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ABSTRACT

An approximate formula which utilizes the concept of conditional power spectral density (PSD) has been employed by several investigators to determine the response PSD of stochastically excited nonlinear systems in numerous applications. However, its derivation has been treated to date in a rather heuristic, even "unnatural" manner, and its mathematical legitimacy has been based on loosely supported arguments. In this paper, a perspective on the veracity of the formula is provided by utilizing spectral representations both for the excitation and for the response processes of the nonlinear system; this is done in conjunction with a stochastic averaging treatment of the problem. Then, the orthogonality properties of the monochromatic functions which are involved in the representations are utilized. Further, not only stationarity but ergodicity of the system response are invoked. In this context, the nonlinear response PSD is construed as a sum of the PSDs which correspond to equivalent response amplitude dependent linear systems. Next, relying on classical excitation–response PSD relationships for these linear systems leads, readily, to the derivation of the formula for the determination of the PSD of the nonlinear system. Related numerical results are also included.

1. Introduction

Popular random vibration techniques, such as the statistical linearization [1,2], have been proven quite successful in determining approximately the response statistics, such as the variance, of nonlinear systems [3,4]. However, this is not the case for quantities such as the response power spectral density (PSD). In fact, response PSD estimation applications that utilize a statistical linearization technique may reflect significant discrepancy from the correct result. Specifically, while the technique yields the right resonance frequencies, it may overestimate the peak values and underestimate the bandwidths [5].

In Ref. [6] an improved linearization technique was proposed, where the stiffness element of the oscillator was treated as a random variable. In Ref. [7] the form of the conditional PSD was presented for the first time and was used to derive an approximate formula for the response PSD. However, the term conditional PSD had not been introduced until Bouc [5] first suggested the terminology and derived an approximate formula for computing the nonlinear PSD by resorting to the stochastic averaging method [8,9]. In general, the technique developed in [5,7] utilizes a family of equivalent linear systems whose elements are response amplitude envelope dependent. Specifically, the response PSD is estimated as a weighted sum of the response PSDs of the linear systems. The weight function is merely the probability density function (PDF) of the amplitude process. In Ref. [10] the response amplitude PSD of a randomly excited Preisach system was computed accurately using the discussed formula. This kind of approach was also adopted and applied in the context of nonlinear system identification [11,12]. Moreover, in Ref. [13] and subsequent papers [14,15] an alternative formulation was presented relying on equivalent linear systems whose elements are response energy envelope dependent. In this manner, additional information related to the occurrence of higher harmonics was captured.

Taking into account the versatility and the accuracy of this technique as it has been reported in a large number of applications, it is desirable to attempt to establish its veracity on a firm mathematical basis; thus, the derivation of a concrete proof and the presentation of an alternative perspective on the veracity of the technique constitute this paper. To this aim, spectral representations [16,17] for both the stochastic excitation and the response processes,

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the orthogonality conditions of the monochromatic functions and stochastic averaging are combined to derive the discussed formula.

2. Mathematical formulation

2.1. Equivalent linear system interpretation

Consider a nonlinear single degree of freedom system whose motion is governed by the differential equation

$$\ddot{x} + \beta\dot{x} + z(t, x, \dot{x}) = w(t). \tag{1}$$

In this equation a dot over a variable denotes differentiation with respect to time (t); $(z(t, x, \dot{x}))$ is the restoring force which can be either hysteretic or depend only on the instantaneous values of (x) and $(\dot{x})$; (β) is a linear damping coefficient; and $(w(t))$ represents a zero-mean stationary random process possessing a broad-band power spectrum $S_w(\omega)$. It is further assumed that the system is lightly damped and thus the response can be treated as a narrow-band process with slowly varying effective amplitude (A), and frequency $(\omega(A))$.

A statistical linearization/stochastic averaging technique may next be adopted to determine response amplitude dependent equivalent damping and stiffness elements. In fact, several different implementations of the linearization technique exist in the literature, which are presented and discussed in [1,2]. Following the technique developed in [18], the linearized counterpart of Eq. (1) becomes

$$\ddot{x} + \beta(A)\dot{x} + \omega^2(A)x = w(t), \tag{2}$$

where the equivalent damping element and natural frequency are taken to be functions of the amplitude (A) of the response to account for the effect of the nonlinearity. Due to the overall light damping of the system, the amplitude (A) is a slowly varying function with respect to time and therefore is treated as a constant over one cycle of oscillation. Specifically, defining the error between Eqs. (1) and (2) as

$$\varepsilon = z(t, x, \dot{x}) + [\beta - \beta(A)]\dot{x} - \omega^2(A)x, \tag{3}$$

the expressions

$$\beta(A) = \beta + \frac{\oint \dot{x}z\mathrm{d}t}{\oint \dot{x}^2\mathrm{d}t}, \tag{4}$$

and

$$\omega^2(A) = \frac{\oint xz\mathrm{d}t}{\oint x^2\mathrm{d}t} \tag{5}$$

are derived. Eqs. (4) and (5) have been obtained by applying an error minimization procedure in the mean square sense, where $(\oint)$ can be interpreted as 'an average over one cycle' operator. Due to the narrow-band attribute of the response Eqs. (4) and (5) yield, respectively,

$$\beta(A) = \beta + \frac{S(A)}{A\omega(A)}, \tag{6}$$

and

$$\omega^2(A) = \frac{C(A)}{A}, \tag{7}$$

where

$$C(A) = \frac{1}{\pi}\int_0^{2\pi} \cos[\psi]z(t, A\cos\psi, -\omega(A)A\sin\psi)\mathrm{d}\psi, \tag{8}$$

and

$$S(A) = -\frac{1}{\pi}\int_0^{2\pi} \sin[\psi]z(t, A\cos\psi, -\omega(A)A\sin\psi)\mathrm{d}\psi. \tag{9}$$

Considering the aforementioned derivation of the equivalent linear system, it can be argued that the PSD of the response process is likely to lend itself to an equivalent amplitude dependent "linearized" representation. Since for a fixed value of the amplitude, i.e. $(A = A^*)$, the oscillator of Eq. (2) takes the form of a linear oscillator with fixed damping and stiffness elements, the PSD of the response is expected to be

$$S_x(\omega) = \frac{S_w(\omega)}{\left(\omega^2(A^*) - \omega^2\right)^2 + (\beta(A^*)\omega)^2}. \tag{10}$$

Consequently, an intuitive estimate of the expression of the response PSD would be a weighted sum of the "linear" PSD components of the form of Eq. (10) for $(A^* \in [0, \infty])$. Furthermore, considering a long time interval, the weighting factors of this sum would be the ratios of the time the process spends at amplitude (A^*) over the total period of time. In mathematical terms, one can set as an approximation for the system response PSD the expression

$$S_x(\omega) = \lim_{\substack{T\to\infty\\ M\to\infty\\ \Delta A\to 0}} \times \left[\sum_{k=0}^{M}\left(\frac{T_k}{T}\frac{S_w(\omega)}{\left(\omega^2(k\Delta A) - \omega^2\right)^2 + (\beta(k\Delta A)\omega)^2}\right)\right]. \tag{11}$$

Note that inherent in the derivation of Eq. (11) is the consideration of two different time scales. Specifically, a short time scale (t_s) is associated with the rapid fluctuations induced by the dynamics of the model, and a relatively long time scale (t_l) is associated with the low variations of the amplitude. In an attempt to confirm the reliability of Eq. (11), a Duffing oscillator of the form

$$\ddot{x} + 2\zeta_0\omega_0\dot{x} + \omega_0^2x + \varepsilon\omega_0^2x^3 = w(t), \quad \varepsilon > 0, \tag{12}$$

is considered, where (ζ_0) is the ratio of critical damping. Using Eqs. (6) and (7), the amplitude dependent equivalent natural frequency and damping term are found to be, respectively,

$$\beta(A) = 2\zeta_0\omega_0, \tag{13}$$

and

$$\omega^2(A) = \omega_0^2\left(1 + \frac{3}{4}\varepsilon A^2\right). \tag{14}$$

Choosing the values $(S_w(\omega) = S_0 = 0.3, \zeta_0 = 0.01, \omega_0 = 3.612$ rad/s, $\varepsilon = 0.2,\ M = 300)$ and utilizing Eq. (11), the response PSD of the Duffing oscillator is estimated in Fig. 1. For this, an adequately long time interval is considered and the time portions the response spends at each amplitude level are determined and substituted into Eq. (11). The derived result is compared with a direct estimation of the response PSD using the Welch method [19,20]. An identical procedure is also applied to a linear oscillator possessing the same values for the natural frequency, and the ratio of critical damping. Examining Eq. (1) the numerical results clearly support the interpretation suggested by Eq. (11). Furthermore, the presented PSD estimations are in agreement with the conclusions drawn in [5,6,21]. Specifically, it can be readily seen that the nonlinear stiffness induces a significant broadening and a shift in resonance peaks in the response PSD.

Returning to Eq. (11) the ergodicity property for the response will be invoked to proceed with. Specifically, the response displacement (x), the response velocity $(\dot{x})$ and the response amplitude (A) processes will be treated as ergodic in the second order sense. Thus, the term $(\lim_{T\to\infty}\frac{T_k}{T})$ may be interpreted as the probability the process maintains the specific amplitude level $(A = k\Delta A)$. Therefore, it can be argued that

$$\lim_{T\to\infty}\frac{T_k}{T} = p(k\Delta A)\Delta A, \tag{15}$$

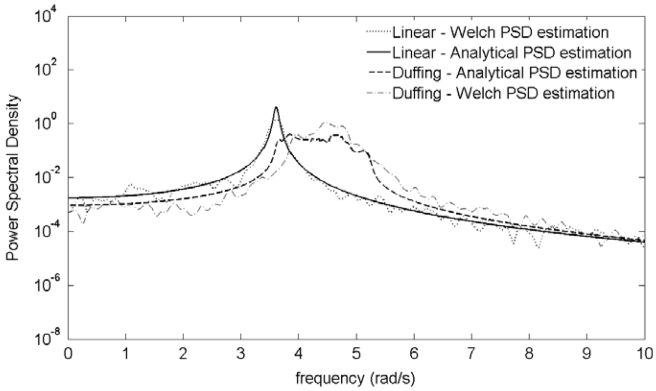

Fig. 1. Response PSD estimation of a Duffing ($S_w(\omega) = S_0 = 0.3$, $\zeta_0 = 0.01$, $\omega_0 = 3.612$ rad/s, $\varepsilon = 0.2$), and of a linear oscillator ($S_w(\omega) = S_0 = 0.3$, $\zeta_0 = 0.01$, $\omega_0 = 3.612$ rad/s); comparison between PSD direct estimation using Welch method and Eq. (11) ($M = 300$).

where ($p(.)$) represents the probability density function of the response amplitude envelope. Substituting Eq. (15) into Eq. (11) leads to the formula

$$S_x(\omega) = \int_0^{\infty} \frac{S_w(\omega)}{\left(\omega^2(A) - \omega^2\right)^2 + (\beta(A)\omega)^2} p(A)\mathrm{d}A \tag{16}$$

for the response PSD as reported in Ref. [7]. Then, the conditional PSD is introduced as

$$S_x(\omega|A) = \frac{S_w(\omega)}{\left(\omega^2(A) - \omega^2\right)^2 + (\beta(A)\omega)^2}. \tag{17}$$

2.2. Response power spectral density determination

In context with the developments of the preceding sections, several efforts have been made in the literature to determine the response power spectrum of a nonlinear system by resorting to the concept of the conditional PSD. For this purpose, the formula

$$S_x(\omega) = \int_0^{\infty} S_x(\omega|A)p(A)\mathrm{d}A, \tag{18}$$

which is equivalent to Eq. (16), has been extensively used by various researchers [5,13,7,10] to derive an approximate estimation of the response PSD of nonlinear oscillators of the form of Eq. (1). In Eq. (18) the variable ($p(A)$) represents the PDF of the response amplitude envelope, whereas the variable ($S_x(\omega|\, A)$) may be viewed as the response PSD of a linear oscillator possessing natural frequency equal to ($\omega(A)$) and damping element equal to ($\beta(A)$). Taking into account the preceding analysis, Eq. (11) can be recast into the form of Eq. (16), or equivalently in the form

$$S_x(\omega) = S_w(\omega)E\left[\frac{1}{\left(\omega^2(A) - \omega^2\right)^2 + (\beta(A)\omega)^2}\right]. \tag{19}$$

Despite the fact that numerous applications of Eq. (18) have been made, no concrete proof of this formula has been reported. In fact, the available treatments vary from the rather heuristic ones to those with indisputable mathematical rigor, which, however, lack proper perspective. In this section, a proof of Eq. (18) is attempted based on a spectral representation [22,17,23,24] of the excitation and of the response processes.

To proceed, adopt spectral representations for the processes ($x(t)$) and ($w(t)$) which utilize monochromatic functions of random amplitude [16,24]. Specifically, set

$$x(t) = \sum_{n=0}^{N-1} A_n \cos\left[(n\Delta\omega)\, t\right] + \sum_{n=0}^{N-1} B_n \sin\left[(n\Delta\omega)\, t\right], \tag{20}$$

and

$$w(t) = \sum_{n=0}^{N-1} C_n \cos\left[(n\Delta\omega)\, t\right] + \sum_{n=0}^{N-1} D_n \sin\left[(n\Delta\omega)\, t\right], \tag{21}$$

where

$$\Delta\omega = \frac{\omega_u}{N}, \tag{22}$$

and (A_n), (B_n) and (C_n), (D_n) are random amplitudes associated with ($x(t)$) and ($w(t)$), respectively.

In Eq. (22), (ω_u) represents an upper cut-off frequency beyond which the corresponding PSD can be assumed to be zero. The sequences of the random variables $\{A_0, A_1, \ldots, A_n\}$, $\{B_0, B_1, \ldots, B_n\}$, $\{C_0, C_1, \ldots, C_n\}$, $\{D_0, D_1, \ldots, D_n\}$ can be shown [17] to be statistically independent with mean value equal to zero and variance equal to

$$\sigma_n^2 = 2S\,(n\Delta\omega)\,\Delta\omega. \tag{23}$$

Moreover, Eqs. (20) and (21) yield stochastic processes which are ergodic in the mean, mean square, correlation function, and first order distribution in the weak sense. A more global and detailed treatment of the properties which are implied by this kind of representation may be found in [16,17]. Note that the approximating processes are periodic with period

$$T_0 = \frac{2\pi}{\Delta\omega} = \frac{2\pi N}{\omega_u}. \tag{24}$$

Taking the first and second derivatives of Eq. (20) with respect to time yields

$$\dot{x}(t) = \sum_{n=0}^{N-1} - (n\Delta\omega)\, A_n \sin\left[(n\Delta\omega)\, t\right] + \sum_{n=0}^{N-1} (n\Delta\omega)\, B_n \cos\left[(n\Delta\omega)\, t\right], \tag{25}$$

and

$$\ddot{x}(t) = \sum_{n=0}^{N-1} - (n\Delta\omega)^2 A_n \cos\left[(n\Delta\omega)\, t\right] + \sum_{n=0}^{N-1} - (n\Delta\omega)^2 B_n \sin\left[(n\Delta\omega)\, t\right]. \tag{26}$$

Considering next the orthogonality conditions of monochromatic functions leads to

$$\int_0^{T_0} \cos[(l\Delta\omega)\, t] \cos[(n\Delta\omega)\, t]\mathrm{d}t = \begin{cases} \frac{1}{2}T_0 & l = n \\ 0 & l \neq n \end{cases}, \tag{27}$$

$$\int_0^{T_0} \sin[(l\Delta\omega)\, t] \sin[(n\Delta\omega)\, t]\mathrm{d}t = \begin{cases} \frac{1}{2}T_0 & l = n \\ 0 & l \neq n \end{cases}, \tag{28}$$

and

$$\int_0^{T_0} \sin[(l\Delta\omega)\, t] \cos[(n\Delta\omega)\, t]\mathrm{d}t = 0. \tag{29}$$

Substituting Eqs. (20), (21), (25) and (26) into Eq. (2) and exploiting Eqs. (27)–(29) yields

$$\begin{cases} -(n\Delta\omega)^2 A_n + \beta(A)(n\Delta\omega)B_n + \omega^2(A)A_n = C_n \\ -(n\Delta\omega)^2 B_n - \beta(A)(n\Delta\omega)A_n + \omega^2(A)B_n = D_n. \end{cases} \tag{30}$$

In producing Eq. (30), it has been assumed that the equivalent damping element and natural frequency are constant over the period (T_0). Further manipulation of Eq. (30) yields

$$A_n^2+B_n^2=\frac{C_n^2+D_n^2}{\left[\omega^2(A)-(n\Delta\omega)^2\right]^2+\left[(n\Delta\omega)\beta(A)\right]^2}. \tag{31}$$

The PSD estimation for the stationary processes $(w(t))$ and $(x(t))$ is given [25,26] by the equations

$$S_w(\omega)=\lim_{T\to\infty}\frac{1}{T}E\left[\left|\int_0^T w(t)\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t\right|^2\right], \tag{32}$$

and

$$S_x(\omega)=\lim_{T\to\infty}\frac{1}{T}E\left[\left|\int_0^T x(t)\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t\right|^2\right]. \tag{33}$$

Substituting Eq. (21) into Eq. (32), taking into account the orthogonality conditions of Eqs. (27)–(29) and assuming a fixed value for the frequency (ω), i.e. $(\omega=\omega^*)$ yields

$$\begin{aligned}S_w(\omega^*) &= \lim_{T\to\infty}\frac{1}{T}E\left[\left|\int_0^T w(t)\mathrm{e}^{-\mathrm{i}\omega^* t}\mathrm{d}t\right|^2\right]\\ &= \cdots \lim_{T\to\infty}\frac{1}{T}E\left[\left|\int_0^T w(t)\cos\left(\omega^* t\right)\mathrm{d}t\right.\right.\\ &\quad \left.\left. -\ \mathrm{i}\int_0^T w(t)\sin\left(\omega^* t\right)\mathrm{d}t\right|^2\right]\\ &= \cdots \lim_{T\to\infty}\frac{1}{T}E\left[\left|\frac{T}{2}C_*-\mathrm{i}\frac{T}{2}D_*\right|^2\right]\\ &= \cdots \lim_{T\to\infty}\frac{1}{T}E\left[\frac{T^2}{4}\left(C_*^2+D_*^2\right)\right].\end{aligned} \tag{34}$$

In this equation the term $(C_*^2+D_*^2)$ is interpreted as the squared amplitude of the process which corresponds to the fixed frequency $(\omega=\omega^*)$. Applying a similar procedure for $S_x(\omega^*)$ yields

$$\begin{aligned}S_x(\omega^*) &= \lim_{T\to\infty}\frac{1}{T}E\left[\left|\int_0^T x(t)\mathrm{e}^{-\mathrm{i}\omega^* t}\mathrm{d}t\right|^2\right]\\ &= \cdots \lim_{T\to\infty}\frac{1}{T}E\left[\left|\int_0^T x(t)\cos\left(\omega^* t\right)\mathrm{d}t\right.\right.\\ &\quad \left.\left. -\ \mathrm{i}\int_0^T x(t)\sin\left(\omega^* t\right)\mathrm{d}t\right|^2\right]\\ &= \cdots \lim_{T\to\infty}\frac{1}{T}E\left[\left|\frac{T}{2}A_*-\mathrm{i}\frac{T}{2}B_*\right|^2\right]\\ &= \cdots \lim_{T\to\infty}\frac{1}{T}E\left[\frac{T^2}{4}\left(A_*^2+B_*^2\right)\right].\end{aligned} \tag{35}$$

Further, combining Eqs. (31) and (35) gives

$$\begin{aligned}S_x(\omega^*) &= \lim_{T\to\infty}\frac{1}{T}E\left[\frac{T^2}{4}\left(A_*^2+B_*^2\right)\right]\\ &= \cdots \lim_{T\to\infty}\frac{1}{T}\\ &\quad \times E\left[\frac{T^2}{4}\left(\frac{C_*^2+D_*^2}{\left(\omega^2(A)-(\omega^*)^2\right)^2+\left((\omega^*)\beta(A)\right)^2}\right)\right]\\ &= \cdots \lim_{T\to\infty}\frac{1}{T}E\left[\frac{T^2}{4}\left(C_*^2+D_*^2\right)\right]\\ &\quad \times E\left[\frac{1}{\left(\omega^2(A)-(\omega^*)^2\right)^2+\left((\omega^*)\beta(A)\right)^2}\right].\end{aligned} \tag{36}$$

Obviously, Eq. (36) can be recast in the form

$$S_x(\omega^*)=S_w(\omega^*)E\left[\frac{1}{\left(\omega^2(A)-(\omega^*)^2\right)^2+\left((\omega^*)\beta(A)\right)^2}\right]. \tag{37}$$

Since Eq. (19) is proven for an arbitrary value of the frequency, i.e. $(\omega=\omega^*)$, it holds true for any value of (ω). Therefore, the validity of Eq. (18) is proven. Note that deriving Eq. (36) independence has been assumed between $(C_*^2+D_*^2)$ and the amplitude (A). This assumption is substantiated using the arguments expounded in the Appendix.

3. Concluding remarks

In this paper a formula which estimates the PSD of the random response of nonlinear systems has been considered. The formula, which is based on the notion of conditional PSD, has been widely used in computing, with remarkable reliability, the response PSDs of a wide range of randomly excited nonlinear oscillators. Despite the popularity and the versatility of the technique, its mathematical legitimacy has been based so far on arguments of limited rigor. In this regard, an effort to provide a concrete proof has been made in this paper by utilizing spectral representations both for the excitation and the response processes of the nonlinear system; this has been done in conjunction with an equivalent linear approximation of the original system. Further, exploiting the orthogonality properties of the monochromatic functions which are involved in the expansions has led to a straightforward verification of the formula. Furthermore, an intuitive interpretation of the concept of the conditional PSD has been included. Specifically, it has been shown that the nonlinear response PSD can be viewed as a sum of the PSDs which correspond to equivalent amplitude dependent linear systems. The necessity of invoking for the system response not only stationarity but ergodicity as well has been pointed out. Related numerical simulations have been included.

Appendix

In this Appendix arguments are expounded to substantiate the assumption of independence between $(C_*^2+D_*^2)$ and the amplitude (A) which is a necessary condition to prove Eq. (37). To elucidate this point, consider a linear system the squared amplitude of the response of which is given by the equation

$$A^2=x^2+\frac{\dot{x}^2}{\omega_0^2}. \tag{A.1}$$

Then, substituting Eqs. (20) and (25) into Eq. (A.1) yields

$$A^2=\left(\sum_{n=0}^{N-1}A_n\cos\left[(n\Delta\omega)\,t\right]+\sum_{n=0}^{N-1}B_n\sin\left[(n\Delta\omega)\,t\right]\right)^2+\cdots$$
$$\frac{\left(\sum_{n=0}^{N-1}-(n\Delta\omega)\,A_n\sin\left[(n\Delta\omega)\,t\right]+\sum_{n=0}^{N-1}(n\Delta\omega)\,B_n\cos\left[(n\Delta\omega)\,t\right]\right)^2}{\omega_0^2}. \tag{A.2}$$

Multiplying Eq. (30) by (A^2) and taking expectations on both sides gives

$$\begin{aligned}&-(n\Delta\omega)^2E\left[A_nA^2\right]+(n\Delta\omega)\beta E\left[B_nA^2\right]+\omega_0^2E\left[A_nA^2\right]\\ &\quad =E\left[C_nA^2\right],\end{aligned} \tag{A.3}$$

and

$$\begin{aligned}&-(n\Delta\omega)^2E\left[B_nA^2\right]-(n\Delta\omega)\beta E\left[A_nA^2\right]+\omega_0^2E\left[B_nA^2\right]\\ &\quad =E\left[D_nA^2\right].\end{aligned} \tag{A.4}$$

Taking into account the statistical independence of the zero-mean random variables $\{A_0, A_1, \ldots, A_n\}$, $\{B_0, B_1, \ldots, B_n\}$, it can be readily seen that the only terms which remain in the expectations ($E[B_nA^2]$) and ($E[A_nA^2]$) when (A^2) is substituted as given by Eq. (A.2), are

$$E\left[A_nA^2\right]=E\left[A_n^3\right]\cos^2\left[(n\Delta\omega)\,t\right]+E\left[A_n^3\right]\frac{(n\Delta\omega)^2}{\omega_0^2}\sin^2\left[(n\Delta\omega)\,t\right]; \tag{A.5}$$

or equivalently

$$E\left[A_nA^2\right]=E\left[A_n^3\right]\left(\cos^2\left[(n\Delta\omega)\,t\right]+\frac{(n\Delta\omega)^2}{\omega_0^2}\sin^2\left[(n\Delta\omega)\,t\right]\right), \tag{A.6}$$

and

$$E\left[B_nA^2\right]=E\left[B_n^3\right]\sin^2\left[(n\Delta\omega)\,t\right]+E\left[B_n^3\right]\frac{(n\Delta\omega)^2}{\omega_0^2}\cos^2\left[(n\Delta\omega)\,t\right]; \tag{A.7}$$

or equivalently

$$E\left[B_nA^2\right]=E\left[B_n^3\right]\left(\sin^2\left[(n\Delta\omega)\,t\right]+\frac{(n\Delta\omega)^2}{\omega_0^2}\cos^2\left[(n\Delta\omega)\,t\right]\right). \tag{A.8}$$

The fact that the variables (A_n) and (B_n) are zero-mean Gaussian variables yields

$$E\left[A_n^3\right]=0, \tag{A.9}$$

and

$$E\left[B_n^3\right]=0. \tag{A.10}$$

Examining Eqs. (A.6) and (A.8)–(A.10) and taking into account Eqs. (A.3) and (A.4) yields

$$E\left[C_nA^2\right]=0, \tag{A.11}$$

and

$$E\left[D_nA^2\right]=0 \tag{A.12}$$

which implies the independence of the quantities (A), (C_n) and (D_n), since the variables (C_n) and (D_n) have a zero-mean value.

Extending this argument to the case of a nonlinear system is not a straightforward task due to the dependence of the natural frequency on the amplitude in Eq. (A.1). However, considering the correlation time (τ_{cor}^{w}) of the excitation process, it can be argued that for a stationary broad-band random process it is approximately equal to

$$\tau_{cor}^{w}\simeq\frac{2\pi}{\omega_u}. \tag{A.13}$$

A similar approximation can be assumed for the response process (x). Taking into account the assumption of light damping, it can be deduced that the main part of the energy of the process is concentrated in a small frequency band around the dominant frequency ($\omega(A)$). Therefore, the correlation time (τ_{cor}^{x}) of the response process can be approximated as

$$\tau_{cor}^{x}\simeq\frac{2\pi}{\omega(A)}. \tag{A.14}$$

Due to the slowly varying nature of the amplitude process (A) it is safe to assume that the correlation time (τ_{cor}^{A}) of the response amplitude is greater than (τ_{cor}^{x}). This argument together with Eqs. (A.13) and (A.14) yields

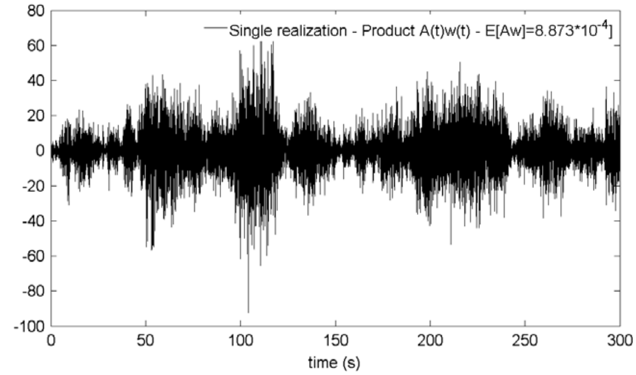

Fig. 2. A single realization of the excitation process multiplied with the corresponding response amplitude process; Duffing oscillator ($S_w(\omega)=S_0=0.3$, $\zeta_0=0.01$, $\omega_0=3.612$ rad/s, $\varepsilon=0.2$).

$$\tau_{cor}^{A}>\tau_{cor}^{x}. \tag{A.15}$$

Thus,

$$\frac{\tau_{cor}^{w}}{\tau_{cor}^{A}}<\frac{\tau_{cor}^{w}}{\tau_{cor}^{x}} \tag{A.16}$$

which leads to

$$\frac{\tau_{cor}^{w}}{\tau_{cor}^{A}}<\frac{\omega(A)}{\omega_u}\ll 1. \tag{A.17}$$

This implies that the terms ($C_*^2+D_*^2$) and (A) in Eq. (36) are uncorrelated, and therefore independence can be assumed. To support the foregoing analysis numerically, the Duffing oscillator (Eq. (12)) is considered possessing identical parameters ($S_w(\omega)=S_0=0.3$, $\zeta_0=0.01$, $\omega_0=3.612$ rad/s, $\varepsilon=0.2$). To show that ($E(A\cdot w)=0$) the ergodicity property for the two processes is invoked. Thus,

$$E\left(A\cdot w\right)=\lim_{T\to\infty}\frac{\int_0^T A(t)\cdot w(t)\mathrm{d}t}{T}. \tag{A.18}$$

To calculate the limit in Eq. (A.18) realizations of the processes (w) and (A) are produced. In Fig. 2 the product ($A(t)w(t)$) is plotted. Using Eq. (A.18) yields the value ($E(A\cdot w)=8.873\cdot 10^{-4}$) suggesting the independence of the two terms in Eq. (36).

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