Random matrix theory and random uncertainties modeling

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ABSTRACT: Random matrix theory was intensively studied in the context of nuclear physics. For physical applications, the most important ensemble is the Gaussian Orthogonal Ensemble (GOE) whose elements are real symmetric random matrices with statistically independent entries and are invariant under orthogonal linear transformations. Recently, a new approach, called a nonparametric model of random uncertainties, has been introduced by the author for modeling random uncertainties in vibration analysis. This approach has been developed in introducing a new ensemble of random matrices constituted of symmetric positive-definite real random matrices, called the "positive-definite" ensemble, which differs from the GOE. The first objective of this paper is to compare the GOE with the "positive-definite" ensemble of random matrices in the context of the nonparametric approach of random uncertainties in dynamic systems for the low-frequency range. The second objective of this paper is to give a new validation for the nonparametric model of random uncertainties in dynamic systems in comparing, in the low-frequency range, the dynamical response of a simple system having random uncertainties modeled by the parametric and the nonparametric methods. It is proved that the "positive-definite" ensemble of random matrices, which has been introduced in the context of the development of this nonparametric approach, is well adapted to the low-frequency vibration analysis, while the use of the Gaussian orthogonal ensemble (GOE) is not.

1 INTRODUCTION

The random matrix theory was intensively studied by physicists and mathematicians in the context of nuclear physics. These works began with Wigner in the 1950s and received an important effort in the 1960s by Wigner (1962), Dyson (1962), Mehta and others. In 1967, Mehta published the first edition of a book whose second edition (see Mehta (1991)) published in 1991 is an excellent synthesis of the random matrix theory. For physical applications, the most important ensemble of the random matrix theory, is the Gaussian Orthogonal Ensemble (GOE) whose elements are constituted of real symmetric random matrices with statistically independent entries and which are invariant under orthogonal linear transformations. The random matrix theory has been used in other domains that nuclear physics. In 1984, Bohigas & Giannoni & Schmit (1984) found that the level fluctuations of the quantum Sinai's billard were able to be predicted with the GOE of random matrices. In 1989, Weaver (1989) show that the higher frequencies of elastodynamic structures constituted of small aluminium blocks have the behavior of the eigenvalues of a matrix belonging to the GOE. Then, Legrand & Schmit & Sornette (1992) studied the high-frequency spectral statistics with the GOE for elastodynamics and vibration problems in the high-frequency range. All these results have clearly been validated for the high-frequency range in elastodynamics but not at all for the low- and medium-frequency ranges.

Recently, a new approach, called a nonparametric model of random uncertainties, has been introduced by Soize (2000, 2001a & 2001b) for modeling random uncertainties in linear and nonlinear elastodynamics in the modal range, that is to say, in the low-frequency range. This nonparametric approach differs from the parametric and stochastic finite element methods for random uncertainties modeling and has been developed in introducing a new ensemble of random matrices constituted of symmetric positive-definite real random matrices (see Soize (2000 & 2001a)), that we call the "positive-definite" ensemble of random matrices. This ensemble differs from the GOE and from the other known ensembles of the random matrix theory.

The first objective of this paper is to compare the GOE with the "positive-definite" ensemble of random matrices in the context of the nonparametric approach of random uncertainties in dynamic systems for the low-

frequency range. This comparison will be limited to the case for which the generalized stiffness matrix of the dynamic system is random while its generalized mass and damping matrices are deterministic. This limitation is due to the fact that, in the state of the art, the GOE does not allow a damped dynamic system to be modeled while the new ensemble allows mass, damping and stiffness random uncertainties to be modeled. The second objective of this paper is to give a new validation for the nonparametric model of random uncertainties in dynamic systems in comparing, in the low-frequency range, the dynamical response of a simple system having random uncertainties modeled by the parametric method and by the nonparametric method.

1.1 *Notation*

In this paper, $\mathbb{M}_n(\mathbb{R})$, $\mathbb{M}_n^S(\mathbb{R})$ and $\mathbb{M}_n^+(\mathbb{R})$ are the set of all the $(n\times n)$ real matrices, the set of all the symmetric $(n\times n)$ real matrices and the set of all the positive-definite symmetric $(n\times n)$ real matrices, respectively. One has $\mathbb{M}_n^+(\mathbb{R}) \subset \mathbb{M}_n^S(\mathbb{R}) \subset \mathbb{M}_n(\mathbb{R})$. If $[A]$ belongs to $\mathbb{M}_n(\mathbb{R})$, $\|[A]\|_F = (\mathrm{tr}\{[A]\,[A]^T\})^{1/2}$ is the Frobenius norm of matrix $[A]$, where tr is the trace of the matrices, det is the determinant of the matrices and $[A]^T$ is the transpose of matrix $[A]$. The indicatrix function $\mathbb{1}_{\mathcal{B}}(b)$ of any set $\mathcal{B}$ is such that $\mathbb{1}_{\mathcal{B}}(b)$ is equal to 1 if $b \in \mathcal{B}$ and is equal to zero if $b \notin \mathcal{B}$. The gamma function is defined for $z > 0$ by $\Gamma(z) = \int_0^{+\infty} t^{z-1}\, e^{-t}\, dt$. All the random variables are defined on a probability space $(\mathcal{A}, \mathcal{T}, \mathcal{P})$ and E is the mathematical expectation.

1.2 *A nonparametric model of random uncertainties in vibration analysis*

We briefly recall the main ideas introduced in Soize (2000 & 2001a) concerning the nonparametric model in elastodynamics and vibrations for the low-frequency range. We limit the developments to the case of linear dynamic systems for which only the generalized stiffness matrix is uncertain. The main available information is constituted of the mean reduced model constructed with the n generalized coordinates of the mode-superposition method associated with the elastic modes corresponding to the n lowest eigenfrequencies of the linear dynamic system, presently assumed to be fixed, damped and stable. The nonparametric probabilistic model of random uncertainties consists in replacing the generalized diagonal stiffness matrix $[\underline{K}_n] \in \mathbb{M}_n^+(\mathbb{R})$ of the mean reduced model by the full random matrix $[\mathbf{K}_n]$ with values in $\mathbb{M}_n^+(\mathbb{R})$, for which the mean value is known and such that

$$E\{[\mathbf{K}_n]\} = [\underline{K}_n] \quad . \tag{1}$$

The probability model of random matrix $[\mathbf{K}_n]$ was constructed in Soize (2000 & 2001a). Random matrix $[\mathbf{K}_n]$ has to be with values in $\mathbb{M}_n^+(\mathbb{R})$ in order to obtain a mechanical system with random uncertainties, which models a fixed and stable dynamic system. This probability model is summarized in Section 1.3.

1.3 *Probability model for a random matrix belonging to the "positive-definite" ensemble of random matrices*

In this subsection, we summarize a part of the results developed in Soize (2000 & 2001a), concerning the probability model of random matrix $[\mathbf{K}_n]$ belonging to the "positive-definite" ensemble of random matrices. Consequently, random matrix $[\mathbf{K}_n]$ is a random matrix with values in $\mathbb{M}_n^+(\mathbb{R})$. Its mean value $[\underline{K}_n]$ is a positive-definite real matrix. There is an upper triangular matrix $[\underline{L}_{K_n}]$ in $\mathbb{M}_n(\mathbb{R})$ (Cholesky factorization) such that

$$[\underline{K}_n] = [\underline{L}_{K_n}]^T\, [\underline{L}_{K_n}] \quad . \tag{2}$$

Random matrix $[\mathbf{K}_n]$ is written as

$$[\mathbf{K}_n] = [\underline{L}_{K_n}]^T\, [\mathbf{G}_{K_n}]\, [\underline{L}_{K_n}] \quad , \tag{3}$$

in which matrix $[\mathbf{G}_{K_n}]$ is a random variable with values in $\mathbb{M}_n^+(\mathbb{R})$ such that

$$[\underline{G}_{K_n}] = E\{[\mathbf{G}_{K_n}]\} = [\,I_n] \quad , \tag{4}$$

where $[\,I_n]$ is the $(n\times n)$ identity matrix. Let $\delta_K > 0$ be the real parameter defined by

$$\delta_K = \left\{ \frac{E\{\|\,[\mathbf{G}_{K_n}] - [\underline{G}_{K_n}]\,\|_F^2\}}{\|\,[\underline{G}_{K_n}]\,\|_F^2} \right\}^{1/2} \quad . \tag{5}$$

Parameter δ_K allows the dispersion of the probability model of random matrix $[\mathbf{K}_n]$ to be controlled. If $n_0 \geq 1$ is a given and fixed integer, then the dispersion of the probability model is defined by giving parameter δ_K, independent of n, a value such that $0 < \delta_K < \{(n_0+1)/(n_0+5)\}^{1/2}$. Random matrix $[\mathbf{G}_{K_n}]$ is written as

$$[\mathbf{G}_{K_n}] = [\mathbf{L}_{K_n}]^T\, [\mathbf{L}_{K_n}] \quad , \tag{6}$$

in which $[\mathbf{L}_{K_n}]$ is an upper triangular random matrix with values in $\mathbb{M}_n(\mathbb{R})$ such that:

(1) random variables $\{[\mathbf{L}_{K_n}]_{jj'}, j \leq j'\}$ are independent;

(2) for $j < j'$, real-valued random variable $[\mathbf{L}_{K_n}]_{jj'}$ is written as $[\mathbf{L}_{K_n}]_{jj'} = \sigma_n U_{jj'}$ in which $\sigma_n = \delta_K (n+1)^{-1/2}$ and where $U_{jj'}$ is a real-valued Gaussian random variable with zero mean and variance equal to 1;

(3) for $j = j'$, positive-valued random variable $[\mathbf{L}_{K_n}]_{jj}$ is written as $[\mathbf{L}_{K_n}]_{jj} = \sigma_n \sqrt{2V_j}$ in which σ_n is defined above and where V_j is a positive-valued gamma random variable whose probability density function $p_{V_j}(v)$ with respect to dv is written as

$$p_{V_j}(v) = \mathbb{1}_{\mathbb{R}^+}(v) \frac{1}{\Gamma\left(\frac{n+1}{2\delta_K^2} + \frac{1-j}{2}\right)}\; v^{\frac{n+1}{2\delta_K^2} - \frac{1+j}{2}}\; e^{-v} \quad . \tag{7}$$

The variance $V_{jk}^{G_{K_n}} = E\{([\mathbf{G}_{K_n}]_{jk} - [\underline{G}_{K_n}]_{jk})^2\}$ of random variable $[\mathbf{G}_{K_n}]_{jk}$ is such that

$$V_{jk}^{G_{K_n}} = \frac{\delta_K^2}{n+1}(1+\delta_{jk}) \quad , \tag{8}$$

in which $\delta_{jk} = 0$ if $j \neq k$ and $\delta_{jj} = 1$.

1.4 *Probability model for a random matrix belonging to the GOE*

In this subsection, the random matrix $[\mathbf{K}_n^{\mathrm{GOE}}]$ with values in $\mathbb{M}_n^S(\mathbb{R})$ is constructed by using the Gaussian Orthogonal Ensemble (GOE) (concerning the GOE, see for instance Mehtah (1991)). In order to perform the comparisons with the model summarized in Section 1.3, it is assumed that the mean value of random matrix $[\mathbf{K}_n^{\mathrm{GOE}}]$ is the positive-definite symmetric real matrix $[\underline{K}_n]$ defined by Eq. (1), that is to say

$$E\{[\mathbf{K}_n^{\mathrm{GOE}}]\} = [\underline{K}_n] \in \mathbb{M}_n^+(\mathbb{R}) \quad . \tag{9}$$

Random matrix $[\mathbf{K}_n^{\mathrm{GOE}}]$ is written as

$$[\mathbf{K}_n^{\mathrm{GOE}}] = [\underline{L}_{K_n}]^T \, [\mathbf{G}_{K_n}^{\mathrm{GOE}}] \, [\underline{L}_{K_n}] \quad , \tag{10}$$

in which matrix $[\mathbf{G}_{K_n}^{\mathrm{GOE}}]$ is a random variable with values in $\mathbb{M}_n^S(\mathbb{R})$ such that

$$[\underline{G}_{K_n}^{\mathrm{GOE}}] = E\{[\mathbf{G}_{K_n}^{\mathrm{GOE}}]\} = [\, I_n] \quad . \tag{11}$$

The dispersion parameter of random matrix $[\mathbf{K}_n^{\mathrm{GOE}}]$ is taken as parameter δ_K of random matrix $[\mathbf{K}_n]$, defined by Eq. (5). One then has

$$\frac{E\{\| \, [\mathbf{G}_{K_n}^{\mathrm{GOE}}] - [\underline{G}_{K_n}^{\mathrm{GOE}}] \, \|_F^2\}}{\| \, [\underline{G}_{K_n}^{\mathrm{GOE}}] \, \|_F^2} = \delta_K^2 \quad . \tag{12}$$

It should be noted that $\| \, [\underline{G}_{K_n}^{\mathrm{GOE}}] \, \|_F^2 = n$. As the mean value of a random matrix $[\mathbf{H}_n^{\mathrm{GOE}}]$ belonging to the GOE is such that $E\{[\mathbf{H}_n^{\mathrm{GOE}}]\} = [\,0]$, random matrix $[\mathbf{G}_{K_n}^{\mathrm{GOE}}]$ constructed with the GOE has to be written as

$$[\mathbf{G}_{K_n}^{\mathrm{GOE}}] = [\, I_n] + [\mathbf{H}_n^{\mathrm{GOE}}] \quad , \tag{13}$$

in which $[\mathbf{H}_n^{\mathrm{GOE}}]$ belongs to the GOE, that is to say, is a random matrix with values in $\mathbb{M}_n^S(\mathbb{R})$. Real-valued random variables $\{[\mathbf{H}_n^{\mathrm{GOE}}]_{jk}, j \leq k\}$ are mutually independent, second-order, centered, Gaussian and such that

$$\begin{aligned} &E\big\{[\mathbf{H}_n^{\mathrm{GOE}}]_{jk}\big\} = [\,0] \quad , \\ &E\big\{([\mathbf{H}_n^{\mathrm{GOE}}]_{jk})^2\big\} = \frac{\delta_K^2}{n+1}(1+\delta_{jk}) \quad . \end{aligned} \tag{14}$$

Therefore, random matrices $[\mathbf{H}_n^{\mathrm{GOE}}]$, $[\mathbf{G}_{K_n}^{\mathrm{GOE}}]$ and $[\mathbf{K}_n^{\mathrm{GOE}}]$ are not positive matrices almost surely. The variance

$$V_{jk}^{G_{K_n}^{\mathrm{GOE}}} = E\big\{([\mathbf{G}_{K_n}^{\mathrm{GOE}}]_{jk} - [\underline{G}_{K_n}^{\mathrm{GOE}}]_{jk})^2\big\} = E\big\{([\mathbf{H}_n^{\mathrm{GOE}}]_{jk})^2\big\}$$

of the random variable $[\mathbf{G}_{K_n}^{\mathrm{GOE}}]_{jk}$ is such that, for all j and k in $\{1, \ldots, n\}$,

$$V_{jk}^{G_{K_n}^{\mathrm{GOE}}} = V_{jk}^{G_{K_n}} = \frac{\delta_K^2}{n+1}(1+\delta_{jk}) \quad . \tag{15}$$

2 NONPARAMETRIC MODEL OF RANDOM UNCERTAINTIES IN VIBRATION ANALYSIS

2.1 *Definition of the mean reduced model of the dynamic system*

We consider a fixed stable linear mean dynamic system whose Fourier transform $\underline{\mathbf{u}}(\mathbf{x}, \omega)$ with respect to t of the vector-valued displacement field $\underline{\mathbf{u}}(\mathbf{x}, t)$, is defined on a bounded domain $\underline{\Omega} \subset \mathbb{R}^d$ with $d \geq 1$, equipped with the measure denoted $d\mathbf{x}$ and such that $|\underline{\Omega}| = \int_{\underline{\Omega}} d\mathbf{x}$ is the "volume" of domain $\underline{\Omega}$. For all ω belonging to the frequency band of analysis $[\,0\,, \omega_{\max}]$ with $\omega_{\max} > 0$, the mean reduced model of dimension $n \geq 1$ of this mean dynamic system is written as

$$\mathbf{u}(\mathbf{x}, \omega) = \sum_{\alpha=1}^{n} q_\alpha(\omega) \, \underline{\boldsymbol{\varphi}}_\alpha(\mathbf{x}) \quad , \quad \mathbf{x} \in \underline{\Omega} \quad , \tag{16}$$

$$(-\omega^2 \, [\underline{M}_n] + i\omega \, [\underline{D}_n] + [\underline{K}_n]) \, \mathbf{q}(\omega) = \mathbf{f}(\omega) \quad , \tag{17}$$

in which $\underline{\boldsymbol{\varphi}}_1, \ldots, \underline{\boldsymbol{\varphi}}_n$ are the elastic modes corresponding to the n lowest eigenfrequencies $0 < \underline{\omega}_1 \leq \underline{\omega}_2 \leq \ldots \leq \underline{\omega}_n$ of the mean dynamic system, $\mathbf{q}(\omega) = (q_1(\omega), \ldots, q_n(\omega)) \in \mathbb{C}^n$ is the complex vector of the generalized coordinates, $\mathbf{f}(\omega) = (f_1(\omega), \ldots, f_n(\omega)) \in \mathbb{C}^n$ is the complex vector of the generalized external forces, $[\underline{M}_n]$, $[\underline{D}_n]$ and $[\underline{K}_n]$ belong to $\mathbb{M}_n^+(\mathbb{R})$ and represent the generalized diagonal mass matrix, the generalized full damping matrix and the generalized diagonal stiffness matrix, respectively. It is assumed that

(1) the mass density of the mean dynamic system is a constant equal to 1,

(2) $\int_{\underline{\Omega}} \underline{\boldsymbol{\varphi}}_\alpha(\mathbf{x}) \cdot \underline{\boldsymbol{\varphi}}_\beta(\mathbf{x}) \, d\mathbf{x} = \delta_{\alpha\beta}$ and

(3) the generalized damping matrix is a diagonal matrix.

We then have

$$[\underline{M}_n] = \delta_{\alpha\beta} \ , \ [\underline{D}_n] = 2\,\underline{\xi}\,\omega_{\mathrm{ref}}\,\delta_{\alpha\beta} \ , \ [\underline{K}_n] = \underline{\omega}_\alpha^2 \delta_{\alpha\beta} \ , \tag{18}$$

in which $\underline{\xi} > 0$ and $\omega_{\mathrm{ref}} > 0$ are given positive constants. The observation of the mean dynamic system is defined by the positive-valued function $\omega \mapsto \underline{e}_n(\omega)$ such that

$$\underline{e}_n(\omega) = \|\omega^2 \, [\widehat{h}_n(\omega)]\|_F \quad , \tag{19}$$

in which

$$[\widehat{h}_n(\omega)] = (-\omega^2 \, [\underline{M}_n] + i\omega \, [\underline{D}_n] + [\underline{K}_n])^{-1}$$

is the generalized frequency response function of the mean dynamic system and $\|[A]\|_F = (\mathrm{tr}\{[A]\,[A]^*\})^{1/2}$ with $[A]^* = \overline{[A]}^T$.

2.2 *Nonparametric model of random uncertainties*

The nonparametric model of random uncertainties is introduced as explained in Section 1.2. For the reasons given in Section 1, it is assumed that only the stiffness operator is uncertain. Consequently, the use of the "positive-definite" ensemble for the nonparametric modeling of random uncertainties leads us to the following random generalized frequency response function of the random dynamic system,

$$[\mathbf{H}_n(\omega)] = (-\omega^2\,[\underline{M}_n] + i\omega\,[\underline{D}_n] + [\mathbf{K}_n])^{-1} \quad , \tag{20}$$

in which the probability model of random matrix $[\mathbf{K}_n]$ is defined in Section 1.3. The random observation associated with Eq. (19), is the positive-valued random variable $\mathcal{E}_n(\omega)$ defined by

$$\mathcal{E}_n(\omega) = \|\omega^2\,[\mathbf{H}_n(\omega)]\|_F \quad . \tag{21}$$

If the generalized stiffness matrix is modeled by the GOE, then $[\mathbf{K}_n]$ is replaced by $[\mathbf{K}_n^{\mathrm{GOE}}]$ whose probability model is defined in Section 1.4. In this case, $\mathcal{E}_n(\omega)$ is denoted $\mathcal{E}_n^{\mathrm{GOE}}(\omega)$.

2.3 *Comparison of the two ensembles of random matrices*

It is assumed that the mean reduced model of the dynamic system is such that dimension $n = 30$ and such that, for all α and β in $\{1, \ldots, 30\}$,

$$\begin{aligned} [\underline{M}_n]_{\alpha\beta} &= \delta_{\alpha\beta} \quad , \\ [\underline{K}_n]_{\alpha\beta} &= \underline{k}_1 \int_{-1}^{+1} \underline{\varphi}''_\alpha(x)\underline{\varphi}''_\beta(x)\,dx + \underline{k}_2 \int_{-1}^{+1} \underline{\varphi}_\alpha(x)\underline{\varphi}_\beta(x)\,dx\,, \end{aligned} \tag{22}$$

in which $\underline{\varphi}_\alpha(x) = \sin(\pi\alpha(1+x)/2)$ and $\underline{\varphi}''_\alpha$ is the second derivative of $\underline{\varphi}_\alpha$ with respect to x. This model corresponds to an Euler beam in bending mode, with length 2, simply supported at its ends, attached to a continuous elastic support along its length, and for which elastic bending modes $\underline{\varphi}_\alpha$ are associated with the 30 lowest eigenfrequencies $\underline{\omega}_\alpha$ such that

$$\underline{\omega}_\alpha^2 = \underline{\lambda}_\alpha = \underline{k}_1 \left(\frac{\alpha\pi}{2}\right)^4 + \underline{k}_2 \quad . \tag{23}$$

For all the numerical examples considered in this paper, one takes $\underline{k}_1 = 0.999\,998\,7$ and $\underline{k}_2 = 2.027\,850\,8 \times 10^{-7}$. Consequently, one has $1 \leq \underline{\lambda}_\alpha \leq 2$ with $\underline{\lambda}_1 = 1$, $\underline{\lambda}_2 = 1.000\,185$, ..., $\underline{\lambda}_{29} = 1.873\,186$, $\underline{\lambda}_{30} = 2$. In addition, it is assumed that $\underline{\xi} = 0.01$ and $\omega_{\mathrm{ref}} = 2\pi \times 0.02\,rad/s$. The frequency band of analysis is such that $\omega_{\max} = 2\pi \times 0.22\,rad/s$. The value of the dispersion parameter is $\delta_K = 0.50$. The Monte Carlo numerical simulation method is carried out with $n_s = 40\,000$ realizations, denoted $\theta_1, \ldots \theta_{n_s}$, for which the realizations $\omega \mapsto \mathcal{E}_n(\omega;\theta_1)$, ..., $\omega \mapsto \mathcal{E}_n(\omega;\theta_{n_s})$ are numerically calculated for the two ensembles of random matrices, with a sampling frequency step $\Delta\omega = \omega_{\max}/300$. For ω fixed in $[0\,, \omega_{\max}]$, the mean values $E\{\mathcal{E}_n(\omega)\}$ and $E\{\mathcal{E}_n^{\mathrm{GOE}}(\omega)\}$, and the standard deviations $\sigma_{\mathcal{E}_n}(\omega)$ and $\sigma_{\mathcal{E}_n^{\mathrm{GOE}}}(\omega)$, of random variables $\mathcal{E}_n(\omega)$ and $\mathcal{E}_n^{\mathrm{GOE}}(\omega)$ respectively, are usually estimated. For the comparisons, we define the functions $\nu \mapsto \mathrm{dB}(\nu)$ and $\nu \mapsto \mathrm{dB}^{\mathrm{GOE}}(\nu)$ such that

$$\begin{aligned} \mathrm{dB}(\nu) &= \log_{10}(E\{\mathcal{E}_n(2\pi\nu)\}) \quad , \\ \mathrm{dB}^{\mathrm{GOE}}(\nu) &= \log_{10}(E\{\mathcal{E}_n^{\mathrm{GOE}}(2\pi\nu)\}) \quad . \end{aligned} \tag{24}$$

Finally, for the "positive-definite" ensemble and for the GOE, for all ν fixed in $[0\,, \omega_{\max}/2\pi]$, the extreme value statistics associated with realizations $\theta_1, \ldots, \theta_{n_s}$ are defined by

$$\begin{aligned} \mathrm{dB}_{\max}(\nu) &= \log_{10}\{\max_k \mathcal{E}_n(2\pi\nu;\theta_k)\} \quad , \\ \mathrm{dB}_{\min}(\nu) &= \log_{10}\{\min_k \mathcal{E}_n(2\pi\nu;\theta_k)\}\,, \end{aligned} \tag{25}$$

$$\begin{aligned} \mathrm{dB}^{\mathrm{GOE}}_{\max}(\nu) &= \log_{10}\{\max_k \mathcal{E}_n^{\mathrm{GOE}}(2\pi\nu;\theta_k)\} \quad , \\ \mathrm{dB}^{\mathrm{GOE}}_{\min}(\nu) &= \log_{10}\{\min_k \mathcal{E}_n^{\mathrm{GOE}}(2\pi\nu;\theta_k)\} \quad . \end{aligned} \tag{26}$$

Figures 1, 2 and 3 are relative to the frequency band $[0\,, 0.22]\,Hz$. Figure 1 displays (1) the response $\nu \mapsto \log_{10} \underline{\mathcal{E}}_n(2\pi\nu)$ of the mean dynamic system (dashed line), (2) the graphs of functions $\nu \mapsto \mathrm{dB}(\nu)$ (thick solid line) and $\nu \mapsto \mathrm{dB}^{\mathrm{GOE}}(\nu)$ (thin solid line).

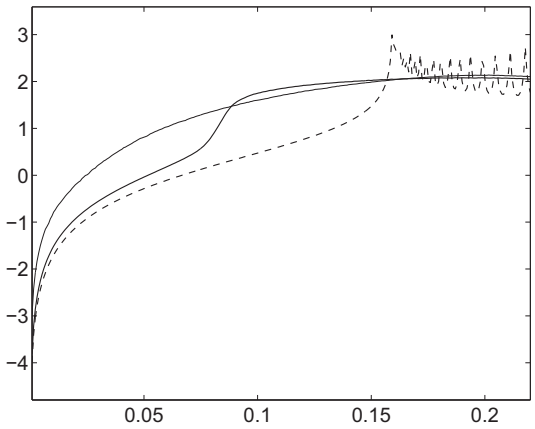

Figure 1. Nonparametric approach. Dispersion parameter $\delta_K = 0.50$. Frequency band $[0\,, 0.22]\,Hz$ (horizontal axis). Graphs of functions $\nu \mapsto \log_{10} \underline{\mathcal{E}}_n(2\pi\nu)$ (dashed line), $\nu \mapsto \mathrm{dB}(\nu)$ (thick solid line) and $\nu \mapsto \mathrm{dB}^{\mathrm{GOE}}(\nu)$ (thin solid line).

Figure 2 displays the graphs of functions $\nu \mapsto \sigma_{\mathcal{E}_n}(2\pi\nu)$ (thick solid line) and $\nu \mapsto \sigma_{\mathcal{E}_n^{\mathrm{GOE}}}(2\pi\nu)$ (thin solid line). Figures 1 and 2 show an important difference between the "positive-definite" ensemble and the GOE. For the GOE, the first random eigenvalues (the lowest eigenvalues of the order statistics) have a larger standard deviation than for the "positive-definite" ensemble and their probability distributions are different. This is the reason why the mean value and the standard deviation of the random response are very different in the frequency band $[0\,, 0.1]\,Hz$ for the two ensembles of random matrices.

These differences can also be seen in Figure 3 which shows (1) for the "positive-definite" ensemble, the

graphs of functions $\nu \mapsto \mathrm{dB}(\nu)$ (thick dashed line), $\nu \mapsto \mathrm{dB}_{\max}(\nu)$ (upper thick solid line), $\nu \mapsto \mathrm{dB}_{\min}(\nu)$ (lower thick solid line), (2) for the GOE, the graphs of functions $\nu \mapsto \mathrm{dB}^{\mathrm{GOE}}(\nu)$ (thin dashed line), $\nu \mapsto \mathrm{dB}^{\mathrm{GOE}}_{\max}(\nu)$ (upper thin solid line), $\nu \mapsto \mathrm{dB}^{\mathrm{GOE}}_{\min}(\nu)$ (lower thin solid line).

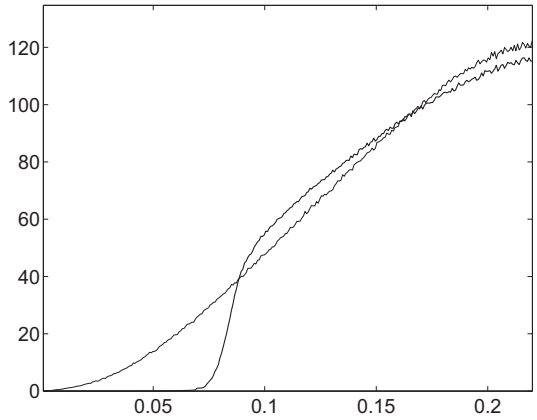

Figure 2. Nonparametric approach. Dispersion parameter $\delta_K = 0.50$. Frequency band $[0\,,0.22]\ Hz$ (horizontal axis). Graphs of functions $\nu \mapsto \sigma_{\mathcal{E}_n}(2\pi\nu)$ (thick solid line) and $\nu \mapsto \sigma_{\mathcal{E}_n^{\mathrm{GOE}}}(2\pi\nu)$ (thin solid line).

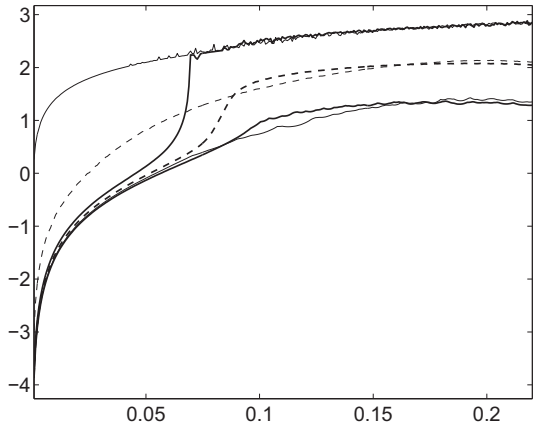

Figure 3. Nonparametric approach. Dispersion parameter $\delta_K = 0.50$. For the "positive-definite" ensemble: graphs of functions $\nu \mapsto \mathrm{dB}(\nu)$ (thick dashed line), $\nu \mapsto \mathrm{dB}_{\max}(\nu)$ (upper thick solid line), $\nu \mapsto \mathrm{dB}_{\min}(\nu)$ (lower thick solid line). For the Gaussian orthogonal ensemble, graphs of functions $\nu \mapsto \mathrm{dB}^{\mathrm{GOE}}(\nu)$ (thin dashed line), $\nu \mapsto \mathrm{dB}^{\mathrm{GOE}}_{\max}(\nu)$ (upper thin solid line), $\nu \mapsto \mathrm{dB}^{\mathrm{GOE}}_{\min}(\nu)$ (lower thin solid line).

3 VALIDATION OF THE NONPARAMETRIC MODEL

The nonparametric model of random uncertainties (introduced in Section 1.2) in vibration analysis has been introduced to replace the usual parametric model for complex dynamic systems when the number of uncertain local parameters is large and above all, to take into account the model uncertainties which cannot be modeled with the parametric models. Nevertheless, as Section 2 shows that the results given by the two ensembles of random matrices are very different, it is interesting to analyze a simple dynamic system with random uncertainties which can easily be modeled by using the usual parametric approach, in order to conclude if the use of the "positive-definite" ensemble is better than the use of the GOE for the nonparametric model of random uncertainties in low-frequency vibration analysis. Consequently, we consider the dynamic system with parametric random uncertainties whose associated mean dynamic system is defined in Sections 2.1 and 2.3.

3.1 *Defining the dynamic system with parametric random uncertainties*

We consider a dynamic system with parametric random uncertainties on the stiffness operator, for which the mean dynamic system is defined in Sections 2.1 and 2.3. In the frequency domain, the weak formulation of the corresponding boundary value problem is written as

$$-\omega^2 \int_{-1}^{+1} U(x,\omega)\, v(x)\, dx + 2i\omega\underline{\xi}\omega_{\mathrm{ref}} \int_{-1}^{+1} U(x,\omega)\, v(x)\, dx$$

$$+\underline{k}_1 \int_{-1}^{+1} Y(x)\, U''(x,\omega)\, v''(x)\, dx + \underline{k}_2 \int_{-1}^{+1} T(x)\, U(x,\omega)\, v(x)\, dx$$

$$= \int_{-1}^{+1} g(x,\omega)\, v(x)\, dx \quad , \qquad (27)$$

in which v'' is the second derivative of v with respect to x and where the test function v belongs to the admissible function space constituted of the "sufficiently differentiable" real-valued functions v defined on $\underline{\Omega} =]-1\,,+1[$ and such that $v(-1) = v(+1) = 0$ and $v''(-1) = v''(+1) = 0$. The external excitation is represented by the complex-valued force field $x \mapsto g(x,\omega)$ define on $\underline{\Omega}$. In Eq. (27), $\underline{\xi}$, ω_{ref}, $\underline{k}_1$ and $\underline{k}_2$ are defined in Section 2.3. Parameters $Y(x)$ and $T(x)$ are second-order stochastic processes indexed by $\underline{\Omega}$ with values in $\mathbb{R}^+$, statistically independent, such that, for all x in $\underline{\Omega}$, $E\{Y(x)\} = 1$ and $E\{T(x)\} = 1$. For x fixed in $\underline{\Omega}$, $Y(x)$ and $T(x)$ are written as

$$Y(x) = \frac{1}{m_Y}\sum_{j=1}^{m_Y} Z_j^Y(x)^2 \ , \ T(x) = \frac{1}{m_T}\sum_{j=1}^{m_T} Z_j^T(x)^2 \ , \quad (28)$$

in which $m_Y \geq 1$ and $m_T \geq 1$ are two finite positive integers and where $Z_1^Y, \ldots, Z_{m_Y}^Y$, $Z_1^T, \ldots, Z_{m_T}^T$ are $m_Y + m_T$ independent copies of a stochastic process Z_b defined as follows. Stochastic process $\{Z_b(x), x \in \mathbb{R}\}$ is indexed by $\mathbb{R}$, with values in $\mathbb{R}$, second-order, centered, Gaussian and stationary, such that $E\{Z_b(x)\} = 0$ and $E\{Z_b(x)^2\} = 1$. Let $S_{Z_b}(k)$ be its power spectral density function defined on $\mathbb{R}$ with values in $\mathbb{R}^+$, related to its autocorrelation function $R_{Z_b}(\eta) = E\{Z_b(x+\eta)Z_b(x)\}$ by the equation $R_{Z_b}(\eta) = \int_{\mathbb{R}} e^{ik\eta}\, S_{Z_b}(k)\, dk$. Power spectral density function is defined by

$$S_{Z_b}(k) = \frac{L}{\pi a}\frac{1}{(1 + L^2k^2)}\, \mathbb{1}_{[-b,b]}(k) \quad , \qquad (29)$$

in which $0 < b < +\infty$ is a finite positive real constant and where a is such that $a = \frac{2}{\pi}\arctan(bL)$. It can then verified that

$$\sigma_Y^2 = E\{Y(x)^2\} - 1 = \frac{2}{m_Y} \quad ,$$
$$\sigma_T^2 = E\{T(x)^2\} - 1 = \frac{2}{m_T} \quad . \tag{30}$$

3.2 *Constructing the random reduced model*

The random reduced model of dimension n is obtained by using the n elastic modes $\underline{\varphi}_1, \ldots, \underline{\varphi}_n$ introduced in Section 2.3, associated with the n lowest eigenfrequencies $\underline{\omega}_1, \ldots, \underline{\omega}_n$ defined by Eq. (23). From Eq. (27), it can be deduced that

$$U(x,\omega) = \sum_{\alpha=1}^{n} Q_\alpha(\omega)\, \underline{\varphi}_\alpha(x) \quad , \quad x \in \underline{\Omega} \quad , \tag{31}$$

$$(-\omega^2\, [\underline{M}_n] + i\omega\, [\underline{D}_n] + [\mathbf{K}_n^{\text{param}}])\, \mathbf{Q}(\omega) = \mathbf{f}(\omega) \quad , \tag{32}$$

in which $[\underline{M}_n] = [I_n]$ (see Eq. (22)), $[\underline{D}_n] = 2\,\underline{\xi}\,\omega_{\text{ref}}\,[I_n]$ (see Eq. (18)) and where $\mathbf{Q}(\omega) = (Q_1(\omega), \ldots, Q_n(\omega))$ is the random vector of the generalized coordinates and where $\mathbf{f}(\omega) = (f_1(\omega), \ldots, f_n(\omega))$ is the complex vector of the generalized external forces which is such that $f_\alpha(\omega) = \int_{-1}^{+1} g(x,\omega)\, \underline{\varphi}_\alpha(x)\, dx$. Let $[\underline{K}_n]_{\alpha\beta} = \underline{\omega}_\alpha^2\, \delta_{\alpha\beta}$ be the matrix defined by Eq. (18) which can be written as $[\underline{K}_n] = [\underline{L}_{K_n}]^T\, [\underline{L}_{K_n}]$ with $[\underline{L}_{K_n}]_{\alpha\beta} = \underline{\omega}_\alpha\, \delta_{\alpha\beta}$. In Eq. (32), the random matrix $[\mathbf{K}_n^{\text{param}}]$ can be written as

$$[\mathbf{K}_n^{\text{param}}] = [\underline{L}_{K_n}]^T\, [\mathbf{G}_{K_n}^{\text{param}}]\, [\underline{L}_{K_n}] \quad , \tag{33}$$

where the random matrix $[\mathbf{G}_{K_n}^{\text{param}}]$ is such that

$$[\mathbf{G}_{K_n}^{\text{param}}]_{\alpha\beta} = \frac{\underline{k}_1}{\underline{\omega}_\alpha \underline{\omega}_\beta} \int_{-1}^{+1} Y(x) \underline{\varphi}_\alpha''(x) \underline{\varphi}_\beta''(x)\, dx + \frac{\underline{k}_2}{\underline{\omega}_\alpha \underline{\omega}_\beta} \int_{-1}^{+1} T(x) \underline{\varphi}_\alpha(x) \underline{\varphi}_\beta(x)\, dx \quad . \tag{34}$$

We then deduce that

$$[\underline{G}_{K_n}^{\text{param}}] = E\{[\mathbf{G}_{K_n}^{\text{param}}]\} = [I_n] \quad . \tag{35}$$

In order to compare the nonparametric model with the parametric one, we introduce the global dispersion parameter $\delta_K^{\text{param}} > 0$ of random matrix $[\mathbf{G}_{K_n}^{\text{param}}]$ defined (see Eqs. (2) to (5)) by

$$\delta_K^{\text{param}} = \left\{ \frac{E\{\| [\mathbf{G}_{K_n}^{\text{param}}] - [\underline{G}_{K_n}^{\text{param}}] \|_F^2\}}{\| [\underline{G}_{K_n}^{\text{param}}] \|_F^2} \right\}^{1/2} \quad . \tag{36}$$

The random generalized frequency response function associated with Eq. (32) is written as

$$[\mathbf{H}_n(\omega)^{\text{param}}] = (-\omega^2\, [\underline{M}_n] + i\omega\, [\underline{D}_n] + [\mathbf{K}_n^{\text{param}}])^{-1} \quad . \tag{37}$$

Finally, the random observation defined by Eq. (21) is written as

$$\mathcal{E}_n^{\text{param}}(\omega) = \|\omega^2\, [\mathbf{H}_n^{\text{param}}(\omega)]\|_F \quad . \tag{38}$$

3.3 *Numerical parameters and computation*

The Monte Carlo numerical simulation method is carried out with $n_s = 40\,000$ realizations, denoted $\theta_1, \ldots \theta_{n_s}$. The realizations $\omega \mapsto \mathcal{E}_n^{\text{param}}(\omega;\theta_1)$, ..., $\omega \mapsto \mathcal{E}_n^{\text{param}}(\omega;\theta_{n_s})$ are numerically calculated on the frequency band $[0\,, \omega_{\max}]$ with $\omega_{\max} = 2\pi \times 0.22\, rad/s$ and with a sampling frequency step $\Delta\omega = \omega_{\max}/300$. The values of the numerical parameters are $n = 30$, $m_Y = m_T = 4$, $L = 0.076\, m$, $b = 804.25\, m^{-1}$ and $a = 0.9896$. The calculation of parameter δ_K^{param} defined by Eq. (36) yields $\delta_K^{\text{param}} = 0.4942 \simeq 0.50$. Figures 4, 5 and 6 are relative to the frequency band $[0\,, 0.22]\, Hz$. Figure 4 displays the response $\nu \mapsto \log_{10} \underline{e}_n(2\pi\nu)$ of the mean dynamic system (dashed line) calculated in Section 2.3 and the graph of function $\nu \mapsto \text{dB}^{\text{param}}(\nu)$ (thick solid line) such that

$$\text{dB}^{\text{param}}(\nu) = \log_{10}(E\{\mathcal{E}_n^{\text{param}}(2\pi\nu)\}) \quad . \tag{39}$$

Figure 5 displays the function $\nu \mapsto \sigma_{\mathcal{E}_n^{\text{param}}}(2\pi\nu)$ (solid line) in which $\sigma_{\mathcal{E}_n^{\text{param}}}(\omega)$ is the standard deviation of random variable $\mathcal{E}_n^{\text{param}}(\omega)$. Figure 6 shows the graphs of $\nu \mapsto \text{dB}^{\text{param}}(\nu)$ (dashed line), $\nu \mapsto \text{dB}_{\max}^{\text{param}}(\nu)$ (upper solid line) and $\nu \mapsto \text{dB}_{\min}^{\text{param}}(\nu)$ (lower solid line) in which

$$\text{dB}_{\max}^{\text{param}}(\nu) = \log_{10}\{\max_k \mathcal{E}_n^{\text{param}}(2\pi\nu;\theta_k)\} \quad ,$$
$$\text{dB}_{\min}^{\text{param}}(\nu) = \log_{10}\{\min_k \mathcal{E}_n^{\text{param}}(2\pi\nu;\theta_k)\} \quad . \tag{40}$$

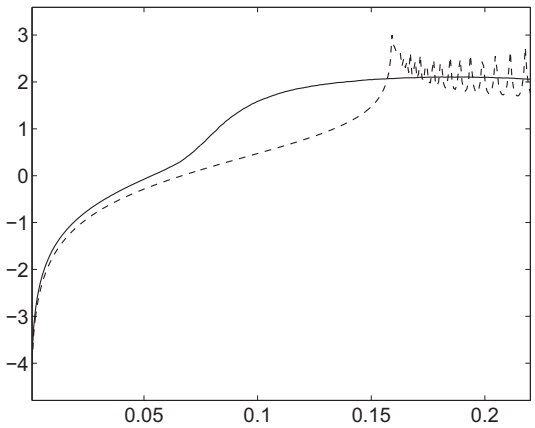

Figure 4. Parametric approach. Dispersion parameter $\delta_K^{\text{param}} = 0.4942$. Frequency band $[0\,, 0.22]\, Hz$ (horizontal axis). Graphs of $\nu \mapsto \log_{10} \underline{e}_n(2\pi\nu)$ (dashed line) and $\nu \mapsto \text{dB}^{\text{param}}(\nu)$ (thick solid line).

3.4 *Comparison of the parametric model with the nonparametric model*

Due to the fact that $\delta_K = 0.50 \simeq \delta_K^{\text{param}} = 0.4942$, we can compare the results given by the nonparametric approach (Figures 1, 2 and 3) with the results given by the parametric approach (Figures 4, 5 and 6). These figures cleary prove that the nonparametric results look like the parametric results when the "positive-definite" ensemble is used and is very different in the low-frequency domain when the GOE is used. Consequently, the present results give an additional validation point of the nonparametric model of random uncertainties whose theory is recalled in Sections 1.2 and 1.3 and which is based on the "positive-definite" ensemble.

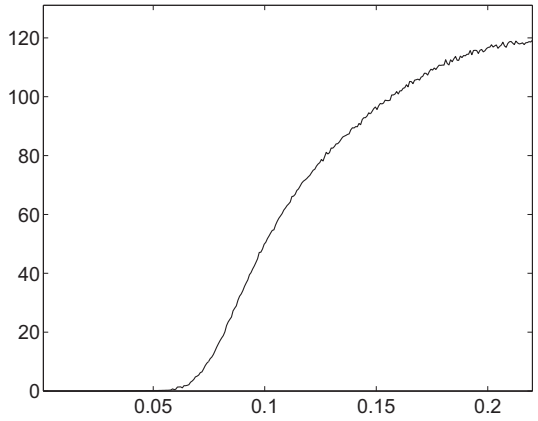

Figure 5. Parametric approach. Dispersion parameter $\delta_K^{\text{param}} = 0.4942$. Frequency band $[0\,,0.22]\,Hz$ (horizontal axis). Graph of function $\nu \mapsto \sigma_{\mathcal{E}_n^{\text{param}}}(2\pi\nu)$ (solid line).

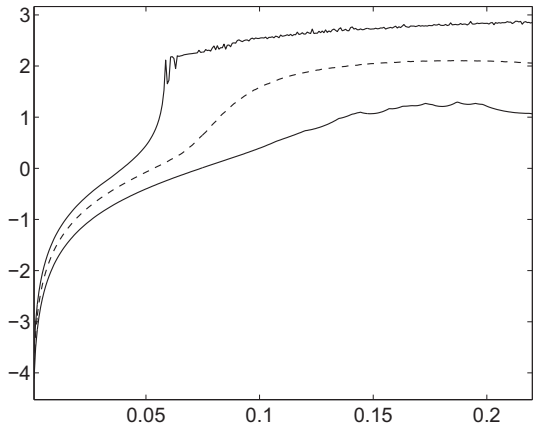

Figure 6. Parametric approach. Dispersion parameter $\delta_K^{\text{param}} = 0.4942$. Frequency band $[0\,,0.22]\,Hz$ (horizontal axis). Graphs of functions $\nu \mapsto \text{dB}^{\text{param}}(\nu)$ (dashed line), $\nu \mapsto \text{dB}_{\text{max}}^{\text{param}}(\nu)$ (upper solid line) and $\nu \mapsto \text{dB}_{\text{min}}^{\text{param}}(\nu)$ (lower solid line).

4 CONCLUSIONS

This paper gives a new validation point of the nonparametric theory of random uncertainties in vibration analysis, recently introduced by the author. It is proved that the "positive-definite" ensemble of random matrices, which has been introduced in the context of the development of this nonparametric approach, is well adapted to the low-frequency vibration analysis, while the use of the Gaussian orthogonal ensemble (GOE) is not. In addition, as it is explained in previous papers devoted to the construction of this nonparametric approach, the "positive-definite" ensemble allows random uncertainties to be modeled for the damping operator while the GOE does not, in the present state of the art.

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