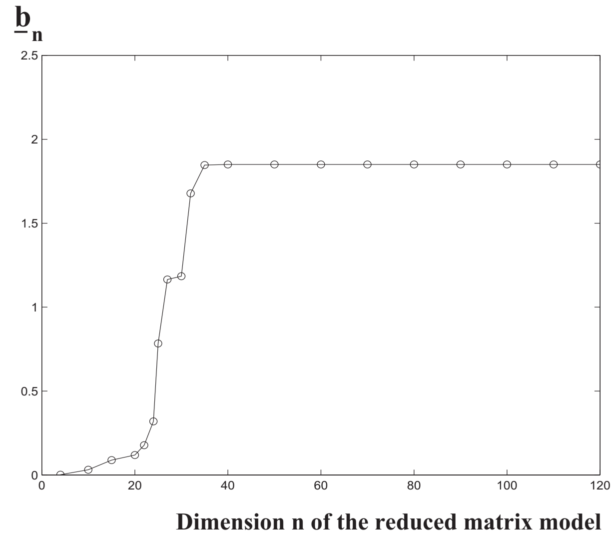

Fig. 5, Christian Soize, J. Acoust. Soc. Am.

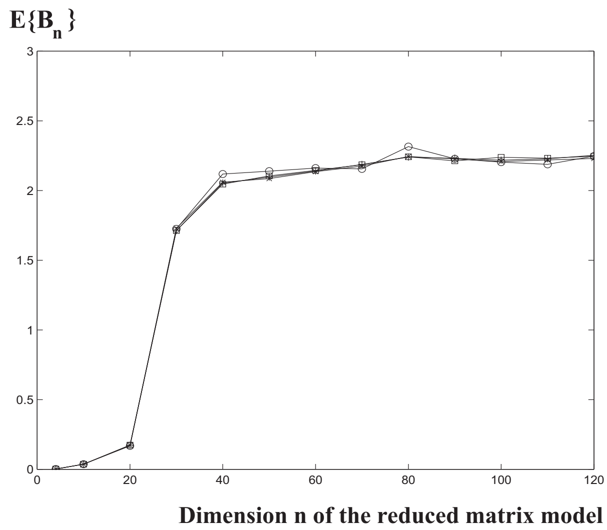

Fig. 7, Christian Soize, J. Acoust. Soc. Am.

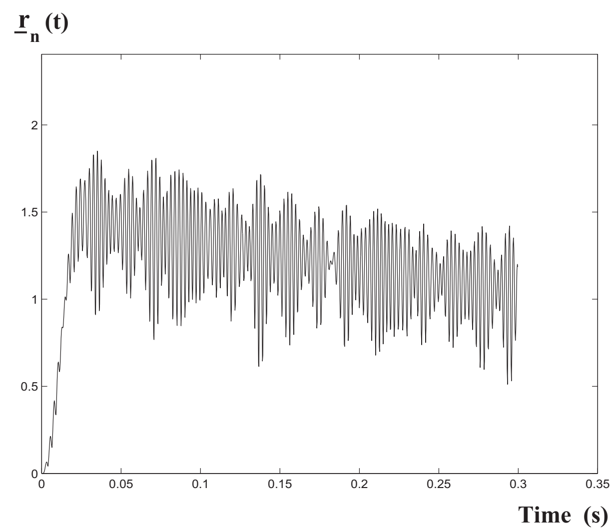

Fig. 6, Christian Soize, J. Acoust. Soc. Am.

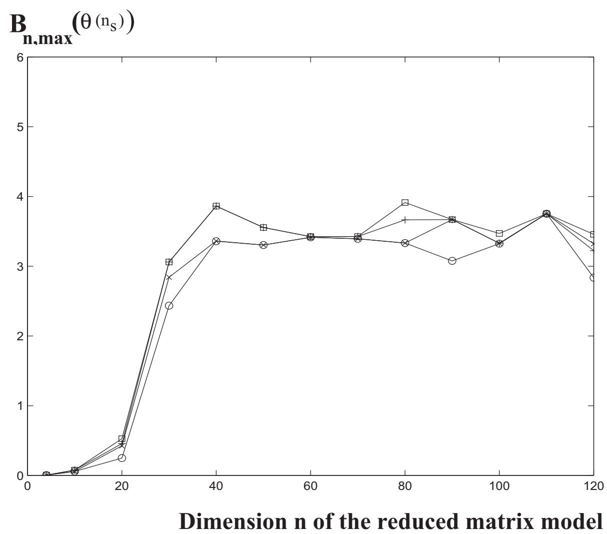

Fig. 8, Christian Soize, J. Acoust. Soc. Am.

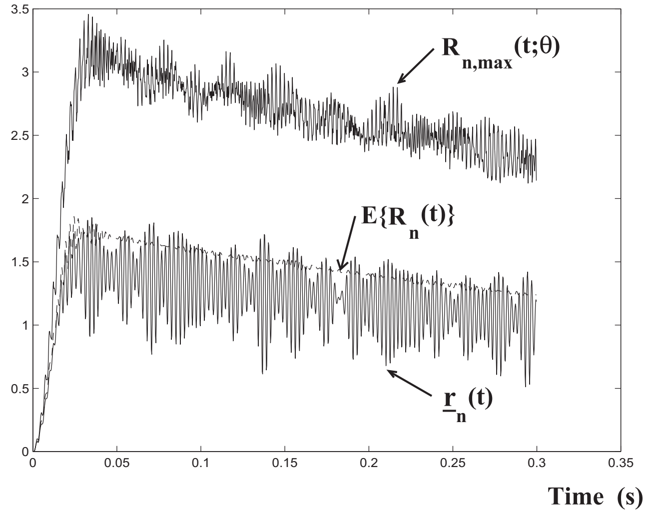

Fig. 9, Christian Soize, J. Acoust. Soc. Am.