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# Stochastic Reduced Order Models for Uncertain Infinite-Dimensional Geometrically Nonlinear Dynamical Systems- Stochastic Excitation Cases

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**Abstract**. The application of the nonparametric stochastic modeling technique to reduced order models of geometrically nonlinear structures recently proposed is here further demonstrated. The complete methodology: selection of the basis functions, determination and validation of the mean reduced order model, and introduction of uncertainty is first briefly reviewed. Then, it is applied to a cantilevered beam to study the effects of uncertainty on its response to a combined loading composed of a static in-plane load and a stochastic transverse excitation representative of earthquake ground motions. The analysis carried out using a 7-mode reduced order model permits the efficient determination of the probability density function of the buckling load and of the uncertainty bands on the power spectral densities of the stochastic response, transverse and inplane, of the various points of the structure.

**Key words:** Uncertainty, reduced order models, random matrices, geometrically nonlinear srructures, nonparametric stochastic modeling

# 1 Introduction

The sharp increase in computational capabilities of the last 10-15 years has led to very satisfactory solutions for many complex structural dynamic problems *for given values of the structural parameters*. Further, these same analyses have also demonstrated that these solutions can be very sensitive to small variations of the structural parameters, thereby emphasizing the need to consider structural uncertainty. Several approaches have been devised to model this uncertainty and estimate its effects on the structural response; among those are the polynomial chaos methodology (e.g. Ghanem and Spanos [1]) and the nonparametric approach initially proposed by Soize [2,3]. The latter approach is particularly computationally attractive as it applies to reduced order models of the structure, seeking the distribution of the uncertain parameters that maximizes their statistical entropy under given physical constraints.

The nonparametric method has been applied to a broad class of problems including a recent extension to nonlinear geometric structural dynamic problems [4] by relying on novel developments in the formulation of reduced order models for such structures (see Kim et al. [5] and references therein). According to these reduced order models, the structural response is expressed in a time-invariant basis with time-varying generalized coordinates satisfying coupled Duffing-type differential equations. Further, the parameters of this reduced order model, i.e. mass and linear, quadratic, and cubic stiffness coefficients, are identified directly from a full finite element model of the structure rendering the approach applicable to infinite-dimensional systems.

The earlier investigation of [4] is here further extended and validated to different infinite-dimensional structural models under stochastic excitations. A key element of the approach is the existence of a positive definite matrix $\mathbf{K_B}$ that regroups the linear, quadratic, and cubic stiffness coefficients. It is that matrix which is randomized in the nonparametric formulation while maintaining the positive definiteness so that the simulated stiffness properties, linear and nonlinear, are rendered uncertain in a physically admissible manner.

The complete process, reduced order modeling strategy and application of the nonparametric methodology, is presented on a cantilevered beam subjected to a static compressive load near the buckling limit and a transverse excitation corresponding to ground motions.

# 2 Reduced Order Modeling of Geometrically Nonlinear Structures

The formulation of reduced order model of geometrically nonlinear structures involves three specific issues: (i) the selection of the basis functions used to represent the motion of the structure, (ii) the determination of the form of the equations governing the generalized coordinates, and (iii) the determination of the coefficients of these equations. The resolution of these issues is briefly reviewed below.

## *2.1 Basis functions selection*

In parallel with modal analysis of linear systems, the displacement field of the structure will be expressed in a modal expansion-type representation, i.e., as

$$u_i(\mathbf{X},t) = q_n(t)\, U_i^{(n)}(\mathbf{X}) \quad i = 1,2,3 \tag{1}$$

(summation over repeated indices, $n$ here, is implied). In this representation, $U_i^{(n)}(\mathbf{X})$ denote time-invariant, spatially varying basis functions while $q_n(t)$ are the corresponding time-dependent generalized coordinates. Note that the spatial domain $\Omega$ to which $\mathbf{X}$ belongs is the undeformed configuration of the structure, see section 2.2.

In selecting the functions $U_i^{(n)}(\mathbf{X})$, it is first expected that the nonlinear reduced order model (1) should reduce naturally to a modal model in the limit of small motions. Thus, the nonlinear basis should completely include its linear counterpart, i.e. the set of linear modes significantly excited. This is however not enough and a complete representation of the structural response requires additional basis functions. As an example, consider the response to transverse loads of a flat, symmetric beam or plate subjected to a purely transverse loading. In the linear, infinitesimal case, the decoupling of the inplane and transverse modes implies that only the latter ones are necessary and no inplane motion takes place. However, finite deformations can only occur with a stretching of the beam or plate and, accordingly, with inplane deformations. Thus, the nonlinear basis required for a full representation must include both transverse linear modes and functions describing the inplane motions.

The basis functions selected here to complement the linear modes are the "dual" modes of [5], i.e. a set of static nonlinear displacement fields induced by external loads such that the response they would induce in the structure would be proportional to either one of the linear modes or a linear combination of two of them. Constructed in this manner, the dual modes capture the nonlinear effects corresponding to motions that would take place if the structure was behaving linearly.

## 2.2 Form of the reduced order model equations

The derivation of the form of the ordinary differential equations governing the evolution of the generalized coordinates $q_n(t)$ is next derived from the equations of finite deformation elasticity in a Galerkin procedure. To this end, note first that the time-invariance of the functions $U_i^{(n)}(\mathbf{X})$ is most easily achieved when the spatial domain $\Omega$ occupied by the structure is constant. This situation occurs when the displacement field $\mathbf{u}$ is expressed in the undeformed configuration in which the equations of elasticity are (see [6,7])

$$\frac{\partial}{\partial X_k}\left(F_{ij}\, S_{jk}\right)+\rho_0\, b_i^0 = \rho_0\, \ddot{u}_i \qquad \text{for } \mathbf{X} \in \Omega_0 \qquad (2)$$

where $\mathbf{S}$ denotes the second Piola-Kirchhoff stress tensor, $\rho_0$ is the density in the reference configuration, and $\underline{b}^0$ is the vector of body forces. Further, in Eq. (2), the deformation gradient tensor $\mathbf{F}$ is defined by its components $F_{ij}$ as

$$F_{ij} = \delta_{ij} + \frac{\partial u_i}{\partial X_j} \qquad (3)$$

where $\delta_{ij}$ denotes the Kronecker symbol. Associated to Equation (2) are appropriate boundary conditions, e.g. specification of displacement and/or tractions on the boundary $\partial\Omega_0$ of the reference configuration domain.

To complete the formulation of the elastodynamic problem, it remains to specify the constitutive behavior of the material. In this regard, it will be assumed here that the second Piola-Kirchhoff stress tensors **S** is linearly related to the Green strain tensor **E**, i.e.

$$S_{ij} = C_{ijkl}\, E_{kl} \qquad \text{where} \qquad E_{ij} = \frac{1}{2}\left(F_{ki}\, F_{kj} - \delta_{ij}\right) \tag{4),(5}$$

where $C_{ijkl}$ denotes the fourth order elasticity tensor.

Introducing the assumed displacement field of Equation (1) in Eqs (2)-(5) and proceeding with a Galerkin approach leads to the desired governing equations, i.e.

$$M_{ij}\ddot{q}_j + D_{ij}\dot{q}_j + K_{ij}^{(1)}\, q_j + K_{ijl}^{(2)} q_j q_l + K_{ijlp}^{(3)} q_j q_l q_p = F_i \tag{6}$$

in which a damping term $D_{ij}\,\dot{q}_j$ has been included to collectively represent various dissipation mechanisms. In Equation (6), $M_{ij}$, $K_{ij}^{(1)}$, and $F_i$ denote the coefficients of the linear mass and stiffness matrices and the modal forces while $K_{ijl}^{(2)}$ and $K_{ijlp}^{(3)}$ are nonlinear stiffness coefficients.

## 2.3 Identification of the stiffness coefficients

The *form* of the reduced order model, derived in the previous section as Equation (6), involves a series of structure and loading dependent coefficients, i.e. $M_{ij}$, $F_i$, $K_{ij}^{(1)}$, $K_{ijl}^{(2)}$, and $K_{ijlp}^{(3)}$. While the modal masses ($M_{ij}$) and modal forces ($F_i$) can be evaluated as in linear modal models, the stiffness coefficients, linear, quadratic, and cubic, necessitate a dedicated identification strategy. The specific methodology used here was initially proposed in [8] and further modified in [5], it is based on the availability of a series of static nonlinear solutions (usually from a finite element of the structure) in which the (static) displacement field is imposed and the corresponding necessary forces are determined.

The identification procedure starts with the imposition of static displacement fields that are proportional to each of the basis functions $U_i^{(n)}(\mathbf{X})$, i.e.

$$u_i(\mathbf{X}) = q^{(j)}\; U_i^{(n)}(\mathbf{X}) \quad i = 1,2,3\ . \tag{7}$$

In fact, three such cases, with different values of the factor $q^{(j)} = q^{(1)}$, $q^{(2)}$ (typically -$q^{(1)}$), and $q^{(3)}$, are considered for each $U_i^{(n)}(\mathbf{X})$ and the corresponding necessary forces $\mathbf{F}^{(j)}(\mathbf{X})$ are determined from the finite element model and projected onto the basis functions $U_i^{(m)}(\mathbf{X})$ to yield the modal forces $F_m^{(j)}$. Introducing this data into the reduced order model equations (6) yields the conditions

$$K_{in}^{(1)}\, q^{(j)} + K_{inn}^{(2)}\left[q^{(j)}\right]^2 + K_{innn}^{(3)}\, \left[q^{(j)}\right]^2 = F_i^{(j)}\,. \tag{8}$$

Considering Equation (8) for $j$ = 1, 2, and 3 leads, for each pair of indices $i$ and $n$, to 3 linear algebraic equations in the 3 unknowns $K_{in}^{(1)}$, $K_{inn}^{(2)}$, and $K_{innn}^{(3)}$ from which these coefficients are determined.

The identification of the remaining stiffness coefficients proceed in a similar manner by imposing static displacement fields which are linear combinations of 2 and then 3 of the modal bases, see [4,5,8] for complete details.

The above identification procedure has successfully been applied to a variety of problems (e.g. see [5]) but was found to be too sensitive to small errors in the predicted modal forces in connection with cantilevered structures (see [10] for discussion). This difficulty led to a modification of the estimation procedure in which the linear and quadratic stiffness coefficients of the final model were indeed estimated as above but with cubic coefficients selected (the decondensation technique of [10]) to match the corresponding coefficients of a reduced order model in which only the transverse motions are used with the inplane ones condensed. This two-step approach was employed here for the cantilevered beam of section 3.

## *2.4 Nonparametric uncertainty modeling*

Two different methodologies have been proposed for the consideration of uncertainty in linear structural dynamic systems. The first one, referred to here as parametric, introduces the uncertainty at the level of the full computational model (e.g. finite element model) through the randomization of some or all of its material properties (Young's modulus, Poisson's ratio, etc., e.g see [1]). This approach is particularly well suited for the consideration of *data uncertainty*, i.e. lack of knowledge or variability in the system properties but not for *model uncertainty* which is associated with deviations of the structure from its computational model. As example, for a beam that is nominally straight, such deviations include the presence of a curvature, a twist, or any other variation of geometry that would require a change of mesh in the finite element model.

A computationally efficient approach for the consideration of data and model uncertainty, referred to as the nonparametric method, has been proposed a few years ago, e.g. [2,3]. In this approach, the uncertainty is introduced directly at the level of the reduced order model by allowing the matrices it involves (e.g. mass, stiffness, and/or damping matrices) to be random. Further, the probability density functions of these matrices is derived (not chosen) to provide the maximum of its statistical entropy under mathematical and physical constraints, i.e. that it leads to a total unit probability, that all matrices be positive definite if physically required (for the mass, damping, and stiffness matrices), that their means be equal to the matrices of the mean reduced order model, and finally that the expected value of the square Frobenius norm of their inverse be finite, see [2,3] for complete discussion.

Key to the implementation of this approach is the simulation of random matrices according to the derived probability density function which is conveniently achieved as follows. Denote by $\overline{\mathbf{A}}$ the mean reduced order matrix considered and let $\overline{\mathbf{L}}$ be any decomposition (e.g. Cholesky) satisfying $\overline{\mathbf{A}} = \overline{\mathbf{L}}\,\overline{\mathbf{L}}^T$ . Then, random matrices $\mathbf{A}$ may then be simulated as

$$\overline{\mathbf{A}} = \overline{\mathbf{L}}\,\mathbf{H}\,\mathbf{H}^T\,\overline{\mathbf{L}}^T \tag{9}$$

where the random matrix **H** is lower triangular. Further, its elements were shown [2,3] to be independent random variables with those located off diagonal being normally distributed with zero mean and common variance. Finally, the diagonal elements of **H** are proportional to the square root of Gamma distributed random variables [2,3]. A single free parameter exists in this strategy which can be selected to match a particular information on the level of variability, such as coefficient of variation of natural frequencies or the overall measure of uncertainty $\delta$ introduced in [2,3].

The above discussion was first carried out in the context of linear structural dynamic systems but it was recently extended [4] to reduced order models of nonlinear geometric problems of the form of Eq. (6). Pivotal in this extension is the property (e.g. see [4]) that the linear, quadratic, and cubic stiffness coefficients $K_{ij}^{(1)}$, $K_{ijl}^{(2)}$, and $K_{ijlp}^{(3)}$ can be combined to form a matrix $\mathbf{K_B}$ which is symmetric and positive definite. Then, random coefficients $K_{ij}^{(1)}$, $K_{ijl}^{(2)}$, and $K_{ijlp}^{(3)}$ can be obtained from random matrices $\mathbf{K_B}$ generated from their mean model counterpart $\overline{\mathbf{K}}_{\mathbf{B}}$ according to Eq. (6).

# 3 Effects of Uncertainty on a Cantilevered Structure

The methodology developed in the previous sections was applied to a cantilevered beam of length 0.2286m, width 0.0127m, and thickness $7.75\times10^{-4}$m which was discretized by the finite element method (with MSC NASTRAN) into 40 CBEAM elements of equal lengths. The beam material was high-carbon steel with a Young's modulus of 205,000 MPa, a shear modulus of 80,000 MPa, and a mass density of 7,875 kg/m$^3$ leading to natural frequencies of the first transverse modes of 12.4, 77.9, 218, and 427 Hz. A Rayleigh damping model was assumed that yielded damping ratios of 1.07%, 0.47%, 0.91%, and 1.69%, respectively. Finally, the beam was subjected to a combined loading: random time-varying transverse motions of its support and an inplane compressive static force.

The development of an accurate reduced order model represented the first step of the uncertainty analysis. The ground motions selected here exhibited a Kanai-Tajimi spectrum (see [10]) of characteristic frequency equal to 5Hz ( $\omega_g$ =10π) and damping ratio of $\zeta_g$ =0.3. Given this low frequency excitation, only the first three linear, purely transverse modes were considered for the linear part of the basis. To these functions, 4 dual modes were added that exhibited only inplane motions thereby forming a 7 mode reduced order model.

The first validation of this reduced order model focused on the power spectral densities of the transverse and inplane relative displacements of the beam tip. A comparison of the spectra obtained from a full finite element computation (Nastran SOL 400) and from the reduced order model equations is presented in Figure 1 for two different standard deviations of the ground motions and a common compressive inplane force equal to 80% of its buckling limit (i.e. 4N). The standard deviations of the transverse tip deflections corresponding to this loading were found to be 10% and 18% of the beam length. Owing to the long Nastran computations, this comparison was achieved with records of 40 seconds from which the first 20 were removed as transient. The remaining 20 seconds of data may not be sufficient for an accurate capture of the low frequency response but are sufficient here for the validation of the reduced order model the response of which was similarly treated.

Clearly, the matching between full finite element and reduced order model predictions is excellent except at the very low frequencies for the inplane motions at the highest loading level. As the response levels increases, the Nastran and reduced order models will differ, see [9], because of the difference in the definitions of linear elasticity used in these methodologies, in a total Lagrangian in the latter while the former is believed to proceed in an updated Lagrangian framework. The results of Figure 1 demonstrate the appropriateness of the reduced order model for the prediction of the mean and uncertain beam responses.

The response of the uncertain beam to the specified combined loading was considered next using the nonparametric methodology of section 2.4. The free parameter was selected to achieve a coefficient of variation of 2% of the first transverse natural frequencies of the beam. With this level of uncertainty, the response of 300 random beams was computed using the stochastic reduced order model for 90 seconds with the first 20 seconds considered as transient. The uncertainty bands corresponding to the 2nd and 98th percentiles of the generated power spectra of the response were then determined and are shown in Figure 2 for the two excitation levels and for both transverse and inplane motions. Note that the power spectrum of the mean model is within the 2nd-98th percentiles band for the lowest excitation levels but it reaches the 98th percentile (for the transverse motions) or exceeds it (for the inplane motions) at the highest excitation level. This finding is justified by the inclusion in the nonparametric methodology of *model uncertainty*, i.e. the presence of coupling terms in the stochastic reduced order model which are not present for the mean model owing to its symmetry. Thus, the simulated reduced order models would be representative of typically curved beams for which the response is typically smaller than for the straight beam of the mean reduced order model. Uncertainty in the mass matrix was also considered with the non-parametric method but its effects appear very small and thus are not presented here for brevity.

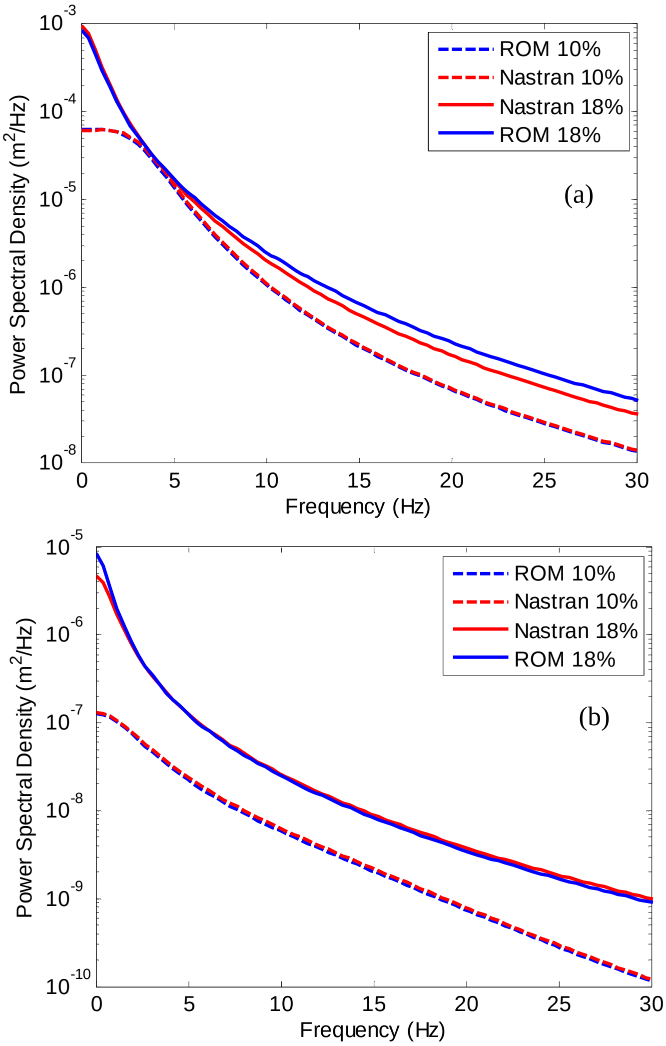


**Fig. 1** Power spectra of the tip displacements, Nastran vs. reduced order model (ROM) for the two loading cases with standard deviation of transverse response = 10% and 18% of span (a) Transverse. (b) Inplane.

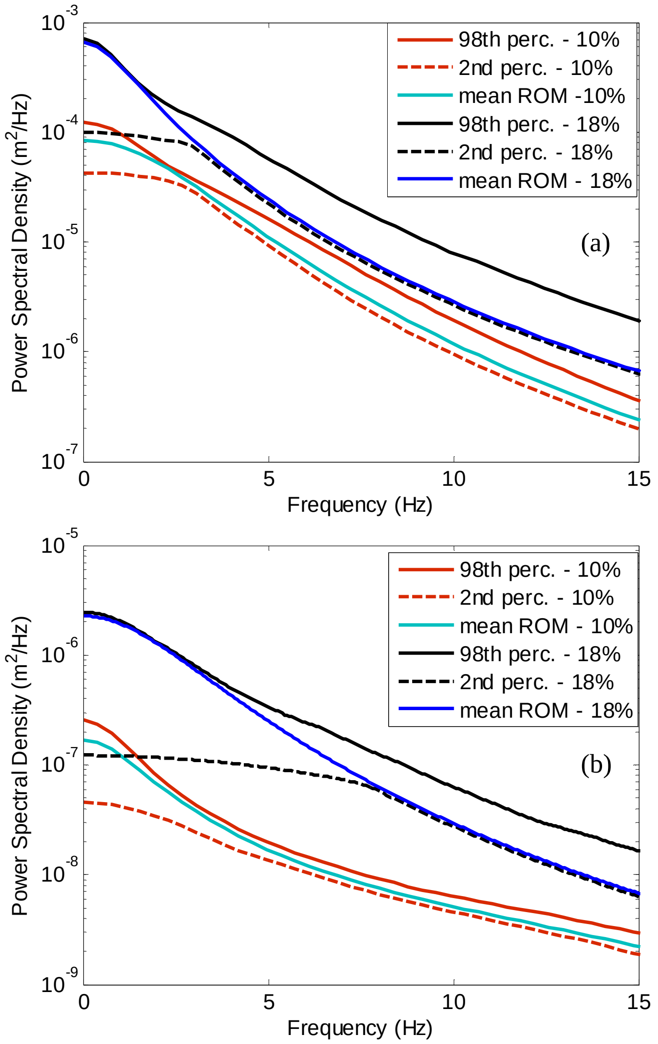


**Fig. 2** Power spectra of the tip displacements, uncertainty in stiffnesses for the two loading cases with standard deviation of transverse response = 10% and 18% of span. Uncertainty bands corresponding to the 2nd and 98th percentiles and mean ROM. (a) Transverse. (b) Inplane.

# 4 Summary

The present paper reported on a continued investigation of the effects of uncertainty on the response of nonlinear geometric structures. Owing to the computationally expensive Monte Carlo simulations involved in such investigations, a nonlinear reduced order modeling strategy was adopted for the mean model, see sections 2.1-2.3 and references therein for details. Next, uncertainty was introduced in this mean model according to the nonparametric methodology (see section 2.4) which allows the consideration of uncertainty in both structural properties (data uncertainty) and geometry (as example of model uncertainty).

This framework was demonstrated on a cantilevered beam subjected to the combined action of a static compressive inplane load and a transverse random excitation typical of ground motions. The mean model was first constructed from a full finite element model and its predictive capabilities validated vs. this full model at significant displacement levels (tip deflections of up to 18% of beam length). Uncertainty was then introduced leading to a stochastic reduced order model the stationary response of which was determined. The uncertainty bands associated with the 2nd and 98th percentiles of the power spectrum of the tip displacements were determined and it was found that the mean model power spectrum fits well within these bands at lower response levels but shifts to the 98th percentile as the response level increases owing to model uncertainty effects.

## References

1. Ghanem, R. and Spanos, P.D.: Stochastic Finite Elements: A Spectral Approach, Springer-Verlag (1991)
2. Soize, C.: A Nonparametric Model of Random Uncertainties on Reduced Matrix Model in Structural Dynamics. Probabilistic Engineering Mechanics,15, 277-294 (2000)
3. Soize, C.: Maximum Entropy Approach for Modeling Random Uncertainties in Transient Elastodynamics. Journal of the Acoustical Society of America, 109, 1979-1996 (2001)
4. Mignolet, M.P. and Soize, C.: Stochastic Reduced Order Models for Uncertain Geometrically Nonlinear Dynamical Systems. Computer Methods in Applied Mechanics and Engineering, 197, 3951-3963 (2008)
5. Kim, K., Wang, X.Q. and Mignolet, M.P.: Nonlinear Reduced Order Modeling of Isotropic and Functionally Graded Plates. Proceedings of the 49th Structures, Structural Dynamics, and Materials Conference, AIAA Paper AIAA-2008-1873 (2008)
6. Fung, Y.C. and Tong, P.: Classical and Computational Solid Mechanics, World Scientific (2001)
7. Bonet, J. and Wood, R.D.: Nonlinear Continuum Mechanics for Finite Element Analysis, Cambridge University Press (1997)
8. Muravyov, A.A. and Rizzi, S.A.: Determination of nonlinear stiffness with application to random vibration of geometrically nonlinear structures. Computers and Structures, 81, 1513-1523 (2003)
9. Kim, K., Khanna, V., Wang, X.Q. and Mignolet, M.P.: Nonlinear reduced order modeling of flat cantilevered structures. Proceedings of the 50th Structures, Structural Dynamics, and Materials Conference, AIAA Paper AIAA-2009-2492 (2009)
10. Madsen, H.O., Krenk, S. and Lind, N.C.: Methods of Structural Safety, Dover (2006)