Weighted fusion graphs: merging properties and watersheds

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Abstract

This paper deals with mathematical properties of watersheds in weighted graphs linked to region merging methods, as used in image analysis.

In a graph, a cleft (or a binary watershed) is a set of vertices that cannot be reduced, by point removal, without changing the number of regions (connected components) of its complement. To obtain a watershed adapted to morphological region merging, it has been shown that one has to use the topological thinnings introduced by M. Couprie and G. Bertrand. Unfortunately, topological thinnings do not always produce thin clefts.

Therefore, we introduce a new transformation on vertex weighted graphs, called C-watershed, that always produces a cleft. We present the class of perfect fusion graphs, for which any two neighboring regions can be merged, while preserving all other regions, by removing from the cleft the points adjacent to both. An important theorem of this paper states that, on these graphs, the C-watersheds are topological thinnings and the corresponding divides are thin clefts. We propose a linear-time immersion-like algorithm to compute C-watersheds on perfect fusion graphs, whereas, in general, a linear-time topological thinning algorithm does not exist. Furthermore, we prove that this algorithm is monotone in the sense that the vertices are processed in increasing order of weight. Finally, we derive some characterizations of perfect fusion graphs based on thinness properties of both C-watersheds and topological watersheds.

Key words: Graph theory, regions merging, watershed, fusion graphs, image segmentation, image processing

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Introduction

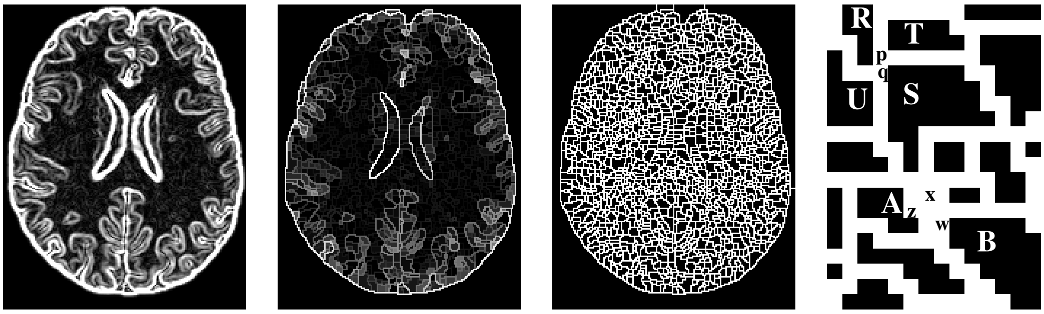
Image segmentation is the task of delineating objects of interest that appear in an image. In many cases, the result of such a process, also called a segmentation, can be viewed as a set of connected regions lying in a background which constitutes the separation between regions. In order to define regions, an image is often considered as a graph whose vertex set is made of the pixels and whose edge set is given by an adjacency relation between them. For instance, with the well-known 8-adjacency relation [1] each vertex is adjacent to its 8 closest neighbors. Then, the regions are simply the connected components of the set of foreground pixels. A popular approach to image analysis, called region merging [2,3], consists of progressively merging pairs of regions, starting from an initial segmentation that contains too many regions.

Given a grayscale image, or more generally a vertex-weighted graph (*i.e.*, a graph and a map that assigns a scalar value to each vertex), how is it possible to obtain an initial segmentation for a region merging procedure? The watershed transform [4–8] is a powerful tool for solving this problem. Let us consider a 2D grayscale image as a topographical relief, where the dark pixels correspond to basins and valleys, whereas bright pixels correspond to hills and crests. Suppose that we are interested in segmenting "dark" regions. Intuitively, the watersheds of an image are constituted by the crests which separate the basins corresponding to regional minima. This notion is illustrated in Fig. 1, where the white points in Fig. 1c constitute a watershed of the image in Fig. 1a equipped with the 8-adjacency relation. Due to noise and texture, real-world images often have a huge number of regional minima, hence the "mosaic" aspect of Fig. 1c. Nevertheless, it has been shown in numerous applications that this segmentation is an interesting starting point for a region merging process (see, *e.g.*, [9–11]) [1].

In order to identify the next pair of neighboring regions to be merged, many methods are based on the values of the points that belong to the initial separation between regions. In particular, in mathematical morphology, several methods [13–15] are implicitly based on the assumption that the initial separation satisfies a fundamental constraint: the values of the points in the separation must convey a notion of contrast, called connection value, between the minima of the original image. The connection value between two minima A and B is the minimal value k such that there exists a path from A to B the maximal value of which is k. From a topographical point of view, this value

[1] Note that, in the framework of mathematical morphology, there exists another approach, called *watershed from markers* [7], to reduce the so-called over-segmentation problem. In many cases, this approach may be seen as a region merging procedure (see [12]).

can be intuitively interpreted as the minimal altitude that a global flooding of the relief must reach in order to merge the basins that flood A and B.



(a) (b) (c) (d)

Fig. 1. (a): Original image (cross-section of a brain, after applying a gradient operator); (b): a topological watershed of (a) with the 8-adjacency; (c): the divide X (white points) of the topological watershed shown in b; (d): a zoom on a part of c. The point x is interior for X and w and z are adjacent to a unique connected component of $\overline{X}$.

In the topological approach to the watershed [8,16–18], we consider a transformation that iteratively lowers the value of a map F while preserving some topological properties, namely the number of connected components of each lower threshold of F. This transform and its result are called *W-thinning*; a *topological watershed* being a W-thinning that is minimal for the relation $\leq$ on maps. This notion is illustrated in Fig. 1 where the map H (Fig. 1b) is a topological watershed of F (Fig. 1a) equipped with the 8-adjacency relation. The *divide* of a map is the set of points that do not belong to any regional minimum (see the divide of H in Fig. 1c). It has been proved in [16,18] that the values of the points in the divide of a W-thinning convey the connection value between the minima of the original map. More remarkably, any set of points that verifies this property can be obtained by a W-thinning. Therefore, the divide of a W-thinning is a good choice for the initial segmentation in many region merging methods.

The divide of a W-thinning, and, in particular, of a topological watershed is not necessarily thin. Firstly, we observe that the divide of a topological watershed can contain some points adjacent to a unique connected component of its complement (see points w and z in Fig. 1d, which depicts a zoom on a part of Fig. 1c). Secondly, it may also contain some *inner* points, *i.e.*, points that are not adjacent to any point outside the divide (see point x in Fig. 1d). For implementing region merging schemes, these two kinds of thickness are problems. For instance, in Fig. 1d, regions A and B could hardly be considered as “candidate” to be merged since there is no point in the divide which is adjacent to both.

To solve the first problem, we want any divide to be a *cleft*, that is a set of vertices that does not contain any point adjacent to a unique connected component of its complement. In the case of an image, this notion corresponds to the intuitive idea of a frontier that separates connected regions. However, in general, a cleft is not necessarily thin. It can indeed contain some inner points and thus the second problem remains. In [19,20], we provide a framework to study the properties of thinness of clefts in any kind of graph, and characterize the class of graphs in which any cleft is thin. This class is strongly linked to a merging property. If we want to merge a pair of neighboring regions, what happens if each point adjacent to these two regions is also adjacent to a third one, which is not wanted in the merging? Fig. 1d illustrates such a situation, where R and S are neighbor and where p is adjacent to regions R, S, T and q to R, S, U. A major contribution of [20] is the definition and the study of four classes of graphs, with respect to the possibility of "getting stuck" in a merging process. In particular, we say that a graph is a *perfect fusion graph* if any two neighboring regions can be merged, while preserving all other regions, by removing from the divide all the points adjacent to both.

Let us now turn back to W-thinnings in vertex weighted graphs. Are the divides of topological watersheds always thin clefts on perfect fusion graphs? In this paper, we show that this is indeed true (Th. 18). This constitutes one of our main results. In addition, the paper also contains the following original contributions:

- we introduce a notion of thinness for maps and characterize, thanks to a merging property, the class of graphs in which any topological watershed is thin (Th. 9);
- we introduce a transformation, called C-watershed, that necessarily produces a map whose divide is a cleft. We give a local characterization (Th. 17) of the class of graphs in which any C-watershed is a W-thinning and deduce Th. 18 from this characterization;
- we introduce a linear-time immersion-like monotone algorithm to compute C-watersheds on perfect fusion graphs, whereas, in general, a linear-time W-thinning algorithm does not exist;
- finally, we derive some characterizations of perfect fusion graphs based on thinness properties of both C-watersheds and topological watersheds (Prop. 21).

This paper extends a preliminary version published in a conference [21]. It includes the proofs of the properties presented in [21] and two original theorems (Th. 9 and Th. 17).

1 Clefts and fusion graphs

1.1 Basic notions and notations

In this paper E stands for a finite nonempty set. We denote by $|E|$ the number of elements of E and by 2^E the set composed of all the subsets of E. Let $X \subseteq E$, we write $\overline{X}$ the complementary set of X in E, *i.e.*, $\overline{X} = E \setminus X$.

We define a graph as a pair (E, Γ) where E is a finite set and Γ is a binary relation on E (*i.e.*, $\Gamma \subseteq E \times E$), which is reflexive (for all x in E, $(x, x) \in \Gamma$) and symmetric (for all x, y in E, $(y, x) \in \Gamma$ whenever $(x, y) \in \Gamma$). Each element of E (resp. Γ) is called a *vertex* or a *point* (resp. *an edge*). We will also denote by Γ the map from E to 2^E such that, for all $x \in E$, $\Gamma(x) = \{y \in E \mid (x, y) \in \Gamma\}$. If $y \in \Gamma(x)$, we say that y is *adjacent to* x. We denote by $\Gamma^\star$ the binary relation on E defined by $\Gamma^\star = \Gamma \setminus \{(x, x) \mid x \in E\}$. Let $X \subseteq E$, we define $\Gamma(X) = \cup_{x \in X} \Gamma(x)$, and $\Gamma^\star(X) = \Gamma(X) \setminus X$. If $y \in \Gamma(X)$, we say that y *is adjacent to* X. If $X, Y \subseteq E$ and $\Gamma(X) \cap Y \neq \emptyset$, we say that Y *is adjacent to* X.

Let $G = (E, \Gamma)$ be a graph and let $X \subseteq E$, we define the *subgraph of* G *induced by* X as the graph $G_X = (X, \Gamma \cap [X \times X])$. In this case, we also say that G_X *is a subgraph of* G. Let $G' = (E', \Gamma')$ be a graph, we say that G *and* G' *are isomorphic* if there exists a bijection f from E to E' such that, for all $x, y \in E$, y belongs to $\Gamma(x)$ if and only if $f(y)$ belongs to $\Gamma'(f(x))$.

Let (E, Γ) be a graph and let $X \subseteq E$. A *path (of length* ℓ*) in* X is a sequence $\pi = \langle x_0, ..., x_\ell \rangle$ such that $x_i \in X$, $i \in [0, l]$, and $x_i \in \Gamma(x_{i-1})$, $i \in [1, \ell]$. We also say that π is a *path from* x_0 *to* x_ℓ *in* X and that x_0 and x_ℓ are *linked for* X. We say that X *is connected* if any x and y in X are linked for X.

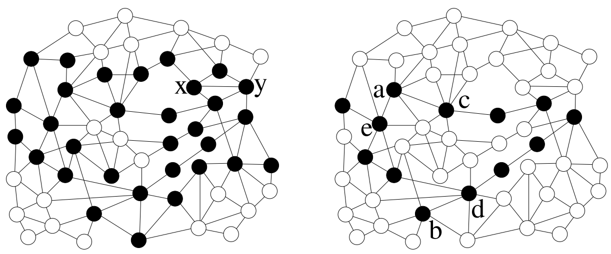
Important Remark. *From now,* (E, Γ) *denotes a graph, and we furthermore assume for simplicity that* E *is connected.*
Notice that, nevertheless, the subsequent definitions and properties may be easily extended to non-connected graphs.

Let $X \subseteq E$ and $Y \subseteq X$. We say that Y *is a (connected) component of* X, or simply *a component of* X, if Y is connected and if Y is maximal for this property, *i.e.*, if $Z = Y$ whenever $Y \subseteq Z \subseteq X$ and Z connected. We denote by $\mathcal{C}(X)$ the set of all connected components of X.

In a graph, a cleft is a set of vertices which cannot be reduced without changing the number of connected component of its complementary set. In image analysis, this notion corresponds to the intuitive idea of a frontier that separates connected regions. Therefore, many segmentation algorithms are expected to compute a cleft. We give in this section some formal definitions of these concepts (see [8,16,20]).

Definition 1 (cleft [16]) *Let $X \subseteq E$ and let $x \in X$. We say that x is* uniconnected for X *if x is adjacent to exactly one component of $\overline{X}$.*
The set X is a cleft *if there is no uniconnected point for X.*

In Fig. 2a, y is uniconnected for the set constituted by the black vertices, whereas x is not. Observe that the set Y of black points in Fig. 2b is a cleft since it contains no uniconnected point for Y. On the contrary, the bold sets in Figs. 2a and c are not clefts.



(a) (b) (c)

Fig. 2. (a): A graph (E, Γ) and a subset X (black points) of E; (b): the set Y of black points is a cleft; (c): the set obtained after merging two components of $\overline{Y}$ through the set $\{a, c\}$.

Definition 2 (thin set, Def. 3 in [20]) *Let $X \subseteq E$ and let $x \in X$. We say that x is* an inner point for X *if x is not adjacent to $\overline{X}$. The* interior *of X is the set of all inner points for X, denoted by $int(X)$. If $int(X) = \emptyset$, we say that X is* thin.

For example, the point x in Fig. 2a is an inner point for the set of black vertices. In Fig. 3a, the set of black vertices is thin whereas the set made of black and gray points is not thin: its interior, depicted in gray, is not empty. However, observe that this set is a cleft since it does not contain any uniconnected point.

Important Remark. In previous papers [16,20,21] by the same authors the notion of cleft was called *(binary) watershed*. For the sake of clarity, we chose, in this paper, to keep the term of watershed only for the notion of topological watershed. Note also that, in previous references [8,16,18,17,20,21], uniconnected points were called *W-simple points* .

1.3 Fusion Graphs

The theoretical framework set up in [19,20] enables to study the properties of region merging methods in graphs, as used in image analysis. In particular, one of the most striking outcomes of [20] links region merging properties and a thinness property of clefts. In this section, we recall some major definitions and results of [20] that are useful in the sequel of this paper.

In the following definition the prefix "F-" stands for fusion.
Let $X \subseteq E$. Let $x \in X$, we say that *x is F-simple (for X)*, if x is adjacent to exactly two components of $\overline{X}$. Let $S \subseteq X$. We say that *S is F-simple (for X)* if S is adjacent to exactly two components $A, B \in \mathcal{C}(\overline{X})$ such that $A \cup B \cup S$ is connected.

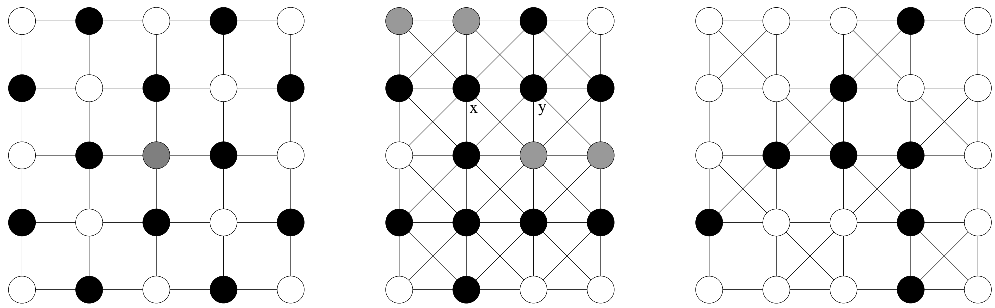
Let us look at Fig. 2b. The set made of the black vertices separates its complementary set into four components. The points a and c are F-simple for the black vertices whereas b and d are not. The sets $\{a, c, e\}$ and $\{c, e\}$ are F-simple and the sets $\{b, d\}$ and $\{a, c, b\}$ are not. Let X be a set that separates its complementary $\overline{X}$ into k components. If we remove from X an F-simple set, then we obtain a set that separates its complementary into $k - 1$ components. For instance, if we remove from the black vertices in Fig. 2b the F-simple set $\{a, c\}$ or $\{c, e\}$, then we obtain a new set that separates its complementary into three components (see Fig. 2c). This operation may be seen as an elementary merging in the sense that only two components are merged.

Let $X \subseteq E$ and let A and B be two elements of $\mathcal{C}(\overline{X})$ with $A \neq B$. We say that *A and B can be merged (for X) through S* if S is F-simple and A and B are precisely the two components of $\overline{X}$ adjacent to S. We say that A *can be merged* (for X) if there exists $B \in \mathcal{C}(\overline{X})$ and $S \subseteq X$ such that A and B can be merged through S.

Definition 3 (fusion graph, Def. 26 in [20]) *We say that (E, Γ) is a* fusion graph *if for any subset of vertices $X \subseteq E$ such that $|\mathcal{C}(\overline{X})| \geq 2$, any component of $\overline{X}$ can be merged.*

Notice that all graphs are not fusion graphs. For instance, the graph depicted in Fig. 3a is not a fusion graph: none of the components of the complementary set of the black vertices can be merged. On the other hand, it can be verified that the graph depicted in Fig. 3b is an example of a fusion graph.

Let A and B be two subsets of E. We set $\Gamma^\star(A, B) = \Gamma^\star(A) \cap \Gamma^\star(B)$ and if $\Gamma^\star(A, B) \neq \emptyset$, we say that *$A$ and B are neighbors.*



(a) (b) (c)

Fig. 3. (a): A graph induced by the 4-adjacency relation; a cleft X (black vertices) for which none of the components of $\overline{X}$ can be merged; a cleft Y (gray and black vertices) which is not thin, its interior is depicted in gray. (b): A graph induced by the 8-adjacency relation; a cleft X (black vertices); in gray two components of $\overline{X}$, A and B, which are neighbor and cannot be merged through $\Gamma^\star(A, B) = \{x, y\}$; (c): A graph induced by one of the two perfect fusion grids over $\mathbb{Z}^2$ and a cleft (black vertices).

Definition 4 (perfect fusion graph, Def. 28 in [20]) *We say that* (E, Γ) *is a* perfect fusion graph *if, for any* $X \subseteq E$*, any neighbors* A *and* B *in* $\mathcal{C}(\overline{X})$ *can be merged through* $\Gamma^\star(A, B)$.

In other words, the perfect fusion graphs are the graphs in which merging two neighboring regions can always be performed by removing from the frontier set all the points which are adjacent to both regions. This class of graphs permits, in particular, to rigorously define hierarchical schemes based on region merging and to implement them in a straightforward manner. It has been shown [20] that *any perfect fusion graph is a fusion graph and that the converse is not true.*

In image analysis, there are two fundamental adjacency relations defined over $\mathbb{Z}^2$. The 4-adjacency, denoted by Γ_4, is defined by: $\forall x, y \in \mathbb{Z}^2, (x, y) \in \Gamma_4$ iff $|x_1 - y_1| + |x_1 - y_2| \leq 1$, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. The 8-adjacency, denoted by Γ_8 is defined by: $\forall x, y \in \mathbb{Z}^2, (x, y) \in \Gamma_8$ iff $\max\{|x_1 - y_1|, |x_1 - y_2|\} \leq 1$. Examples of graphs induced by the 4- and 8-adjacency are shown in respectively Figs. 3a and b.

The graphs induced by the 4-adjacency are not in general fusion graphs (see, for instance, the counter-example of Fig. 3a). On the contrary, the graphs induced by the 8-adjacency are fusion graphs (see Prop. 48 in [20]). Nevertheless, in general, they are not perfect fusion graphs. Consider for instance the graph induced by the 8-adjacency depicted in Fig. 3b and the set Y made of the black vertices. The two components of $\overline{Y}$, depicted in gray, are neighbors since the points x and y are adjacent to both but they cannot be merged. In [20] the authors introduce a family of adjacency relations on $\mathbb{Z}^n$, which can be used in image processing, that induce perfect fusion graphs. For instance,

(a) (b)

Fig. 4. Two graphs used in Th. 6 and Th. 17.

the graph depicted in Fig. 3c is induced by one of the two perfect fusion grids on $\mathbb{Z}^2$. The other perfect fusion grid on $\mathbb{Z}^2$ is equivalent to this one up to a unit translation.

The following theorem, which is a fundamental result of [20], establishes the links between fusion graphs and thin clefts.

Theorem 5 (Th. 32 in [20]) *A graph G is a fusion graph if and only if any cleft X in G, such that $|\mathcal{C}(\overline{X})| \geq 2$, is thin.*

The four following necessary and sufficient conditions for perfect fusion graphs show the relation existing between perfect fusion graphs, fusion graphs and a sub-class of thin clefts.

We denote by $G^{\blacktriangle}$ the graph of Fig. 4a.

Theorem 6 (from Th. 41 in [20]) *The five following statements are equivalent:*
i) (E, Γ) is a perfect fusion graph;
ii) the graph $G^{\blacktriangle}$ is not a subgraph of (E, Γ);
iii) for any cleft X in E such that $\mathcal{C}(\overline{X}) \geq 2$, each point x in X is F-simple for X;
iv) for any connected subset A of E, the subgraph of (E, Γ) induced by A is a fusion graph;
v) for any $x \in E$, any $X \subseteq \Gamma(x)$ contains at most two connected components.

2 W-thinnings and topological watersheds

As seen in the introduction, the watershed transform [5–8] of a grayscale image, or more generally of a vertex-weighted graph, produces a set of connected regions separated by a divide. Such a divide has often been used, in image analysis, as an entry point for region merging methods. In this section we recall the definitions of W-thinnings and topological watersheds [8,16].

Let $k_{\min}$ and $k_{\max}$ be two elements of $\mathbb{Z}$ such that $k_{\min} < k_{\max}$. We set $\mathbb{K} = \{k \in \mathbb{Z}; k_{\min} \leq k < k_{\max}\}$ and $\mathbb{K}^+ = \mathbb{K} \cup \{k_{\max}\}$. We denote by $\mathcal{F}(E)$ the set composed of all maps from E to $\mathbb{K}$.

Let $F \in \mathcal{F}(E)$. If $x \in E$, $F(x)$ is called the *altitude of* x. Let $k \in \mathbb{K}^+$. We denote by $F[k]$ the set $\{x \in E; F(x) \geq k\}$ and by $\overline{F}[k]$ its complementary set; $F[k]$ is called an *upper section of* F and $\overline{F}[k]$, a *lower section of* F. A connected component of $\overline{F}[k]$ which does not contain a connected component of $\overline{F}[k-1]$ is a *(regional) minimum* of F. We denote by $M(F) \subseteq E$ the union of all minima of F. We say that $\overline{M(F)}$ is the *divide of* F. A subset X of E is *flat for* F if any two points x, y in X are such that $F(x) = F(y)$. If X is flat for F, the *altitude of* X is the altitude of any point of X.

By the mean of upper sections, the definitions of uniconnected points and clefts can be extended to the case of maps [8,16–18].

Let $F \in \mathcal{F}(E)$. The lowering of F at x, denoted by $[F \setminus x]$, is the map in $\mathcal{F}(E)$ such that:
- $[F \setminus x](x) = F(x) - 1$; and
- $[F \setminus x](y) = F(y)$ for any $y \in E \setminus \{x\}$.

Definition 7 (watershed, Def. 10 in [16]) *Let $F \in \mathcal{F}(E)$. Let $x \in E$ and $k = F(x)$. We say that x is* W-destructible for F *if x is uniconnected for $F[k]$. If there is no W-destructible point for F we say that F is a* (topological) watershed.
Let $H \in \mathcal{F}(E)$. We say that H is a W-thinning of F *if:*
i) $H = F$; or if:
ii) there exists a W-thinning $I \in \mathcal{F}(E)$ of F such that H is the lowering of I at a W-destructible point for I.
If H is both a W-thinning of F and a watershed, we say that H is a (topological) watershed of F.

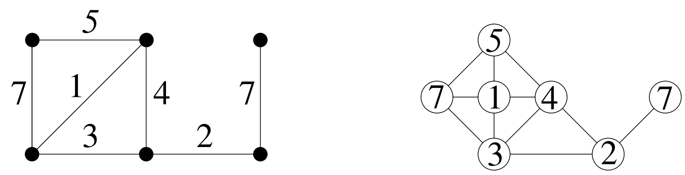
In other words, a point x such that $F(x) = k$ is W-destructible for F if x is adjacent to exactly one component of $\overline{F}[k] = \{y \in E \mid F(y) < k\}$. A map H is a W-thinning of F, if there exists a (possibly empty) sequence of maps $\langle F_0, \ldots, F_\ell \rangle$ such that $F_0 = F$, $F_\ell = H$ and, for any $i \in [1, \ell]$, F_i is the lowering of F_{i-1} at a W-destructible point for F_{i-1}. Furthermore, H is a topological watershed of F, if H is a W-thinning of F which has no W-destructible point.

In Fig. 5a and b, assume that the images are equipped with the 8-adjacency relation. In both Fig. 5a and b, it may be seen that there are three minima which are the components with levels 0,1 and 2. In Fig. 5a, the point labeled r is W-destructible. In Fig. 5b, no point is W-destructible. The map depicted in Fig. 5b is a topological watershed of the map in Fig. 5a.

The divide of a topological watershed constitutes an interesting image segmentation [16,18] which possesses important properties not guaranteed by most watershed algorithms [5,6]. In particular, it preserves the connection value be-

(*a*) (*b*)

Fig. 5. The depicted images are equipped with the 8-adjacency relation. (*a*): A map F; (*b*): a topological watershed of F.



(*a*) (*b*)

Fig. 6. (*a*): A graph G. The values constitute a map F that weights the edges of G. (*b*): A graph G' that is the line graph of G. The map F weights the vertices of G'.

tween the minima of the original map; intuitively, the connection value (see [16,18,22,23]) between two minima can be thought of as the minimal altitude at which one need to climb in order to reach one minimum from the other. It has been shown (Th. 7 in [16]) that a topological watershed can be equivalently defined as a transformation which extends the lower sections (and hence the minima) of the original map as much as possible while preserving the connection value between all pairs of minima. As said in the introduction, this contrast preservation property is a fundamental property on which rely many popular region merging methods based on watersheds [13,15,14].

In image analysis applications, we sometimes deal with graphs whose edges, rather than vertices, are weighted by a cost map [12,24–27]. To finish this section, we recall the definition of *line graphs* (see, *e.g.*, [28]). This class of graphs allows us to highlight that the approaches of watershed and region merging based on edge-weighted graphs are particular cases of the approaches based on vertex-weighted graphs developed in this paper.

The *line graph* of (E, Γ) is the graph (E', Γ') such that $E' = \Gamma^\star$ and (u, v) belongs to Γ' whenever $u \in \Gamma^\star$, $v \in \Gamma^\star$, $u \neq v$ and u, v share a common vertex of E.
We say that the graph (E', Γ') is a line graph if there exists a graph (E, Γ) such that (E', Γ') is isomorphic to the line graph of (E, Γ).

For instance, the graph depicted in Fig. 6b is the line graph of the one depicted in Fig. 3a. It has been proved [20] that any line graph is a perfect fusion graph and that the converse is not true. Thus, all definitions, properties and algorithm for watershed on perfect fusion graphs developed in Secs. 4 and 3 also hold for watershed approaches based on edge-weighted graphs. A more detailed presentation of watersheds in edge-weighted graphs can be found in [26,27].

3 Thinness of topological watersheds

The divides produced by watershed algorithms [5–7], and in particular by topological watershed algorithms [17], are not always clefts and can sometimes be thick, even on fusion graphs. Consider, for instance, the digital image F depicted in Fig. 5b and assume that it is equipped with the graph induced by the 8-adjacency. Although the map F is a topological watershed and the considered graph is a fusion graph (see Prop. 48 in [20]), the point labeled s is inner for $\overline{M(F)}$. As said in the introduction, such thickness is a problem for defining and implementing region merging schemes. In Sec. 1.3, we presented a result of [20] which characterizes, thanks to a merging property, the class of graphs in which any cleft is thin (Th. 5). In this section, we provide a similar result for the case of topological watersheds.

To this end, we introduce a notion of thinness for maps which extends the one for sets by the mean of upper sections.

Definition 8 (thin map) *Let $F \in \mathcal{F}(E)$, let $x \in E$ and $k = F(x)$. We say that x is an* inner point *for F if x is inner for $F[k]$. The* interior *of F, denoted by $int(F)$, is the set of points in $\overline{M(F)}$ that are inner for F. We say that F is* thin *if $int(F) = \emptyset$.*

In other words, a point is inner for a map if all its neighbors have an altitude greater than or equal to its own altitude. Thus, a map is thin if any point in its divide has at least one neighbor of strictly lower altitude. It may be seen that the topological watershed depicted in Fig. 8b is thin whereas the one in Fig. 5b is not.

From the very definition of a topological watershed and thanks to Th. 5, it may be seen that any graph in which all topological watersheds are thin is necessarily a fusion graph. Indeed, if we consider a graph (E, Γ) which is not a fusion graph, then, by Th. 5 there exists a cleft $X \subseteq E$ (with $|\mathcal{C}(\overline{X})| \geq 2$), which is not thin (in the sense of Def. 2). From this set, we can construct the map $F \in \mathcal{F}(E)$ such that $F(x) = 1$ for any $x \in X$ and $F(x) = 0$ otherwise. Clearly, by definitions of a cleft and of a watershed, F is not thin (in the sense of Def. 8). Thus, if all topological watersheds are thin in (E, Γ), then (E, Γ) must be a fusion graph. The map in Fig. 5b shows that, contrarily to the case of clefts (Th. 5), the converse is, in general, not true. However, as established by the following characterization theorem, deep links exist between these two classes of graphs.

Theorem 9 *Any topological watershed in (E, Γ) is thin if and only if for any cleft $X \subseteq E$, for any $A \in \mathcal{C}(\overline{X})$, the subgraph of (E, Γ) induced by A is a fusion graph.*

The proof of Th. 9 can be found in annex.

We remark that the above condition, that characterizes the graphs in which any topological watershed is thin, is a weakening of condition *iv* of Th. 6 which characterizes the perfect fusion graphs. Thus, any topological watershed on a perfect fusion graph is thin. On the other hand, there exists some graphs which are not perfect fusion graphs and in which any topological watershed is thin (see for instance Fig. 7).

In image analysis, a cleft (Sec. 1.2) can be seen as a frontier between connected regions and the divide of a topological watershed constitutes an interesting segmentation (Sec. 2). Therefore, a desirable property is that the divide of a topological watershed is a cleft. Unfortunately, such a property does not hold even in the case of a graph in which any topological watershed is thin. Consider, for instance, the graph (E, Γ) and the topological watershed F depicted in Fig. 7a. It may be verified, thanks to Th. 9, that in (E, Γ) any topological watershed is thin; this is, in particular, the case of F. However, it may be checked that the points which are bold circled are uniconnected for the divide of F. Hence $\overline{M(F)}$ is not a cleft. Observe also that the black vertex is inner (in the sense of Def. 2) for $\overline{M(F)}$ and thus that $\overline{M(F)}$ is not thin. Thus, the notion of thinness defined in this section does not lead to topological watersheds adapted for region merging schemes. In the next sections, we study watersheds in perfect fusion graphs and show that this kind of problems cannot happen.

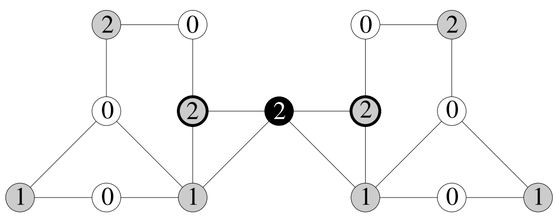


Fig. 7. A graph in which any topological watershed is thin. The values define a map that is a topological watershed. The black point is inner for the divide in the sense of Def. 2. The circled points are uniconnected for the divide.

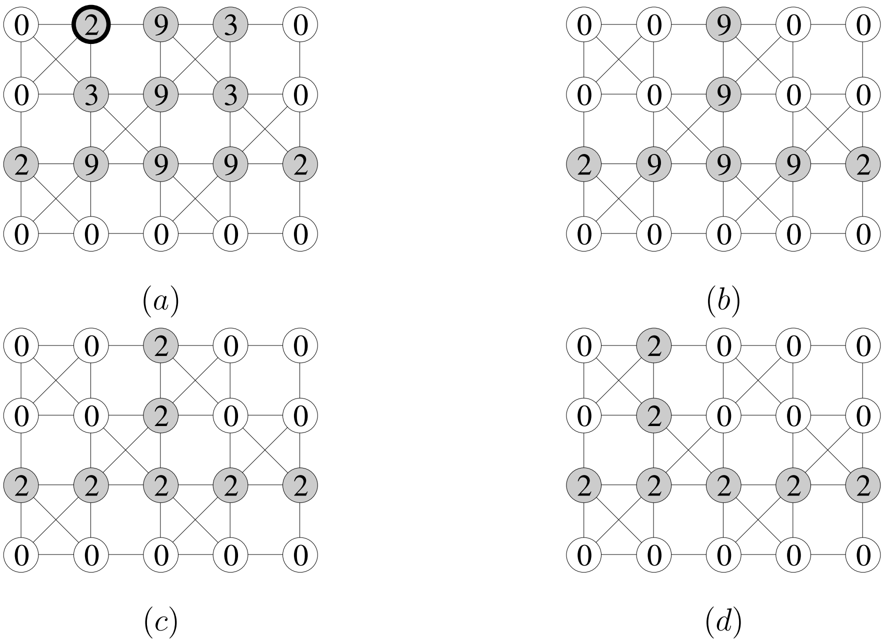


Fig. 8. Example of maps on perfect fusion graphs, the minima are in white; (a): the bold circled vertex is M-cliff; (b): a C-watershed of (a); (c): a topological watershed of both (a) and (b); (d): a topological watershed of (a).

4 C-watersheds in perfect fusion graphs

In this section, we introduce a new grayscale transformation, called *C-watershed*, that always produces a map whose divide is a cleft. An important result (Th. 13) is that, on a perfect fusion graph, any C-watershed of a map is a W-thinning of this map whose divide is a thin cleft. Furthermore, we propose and prove the correctness of a linear time algorithm to compute C-watersheds on perfect fusion graphs.

Definition 10 (M-cliff point) *Let $F \in \mathcal{F}(E)$ and let $x \in E$. We say that x* is a cliff point (for F) *if x is uniconnected for $\overline{M(F)}$. We say that x* is M-cliff (for F) *if x is a cliff point of minimal altitude (*i.e., $F(x) = \min\{F(y) \mid y \in E$ *is a cliff point for $F\}$).*

In other words, a cliff point for a map $F \in \mathcal{F}(E)$ is a point in $\overline{M(F)}$ which is adjacent to a single minimum of F. A point x is M-cliff for F if no other point of $\overline{M(F)}$ adjacent to a single minimum has an altitude strictly lower than the altitude of x.

In Fig. 8a, the points at altitude 3 are cliff points and the bold circled point is the only M-cliff point. In Figs. 8b,c and d, it can be seen that there is no M-cliff point and no cliff point.

Let $F \in \mathcal{F}(E)$. Let x be a W-destructible point. Let $\ell \in \mathbb{K}$. The point x *is W-destructible with lowest value* ℓ *(for* F*)* if for any $h \in \mathbb{K}$ such that $\ell < h \leq F(x)$, x is uniconnected for $F[h]$ and if x is not uniconnected for $F[\ell]$.

Lemma 11 *Let $F \in \mathcal{F}(E)$. Let $x \in E$ be M-cliff for F and let $\ell \in \mathbb{K}$ be the level of the only minimum adjacent to x. If (E, Γ) is a perfect fusion graph then x is W-destructible with lowest value ℓ for F.*

The proof of Lem. 11 can be found in Annex 7.2.

Remark that on non-perfect fusion graphs, the points which are M-cliff are not necessarily W-destructible. Indeed, it can be verified that the graph of Fig. 7 is not a perfect fusion graph and that any of the circled vertices is an M-cliff point that is not W-destructible (since it is adjacent to exactly two components of $\overline{F}[2]$, the set of vertices whose altitude is strictly less than 2).

Lem. 11 invites us to investigate a particular kind of W-thinning which consists of iteratively lowering the values of M-cliff points.

Let $F \in \mathcal{F}(E)$ and $\ell \in \mathbb{K}$, we denote by $[F_{x,\ell}]$ the map in $\mathcal{F}(E)$ such that $[F_{x,\ell}](x) = \ell$ and $[F_{x,\ell}](y) = F(y)$ for any $y \in E \setminus \{x\}$.

Definition 12 (C-watershed) *Let F and H be in $\mathcal{F}(E)$. We say that H* is a C-thinning of F *if:*
i) $H = F$, or if
ii) there exists a map I that is a C-thinning of F and there exists a point x M-cliff for I such that $H = [I_{x,\ell}]$, where ℓ is the altitude of the only minimum of I adjacent to x.
We say that F is a C-watershed *if there is no M-cliff point for F. If H is both a C-thinning of F and a C-watershed, we say that H* is a C-watershed of F*.*

Note that, similarly to W-thinnings and watersheds, the C-thinnings and C-watersheds can be equivalently defined by sequences of lowerings at M-cliff points.

It follows from Lem. 11, Def. 10 and Def. 12 that on a perfect fusion graph any C-thinning of a map is a W-thinning of this map. Furthermore, a map is a C-watershed if and only if its divide is a cleft. Indeed, a map is a C-watershed iff it has no M-cliff point hence no cliff point and thus, by definition, no uniconnected point for its divide. Any perfect fusion graph is a fusion graph. Therefore, by Th. 5, any cleft on a perfect fusion graph is thin. Hence, we deduce the following theorem which establishes that the divides of the C-watersheds constitute interesting segmentations in perfect fusion graphs.

Theorem 13 *Let F be a map in $\mathcal{F}(E)$ and suppose that (E, Γ) is a perfect fusion graph.*

Let H be a C-watershed of F. Then, H is a W-thinning of F. Furthermore, the divide of H is a thin cleft.

To illustrate the previous theorem, let us look at Fig. 8. The map H depicted in (b) is a C-watershed of the map F depicted in (a). It can be verified that H is a W-thinning of F and that the divide of H is a thin cleft. In general, a C-watershed is not a topological watershed. For instance, the map H is a C-watershed, but the points at altitude 9 are W-destructible. Nevertheless, as implied by the following property, the divide of any C-watershed of F is equal to the divide of a topological watershed of F.

A W-thinning of $F \in \mathcal{F}(E)$ is a *lowering* (*i.e.*, a map H such that for any $x \in E$, $H(x) \leq F(x)$) of F which preserves the number of components of all lower sections of F. In particular, it preserves the number of minima of F. By the preceding theorem, the divide of any C-watershed is a cleft. Hence, the minima of a C-watershed cannot be further "extended" while preserving all of them. As a consequence, we deduce the following property.

Property 14 *Let $F \in \mathcal{F}(E)$ be a C-watershed. The divide of any W-thinning of F is equal to the divide of F.*

The algorithms to compute (the divide of) a topological watershed [17] are not linear and require the computation of an auxiliary data structure called component tree [29]. It is possible to reach a better complexity for the computation of a C-watershed on a perfect fusion graph.

In a C-thinning sequence, the points which are in a minimum at a given step never become M-cliff further in the sequence. This observation leads us to the definition of Algorithm 1, a very simple algorithm for C-watersheds.

At each iteration of the main loop (line 6) of Algorithm 1, it may be seen that any point adjacent to a unique minimum of F is in the set L. Thus, it may be easily deduced that at each iteration of the main loop, F is a C-thinning (hence, by Lem. 11, a W-thinning) of the input map.
At the end of Algorithm 1, the set L is empty. Thus there is no point adjacent to a unique minimum of F, in other words, there is no point M-cliff or cliff for F. As a consequence of the preceding remarks, at the end of Algorithm 1, the map F is a C-watershed of the input map.

In Algorithm 1, the operations performed on the set L are the insertion of an element and the extraction of an element with minimal altitude. Thus, L may be managed as a priority queue. In [30], an efficient priority queue algorithm has been proposed. It supports the operation of insertion, extraction of a minimal element or deletion in worst case time $O(\log \log m)$ where m is the numbers of elements in L. In fact, for computing a C-watershed, we can use a faster data structure. To reach this goal, we first need to establish the following

Algorithm 1: C-watershed

Data: a perfect fusion graph (E, Γ), a map $F \in \mathcal{F}(E)$
Result: F
1 $L := \emptyset$; $K := \emptyset$;
2 Attribute distinct labels to all minima of F and label the points of $M(F)$ with the corresponding labels;
3 **foreach** $x \in E$ **do**
4   **if** $x \in M(F)$ **then** $K := K \cup \{x\}$;
5   **else if** *x is adjacent to $M(F)$* **then** $L := L \cup \{x\}$; $K := K \cup \{x\}$;
6 **while** $L \neq \emptyset$ **do**
7     $x :=$ an element of L with minimal altitude for F;
8     $L := L \setminus \{x\}$;
9   **if** *x is adjacent to exactly one minimum of F* **then**
10   Set $F[x]$ to the altitude of the only minimum of F adjacent to x;
11   Label x with the corresponding label;
12   **foreach** $y \in \Gamma^\star(x) \cap \overline{K}$ **do** $L := L \cup \{y\}$; $K := K \cup \{y\}$;

fundamental theorem. It states that in a C-thinning sequence the points are lowered down by increasing order of altitude (for the original map F).

Theorem 15 (monotony) *Let $F \in \mathcal{F}(E)$ and suppose that (E, Γ) is a perfect fusion graph. Let H be a C-thinning of F. Any point M-cliff for H has an altitude greater than or equal to the altitude of any point M-cliff for F.*

A proof of Prop. 15 is given in Annex 7.3.

From Th. 15, we deduce that in Algorithm 1, when the map F is lowered at a point x with altitude k, any point inserted further in the set L has a level greater than or equal to k. Thus, the set L may be managed by a monotone priority queue, that is, a priority queue whose minimum value is non-decreasing over time (see [31] for an application of such a queue). M. Thorup [30] proved that if we can sort n keys in time $n.s(n)$, then, and only then, there is a monotone queue with capacity n, supporting the *insert* and *extract-min* operations in $s(n)$ amortized time.

In Algorithm 1, the set K is used to avoid multiple insertions of a same point in the set L and can be managed as a Boolean array. Thus, the main loop (line 6) is executed at most $|E|$ times. Furthermore, the minima of a map can be extracted in linear time thanks to well known algorithms (see [32]). We deduce Prop. 16 from these observations.

Property 16 *If the elements of E can be sorted according to F in linear time with respect to $|E|$, then Algorithm 1 terminates in linear time with respect to $(|E| + |\Gamma|)$.*

Since Algorithm 1 possesses the monotone property discussed above, it can be classified in the group of immersion algorithms (see [5–7] for examples). On non-perfect fusion graphs, Prop. 15 is in general not true. Consider, for instance, the map F in Fig. 5b. The point labeled t, with altitude 9, is M-cliff for F, but is not W-destructible. Let $H = [F_{t,2}]$, H is a C-thinning of F. We can remark that the point labeled s is the only M-cliff point for H and its altitude is strictly less than 9. Thus, on non-perfect fusion graphs a C-thinning sequence is in general not monotone. Moreover, it has been shown in [18] that in the case of a non-perfect fusion graph, some immersion algorithms are not monotone and a monotone W-thinning algorithm does not, in general, produce a divide that satisfies Prop. 14.

In this section, we have introduced the C-watershed transformation and have shown interesting properties on perfect fusion graphs. We may wonder whether it is possible to extend (some of) these properties to other kinds of graphs. In other words: what is the largest class of graphs such that Lem. 11 holds?

Let us denote by G^λ the graph depicted in Fig. 4b.

Theorem 17 *The three following statements are equivalent:*
i) for any $F \in \mathcal{F}(E)$, any point M-cliff for F is W-destructible;
ii) for any $F \in \mathcal{F}(E)$, any C-watershed of F is a W-thinning of F;
iii) the graph G^λ is not a subgraph of (E, Γ).

The proof of Th. 17 can be found in Annex 7.2.

5 Topological watersheds on perfect fusion graphs

As seen in the previous section, on a perfect fusion graph, a C-watershed is always a W-thinning whose divide is a thin cleft. In this section, we extend this result to divides of topological watersheds. Then, we define a particular type of topological watersheds that derive from the C-watersheds. We show that this family of topological watershed satisfies an interesting additional property. Finally, we derive some new characterizations of perfect fusion graphs based on both thinness properties of C-watersheds and topological watersheds.

If a map F is a topological watershed, then there is no point in E which is W-destructible for F. Since any point M-cliff for F is W-destructible for F, we deduce the following property.

Theorem 18 *Let $F \in \mathcal{F}(E)$ and assume that (E, Γ) is a perfect fusion graph. If F is a topological watershed, then F is a C-watershed. Furthermore, the divide of any topological watershed is a thin cleft.*

As shown in the previous section a C-watershed is not necessarily a topological watershed. Thus, the converse of Th. 18 is, in general, not true.

Nevertheless, as stated by the next property, the C-watershed can be used to design an interesting strategy for computing a topological watershed in a perfect fusion graph.

Definition 19 (C-topological watershed) *Let $F \in \mathcal{F}(E)$ and assume that (E, Γ) is a perfect fusion graph. We say that H is a* C-topological watershed *of F, if there exists a C-watershed I of F such that H is a topological watershed of I.*

In the case of a perfect fusion graph, it is proved by Th. 13 that a C-topological watershed of a map is necessarily a topological watershed of this map. On the contrary, as we will see a little later, all topological watersheds are not C-topological watersheds. Among all topological watersheds of a map, the C-topological watersheds satisfy an additional property (Prop. 20). Informally speaking, it states that the divides of C-topological watersheds are located on the "highest crests" that separate the minima of the original map. The following property is a direct consequence of the monotony theorem (Th. 15) on C-watersheds.

Let $F \in \mathcal{F}(E)$. We say that a path $\pi = \langle x_0, \ldots, x_\ell \rangle$ in E is descending for F if for any $k \in [1, \ell]$, we have $F(x_k) \leq F(x_{k-1})$.

Property 20 *Let F and H be two maps in $\mathcal{F}(E)$ and assume that (E, Γ) is a perfect fusion graph. If H is a C-topological watershed of F, then for any point x in $\overline{M(H)}$, there exist two minima of F which can be reached from x by a descending path for F.*

The map H, depicted on Fig. 8c, is a C-topological watershed of the map F (Fig. 8a). It can be checked that from any point in $\overline{M(H)}$, two distinct minima of F can be reached by a descending path for F. On the contrary, the previous property is not verified by all topological watersheds. For instance, let us analyze the map I of Fig. 8d that is a topological watershed. We denote by x the vertex circled in Fig. 8a. Note that $x \in \overline{M(I)}$. The only minimum of F which can be reached from x by a descending path is the one at the top left of the figure. Thus, the topological watershed I of F, which is not a C-topological watershed of F, does not verify Prop. 20.

From the preceding results, we derive some grayscale characterizations of perfect fusion graphs based on thinness properties of both C-watersheds and topological watersheds. Their proof can be found in Annex 7.4.

Let $H \in \mathcal{F}(E)$, $x \in E$ and let $k = H(x)$. If x is F-simple for $H[k]$, *i.e.*, x is adjacent to exactly two components of $H[k]$, we say that x is *F-simple for H*.

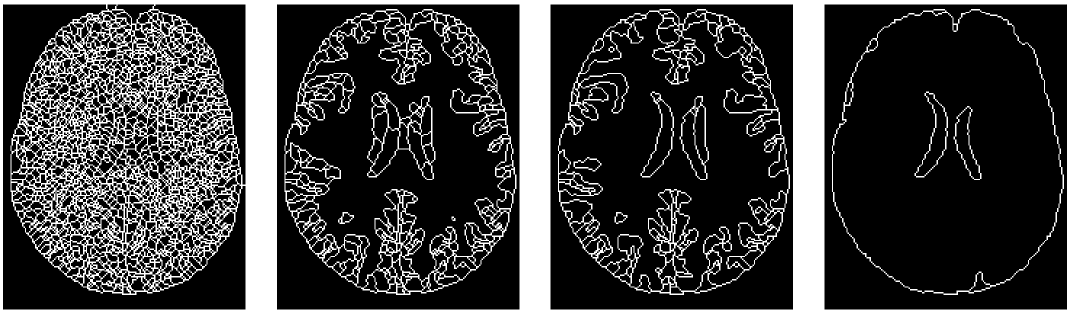
Property 21 *The four following statements are equivalent:*
i) (E, Γ) *is a perfect fusion graph;*
ii) for any C-watershed $F \in \mathcal{F}(E)$*, any point in* $\overline{M(F)}$ *is F-simple for* $\overline{M(F)}$*;*
iii) for any topological watershed $F \in \mathcal{F}(E)$*, any point in* $\overline{M(F)}$ *is F-simple for* $\overline{M(F)}$*;*
iv) for any topological watershed $F \in \mathcal{F}(E)$*, any point in* $\overline{M(F)}$ *is F-simple for* F*.*

6 Perspectives: perfect fusion grids and hierarchical schemes

With the counter-examples depicted in this paper, we have seen that there exist topological watersheds whose divides are not thin clefts in 2D on the graphs induced by the 4- and 8-adjacency relation. In 3D, similar counter-examples (see [20]) can be found for the graphs induced by the 6- and 26-adjacency relations that are the extensions of the 4- and 8-adjacency to $\mathbb{Z}^3$. On the contrary, we have shown that, on perfect fusion graphs, the divide of any topological watershed is a thin cleft. On these graphs, region merging schemes are easy to rigorously define and straightforward to implement. Thus, the framework of perfect fusion graphs is adapted for region merging methods based on topological watersheds.

In [20], we introduced the family of perfect fusion grids over $\mathbb{Z}^n$, for any $n \in \mathbb{N}$. Any element of this family is indeed a perfect fusion graph. We proved that any of these grids is "between" the direct adjacency graph (which generalizes the 4-adjacency to $\mathbb{Z}^n$) and the indirect adjacency graph (which generalizes the 8-adjacency to $\mathbb{Z}^n$). These n-dimensional grids are all equivalent (up to a unit translation) and, in a forthcoming paper, we intend to prove that they are the only graphs that possess these two properties [33]. However, we must notice that watersheds obtained from the same digital image using these different grids, may indeed be different. An example of (a restriction of) a 2-dimensional perfect fusion grid is presented in Fig. 8.

Perfect fusion grids thus constitute an interesting alternative to classical grids for watershed-based region merging methods. An example of such a procedure could be described, starting from the divide of a C-watershed, by the iteration of the following three steps: i) select the most significant region A according to a given criterion (*e.g.*, , the dynamics described in [34,22]); ii) select a region B such that A and B are neighbors and the minimal value of the points in $\Gamma^\star(A, B)$ is minimal; and iii) merge A and B through $\Gamma^\star(A, B)$. An illustration of such a scheme using the dynamics is shown in Fig. 9. Future work will include revisiting hierarchical segmentation methods [13,15] on perfect fusion grids and comparing these new schemes with previously published algorithms.



(a) (b) (c) (d)

Fig. 9. Region merging on a perfect fusion grid; (a), the divide of a C-watershed of Fig. 1a; and (b, c, d), several steps of region merging starting from (b).

References

[1] T. Kong, A. Rosenfeld, Digital topology: Introduction and survey, Computer Vision, Graphics, and Image Processing 48 (3) (1989) 357–393.

[2] A. Rosenfeld, A. Kak, Digital picture processing, Vol. 2, Academic Press, 1982, Ch. 10, section 10.4.2.d (region merging).

[3] T. Pavlidis, Structural Pattern Recognition, Vol. 1 of Springer Series in Electrophysics, Springer-Verlag, 1977, Ch. 4–5, pp. 90–123, (segmentation techniques).

[4] S. Beucher, C. Lantuéjoul, Use of watersheds in contour detection, in: procs. Int Workshop on Image Processing Real-Time Edge and Motion Detection/Estimation, 1979.

[5] L. Vincent, P. Soille, Watersheds in digital spaces: An efficient algorithm based on immersion simulations, IEEE Trans. Pattern Analysis and Machine Intelligence 13 (6) (1991) 583–598.

[6] F. Meyer, Un algorithme optimal de ligne de partage des eaux, in: Actes du 8ème Congrès AFCET, Lyon-Villeurbanne, France, 1991, pp. 847–859.

[7] S. Beucher, F. Meyer, The morphological approach to segmentation: the watershed transformation, E. Dougherty (Ed.), Mathematical Morphology in Image Processing, Marcel Decker (1993) 443–481.

[8] M. Couprie, G. Bertrand, Topological grayscale watershed transform, in: SPIE Vision Geometry V Proceedings, Vol. 3168, 1997, pp. 136–146.

[9] P. Jasiobedzki, C. Taylor, J. Brunt, Automated analysis of retinal images, Image and Vision Computing 1 (3) (1993) 139–144.

[10] K. Haris, S. N. Efstratiadis, N. Maglaveras, A. K. Katsaggelos, Hybrid image segmentation using watersheds and fast region merging, IEEE Trans. on Image Processing 7 (12) (1998) 1684–1699.

[11] A. Bleau, L. Leon, Watershed-based segmentation and region merging, Computer Vision and Image Understanding 77 (3) (2000) 317–370.

[12] F. Meyer, Minimum spanning forests for morphological segmentation, in: Procs. of the second international conference on Mathematical Morphology and its Applications to Image Processing, 1994, pp. 77–84.

[13] S. Beucher, Watershed, hierarchical segmentation and waterfall algorithm, in: Mathematical Morphology and its Applications to Image Processing, 1994, pp. 69–76.

[14] F. Meyer, The dynamics of minima and contours, in: P. Maragos, R. W. Schafer, M. A. Butt (Eds.), Mathematical Morphology and its Application to Image and Signal Processing, Kluwer Academic Publishers, Boston, 1996, pp. 329–336.

[15] L. Najman, M. Schmitt, Geodesic saliency of watershed contours and hierarchical segmentation, IEEE Trans. Pattern Analysis and Machine Intelligence 18 (12) (1996) 1163–1173.

[16] G. Bertrand, On topological watersheds, Journal of Mathematical Imaging and Vision 22 (2-3) (2005) 217–230.

[17] M. Couprie, L. Najman, G. Bertrand, Quasi-linear algorithms for the topological watershed, Journal of Mathematical Imaging and Vision 22 (2-3) (2005) 231–249.

[18] L. Najman, M. Couprie, G. Bertrand, Watersheds, mosaics and the emergence paradigm, Discrete Applied Mathematics 147 (2-3) (2005) 301–324.

[19] J. Cousty, G. Bertrand, M. Couprie, L. Najman, Fusion graphs, region merging and watersheds, in: Discrete Geometry for Computer Imagery, procs. of DGCI, Vol. 4245 of Lecture Notes in Computer Science, Springer, 2006, pp. 343–354.

[20] J. Cousty, G. Bertrand, M. Couprie, L. Najman, Fusion graphs: merging properties and watersheds, Journal of Mathematical Imaging and Vision Accepted. Also in technical report IGM2005-04, http://igm.univ-mlv.fr/LabInfo/rapportsInternes/2005/04.v2.pdf.

[21] J. Cousty, M. Couprie, L. Najman, G. Bertrand, Grayscale watersheds on perfect fusion graphs, in: Combinatorial Image Analysis, procs. of IWCIA, Vol. 4040 of Lecture Notes in Computer Science, Springer, 2006, pp. 60–73.

[22] G. Bertrand, On the dynamics, Image and Vision Computing 25 (4) (2007) 447–454.

[23] A. Rosenfeld, The fuzzy geometry of image subsets, Pattern Recognition Letters 2 (1984) 311–317.

[24] J. K. Udupa, S. Samarsekara, Fuzzy connectedness and object definition: Theory, algorithms, and applications in image segmentation, Graphical Models and Image Processing 58 (1996) 246–261.

[25] A. X. Falcão, J. Stolfi, R. de Alencar Lotufo, The image foresting transform: Theory, algorithm and applications, IEEE Trans. Pattern Analysis and Machine Intelligence 26 (2004) 19–29.

[26] J. Cousty, G. Bertrand, L. Najman, M. Couprie, Watershed cuts, in: Mathematical Morphology and its Applications to Signal and Image Processing, Procs. of ISMM, 2007, pp. 301–312.

[27] J. Cousty, G. Bertrand, L. Najman, M. Couprie, Watersheds, minimum spanning forests, and the drop of water principle, Submitted. Also in technical report IGM2007-01, http://igm.univ-mlv.fr/LabInfo/rapportsInternes/2007/01.pdf.

[28] C. Berge, Graphes et hypergraphes, Dunod, 1970.

[29] L. Najman, M. Couprie, Building the component tree in quasi-linear time, IEEE Trans. on Image Processing 15 (11) (2006) 3531–3539.

[30] M. Thorup, On RAM priority queues, SIAM Journal on Computing 30 (1) (2000) 86–109.

[31] R. Ahuja, K. Mehlhom, J. Orlin, R. Tarjan, Faster algorithms for the shortest path problem, SIAM Journal on Computing 37 (2) (1990) 213–223.

[32] P. Soille, Morphological Image Analysis, Springer-Verlag, 1999.

[33] J. Cousty, G. Bertrand, Uniqueness of the perfect fusion grid on $\mathbb{Z}^d$ In preparation.

[34] M. Grimaud, New measure of contrast: dynamics, in: P. Gader, E. Dougherty, J. Serra (Eds.), Image algebra and morphological image processing III, Vol. SPIE-1769, 1992, pp. 292–305.

7 Annex

7.1 Proof of Th. 9

Let us recall a characterization of topological watersheds (Th. 22) that will help us to prove Th. 9.

Let X and Y be two subsets of E. We say that Y is a W-thinning of X if:
i) $Y = X$; or if
ii) there exists a set Z that is a W-thinning of X and there exists a uniconnected point x for Z such that $Y = X \setminus \{x\}$.
Let $C \subseteq X$ and assume that Y is a W-thinning of X. We say that Y is *a cleft of* X *constrained by* C if $Z = Y$ whenever Z is a W-thinning of Y and $C \subseteq Z$.

In other words, Y is a cleft of X constrained by C if Y is a W-thinning of X which contains C and if any point in $Y \setminus C$ is not uniconnected for Y.

Theorem 22 (Th. 2 in [16]) *Let F be in $\mathcal{F}(E)$. The map F is a topological watershed if and only if, for each $k \in \mathbb{K}$, $F[k]$ is a cleft constrained by $F[k+1]$.*

We are now ready to prove Th. 9.

Proof of Th. 9

i): Suppose that there exists a cleft $X \subseteq E$, and that there exists $A \in \mathcal{C}(\overline{X})$ such that (A, Γ_A) is not a fusion graph. Then, by Th. 5, there exists a cleft $Y \subseteq A$ on (A, Γ_A), such that $|\mathcal{C}(\overline{Y})| \leq 2$, that is not thin. Let us define the map $F \in \mathcal{F}(E)$ as follows: for any $x \in E$, if $x \in X$, $F(x) = 2$, if $x \in Y$, $F(x) = 1$ and if $x \in (E \setminus (X \cup Y))$, $F(x) = 0$. It can be seen that F is a topological watershed which is not thin.

ii): Suppose that F is a topological watershed which is not thin. There exists $x \in int(F)$ with maximal altitude. Let k be the altitude of x. Necessarily $F[k+1] \cap int(F) = \emptyset$. Let $A \in \mathcal{C}(\overline{F}[k+1])$ such that $x \in A$. Let $Y = A \cap F[k]$. Since $A \cap F[k+1] = \emptyset$ (as $A \in \mathcal{C}(\overline{F}[k+1])$), by Th. 22, there is no point uniconnected for $F[k]$ in Y. Then any point $y \in Y$ which is adjacent to $\overline{F}[k]$ is adjacent to at least two connected components of $\overline{F}[k]$. Furthermore it can be seen that these connected components are all included in A. Thus Y is a cleft on (A, Γ_A).

Remark that $x \in Y$. From the very definition of $int(F)$ and since $x \in int(F)$, we have: $x \in int(F[k])$ and $x \in int(Y)$. Hence Y is a cleft on (A, Γ_A) which is not thin and by Th. 5 (A, Γ_A) is not a fusion graph.

We are now going to prove that there exists a cleft $X \subseteq E$ on (E, Γ) such that $A \in \mathcal{C}(\overline{X})$. More precisely, we are going to show that $X = \Gamma^\star(A)$ is a cleft (and obviously $A \in \mathcal{C}(\overline{X})$).

If $X = \emptyset$ the proof is done. Otherwise, let $z \in X$ and let $k' = F(z)$. Since $A \in \mathcal{C}(\overline{F}[k+1])$ we know that $k' > k$. Since $k' > k$ and since x is the highest point of $int(F)$, we know that z cannot be inner for F. Hence, since F is a watershed, we deduce that z is adjacent to at least two connected components of $\overline{F}[k']$. One of these components, say A', must contain A. Let B be any other one of these components. It must contain a point y adjacent to z such that $y \notin \Gamma^(A')$ (otherwise we would have $B = A'$). Since $A \subseteq A'$, we have $y \notin \Gamma(A)$. Thus, z is adjacent to at least two components of $\overline{X}$, which are namely A and the component of $\overline{X}$ which contains y. Hence, z is not a uniconnected for X and therefore, by the very definition of a cleft, we deduce that $X = \Gamma^\star(A)$ is a cleft on (E, Γ) such that $A \in \mathcal{C}(\overline{X})$, hence Th. 9. $\square$

Observe that Lem. 11 is a corollary of Th. 17 (as shown below, after the proof of Th. 17). Thus we start by the proof of Th. 17.

Remark 23 *Let F be a map in $\mathcal{F}(E)$. From the very definition of W-destructible points and topological watersheds, it may be seen that if H is a W-thinning of F, then, for any $k \in \mathbb{K}^+$, we have $|\mathcal{C}(\overline{F}[k])| = |\mathcal{C}(\overline{H}[k])|$.*

Proof of Th. 17

$i \Rightarrow ii$: Follows straightforwardly the definition of a C-watershed.
$\overline{iii} \Rightarrow \overline{ii}$: Suppose that there exists five points p, q, r, s and t in E such that the subgraph G' of (E, Γ) induced by $\{p, q, r, s, t\}$ is isomorphic to G^λ. Let us consider $F \in \mathcal{F}(E)$ such that: (a), $F(p) = F(r) = F(t) = 0$; (b), $F(q) = 1$; (c), $F(s) = 2$; and (d) for any $x \in E \setminus \{p, q, r, s, t\}$, $F(x) = 4$. Note that the following arguments hold if either $[E \setminus \{p, q, r, s, t\}] \neq \emptyset$ or $[E \setminus \{p, q, r, s, t\}] = \emptyset$. It may be easily seen that $\overline{F}[2]$ is made of exactly two components. Observe that the point s is the only M-cliff point for F and that it is adjacent to $\{t\}$ a minimum of F at altitude 0. Thus, for any C-watershed H of F ($H \neq F$), we have $H(s) = 0$. It may be seen that $|\mathcal{C}(\overline{H}[2])| = 1$ whereas $|\mathcal{C}(\overline{F}[2])| = 2$. Thus we establish $\overline{ii}$ by the converse of Rem. 23.
$\overline{i} \Rightarrow \overline{iii}$: Suppose that there exists a map $F \in \mathcal{F}(E)$ such that there is a point $x_1 \in E$ that is M-cliff for F and not W-destructible for F. Let $h = F(x_1)$. Let $x_0 \in \Gamma(x_1)$ be any point in $M \subseteq M(F)$ the only minimum of F adjacent to x_1. It may be seen that $x_0 \in \overline{F}[h]$. Thus, since x_1 is not W-destructible, x_1 must be adjacent to X a connected component of $\overline{F}[h]$ which contains neither $\{x_0\}$ nor any point of M. Let $Y = M(F) \cap X$. As a consequence of the definition of a minimum, Y is not empty. Let Π be the set of all paths $\langle x_0, x_1, x_2, \ldots x_k \rangle$ such that $x_2, \ldots x_k$ are in X and x_k is adjacent to Y. Let $\pi = \langle x_0, x_1, x_2, \ldots x_\ell \rangle$ be a shortest path among the paths in Π (*i.e.*, the length of any path in Π is greater than or equal to the length ℓ of π). Since x_1 is adjacent to a unique minimum M of F and since M does not intersect X, x_1 cannot be adjacent to Y. Hence, $\ell > 1$ and, by construction, it follows that $x_\ell \in X$. Thus, $F(x_\ell) < h$ because $X \subseteq \overline{F}[h]$. Suppose that x_ℓ is adjacent to a single minimum of F, then x_ℓ would be a cliff point, the altitude of which is less than the altitude of x_1, a contradiction since x_1 is M-cliff for F. Thus, there exist two points in $\Gamma(x_\ell)$, say y and z, which are in two distinct minima of F. To establish ($\overline{iii}$), we are now going to prove that the subgraph of (E, Γ) induced by $\{y, z, x_\ell, x_{\ell-1}, x_{\ell-2}\}$ is isomorphic to G^λ. Since y and z are in two distinct minima of F, $y \notin \Gamma(z)$. Furthermore, by construction, $x_{\ell-1} \in \Gamma(x_{\ell-2})$, $x_\ell \in \Gamma(x_{\ell-1})$ and y and z both belong to $\Gamma(x_\ell)$. It is thus sufficient to prove that $y \notin \Gamma(x_{\ell-1})$, $y \notin \Gamma(x_{\ell-2})$ (and also $z \notin \Gamma(x_{\ell-1})$, $z \notin \Gamma(x_{\ell-2})$) and that $x_\ell \notin \Gamma(x_{\ell-2})$. If $\ell = 2$, then $x_{\ell-2} = x_0$ and $x_{\ell-1} = x_1$. In this case, if $x_\ell \in \Gamma(x_{\ell-2})$, then x_0 must

belong to X since $x_\ell \in X$ and $F(x_0) < h$. This is by construction impossible. If $\ell > 2$ and $x_\ell \in \Gamma(x_{\ell-2})$, then π is not a shortest among the paths in Π since $\langle x_0, \ldots, x_{\ell-2}, x_\ell \rangle$ would also belong to Π. Therefore, in all cases, $x_{\ell-2} \notin \Gamma(x_\ell)$. If $y \in \Gamma(x_{\ell-1})$, it may be seen that $\pi' = \langle x_0, \ldots x_{\ell-1} \rangle$ would belong to Π, which is impossible since the length of π' is less that the one of π. In the same manner, we can see that $y \notin \Gamma(x_{\ell-2})$, $z \notin \Gamma(x_{\ell-1})$, and $z \notin \Gamma(x_\ell)$. Thus $\overline{iii}$. $\square$

Proof of Lem. 11 Since (E, Γ) is a perfect fusion graph, from statement *ii* of Th. 6, $G^\blacktriangle$ is not a subgraph of (E, Γ). Remark that $G^\blacktriangle$ is a subgraph of G^λ. Thus G^λ is not a subgraph of (E, Γ). Let $x \in E$ be any M-cliff point for F. From Th. 17, x is W-destructible for F.

Let ℓ be the level of the only minimum of F adjacent to x. When x has been lowered down to $F(x) - 1$ and while it has not been lowered down to ℓ, x is the only point M-cliff for F^i (where $F^i = [F^{i-1} \setminus x]$, $F^0 = F$). When x has been lowered down to ℓ it is not W-destructible any more, hence ℓ is the lowest value of x. $\square$

7.3 Proof of Th. 15

Lemma 24 *Assume that (E, Γ) is a perfect fusion graph. Let $X \subseteq E$ be a nonempty connected set and let Y be a nonempty subset of X. If $int(X \setminus Y) \neq \emptyset$, then there exists $y \in X \setminus Y$ adjacent to a single connected component of Y.*

Proof Let us consider (X, Γ_X) the subgraph of (E, Γ) induced by X. It is a connected graph. By Th. 6iv, (X, Γ_X) is a fusion graph. If $|\mathcal{C}(Y)| < 2$, the result follows from the connectedness of X. Otherwise, since $int(X \setminus Y) \neq \emptyset$, we deduce from Th. 5 that $X \setminus Y$ is not a cleft on (X, Γ_X), hence there exists $y \in [X \setminus Y]$ which is uniconnected for $[X \setminus Y]$. Therefore, by definition of a uniconnected point, y is adjacent to a single connected component of $\overline{(X \setminus Y)}$ on (X, Γ_X), hence y is adjacent to exactly one connected component of Y. $\square$

Proof of Th. 15 Let x be any point M-cliff for F. Let $H_0 = [F_{x,\ell}]$, where ℓ is the altitude of the only minimum of F adjacent to x. Let y be any point M-cliff for H_0. It may be seen that y is adjacent to zero or one minimum of F. If y is adjacent to one minimum of F, y is a cliff point for F, and then $F(y) \geq F(x)$ since x is M-cliff for F. Suppose now that y is not adjacent to any minimum of F (*i.e.*, $y \in int(\overline{M(F)})$). Let $k = F(y)$. Let X be the component of $\overline{F}[k+1]$ that contains y. From the definition of a minimum, it may be seen that the family $\mathcal{M}$ of all minima of F included in X is not empty. Let $Y = \cup\{M \in \mathcal{M}\}$. Since $y \in int(\overline{M(F)})$, it can be seen that y is inner for $[X \setminus Y]$. As a consequence, by Lem. 24, there exists $z \in [X \setminus Y]$ adjacent to a unique component of Y, that is z is a cliff point for F. As x is an M-cliff point for F, $F(x) \leq F(z)$. Since $z \in \overline{F}[k+1]$, $F(z) \leq F(y)$. We thus get $F(x) \leq F(y)$. By induction, Th. 15 is established. $\square$

7.4 Proof of Prop. 21

Proof of Prop. 21
$i \Rightarrow ii$: The fact that (E, Γ) is a perfect fusion graph implies (from Th. 6iii) that any cleft $X \subseteq E$ is such that any $x \in X$ is F-simple for X. By definition, a map $F \in \mathcal{F}(E)$ is a C-watershed if and only if $\overline{M(F)}$ is a cleft, hence ii.
$ii \Rightarrow iii$: By Th. 18, any topological watershed is a C-watershed hence iii.
$(\overline{i} \Rightarrow \overline{iii})$ and $(\overline{i} \Rightarrow \overline{iv})$: Suppose that (E, Γ) is not a perfect fusion graph. Then, by Th. 6.iii, there exists a cleft $Y \subseteq E$ such that there exists $y \in Y$ which is not F-simple for Y. We define $F \in \mathcal{F}(E)$ such that for any x in Y, $F(x) = 1$ and for any $x \in \overline{Y}$, $F(x) = 0$. Remark that $Y = F[1] = \overline{M(F)}$, and that $\overline{F}[1] = M(F)$. Thus, since Y is a cleft, there is no uniconnected point for $\overline{M(F)}$ and there is no W-destructible point for F. Thus, F is both a topological watershed and a C-watershed. Since y is not F-simple for Y, according to the preceding remarks, we deduce that y is neither F-simple for F nor F-simple for $\overline{M(F)}$. Hence $\overline{iii}$ and $\overline{iv}$ both hold.
$i \Rightarrow iv$: Let $F \in \mathcal{F}(E)$ be any topological watershed. Let $x \in \overline{M(F)}$ and let $k = F(x)$. Since $(i \Rightarrow ii \Rightarrow iii)$, x is adjacent to exactly two minima of F. Hence, x is adjacent to at least one component of $\overline{F}[k]$. Since F is a topological watershed, x is not uniconnected for $F[k]$. Therefore, x is adjacent to at least two components of $\overline{F}[k]$. Since (E, Γ) is a perfect fusion graph, by Th. 6.v, x is adjacent to at most two components of $\overline{F}[k]$, hence x is F-simple for F. $\square$