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ABSTRACT: Construction codes demand ever increasing earthquake-resisting features for strategic buildings such as dams and nuclear plants and the incorporation of uncertainty in the design models for these structures, particularly in soil domains, becomes a major issue. Parametric methods and a recent non-parametric method are considered for the construction of a probabilistic model of the soil impedance matrix, the latter supplying interesting features provided that the matrices of a certain mean model can be identified. This difficulty is tackled using a hidden state variables model, ensuring causality of the impedance and necessary positive definiteness conditions on the generated matrices. The identification of the uncertain parameters in the soil and the difficult task of quantifying their variability are not required and computational costs are significantly reduced.

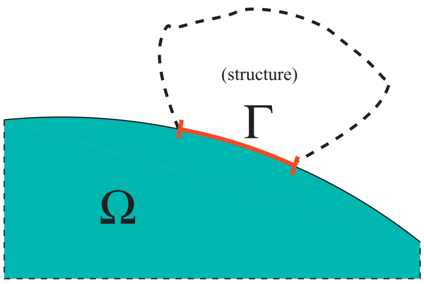

Figure 1. unbounded domain Ω and coupling boundary Γ

1 INTRODUCTION

Let Ω be a three-dimensional open half space of $\mathbb{R}^3$ with a smooth boundary $\partial\Omega$ (Fig. 1). Let Γ be a bounded part of $\partial\Omega$, coupling Ω with another domain. Let $\boldsymbol{\tau}$ be a stress field defined on Γ and $\boldsymbol{u}$ the corresponding displacement field. The primal formulation of the local problem associated to Ω leads to the following linear operator equation

$$\mathcal{Z}\boldsymbol{u} = \boldsymbol{\tau}, \tag{1}$$

where $\mathcal{Z}$ is the Steklov-Poincaré operator, corresponding to the condensation of the dynamic stiffness operator of the domain Ω. The dual formulation leads to $\mathcal{S}\boldsymbol{\tau} = \boldsymbol{u}$, where $\mathcal{S}$ is the flexibility operator, formally verifying $\mathcal{S} = \mathcal{Z}^{-1}$.

Particularly important in earthquake engineering, as well as in many other civil engineering and aerospace engineering applications, the computation of the impedances of unbounded domains has been extensively studied in a deterministic framework. For a bounded Γ, this operator can be approximated with finite error by an impedance matrix $[Z]$ (Wolf 1985). Quantification of uncertainty on these operators has been addressed more recently (Schuëller 1997, Manolis 2002), leading to the following linear stochastic operator equation

$$\mathcal{Z}(\theta)\,\boldsymbol{u} = \boldsymbol{\tau}(\theta), \tag{2}$$

where $\boldsymbol{\tau}(\theta)$ is a stochastic field and $\mathcal{Z}(\theta)$ is the stochastic stiffness operator. Considering uncertainty only in a bounded volume Ω_δ of Ω (Fig. 2), $\mathcal{Z}(\theta)$ is then a perturbation of a deterministic operator and therefore, all realisations of $\mathcal{Z}(\theta)$ can be approximated with finite error on a common basis, leading to

$$[Z(\theta)]\,\mathbf{u} = \mathbf{t}(\theta), \tag{3}$$

where $\mathbf{u}$ is the displacement vector, $\mathbf{t}(\theta)$ is an approximation of the stochastic stress vector and $[Z(\theta)]$ is the stochastic stiffness matrix.

This paper presents existing methods to compute the soil impedance matrix (section 2), stressing the

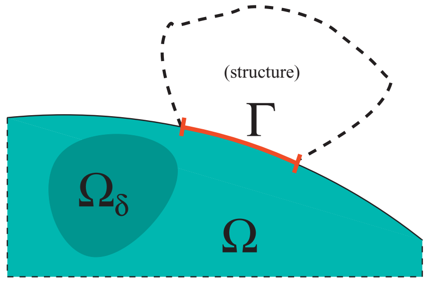

Figure 2. bounded uncertain domain Ω_δ in unbounded deterministic domain Ω and coupling boundary Γ

appeal of the non-parametric method, which requires the construction of a probabilistic model (section 3) based on real positive definite matrices and for which causality is enforced. The identification of the mean matrices for this model is then presented (section 4), enabling application of this method in a simple case (section 5).

2 COMPUTATIONAL METHODS

In this section two classes of computational methods to account for the uncertainty in $[Z(\theta)]$ are presented: the classical parametric methods and a more recent non-parametric method.

2.1 *Parametric methods*

The Stochastic Finite Element Method (SFEM) is a classical tool to compute $[Z(\theta)]$ (Cornell 1971). Like its deterministic equivalent, it requires the discretization of domains Ω and Ω_δ and therefore the creation of an artificial boundary, bringing along problems of unphysical wave reflections. Using a Finite Element Method (FEM) - Boundary Element Method (BEM) coupling approach to model the problem (Savin and Clouteau 2002) allows for correct consideration of the unboundedness of domain Ω and reduces the size of the discretized domain. However, it still implies, in practical situations, a very large number of variables, and *in fine* high computational costs.

Besides these classical drawbacks of the FEM, these and other parametric methods require the identification of the uncertain parameters and the quantification of that uncertainty, that is to say appropriate probabilistic models of these uncertain parameters have to be constructed, based on given statistics. Such problems are not simple, particularly in the case of soils, where sources of uncertainty are numerous and measurement difficulties hinder the recollection of accurate data (Favre 1998).

The propagation of the uncertainty from the parameters to the response of the system is then usually performed via Monte-Carlo simulations, leading to prohibitive costs, particularly when uncertainty on several parameters has to be considered. Also, since the correlation between these parameters is difficult to assess, physically unsound systems can be computed.

2.2 *Non-parametric method*

Recently, Soize introduced a non-parametric method (Soize 2000) where the application of the maximum entropy principle (Jaynes 1957) to the reduced matrix model of a system leads to a probabilistic model using only the information available. This method is based on the direct construction of a probabilistic model of the generalized mass, damping and stiffness matrices, obviating the identification of the uncertain local parameters and the construction of their probabilistic model. Coarse statistical studies on the parameters are therefore not needed and physically unsound results are avoided, as long as the physics were correctly introduced in the model. The non-parametric method also accounts for modelling errors.

In the case of bounded uncertain domains (let it be Ω_b), the stiffness matrix can be written as a quadratic function of frequency in terms of a positive definite matrix of mass and positive matrices of damping and stiffness.

$$[Z_{\Omega_b}(w;\theta)] = [K(\theta)] + i\omega\,[C(\theta)] - \omega^2\,[M(\theta)]\,. \qquad (4)$$

This ensures causality of the corresponding model in the time domain, since equation (3) with the impedance in the form of (4) is related to a second order differential equation in the time domain. The mean matrices of the probabilistic model are identified with the matrices of the deterministic model.

In the case of a bounded uncertain domain Ω_δ inside an unbounded deterministic domain Ω, the same FEM-BEM coupling approach can be used, introducing the uncertainty in Ω_δ by means of the non-parametric method. This leads to a flexibility matrix in the form:

$$[S(\theta)] = [U_{\Gamma\Gamma}] - [U_{\Gamma\delta}]\,[Z_{\delta\delta}(\theta)]\,[U_\delta]^T\,[Z_\delta(\theta)]\,[U_{\delta\Gamma}]\,, \qquad (5)$$

with

$$[Z_{\delta\delta}(\theta)] = \left([U_{\delta\delta}] + [U_\delta]^T\,[Z_\delta(\theta)]\,[U_\delta]\right)^{-1}. \qquad (6)$$

where $[U_{\Gamma\delta}]$, $[U_{\Gamma\Gamma}]$, $[U_{\delta\delta}]$ and $U_{\delta\Gamma}$ are matrices following from the discretization of, respectively, the traces on boundary Γ of operators $\mathcal{U}^0_\delta$ and $\mathcal{U}^0_\Gamma$ and the restrictions on domain Ω_δ of the same operators. These operators are defined, if $(\boldsymbol{x},\boldsymbol{y}) \mapsto U^G(\boldsymbol{x},\boldsymbol{y})$ is Green's function of the deterministic domain Ω,

$\boldsymbol{x} \mapsto f_v(\boldsymbol{x})$ a volumic load function defined on Ω_δ and $\boldsymbol{x} \mapsto f_s(\boldsymbol{x})$ a surfacic load function defined on Γ, by

$$\left(\mathcal{U}_\delta^0 f_v\right)(\boldsymbol{x}) = \int_{\Omega_\delta} U^G(\boldsymbol{x}, \boldsymbol{y}) f_v(\boldsymbol{y}) dV_{\boldsymbol{y}}, \tag{7}$$

$$\left(\mathcal{U}_\Gamma^0 f_s\right)(\boldsymbol{x}) = \int_{\Gamma} U^G(\boldsymbol{x}, \boldsymbol{y}) f_s(\boldsymbol{y}) dS_{\boldsymbol{y}}. \tag{8}$$

$[U_\delta]$ derives from the same operator as $[U_{\delta\delta}]$ but is projected on a different basis of functions. $[Z_\delta(\theta)]$ being the impedance of domain Ω_δ, it can then be expanded as in (4), and the non-parametric method can be used to generate the matrices of mass, stiffness and damping, and ultimately condensation on Γ using (5)-(6) to generate $[S(\theta)]$ and $[Z(\theta)]$. Although some difficulties have to be addressed (Soize and Chebli 2003), such as the existence of rigid body modes which take down the positive definiteness of matrices of damping and stiffness, this approach is feasible. Unfortunately, the computational cost is not *a priori* lower than that of the SFEM-BEM method. All internal degrees of freedom (DOFs) of the Finite Element model are considered, when all is needed is their trace on boundary Γ.

The construction of a probabilistic model directly for the soil impedance would add to the advantages of the non-parametric method an important reduction in computational costs. As $[Z(\theta)]$ cannot be *a priori* expanded as in (4) in the case of an unbounded domain, a causal reduced model of the soil impedance has to be constructed and the identification of the mean matrices for this model has to be performed.

3 PROBABILISTIC MODEL FOR THE SOIL IMPEDANCE

The non-parametric method is based on the possibility of generating the analytical probability density function of a frequency independent real positive definite (or positive) matrix given its mean and a certain dispersion parameter. To be able to apply this method, the model for the soil impedance matrix must then be composed only of such matrices and, as stated in section 2.2, causality has to be enforced in order to bound unphysical results. Three methods are described hereafter, beginning with the Kramers-Kronig relations, widely used in experimental physics.

3.1 Kramers-Kronig relations

Initially developped for electromagnetic problems to link the real and imaginary parts of the complex susceptibility (Kramers 1927) and of the complex refraction index (Kronig 1926), the Kramers-Kronig relations have later been recognized a wider range of application. Being built solely on causality, they have to be verified by the frequency response function (FRF) of any physical system, $[S]$ in the present case, and state that

$$\Re\{[S(\omega)]\} = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{\Im\{[S(\omega')]\}}{\omega' - \omega} d\omega', \tag{9}$$

or, equivalently,

$$\Im\{[S(\omega)]\} = -\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{\Re\{[S(\omega')]\}}{\omega' - \omega} d\omega', \tag{10}$$

where $\Re\{[S(\omega)]\}$ and $\Im\{[S(\omega)]\}$ are respectively the real and imaginary parts of $[S(\omega)]$, and $\mathcal{P}$ refers to Cauchy's principal part.

In many applications, $\Im\{[S(\omega)]\}$ can be measured experimentally and equation (9) can then be used to reconstruct $[S(\omega)]$, but numerically, $\Im\{[S(\omega)]\}$ is of no easier access than $\Re\{[S(\omega)]\}$ or $[S(\omega)]$, for which the Kramers-Kronig relations are of no help to construct the probability model of $[S(\omega)]$.

3.2 Hardy functions decomposition

Another method to enforce the causality of the impedance is to expand it, if possible, on a basis of causal functions, like that of the Hardy functions (Pierce 2001). h is said to be a Hardy function on the upper half plane if and only if it is the Laplace transform of some causal function f. Functions $\omega \mapsto e_n(\omega)$ for $n \in \mathbb{N}$

$$e_n(\omega) = \frac{1}{\sqrt{\pi}}\left(\frac{1}{i\omega - 1}\right)\left(\frac{i\omega + 1}{i\omega - 1}\right)^n \tag{11}$$

form an orthonormal basis of the space of Hardy functions. Therefore any causal matrix can be sought on a basis of Hardy functions.

$$[Z(\omega;\theta)] = \sum_{n \geq 0} [Z_n(\theta)]\, e_n(\omega) \tag{12}$$

Unfortunately, very little is known on the *a priori* properties of the $[Z_n(\theta)]$, for $n \geq 0$, such as positive definiteness, impeding application of the non-parametric method. Also the number of terms required, in the right-hand side of (12), to obtain a correct approximation of $[Z(\omega;\theta)]$ might be important.

3.3 Hidden state variables

Ultimately, $[Z(\omega)]$ is sought as the condensation on Γ of a mechanical system governed by a second order differential equation with constant coefficients. As this is in general untrue, if such a structure is to be retained to ensure causality, hidden variables have to be introduced, which will be linked only indirectly to the internal DOFs of the system (Chabas and Soize 1987).

The system is therefore discretized in n_Γ DOFs on the boundary Γ and n_h hidden state variables. This last number has to be accurately chosen so that the model can account for the variations of the FRF. The total number of DOFs of the discretization is $n = n_\Gamma + n_h$. The impedance of this system can be expanded as in (4) in terms of real $n \times n$ positive definite matrices of mass $[M_A]$, damping $[C_A]$ and stiffness $[K_A]$. The impedance $[A(\omega)]$ can be block decomposed in

$$[A(\omega)] = \begin{bmatrix} [Z_{\Gamma\Gamma}(\omega)] & [Z_{\Gamma h}(\omega)] \\ [Z_{\Gamma h}(\omega)]^T & [Z_{hh}(\omega)] \end{bmatrix}, \tag{13}$$

where $[Z_{\alpha\beta}(\omega)] = [K_{\alpha\beta}] + i\omega\,[C_{\alpha\beta}] - \omega^2\,[M_{\alpha\beta}]$, for (α, β) in $\{\Gamma, h\}^2$, $[M_{\Gamma\Gamma}]$, $[C_{\Gamma\Gamma}]$ and $[K_{\Gamma\Gamma}]$ are $n_\Gamma \times n_\Gamma$ real positive definite matrices, $[M_{\Gamma h}]$, $[C_{\Gamma h}]$ and $[K_{\Gamma h}]$ $n_\Gamma \times n_h$ real matrices and $[M_{hh}]$, $[C_{hh}]$ and $[K_{hh}]$ $n_h \times n_h$ real positive definite matrices. Condensation on Γ then leads to

$$[Z(\omega)] = [Z_{\Gamma\Gamma}(\omega)] - [Z_{\Gamma h}(\omega)]\,[Z_{hh}(\omega)]^{-1}\,[Z_{\Gamma h}(\omega)]^T.$$

Assuming, as usually done, that $[C_{hh}]$ is diagonalized by the eigenvectors solutions of the generalized eigenvalue problem $[K_{hh}]\,\psi = \lambda\,[M_{hh}]\,\psi$, $[Z(\omega)]$ can be written

$$[Z(\omega)] = [Z_{\Gamma\Gamma}(\omega)] - \sum_{k=1}^{n_h} \frac{[Z_{\Gamma h}(\omega)]\,\psi_k\psi_k^T\,[Z_{\Gamma h}(\omega)]^T}{-\omega^2 m_k + i\omega c_k + k_k}.$$

In other words, the boundary impedance of this mechanical system has the form

$$[Z(\omega)] = \frac{[N(\omega)]}{d(\omega)}, \tag{14}$$

where $\omega \mapsto [N(\omega)]$ and $\omega \mapsto d(\omega)$ are two polynomials of frequency ω with constant coefficients (matricial for N and scalar for d). The orders of d and N verify $\deg d = 2n_h$ and $\deg N = \deg d + 2$.

This formulation ensures the causality of $[Z(\omega)]$ as (3) then corresponds in the time domain to a differential equation with constant coefficients, and the impedance is computed using only the real positive definite matrices $[M_A]$, $[C_A]$ and $[K_A]$. The non-parametric method can then be applied to this model to obtain a probabilistic model of the soil impedance matrix, provided that the mean matrices $[\underline{M_A}] = E\{[M_A]\}$, $[\underline{C_A}] = E\{[C_A]\}$ and $[\underline{K_A}] = E\{[K_A]\}$ can be identified.

4 IDENTIFICATION OF THE MEAN MODEL

As for the non-parametric method applied to reduced Finite Element uncertain models, where the mean matrices are identified with the matrices of the deterministic model, the mean matrix of the soil impedance

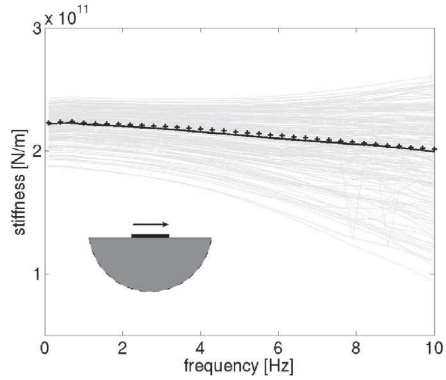

Figure 3. The real part of the x-sway element of the impedance matrix of a homogeneous half space without heterogeneity (+) and with bounded heterogeneity (— all Monte-Carlo trials and — mean of these trials)

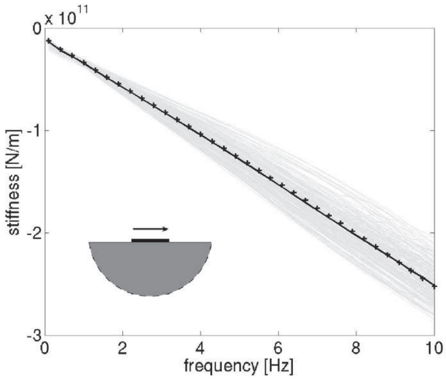

Figure 4. The imaginary part of the x-sway element of the impedance matrix of a homogeneous half space without heterogeneity (+) and with bounded heterogeneity (— all Monte-Carlo trials and — mean of these trials)

will be assimilated to the soil impedance matrix of the domain Ω without the uncertain domain Ω_δ. Although not mathematically sound, this hypothesis is physically appealing and the computations performed with the parametric SFEM-BEM method described in section 2.1 sustain it (Fig. 3-4).

The identification of the mean matrix of the soil impedance with the soil impedance matrix of deterministic soil consists in finding $[\underline{M_A}]$, $[\underline{C_A}]$ and $[\underline{K_A}]$ which minimize

$$\epsilon = \sum_{\ell=1}^{L} \|[\underline{Z}(\omega_\ell)] - [Z_0(\omega_\ell)]\|_F, \tag{15}$$

where the ω_ℓ, for $1 \geq \ell \geq L$, are the frequencies at which the deterministic impedance $[Z_0(\omega_\ell)]$ has been computed and $[\underline{Z}(\omega_\ell)]$ is the condensation on Γ of the mean matrix $[\underline{A}(\omega)] = [\underline{K_A}] + i\omega[\underline{C_A}] - \omega^2[\underline{M_A}]$, given by (13)-(14). This optimization problem is in general nonlinear in the parameters.

Before studying the general case of identifying the mean impedance matrix for n_h hidden state variables, the simple case where the half space with the uncertain domain Ω_δ is modeled by a mass - spring - dashpot system is presented.

4.1 *No hidden state variables ($n_h = 0$)*

In that case, $[\underline{Z}(\omega)] = [\underline{Z_{\Gamma\Gamma}}(\omega)]$ and the minimization problem becomes linear in the parameters. An exact solution for $[\underline{K_A}]$, $[\underline{C_A}]$ and $[\underline{M_A}]$ can then be found.

$$[\underline{K_A}] = \frac{\underline{\omega^4}\,\underline{\Re[Z_0(\omega)]} - \underline{\omega^2}\,\underline{\omega^2\Re[Z_0(\omega)]}}{\underline{\omega^4} - (\underline{\omega^2})^2}, \tag{16}$$

$$[\underline{C_A}] = \frac{\underline{\omega\Im[Z_0(\omega)]}}{\underline{\omega^2}}, \tag{17}$$

$$[\underline{M_A}] = \frac{\underline{\omega^2}\,\underline{\Re[Z_0(\omega)]} - \underline{\omega^2\Re[Z_0(\omega)]}}{\underline{\omega^4} - (\underline{\omega^2})^2}. \tag{18}$$

where $\underline{\omega^4} = \sum_{\ell=1}^{L}\omega_\ell^4$, $\underline{\omega^2} = \sum_{\ell=1}^{L}\omega_\ell^2$, $\underline{\Re[Z_0(\omega)]} = \sum_{\ell=1}^{L}\Re\{[Z_0(\omega_\ell)]\}$, $\underline{\omega^2\Re[Z_0(\omega)]} = \sum_{\ell=1}^{L}\omega_\ell^2\Re\{[Z_0(\omega_\ell)]\}$ and $\underline{\omega\Im[Z_0(\omega)]} = \sum_{\ell=1}^{L}\omega_\ell\Im\{[Z_0(\omega_\ell)]\}$.

4.2 *n_h hidden state variables ($n_h > 0$)*

With the introduction of hidden variables, the minimization of (15) becomes a non-linear problem. Many different methods exist for the resolution of such problems (Heylen et al. 1997), most of these depending on an initial value that has to be guessed to begin the optimization process. This initial value can be sought using a linearized form of (15):

$$\epsilon = \sum_{\ell=1}^{L}\|[\underline{N}(\omega_\ell)] - \underline{d}(\omega_\ell)\,[Z_0(\omega_\ell)]\|_F \tag{19}$$

where, for $1 \leq \ell \leq L$, $[\underline{N}(\omega_\ell)]$ and $\underline{d}(\omega_\ell)$ are defined as in (14) - (14) for $[\underline{Z}(\omega_\ell)]$.The identification on a basis of orthogonal polynomials (Pintelon et al. 2004, Bultheel and Van Barel 1995) is particularly adapted. The minimization process is then completed by a classical non-linear optimization problem starting from the value computed through minimization of (19).

5 EXAMPLE

The application of the non-parametric method to compute the probabilistic model of the soil impedance matrix is performed in four steps:

1. Deterministic $[Z_0(\omega_\ell)]$ is computed using classical computational tools;
2. The mean matrices $[\underline{K_A}]$, $[\underline{C_A}]$ and $[\underline{M_A}]$ of the probabilistic model are identified using the results of section 4;
3. Given a dispersion parameter, the maximum entropy principle gives the probability density function of matrices $[K_A]$, $[C_A]$ and $[M_A]$;
4. Using Monte-Carlo trials, the realisations of matrix $[Z(\omega)]$ are computed and the moments derived.

Using this methodology, for the case of a superficial foundation on a homogenous half space (the results for a parametric method are shown in Fig. 3 and 4), with a dispersion factor $\delta = 0.1$ for all matrices, and considering no hidden variables, the following means and typical deviations can be computed for 500 Monte-Carlo trials.

$$\begin{cases} [K_A]_{xx} = 2.17 \pm 0.12 \times 10^{11}\,N/m, \\ [C_A]_{xx} = 3.56 \pm 0.18 \times 10^{9}\,N/(m.s), \\ [M_A]_{xx} = 0.60 \pm 0.30 \times 10^{7}\,N/(m.s^2) \end{cases}. \tag{20}$$

These values have to be compared to the values obtained with the parametric method.

$$\begin{cases} [K_A]_{xx} = 2.18 \pm 0.13 \times 10^{11}\,N/m, \\ [C_A]_{xx} = 3.54 \pm 0.20 \times 10^{9}\,N/(m.s), \\ [M_A]_{xx} = 0.60 \pm 1.10 \times 10^{7}\,N/(m.s^2) \end{cases}. \tag{21}$$

The only significant difference lies in the mass typical deviation, and is due to the simplicity of the model used (no hidden variables). Considering each Monte-Carlo trial of the impedance matrix independently, the identification in terms of mass, damping and stiffness matrices for the parametric method leads in some cases to a non positive definite matrix of mass. Since these realisations are out of reach for the non-parametric method, the results are necessarily condensed closer around the mean value. Hidden state variables are required in order to take into account more precisely the physics of the impedance matrix, and therefore account for its variations using only positive definite matrices. When considering a layered half space, this need for more DOFs will become even more critical.

6 CONCLUSIONS

The non-parametric method presented in this paper allows for the construction of a probabilistic model of the soil impedance matrix in an objective manner. It does not require a previous identification of the uncertain parameters and the construction of a probabilistic model for each of them, as statistical data is usually scarcely available, and it accounts for modelling errors. Compared to other possible implementations of the non-parametric method in unbounded domains, it achieves a dramatic reduction in computational time.

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