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Inverse problem for the identification of Chaos representations of random fields using experimental vibrational tests

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\section*{Abstract}

This paper deals with the experimental identification of the probabilistic representation of a random field modeling the Young modulus of a non homogeneous isotropic elastic medium by experimental vibration tests. The random field representation is based on the polynomial chaos decomposition. The coefficients of the polynomial chaos are identified setting an inverse problem and then in solving an optimization problem related to the maximum likelihood principle.

\section{Introduction}

We consider an elastic isotropic heterogeneous material at the macroscopic scale. In this paper, this material is modeled by a random medium. Only the Young modulus is modeled by a random field. This random field has to be identified by using an experimental database constituted of vibrational tests related to 100 specimens in the frequency band \([0-50]\)kHz. There are 60 sensors measuring accelerations. All the specimens are excited by the same loads whose spectrum are constant on the frequency band of analysis. The linear elasto-dynamics of the specimen is modeled by the finite element method. A chaos representation (see for instance \([1, 2, 3]\)) of the random field modeling the Young modulus is introduced. This identification problem has been addressed in a recent work proposed in \([4, 5]\). The proposed method in \([5]\) consists in identifying the coefficients of the chaos representation by using the experimental database related to the dynamical response of the specimen. The coefficients related to the finite element model of the specimen are random variables. The first step of the method consists in identifying the related realizations of these random variables by setting an optimization problem for each specimen. This optimization problem consists in minimizing the distance between the frequency response functions of the database and the frequency response function of the finite element model with respect to the finite element coefficients of each specimen. In the second step of the method, the polynomial chaos representation of these random variables is introduced. In the last step of the method, the coefficients of the polynomial chaos representations are estimated by using the maximum likelihood method. A numerical example is presented in order to validate the methodology. The example consists in a viscoelastic random medium occupying a slender geometry. The experimental database is constructed by Monte Carlo numerical simulations of the direct problem. On the presented example, not only the autocorrelation functions of the random field can be identified, but also the first order marginal probability density functions.
2 Construction of an "experimental database" by Monte Carlo numerical simulation of the direct problem

In this paper, the "experimental database" is constructed by numerical simulation. The specimen is constituted of a non-homogeneous isotropic linear elastic medium occupying a three-dimensional bounded domain \( D \) with boundary \( \partial D \) given in a Cartesian system \( Ox_1x_2x_3 \). The geometry of domain \( D \) is a slender rectangular box shown in Fig. 1 whose dimensions along \( x_1, x_2 \) and \( x_3 \) are \( L_1 = 1.3 \times 10^{-1} \text{m}, L_2 = 2 \times 10^{-2} \text{m} \) and \( L_3 = 2 \times 10^{-2} \text{m} \). The structure is fixed on the part \( \Gamma_0 \) of \( \partial D \) for which the displacement field is zero.

![Figure 1: Definition of the specimen](image)

The structure is subjected to an external point force denoted as \( \mathbf{b}(t) \) and applied to the node \( A \) along \( x_1 \)-axis (see Fig. 1). The Fourier transform \( \hat{\mathbf{b}} \) of \( \mathbf{b} \) is the constant vector \((1, 0, 0)\) in the frequency band \([0, 50] \text{kHz}\). The elastic medium is random. It is assumed that the Young modulus is random while the Poisson coefficient is deterministic. This assumption is introduced in order to simplify the presentation. The random Young modulus field is modeled by a positive-valued second-order random field \( Y(x) \) defined by

\[
Y(x) = c_0 \, g(c_1, c_2 \, V(x)), \quad \forall x \in D
\]

in which \( c_0 = 1.6663 \times 10^{10} \text{N.m}^{-2}, c_1 = 1.5625 \) and \( c_2 = 0.2 \). The function \( \theta \mapsto g(\alpha, \theta) \) from \( \mathbb{R} \) into \( [0, +\infty[ \) is such that, for all \( \theta \) in \( \mathbb{R} \),

\[
g(\alpha, \theta) = F^{-1}_{\Gamma_\alpha}(F_\Theta(\theta)),
\]

in which \( \theta \mapsto F_\Theta(\theta) = P(\Theta \leq \theta) \) is the cumulative distribution function of the normalized Gaussian random variable \( \Theta \) and where the function \( p \mapsto F^{-1}_{\Gamma_\alpha}(p) \) from \([0, 1[ \) into \([0, +\infty[ \) is the reciprocal function of the cumulative distribution function \( \gamma \mapsto F_{\Gamma_\alpha}(\gamma) = P(\Gamma_\alpha \leq \gamma) \) of the gamma random variable \( \Gamma_\alpha \) with parameter \( \alpha \). In the right-hand side of Eq. (1), \( \{V(x), x \in D\} \) is a second-order random field such that \( E\{V(x)\} = 0 \) and \( E\{V(x)^2\} = 1 \), defined by

\[
V(x) = \sum_{|\alpha|=1}^3 H_\alpha(Z_1, Z_2, Z_3, Z_4) \sqrt{\gamma_\alpha} |x/2| \psi_\alpha(x/2),
\]

in which \( \{Z_1, Z_2, Z_3, Z_4\} \) are independent normalized Gaussian random variables, \( \alpha \) is a multi-index \((\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{N}^4, |\alpha| = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \) and where \( H_\alpha(z_1, z_2, z_3, z_4) = H_{\alpha_1}(z_1) \times H_{\alpha_2}(z_2) \times \ldots \times H_{\alpha_4}(z_4) \).
The finite element mesh of the structure is shown in Fig. 1 and consists of 8-node isoparametric 3D solid finite elements. The Young modulus is assumed to be the random dynamical stiffness matrix constructed by using the finite element approximation of domain \( \mathcal{D} \) with \( N = 60 \) degrees of freedom and where \( R_1, \ldots, R_N \) are the random coefficients. We introduce the \( \mathbb{R}^N \)-valued random variable \( \mathbf{R} \) such that \( \mathbf{R} = (R_1, \ldots, R_N) \). Let \( [\tilde{\mathbf{A}}(\omega); \mathbf{R}] \) be the random dynamical stiffness matrix constructed by using the finite element approximation \( \tilde{Y}(x) \) of the Young modulus. For each realization \( \mathbf{r}^j = \mathbf{R}(\theta_j) \) of the random variable \( \mathbf{R} \) are constructed by solving the nonlinear optimization problem (see [5])

\[
\min_{\mathbf{r}^j} \ell_{\text{dyn}}(\mathbf{r}^j, \mathbf{u}_f^j),
\]

in which

\[
\ell_{\text{dyn}}(\mathbf{r}^j, \mathbf{u}_f^j) = \sum_{k=1}^{N_{\text{rand}}} \int_{B_k} \left\| \mathcal{P} \left( [\tilde{\mathbf{A}}(\omega; \mathbf{r}^j)]^{-1} \mathbf{f}(\omega) \right) - \mathbf{u}_f^j(\omega) \right\|^2 d\omega.
\]
In the right-hand side of Eq. (5), \( B_k = [\omega_{\text{min},k}, \omega_{\text{max},k}] \) with \( \omega_{\text{min},k} = \omega_k - B_{\text{eq},k}/2 \) and \( \omega_{\text{max},k} = \omega_k + B_{\text{eq},k}/2 \) where \( B_{\text{eq},k} \) is an equivalent bandwidth related to the eigenfrequency \( \omega_k \) of the mean model of the specimens and where \( N_{\text{band}} \) is the number of bands considered for the identification. It should be noted that the optimization problem introduced in [4] in order to solve the inverse problem to calculate the realizations \( r^1, \ldots, r^m \) of random vector \( \mathbf{R} \) is based on an elastostatic problem. In this case, the experimental database is constituted of static measurements and the optimisation problem is

\[
\min_{\mathbf{r}} \ell_{\text{stat}}(\mathbf{r}, \mathbf{u}_T^j) ,
\]

in which

\[
\ell_{\text{stat}}(\mathbf{r}, \mathbf{u}_T^j) = \| \mathcal{P} \left( \left[ \mathbf{A}(0; \mathbf{r}) \right]^{-1} \mathbf{f}(0) \right) - \mathbf{u}_T^j(0) \|^2 .
\]

The optimization problems defined by Eqs. (4) and (6) are solved by using a least-squares estimation of nonlinear parameters (see [6]). Finally, for all \( x \) fixed in \( D \), the realizations \( \tilde{y}^1(x) = \tilde{Y}(x; \theta_1), \ldots, \tilde{y}^m(x) = \tilde{Y}(x; \theta_m) \) of random variable \( \tilde{Y}(x) \) are constructed by using the relation \( \tilde{y}^j(x) = \mathbf{h}(x_1)^T \mathbf{r}^j \) in which \( \mathbf{h}(x_1) = (h_1(x_1), \ldots, h_N(x_1)) \). Figure 4 shows the graph of realization \( x_1 \mapsto \tilde{y}^1(x) \) with \( x_2 = x_3 = 0 \) constructed by solving Eq. (4) (“dynamic inverse problem”) and Eq. (6) (“static inverse problem”).

![Graph](image)

Figure 2: Graph of \( x_1 \mapsto Y(x; \theta_1) \) (thick solid line) and graph of realization \( x_1 \mapsto \tilde{y}^1(x) \) with \( x_2 = x_3 = 0 \) constructed by solving the “dynamic inverse problem” (dash line) with \( N_{\text{band}} = 5 \). Horizontal axis: \( x_1 \). Vertical axis: \( Y(x; \theta_1) \) and \( \tilde{Y}^1(x) \).

### 4 Statistical reduction

The size of the random vector \( \mathbf{R} \) can be reduced. Let \( \lambda_1 \geq \ldots \geq \lambda_N \) be the eigenvalues of the covariance matrix \( [C_{\mathbf{R}}] \) of random vector \( \mathbf{R} \). The normalized eigenvectors associated with the eigenvalues \( \lambda_1, \ldots, \lambda_N \) are denoted by \( \varphi_1, \ldots, \varphi_N \). Consequently, the random vector \( \mathbf{R} \) can be written as

\[
\mathbf{R} = \bar{\mathbf{R}} + \sum_{k=1}^{N} Q_k \sqrt{\lambda_k} \varphi_k ,
\]

in which \( Q_1, \ldots, Q_N \) are \( N \) centered real-valued random variables defined by \( \sqrt{\lambda_k} Q_k = \varphi_k^T (\mathbf{R} - \bar{\mathbf{R}}) \) where \( \bar{\mathbf{R}} = E\{\mathbf{R}\} \) such that for all \( k \) and \( \ell \), \( E\{Q_k\} = 0 \) and \( E\{Q_k Q_\ell\} = \delta_{k\ell} \). From the study of the
function \( n \mapsto \sum_{k=1}^{n} \lambda_k \), it can be deduced that random vector \( \mathbf{R} \) can be approximated by the random vector \( \mathbf{R} + [\Phi] [\Lambda] \mathbf{Q}^\mu \) with \( \mu = 15 < N \) in which the \((\mu \times \mu)\) matrix \([\Lambda]\) and the \((N \times \mu)\) matrix \([\Phi]\) are such that \([\Lambda]_{kk} = \delta_{kk} \sqrt{\lambda_k}\) and \([\Phi]_{kk} = [\varphi_k]_{[k]}\) and where \( \mathbf{Q}^\mu = (Q_1, \ldots, Q_\mu) \). For all \( j = 1, \ldots, m \), the realization \( q_j = \mathbf{Q}^\mu(\theta_j) \) of random vector \( \mathbf{Q}^\mu \) is calculated by \( q_j = [\Lambda]^{-1}[\Phi]^T (\mathbf{r}^j - \mathbf{R}) \).

5 Chaos decomposition

Let \( \mathbf{W}^\nu = (W_1, \ldots, W_\nu) \) be the normalized Gaussian random vector such that \( E\{W_i W_j\} = \delta_{ij} \). The truncated Chaos representation of the \( \mathbb{R}^p \)-valued random variable \( \mathbf{Q}^\mu \) in terms of \( \mathbf{W}^\nu \) is written as

\[
\mathbf{Q}^\mu,\nu = \sum_{\alpha,|\alpha|=1}^{+\infty} a_\alpha H_\alpha(\mathbf{W}^\nu) ,
\]

where \( \alpha \) is a multi-index belonging to \( \mathbb{N}^\nu \) and where \( H_\alpha(\mathbf{W}^\nu) \) is the multi-indexed Hermite polynomials (see section 3). The coefficients \( a_\alpha \) belonging to \( \mathbb{R}^p \) are such that \( \sum_{\alpha,|\alpha|=1}^{+\infty} a_\alpha H_\alpha = [I_\mu] \) in which \([I_\mu]\) is the \((\mu \times \mu)\) unit matrix. The truncated Chaos representation of random vector \( \mathbf{Q}^\mu,\nu \) is denoted by \( \mathbf{Q}^\mu,\nu,\nu,\nu \) and is such that \( \mathbf{Q}^\mu,\nu,\nu,\nu = \sum_{\alpha,|\alpha|=1}^{d} a_\alpha H_\alpha(\mathbf{W}^\nu) \). Consequently, for all \( \mathbf{x} \in \mathcal{D} \), the random Young modulus \( \tilde{Y}(\mathbf{x}) \) can be approximated by the random variable \( \tilde{Y}^\mu,\nu,\nu,\nu(x) = h(x_1)^T (\mathbb{Q}[\mathbb{Q}^\mu,\nu,\nu,\nu] + \mathbf{R}) \).

The maximum likelihood method (see, for instance, [7]) is used to estimate parameters \( a_\alpha \) from realizations \( q^1, \ldots, q^m \). We then have to solve the following problem of optimization: find \( \mathcal{A} = \{ a_\alpha, |\alpha|=1, \ldots, d \} \) such that

\[
\max_{\mathcal{A}} L(q^1, \ldots, q^m; \mathcal{A}) , \quad \text{with} \quad \sum_{\alpha,|\alpha|=1}^{d} a_\alpha a_\alpha^T = [I_\mu]
\]

where \( L(q^1, \ldots, q^m; \mathcal{A}) = p_{\mathbf{Q}^\mu,\nu,\nu,\nu}(q^1, \mathcal{A}) \times \ldots \times p_{\mathbf{Q}^\mu,\nu,\nu,\nu}(q^m, \mathcal{A}) \) is the likelihood function associated with observations \( q^1, \ldots, q^m \) and where \( p_{\mathbf{Q}^\mu,\nu,\nu,\nu} \) is the probability density function of \( \mathbf{Q}^\mu,\nu,\nu,\nu \). However, the optimization problem defined by Eq. (8) yields a very high computational cost induced by the estimation of the joint probability density functions \( p_{\mathbf{Q}^\mu,\nu,\nu,\nu}(q^j, \mathcal{A}) \) (even for reasonable values of the length \( \mu \) of random vector \( \mathbf{Q}^\mu,\nu,\nu,\nu \)). Consequently, it is proposed to substitute the usual likelihood function by the pseudo-likelihood function

\[
\tilde{L}(q^1, \ldots, q^m; \mathcal{A}) = \prod_{k=1}^{\mu} p_{\mathbf{Q}^\mu,\nu,\nu,\nu}(q^j_k, \mathcal{A}) \times \ldots \times p_{\mathbf{Q}^\mu,\nu,\nu,\nu}(q^m_k, \mathcal{A})
\]

where \( q^j = (q^1_j, \ldots, q^\mu_j) \) and \( \mathbf{Q}^\mu,\nu,\nu,\nu = (Q^\mu_1, \ldots, Q^\mu_\mu) \) and where \( p_{\mathbf{Q}^\mu,\nu,\nu,\nu} \) is the probability density function of random variable \( \mathbf{Q}^\mu,\nu,\nu,\nu_k \). Finally, the following problem of optimization is substituted to the problem defined by Eq. (8). Find \( \mathcal{A} = \{ a_\alpha, |\alpha|=1, \ldots, d \} \) such that

\[
\max_{\mathcal{A}} \tilde{L}(q^1, \ldots, q^m; \mathcal{A}) , \quad \text{with} \quad \sum_{\alpha,|\alpha|=1}^{d} a_\alpha a_\alpha^T = [I_\mu]
\]

6 Convergence Analysis

In order to perform a convergence analysis of the method proposed in this paper, the normalized random variables \( \mathcal{Y}(\mathbf{x}) \) and \( \tilde{\mathcal{Y}}^\mu,\nu,\nu,\nu(\mathbf{x}) \) defined by \( \mathcal{Y}(\mathbf{x}) = \mathcal{Y}(\mathbf{x}) / E\{\mathcal{Y}(\mathbf{x})\} \) and \( \tilde{\mathcal{Y}}^\mu,\nu,\nu,\nu(\mathbf{x}) = \tilde{\mathcal{Y}}^\mu,\nu,\nu,\nu(\mathbf{x}) / E\{\tilde{\mathcal{Y}}^\mu,\nu,\nu,\nu(\mathbf{x})\} \),
for all $x \in D$, are introduced. Let $r(x,x')$ and $r^{n,\nu,d}(x,x')$ be the correlation functions of the random fields $Y$ and $\widetilde{Y}^{n,\nu,d}$ such that, for all $x$ and $x'$ in $D$,

$$r(x,x') = \frac{E\{Y(x)Y(x')\} - 1}{\sqrt{(E\{Y(x)^2\} - 1)(E\{Y(x')^2\} - 1)}},$$

and

$$r^{n,\nu,d}(x,x') = \frac{E\{\widetilde{Y}^{n,\nu,d}(x)\widetilde{Y}^{n,\nu,d}(x')\} - 1}{\sqrt{(E\{\widetilde{Y}^{n,\nu,d}(x)^2\} - 1)(E\{\widetilde{Y}^{n,\nu,d}(x')^2\} - 1)}}.$$

The mean-square convergence analysis is performed with respect to the correlation functions and shows that the probabilistic model is converged for $\nu = 4$. The remaining error is due to the truncating of the statistical reduction defined in Section 5.

7 Identification of the probabilistic model

Figure 3 shows the graphs of $x \mapsto r(x,x')$ (thick dashed line) and $x \mapsto r^{n,\nu,d}(x,x')$ (thin solid line) where $x = (x_1,x_2,x_3)$ and $x' = (x'_1,x'_2,x'_3)$ with $x_2 = x_3 = 0$ and $x'_1 = 0.0520, x'_2 = x'_3 = 0$ and with $d = 5, \mu = 15, \nu = 4$.

![Figure 3: Graphs of $x \mapsto r(x,x')$ (thick dashed line) and $x \mapsto r^{n,\nu,d}(x,x')$ (thin solid line) where $x = (x_1,x_2,x_3)$ and $x' = (x'_1,x'_2,x'_3)$ with $x_2 = x_3 = 0$ and $x'_1 = 0.0520, x'_2 = x'_3 = 0$ and with $d = 5, \mu = 15, \nu = 4$. Horizontal axis: $x_1$. Vertical axis: $r(x,x')$ and $r^{n,\nu,d}(x,x')$.](image)

For all $x \in D$, let $y \mapsto p_{Y(x)}(y;x)$ and $y \mapsto p_{\widetilde{Y}^{n,\nu,d}(x)}(y;x)$ be the probability density functions of the random variables $Y(x)$ and $\widetilde{Y}^{n,\nu,d}(x)$. Figure 4 shows the graphs of $y \mapsto \log_{10}(p_{Y(x)}(y;x))$ (thick solid line) and $y \mapsto \log_{10}(p_{\widetilde{Y}^{n,\nu,d}(x)}(y;x))$ (thin solid line) where $x = (x_1,x_2,x_3)$ with $x_2 = x_3 = 0$ and $x_1 = 0.0152$ with $d = 5, \mu = 15$ and $\nu = 4$. It can be seen that the probability density function is accurately identified.

8 Conclusion

A method for solving the stochastic inverse problem using chaos representation of the stochastic field to be identified and an experimental database is proposed. The proposed method uses the maximum likelihood principle to identify the coefficients of the chaos representation. For presented example, this method allows any probabilistic quantities to be identified such as the autocorrelation function of the random field and the marginal probability density functions. It should be noted that the proposed method can easily be extended to the case of a viscoelastic random medium for which the elastic properties depend on frequency.
Figure 4: Graphs of $y \mapsto \log_{10}(p_Y(x)(y|x))$ (thick solid line) and $y \mapsto \log_{10}(p_{\tilde{Y}_{\mu,\nu,d}}(y|x))$ (thin solid line) where $x = (x_1, x_2, x_3)$ with $x_2 = x_3 = 0$ and $x_1 = 0.0152$ with $d = 5$, $\mu = 15$ and $\nu = 4$. Horizontal axis: $y$. Vertical axis: $\log_{10}(p_Y(x)(y|x))$ and $\log_{10}(p_{\tilde{Y}_{\mu,\nu,d}}(y|x))$.

References


