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# Discrete region merging and watersheds

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## Abstract

This paper summarizes some results of the authors concerning watershed divides and their use in region merging schemes.

The first aspect deals with properties of watershed divides that can be used in particular for hierarchical region merging schemes. We introduce the mosaic to retrieve the altitude of points along the divide set. A desirable property is that, when two minima are separated by a crest in the original image, they are still separated by a crest of the same altitude in the mosaic. Our main result states that this is the case if and only if the mosaic is obtained through a topological thinning.

The second aspect is closely related to the thinness of watershed divides. We present fusion graphs, a class of graphs in which any region can be always merged without any problem. This class is equivalent to the one in which watershed divides are thin. Topological thinnings do not always produce thin divides, even on fusion graphs. We also present the class of perfect fusion graphs, in which any pair of neighbouring regions can be merged through their common neighborhood. An important theorem states that the divides of any ultimate topological thinning are thin on any perfect fusion graph.

*Key words:* segmentation, graph, mosaic, (topological) watershed, separation, merging

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## Introduction

A popular approach to image segmentation, called region merging [18, 21], consists of progressively merging pairs of regions until a certain criterion is satisfied. The criterion which is used to identify the next pair of regions which will merge, as well as the stopping criterion are specific to each particular method.

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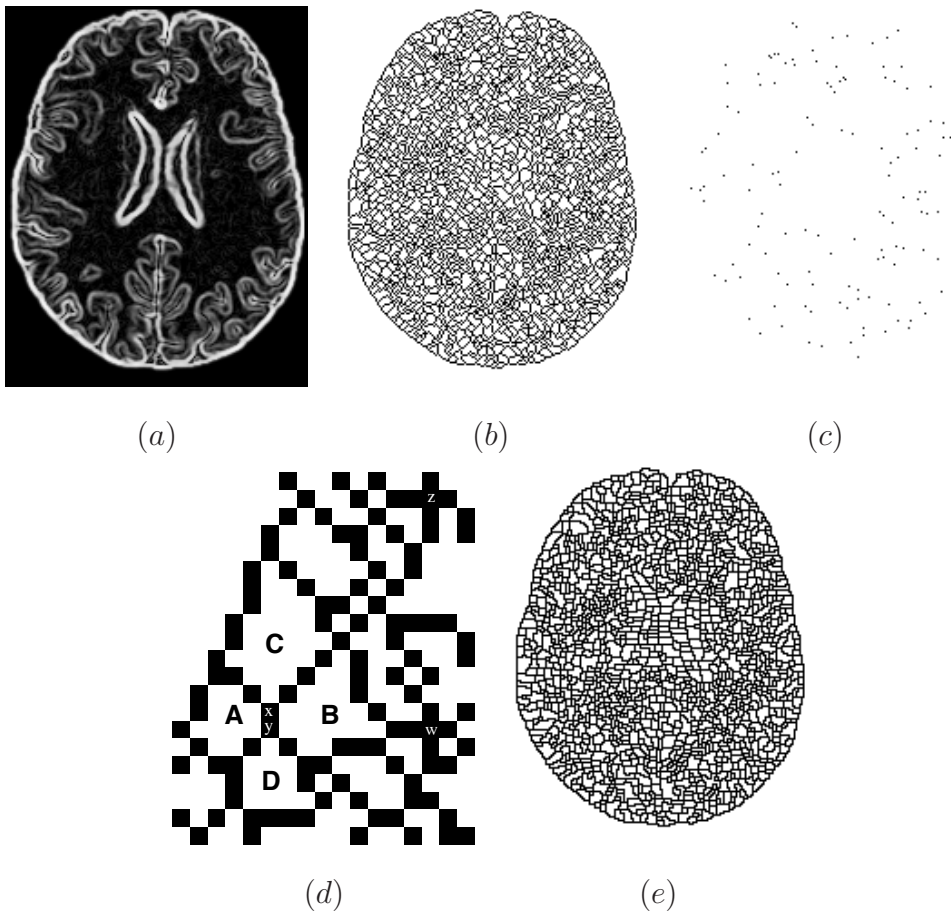


Fig. 1. (a): Original image (cross-section of a brain, after applying a gradient operator). (b): Watershed of (a) with the 4-adjacency (in black). (c): Interior points for the previous image (in black). (d): A zoom on a part of (b). The points  $z$  and  $w$  are interior points. (e): Watershed of (a) with the 8-adjacency (in black). There are no interior points.

Given a grayscale image, how is it possible to obtain an initial set of regions for a region merging process? The watershed transform [4, 14, 22] is a powerful tool for solving this problem. Let us consider a 2D grayscale image as a topographical relief, where dark pixels correspond to basins and valleys, whereas bright pixels correspond to hills and crests. Suppose that we are interested in segmenting “dark” regions. Intuitively, the watersheds of the image are constituted by the crests which separate the basins corresponding to regional minima (see Fig. 1a,b). Due to noise and texture, real-world images often have a huge number of regional minima, hence the “mosaic” aspect of Fig. 1b. In the case of a graph (*e.g.*, an adjacency graph defined on a subset of  $\mathbb{Z}^2$ ), a watershed may be thought of as a “separating set” of vertices which cannot be reduced without merging some connected components of its complementary set.

Surprisingly, for various reasons, some watershed algorithms do not always produce a watershed (in the sense stated above), although they produce a thick divide set that separates the dark regions. For implementing region merging schemes, such

thickness is a problem. On the other hand, the use of watershed divides in hierarchical region merging schemes [3, 17] impose some constraints on the placement of those divides, and we have to check the effectiveness of divides with respect to such constraints.

This paper summarizes some of the main results [1, 5, 6, 8, 11, 12, 16] obtained by the authors regarding these two questions.

To evaluate the effectiveness of divides, we have to consider the altitude of points of the original image along the divide set. We call the greyscale image thus obtained a mosaic. A first goal of this paper is to examine some properties of mosaics related to image segmentation. We say informally that a watershed algorithm produces a “separation” if the minima of the mosaic are of the same altitude as the ones of the original image and if, when two minima are separated by a crest in the original image, they are still separated by a crest of the same altitude in the mosaic. The formal definition relies on the altitude of the lowest pass which separates two minima, named connection value (see also [1, 15, 16]). An important theorem [16] states that a mosaic is a separation *if and only if* it is obtained through a topological thinning [1, 5], a transform that modifies the original image while preserving some topological properties, namely the connectivity of each lower cross-section.

However, a topological thinning does not always give a watershed. Furthermore, even in the case where the result of a watershed algorithm is a watershed, the question of the thinness of the watershed is not solved. Indeed, let us look at the following example. Given a subset  $E$  of  $\mathbb{Z}^2$  and the graph  $(E, \Gamma_1)$  which corresponds to the usual 4-adjacency relation, we observe that a watershed may contain some “interior points”, *i.e.*, points which are not adjacent to any point outside the watershed (see Fig. 1c,d). We can say that a watershed on  $\Gamma_1$  is not necessarily thin. On the other hand, such interior points do not seem to appear in any watershed on  $\Gamma_2$ , which corresponds to the 8-adjacency. Are the watersheds on  $\Gamma_2$  always thin? We will see that it is indeed true. More interestingly, we summarize in this paper a framework that allows the study of the property of thinness of watersheds in any kind of graph. One of the main theorems of [8] is the identification of the class of graphs in which any watershed is necessarily thin (see Th. 24).

Let us now turn back to the region merging problem. What happens if we want to merge a couple of neighboring regions  $A$  and  $B$ , and if each pixel adjacent to these two regions is also adjacent to a third one, which is not wanted in the merging? Fig. 1d illustrates such a situation, where  $x$  is adjacent to regions  $A, B, C$  and  $y$  to  $A, B, D$ . This problem has been identified in particular by T. Pavlidis (see [18], section 5.6: “When three regions meet”), and has been dealt with in some practical ways, but until now a systematic study of properties related to merging in graphs has not been done. A major contribution of [8] is the definition and the study of four classes of graphs, with respect to the possibility of “getting stuck” in a merging process (Sec. 3.1, Sec. 3.3). In this paper, we expose two of these four classes

of graphs. In particular, we say that a graph is a *fusion graph* if any region  $A$  in this graph can always be merged with another region  $B$ , without problems with other regions. The most striking outcome of [8] is that the class of fusion graphs is precisely the class of graphs in which any watershed is thin (Th. 24).

For morphological merging schemes, it is important to obtain a watershed which is both thin and which is a separation. We thus have to obtain a topological thinning whose divide is thin. In one of the four classes of graphs introduced in [8], called the class of *perfect fusion graphs*, any pair of neighboring regions  $A, B$  can always be merged, without problems with other regions, by removing all the pixels which are adjacent to both  $A$  and  $B$ . We will see that the divide of any ultimate topological thinning is thin on perfect fusion graphs [11]. Such properties show that perfect fusion graphs offer an ideal framework for region merging schemes.

In the last section of this paper, we give an intuitive idea of a graph on  $\mathbb{Z}^n$  (for any  $n$ ) that we call the perfect fusion grid [8], which is indeed a perfect fusion graph, and which is “between” the direct adjacency graph (which generalizes the 4-adjacency to  $\mathbb{Z}^n$ ) and the indirect adjacency graph (which generalizes the 8-adjacency). Furthermore, in [7], we prove that this  $n$ -dimensional grid is the unique grid (up to a translation) that possesses those two properties.

## 1 Discrete watersheds: definitions and heuristic

### 1.1 Basic notions and notations

Let  $A$  and  $B$  be two sets, we write  $A \subseteq B$  if  $A$  is a subset of  $B$ , we write  $A \subset B$  if  $A$  is a proper subset of  $B$  (i.e. if  $A$  is a subset of  $B$  and  $A \neq B$ ).

In this paper,  $E$  stands for a finite nonempty set. We denote by  $|E|$  the number of elements of  $E$ , and by  $2^E$  the set composed of all the subsets of  $E$ . We write  $\overline{X}$  the complementary set of  $X$  in  $E$ , i.e.,  $\overline{X} = E \setminus X$ .

We define a graph as a pair  $(E, \Gamma)$  where  $\Gamma$  is a binary relation on  $E$  (i.e.,  $\Gamma \subseteq E \times E$ ), which is reflexive (for all  $x$  in  $E$ ,  $(x, x) \in \Gamma$ ) and symmetric (for all  $x, y$  in  $E$ ,  $(y, x) \in \Gamma$  whenever  $(x, y) \in \Gamma$ ). Each element of  $E$  is called a *vertex* or a *point*. We will also denote by  $\Gamma$  the map from  $E$  to  $2^E$  such that, for all  $x \in E$ ,  $\Gamma(x) = \{y \in E \mid (x, y) \in \Gamma\}$ . If  $y \in \Gamma(x)$ , we say that  $y$  is *adjacent to*  $x$ . We define also the map  $\Gamma^*$  such that for all  $x \in E$ ,  $\Gamma^*(x) = \Gamma(x) \setminus \{x\}$ . Let  $X \subseteq E$ , we define  $\Gamma(X) = \cup_{x \in X} \Gamma(x)$ , and  $\Gamma^*(X) = \Gamma(X) \setminus X$ . If  $y \in \Gamma(X)$ , we say that  $y$  is *adjacent to*  $X$ . If  $X, Y \subseteq E$  and  $\Gamma(X) \cap Y \neq \emptyset$ , we say that  $Y$  is *adjacent to*  $X$  (or that  $X$  is adjacent to  $Y$ , since  $\Gamma$  is symmetric). Let  $G = (E, \Gamma)$  be a graph and let  $X \subseteq E$ , we define the *subgraph of  $G$  induced by  $X$*  as the graph

$G_X = (X, \Gamma \cap [X \times X])$ . In this case, we also say that  $G_X$  is a subgraph of  $G$ .

Let  $(E, \Gamma)$  be a graph, let  $X \subseteq E$ , a path in  $X$  is a sequence  $\pi = \langle x_0, \dots, x_l \rangle$  such that  $x_i \in X, i \in [0, l]$ , and  $x_i \in \Gamma(x_{i-1}), i \in [1, \dots, l]$ . We also say that  $\pi$  is a path from  $x_0$  to  $x_l$  in  $X$ . Let  $x, y \in X$ . We say that  $x$  and  $y$  are linked for  $X$  if there exists a path from  $x$  to  $y$  in  $X$ . We say that  $X$  is connected if any  $x$  and  $y$  in  $X$  are linked for  $X$ .

Let  $Y \subseteq X$ . We say that  $Y$  is a connected component of  $X$ , or simply a component of  $X$ , if  $Y$  is connected and if  $Y$  is maximal for this property, i.e.,  $Z = Y$  whenever  $Y \subseteq Z \subseteq X$  and  $Z$  connected.

We denote by  $\mathcal{C}(X)$  the set of all the connected components of  $X$ .

In image processing, we are often interested by vertex-weighted graphs. We denote by  $\mathcal{F}(E)$  the set composed of all maps from  $E$  to  $\mathbb{Z}$ . A map  $F \in \mathcal{F}(E)$  is also called an image, and if  $x \in E$ ,  $F(x)$  is called the altitude of  $x$  (for  $F$ ). Let  $F \in \mathcal{F}(E)$ . We write  $F_k = \{x \in E \mid F(x) \geq k\}$  with  $k \in \mathbb{Z}$ ,  $F_k$  is called an upper section of  $F$ , and  $\overline{F_k}$  is called a lower section of  $F$ . A non-empty connected component of a lower section  $\overline{F_k}$  is called a (level  $k$ ) lower-component of  $F$ . A level  $k$  lower-component of  $F$  that does not contain a level  $(k - 1)$  lower-component of  $F$  is called a (regional) minimum of  $F$ . We denote by  $\mathcal{M}(F)$  the set of minima of  $F$ .

A subset  $X$  of  $E$  is flat for  $F$  if any two points  $x, y$  of  $X$  are such that  $F(x) = F(y)$ . If  $X$  is flat for  $F$ , we denote by  $F(X)$  the altitude of any point of  $X$  for  $F$ .

**Important remark.** From now,  $(E, \Gamma)$  denotes a graph, and we furthermore assume for simplicity that  $E$  is connected.

Notice that, nevertheless, the subsequent definitions and properties may be easily extended to non-connected graphs.

## 1.2 Watersheds

Informally, in a graph, a watershed may be thought of as a “separating set” of vertices which cannot be reduced without merging some components of its complementary set (see Fig. 2d). We first give formal definitions of these concepts (see [1, 5]) and related ones, then we derive some properties which will be used in the sequel.

**Definition 1.** Let  $X \subseteq E$ , and let  $p \in X$ .

We say that  $p$  is W-simple (for  $X$ ) if  $p$  is adjacent to exactly one component of  $\overline{X}$ .

In this definition and the following ones, the prefix “W-” stands for watershed. In Fig. 2a,  $x$  is a W-simple point for the set  $X$  constituted by the black vertices.

We are now ready to define the notion of watershed which is central to this section.

**Definition 2.** Let  $X \subseteq E$ , let  $Y \subseteq X$ .

We say that  $Y$  is a W-thinning of  $X$ , written  $X \searrow^W Y$ , if

i)  $Y = X$  or if

ii) there exists a set  $Z \subseteq X$  which is a W-thinning of  $X$  and a point  $p \in Z$  which is W-simple for  $Z$ , such that  $Y = Z \setminus \{p\}$ .

A set  $Y \subseteq X$  is a watershed (in  $(E, \Gamma)$ ) if  $Y \searrow^W Z$  implies  $Z = Y$ .

A subset  $Y$  of  $X$  is a watershed of  $X$  if  $Y$  is a W-thinning of  $X$  and if  $Y$  is a watershed.

A watershed  $Y$  is non-trivial if  $Y \neq \emptyset$  and  $Y \neq E$ .

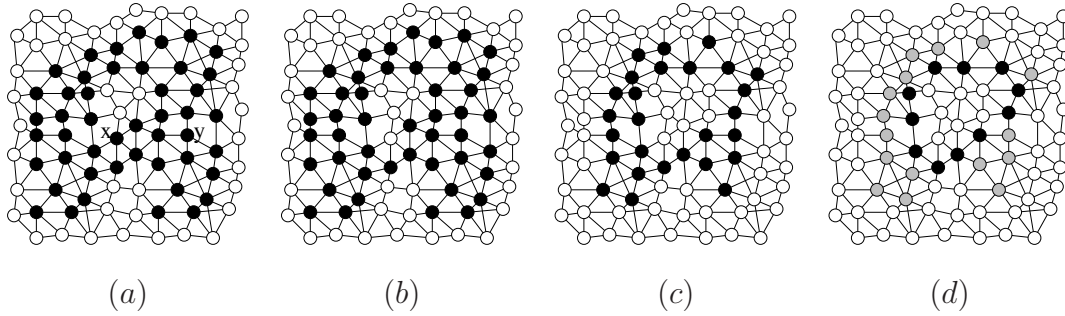


Fig. 2. Illustration of W-thinning and watershed. (a): A graph  $(E, \Gamma)$  and a subset  $X$  (black points) of  $E$ . The point  $x$  is a border point which is W-simple, and  $y \in \text{int}(X)$  (see section 3.2). (b): The set  $Y = X \setminus \{x\}$  (black points) is a W-thinning of  $X$ . (c): The set  $Z$  (black points) is a W-thinning of both  $X$  and  $Y$ . The sets  $Y$  and  $Z$  are not watersheds: some W-simple points exist in both sets. (d): A watershed of  $X$  (black points), which is also a watershed of  $Y$  and of  $Z$ .

**Definition 3.** Let  $X, Y$  be non-empty subsets of  $E$ . We say that  $Y$  is an extension of  $X$  if  $X \subseteq Y$  and if each component of  $Y$  contains exactly one component of  $X$ . We also say that  $Y$  is an extension of  $X$  if  $X$  and  $Y$  are both empty.

**Theorem 4 ([1]).** Let  $X$  and  $Y$  be subsets of  $E$ . The subset  $Y$  is a W-thinning of  $X$  if and only if  $\overline{Y}$  is an extension of  $\overline{X}$ .

### 1.3 Extension-by-flooding: a heuristic for watersheds

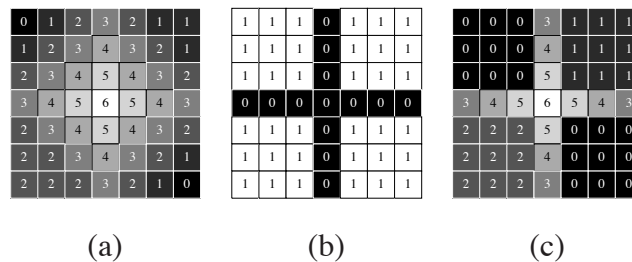


Fig. 3. (a) An image, (b) a minima extension of (a), and (c) the associated mosaic.

Numerous segmentation algorithms associate an influence zone to each minimum of the image, by producing an extension of the set of minima of the image. This is in particular the case for most of the algorithms proposed in the literature to compute watersheds of grayscale maps.

Let  $X$  be a subset of  $E$ , and let  $F \in \mathcal{F}(E)$ . We say that  $X$  is a *minima extension* of  $F$  if  $X$  is an extension of  $\mathcal{M}(F)$ , the set of minima of  $F$ . The complementary set of a minima extension of  $F$  in  $E$  is called a *divide set* of  $F$ . Figure 3 shows a simple example of a minima extension. Excepted when specified otherwise, in the examples of the paper, the underlying graph  $(E, \Gamma)$  corresponds to the 4-adjacency relation on a subset  $E \subset \mathbb{Z}^2$ , i.e., for all  $x = (x_1, x_2) \in E$ ,  $\Gamma(x) = \{(x_1, x_2), (x_1 + 1, x_2), (x_1 - 1, x_2), (x_1, x_2 + 1), (x_1, x_2 - 1)\} \cap E$ .

A popular presentation of the watershed in the morphological community is based on a flooding paradigm. Let us consider the grayscale image as a topographical relief: the grey level of a pixel becomes the elevation of a point, the basins and valleys of the relief correspond to the dark areas, whereas the mountains and crest lines correspond to the light areas. Let us imagine the surface being immersed in a lake, with holes pierced in local minima. Water fills up basins starting at these local minima, and, at points where waters coming from different basins would meet, dams are built. As a result, the surface is partitioned into regions or basins separated by dams, called watershed divides.

Among the numerous algorithms [19, 22] that were developed following this idea, F. Meyer's algorithm [13] (called *flooding algorithm* in the sequel) is probably the simplest to describe and understand. It is based on a simple heuristics that consists in looking at the pixels of the image by increasing grey levels. Starting from an image  $F \in \mathcal{F}(E)$  and the set  $M$  composed of all points belonging to the regional minima of  $F$ , the flooding algorithm expands as much as possible the set  $M$ , while preserving the connected components of  $M$ . This heuristic can be described as follows:

- (1) Attribute to each minimum a label, two distinct minima having distinct labels; mark each point belonging to a minimum with the label of the corresponding minimum. Initialize two sets  $Q$  and  $V$  to the empty set.
- (2) Insert every non-marked neighbor of every marked point in the set  $Q$ ;
- (3) Extract from the set  $Q$  a point  $x$  which has the minimal altitude, that is, a point  $x$  such that  $F(x) = \min\{F(y) \mid y \in Q\}$ . Insert  $x$  in  $V$ . If all marked points in  $\Gamma(x)$  have the same label, then
  - Mark  $x$  with this label; and
  - Insert in  $Q$  every  $y \in \Gamma(x)$  such that  $y \notin Q \cup V$ ;
- (4) Repeat step 3 until the set  $Q$  is empty.

Let  $F \in \mathcal{F}(E)$ , and let  $X$  be the set composed of all the points labeled by the flooding algorithm applied on  $F$ . We call any such set  $X$  produced by the flooding



algorithm an *extension-by-flooding* (of  $F$ ). Note that, in general, there may exist several extension-by-flooding of a given map  $F$ . It is easy to prove the following result: any extension-by-flooding is indeed a minima extension of  $F$ , and furthermore, the complementary set of any extension-by-flooding is a watershed in the sense of definition 2.

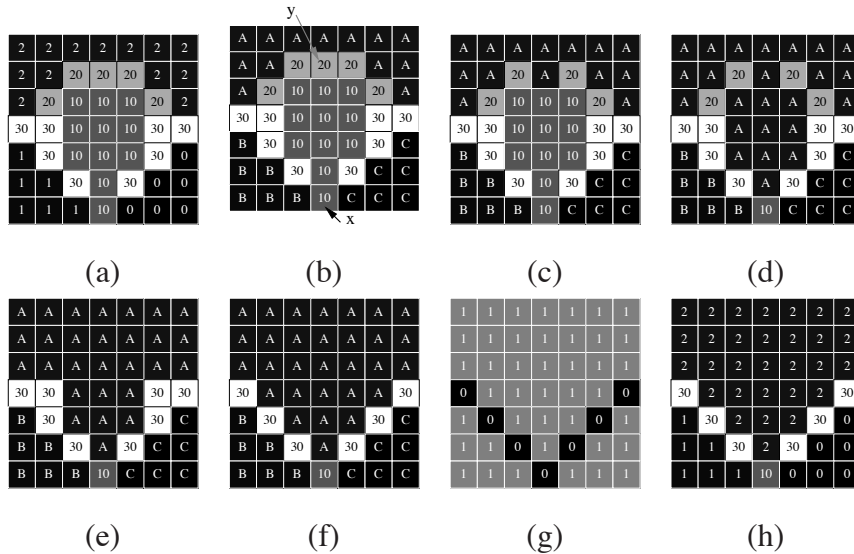


Fig. 4. (a): Original image. (b-f): several steps of the extension algorithm. (g) The extension-by-flooding of (a), and (h) the associated mosaic. One can note that the contour at altitude 20 in the original image (a) is not present in the mosaic (h).

The extension-by-flooding of the image depicted in figure 3.a is the image depicted in figure 3.b. Let us illustrate the behaviour of the algorithm on another example, the one of figure 4.a which presents an image with three minima at altitudes 0, 1 and 2.

- The minima at altitudes 2, 1, 0 are marked with the labels A, B, C respectively (figure 4.b). All the non-marked neighbors of the marked points are put into the set  $Q$ .
- The first point which is extracted from the set  $Q$  is the point  $x$  at altitude 10, which has points marked B and C among its neighbors (figure 4.b). This point cannot be marked.
- The next point to process is one of the points at altitude 20, for instance  $y$  (figure 4.b). The only marked points in the neighborhood of such a point are marked with the label A, and thus  $y$  is marked with the label A (figure 4.c), and the points at altitude 10 which are neighbors of  $y$  are put into the set  $Q$ .
- The next points to process are points at altitude 10. A few steps later, all points at altitude 10 but  $x$  are processed, and marked with the label A (figure 4.d).
- Then the other points at altitude 20 are processed. They are marked with the label A (figure 4.e). The next points to process are those at altitude 30, and we finally obtain the set of labeled points shown in figure 4.f.

Figure 4.g shows the extension-by-flooding of figure 4.a.

#### 1.4 Minima extensions and mosaics

Intuitively, for application to image analysis, the divide set represents the location of points which best separate the dark objects (regional minima), in terms of grey level difference (contrast). In order to evaluate the effectiveness of this separation, we have to consider the values of points along the divide set. This motivates the following definition.

**Definition 5.** *Let  $F \in \mathcal{F}(E)$  and let  $X$  be a minima extension of  $F$ . The mosaic of  $F$  associated with  $X$  is the map  $F_X \in \mathcal{F}(E)$  such that*

- for any  $x \notin X$ ,  $F_X(x) = F(x)$ ; and
- for any  $x \in X$ ,  $F_X(x) = \min\{F(y) \mid y \in C_x\}$ , where  $C_x$  denotes the connected component of  $X$  that contains  $x$ .

The term ‘mosaic’ for this kind of construction, was coined by S. Beucher [2]. The image of figure 3.c is the mosaic associated with the extension-by-flooding of 3.b. The image of figure 4.h is the mosaic associated with the extension by flooding 4.g.

We now state two fundamental remarks that we are going to develop.

**Remark 6.** *We observe that (informally speaking) the extension-by-flooding algorithm does not preserve the “contrast” of the original image. In the original image, to go from, e.g., the minimum at altitude 0 to the minimum at altitude 2, one has to climb to at least an altitude of 20. We observe that such a “contour” is not present in the mosaic produced by the algorithm. Let us emphasize that configurations similar to the examples presented in this paper are found in real-world images, with all usual adjacencies.*

**Remark 7.** *We observe that the extension-by-flooding algorithm is not monotone, in the sense where several pixels at altitude 20 are extracted from the queue  $Q$  after some pixels at altitude 30.*

## 2 Minima extensions, separations and topological watershed

This section introduces the formal framework that leads to a better understanding of the previous observations. In particular, two notions are pivotal in the sequel: minima extension for maps, and separation.

Let  $F$  be a map and let  $F_X$  be the mosaic of  $F$  associated with a minima extension  $X$  of  $F$ . It is natural to try to associate any regional minimum of  $F_X$  to a connected

component of  $X$  and conversely, and to compare the altitude of each minimum of  $F_X$  to the altitude of the corresponding minimum of  $F$ . We will see with forthcoming properties and examples, that both problems are in fact closely linked.

Let  $F$  and  $G$  in  $\mathcal{F}(E)$ . We note  $G \leq F$  if for all  $x \in E$ ,  $G(x) \leq F(x)$ .

The following definition extends to maps the minima extension previously defined for sets.

**Definition 8.** Let  $F$  and  $G$  in  $\mathcal{F}(E)$  such that  $G \leq F$ . We say that  $G$  is a minima extension (of  $F$ ) if:

- i) the set composed by the union of all the minima of  $G$  is a minima extension of  $F$ .
- ii) for any  $X \in \mathcal{M}(F)$  and  $Y \in \mathcal{M}(G)$  such that  $X \subseteq Y$ , we have  $F(X) = G(Y)$ .

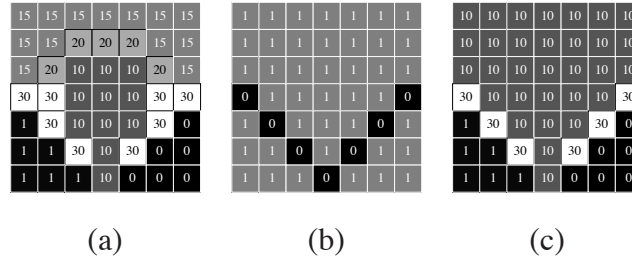


Fig. 5. (a) An image, (b) the extension-by-flooding of (a), and (c) The mosaic of (a) associated to (b).

The image 4.h is an example of a mosaic that is a minima extension of the image 4.a. On the other hand, figure 5.a shows an image  $F$  and figure 5.c shows the mosaic  $F_X$  associated with the extension-by-flooding  $X$  of  $F$  (figure 5.b). One can notice that the connected component of  $X$  which corresponds to the minimum of altitude 15 for  $F$  has an altitude of 10 for  $F_X$ , and is not a minimum of  $F_X$ . Thus, this mosaic  $F_X$  is not a minima extension of  $F$ .

We can now turn back to a more precise analysis of remark 6. To this aim, we present the connection value and the separation. Intuitively, the connection value between two points corresponds to the lowest altitude to which one has to climb to go from one of these points to the other.

**Definition 9.** Let  $F \in \mathcal{F}(E)$ . Let  $\pi = \langle x_0, \dots, x_n \rangle$  be a path in the graph  $(E, \Gamma)$ , we set  $F(\pi) = \max\{F(x_i) \mid i = 0, \dots, n\}$ .

Let  $x, y$  be two points of  $E$ , the connection value for  $F$  between  $x$  and  $y$  is  $F(x, y) = \min\{F(\pi) \mid \pi \in \Pi(x, y)\}$ , where  $\Pi(x, y)$  is the set of all paths from  $x$  to  $y$ .

Let  $X, Y$  be two subsets of  $E$ , the connection value for  $F$  between  $X$  and  $Y$  is defined by  $F(X, Y) = \min\{F(x, y), \text{for any } x \in X \text{ and any } y \in Y\}$ .

A notion equivalent to the connection value up to an inversion of  $F$  (that is, replacing  $F$  by  $-F$ ), has been introduced by A. Rosenfeld [20] under the name of *degree of connectivity* for studying connectivity in the framework of fuzzy sets. Figure 6 illustrates the connection value on the image  $F$  of figure 4.a.

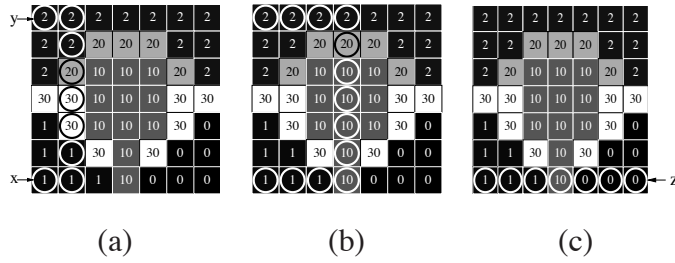


Fig. 6. Illustration of paths and connection values on the image  $F$  of figure 4.a. (a) A path  $\pi_1$  from the point  $x$  to the point  $y$  such that  $F(\pi_1) = 30$ . (b) A path  $\pi_2$  from the point  $x$  to the point  $y$  such that  $F(\pi_2) = 20$ . It is not possible to find a path from  $x$  to  $y$  with a lower maximal altitude, hence  $F(x, y) = 20$ . (c) A path  $\pi_3$  from the point  $x$  to the point  $z$  such that  $F(\pi_3) = 10$ , and we can easily check that  $F(x, z) = 10$ .

Informally, a transformation “preserves the separation” if, when two points are separated by a crest in the original map, they are still separated by a crest of the same “height” in the transform.

**Definition 10** ([1]). *Let  $F \in \mathcal{F}(E)$ , let  $x, y \in E$ . We say that  $x$  and  $y$  are separated (for  $F$ ) if  $F(x, y) > \max\{F(x), F(y)\}$ .*

*We say that  $x$  and  $y$  are  $k$ -separated (for  $F$ ) if they are separated for  $F$  and if  $k = F(x, y)$ .*

*Let  $G \in \mathcal{F}(E)$ , with  $G \leq F$ . We say that  $G$  is a separation of  $F$  if, for all  $x$  and  $y$  in  $E$ , whenever  $x$  and  $y$  are  $k$ -separated for  $F$ ,  $x$  and  $y$  are  $k$ -separated for  $G$ .*

*We say that  $G$  is a strong separation of  $F$  if  $G$  is both a separation of  $F$  and a minima extension of  $F$ .*

**Remark 11.** *We can now restate the remark 6 using the notions introduced in this section. Figure 4 shows that a mosaic produced by the flooding algorithm is not always a minima extension of the original map. Figure 5 shows that a mosaic produced by the flooding algorithm, even in the case where it is a minima extension, is not necessarily a separation of the original map.*

## 2.1 Mosaics and topological watersheds

A different approach to the watershed was presented by M. Couprie and G. Bertrand [5]. The idea is to transform the image  $F$  into an image  $G$  while preserving some topological properties of  $F$ , namely the number of connected components of the lower cross-sections of  $F$ . A minima extension of  $F$  can then be obtained easily from  $G$ , by extracting the regional minima of  $G$ .

Thanks to the notion of cross-sections, we extend W-simple points to grayscale images, and we introduce W-thinnings and topological watersheds.

**Definition 12.** *Let  $F \in \mathcal{F}(E)$ ,  $x \in E$ , and  $k = F(x)$ .*

The point  $x$  is  $W$ -destructible (for  $F$ ) if  $x$  is  $W$ -simple for the upper section  $F_k$ .  
 We say that  $G \in \mathcal{F}(E)$  is a  $W$ -thinning of  $F$  if  $G = F$  or if  $G$  may be derived from  $F$  by iteratively lowering  $W$ -destructible points by one.  
 We say that  $G \in \mathcal{F}(E)$  is a topological watershed of  $F$  if  $G$  is a  $W$ -thinning of  $F$  and if there is no  $W$ -destructible point for  $G$ .

As a consequence of the definition, a topological watershed  $G$  of a map  $F$  is a map which has the same number of regional minima as  $F$ . Furthermore, the number of connected components of any lower cross-section is preserved during this transformation. Quasi-linear algorithms for computing the topological watershed transform can be found in [6].

By the very definition of a  $W$ -destructible point, it may easily be proved that, if  $G$  is a  $W$ -thinning of  $F$ , then the union of all minima of  $G$  is a minima extension of  $F$ . This motivates the following definition.

**Definition 13.** Let  $F \in \mathcal{F}(E)$  and let  $G$  be a  $W$ -thinning of  $F$ . The mosaic of  $F$  associated with  $G$  is the mosaic of  $F$  associated with the union of all minima of  $G$ .

We have the following property.

**Property 14** ([16]). Let  $F \in \mathcal{F}(E)$ , let  $G$  be a  $W$ -thinning of  $F$ , and let  $H$  be the mosaic of  $F$  associated with  $G$ . Then  $H$  is a minima extension of  $F$ .

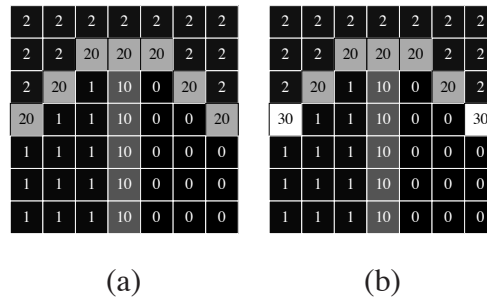


Fig. 7. Example of topological watershed. (a) a topological watershed of figure 4.a (b) the associated mosaic.

Notice that in general, there exist different topological watersheds for a given map  $F$ . Figure 7.a presents one of the possible topological watersheds of figure 4.a, and figure 7.b shows the associated extension map. One can note that both figure 7.a and figure 7.b are separations of figure 4.a.

## 2.2 Mosaics and separations

Recently, G. Bertrand [1] showed that a mathematical key underlying the topological watershed is the *separation*. The following theorem states the equivalence between the notions of  $W$ -thinning and strong separation. The “if” part implies

in particular that a topological watershed of an image  $F$  preserves the connection values between the minima of  $F$ . Furthermore, the “only if” part of the theorem mainly states that if one needs a lowering transformation which is guaranteed to preserve the connection values between the minima of the original map, then this transformation is necessarily a W-thinning.

**Theorem 15** ([1]). *Let  $F$  and  $G$  be two elements of  $\mathcal{F}(E)$ . The map  $G$  is a W-thinning of  $F$  if and only if  $G$  is a strong separation of  $F$ .*

We have proved [16] that the mosaic associated with any W-thinning of a map  $F$  is also a W-thinning of  $F$  (and thus, it is a separation of  $F$ ). Furthermore, we have also proved that an arbitrary mosaic  $F_X$  of a map  $F$  is a separation of  $F$  if and only if  $F_X$  is a W-thinning of  $F$ .

**Theorem 16** ([16]). *Let  $F \in \mathcal{F}(E)$ , let  $X$  be a minima extension of  $F$ , and let  $F_X$  be the mosaic of  $F$  associated with  $X$ . Then  $F_X$  is a separation of  $F$  if and only if  $F_X$  is a W-thinning of  $F$ .*

### 3 Fusion graphs

Region merging [18, 21] is a popular approach to image segmentation. Starting with an initial partition of the image pixels into connected regions, which can in some cases be separated by some boundary pixels, the basic idea consists of progressively merging pairs of regions until some criterion is satisfied. The criterion which is used to identify the next pair of regions which will merge, as well as the stopping criterion are specific to each particular method. Certain methods do not use graph vertices in order to separate regions, nevertheless even these methods fall in the scope of this study through the use of line graphs [9, 10].

#### 3.1 Merging

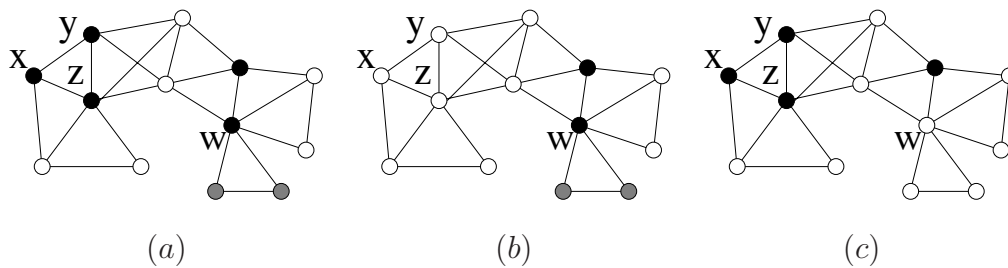


Fig. 8. Illustration of merging. (a): A graph  $(E, \Gamma)$  and a subset  $X$  of  $E$  (black points). (b): The black points represent  $X \setminus S$  with  $S = \{x, y, z\}$ . (c): The black points represent  $X \setminus S'$  with  $S' = \{w\}$ .

Consider the graph  $(E, \Gamma)$  depicted in Fig. 8a, where a subset  $X$  of  $E$  (black vertices) separates its complementary set  $\bar{X}$  into four connected components. If we replace the set  $X$  by, for instance, the set  $X \setminus S$  where  $S = \{x, y, z\}$ , we obtain a set which separates its complementary set into three components, see Fig. 8b: we can also say that we “merged two components of  $\bar{X}$  through  $S$ ”. This operation may be seen as an “elementary merging” in the sense that only two components of  $\bar{X}$  were merged. On the opposite, replacing the set  $X$  by the set  $X \setminus S'$  where  $S' = \{w\}$ , see Fig. 8c, would merge three components of  $\bar{X}$ . We also see that the component of  $\bar{X}$  which is below  $w$  (in light gray) cannot be merged by an “elementary merging” since any attempt to merge it must involve the point  $w$ , and thus also the three components of  $\bar{X}$  adjacent to this point. In this section, we introduce definitions and basic properties related to such merging operations in graphs.

**Definition 17.** *Let  $X \subset E$  and  $S \subseteq X$ . We say that  $S$  is F-simple (for  $X$ ) if  $S$  is adjacent to exactly two components  $A, B \in \mathcal{C}(\bar{X})$  such that  $A \cup B \cup S$  is connected. Let  $p \in X$ . We say that  $p$  is F-simple (for  $X$ ) if  $\{p\}$  is F-simple for  $X$ .*

In this definition, the prefix “F-” stands for fusion. For example, in Fig. 8a, the point  $z$  is F-simple while  $x, y, w$  are not. Also, the sets  $\{z\}, \{x, y\}, \{x, z\}, \{y, z\}, \{x, y, z\}$  are F-simple, but the sets  $\{x\}, \{y\}$  and  $\{w\}$  are not.

**Definition 18.** *Let  $X \subset E$ . Let  $A$  and  $B \in \mathcal{C}(\bar{X})$ , with  $A \neq B$ . We say that  $A$  and  $B$  can be merged (for  $X$ ) if there exists  $S \subseteq X$  such that  $S$  is F-simple for  $X$ , and  $A$  and  $B$  are precisely the two components of  $\bar{X}$  adjacent to  $S$ . In this case, we also say that  $A$  and  $B$  can be merged through  $S$  (for  $X$ ).*

*We say that  $A$  can be merged (for  $X$ ) if there exists  $B \in \mathcal{C}(\bar{X})$  such that  $A$  and  $B$  can be merged for  $X$ .*

For example, in Fig. 8a, the component of  $\bar{X}$  in light gray cannot be merged, but each of the three white components can be merged for  $X$ .

From Def. 2 and Th. 4, if  $X$  is a watershed, any extension of  $\bar{X}$  is equal to  $\bar{X}$ . We have the following corollary.

**Property 19.** *Let  $(E, \Gamma)$  be a graph, let  $X \subset E$  be a watershed and let  $A \in \mathcal{C}(\bar{X})$ . The subset  $A$  can be merged for  $X$  if and only if there exists a vertex  $x \in \Gamma^*(A)$  which is F-simple for  $X$ .*

### 3.2 Thinness, region merging and watersheds

From the definition of an F-simple point, it appears that merging regions will be more or less difficult depending on the thinness of watersheds. The notions of thinness and interior are closely related.

**Definition 20.** Let  $X \subseteq E$ , the interior of  $X$  is the set  $\text{int}(X) = \{x \in X \mid \Gamma(x) \subseteq X\}$ . We say that the set  $X$  is thin if  $\text{int}(X) = \emptyset$ .

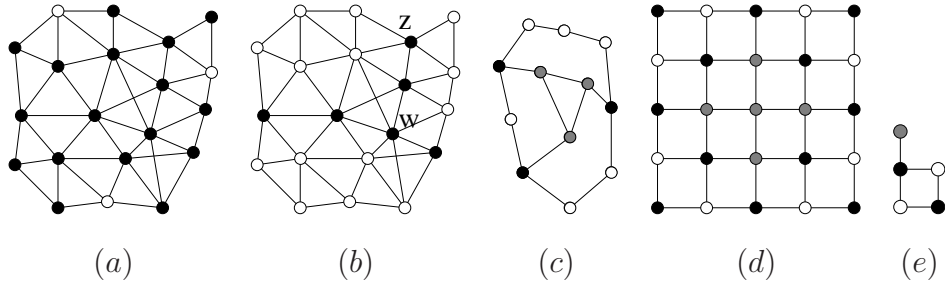


Fig. 9. Illustration of thin and non-thin watersheds. (a): A graph  $(E, \Gamma)$  and a subset  $X$  (black points) of  $E$ . (b): A subset  $Y$  (black points) of  $E$  which is a thin watershed; it is a watershed of the set  $X$  shown in (a). The points  $z$  and  $w$  are both separating for  $Y$ . (c, d, e): The subset  $X$  represented by black and gray points is a watershed which is not thin:  $\text{int}(X)$  is depicted by the gray points.

A watershed is a set which contains no W-simple point, but some of the examples given in Fig. 2 show that such a set is not always thin (in the sense of Def. 20). Fig. 2d and Fig. 9b are two examples of watersheds which are thin: in both cases, the set of black points has no W-simple point and no inner point. Fig. 9c,d show two examples of non-thin watersheds.

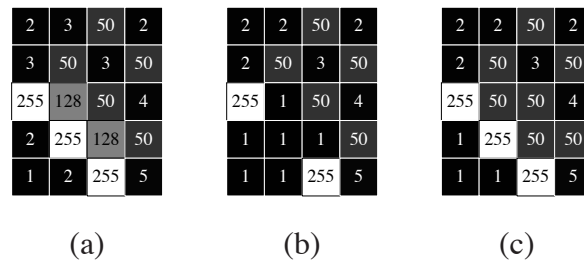


Fig. 10. Divides of topological watershed are not always watersheds. (a): An image, (b): the mosaic of the extension-by-flooding of (a), (c): the topological watershed of (a).

When looking at gray-level images, things get even worse. The preservation of connection values has an impact on the thinness of topological watersheds. As shown in Fig. 10.c, divides of topological watersheds can be thick, and furthermore they are not always watersheds. On the contrary, the extension-by-flooding Fig. 10.b is a watershed, but it does not preserve the connection values.

Section 3.1 and the next one constitute a theoretical basis for the study of region merging methods. The problems encountered by certain of these methods (see [18], section 5.6: “When three regions meet”) can be avoided by using exclusively the notion of merging introduced in section 3.1. In the sequel, we investigate several classes of graphs with respect to the possibility of “getting stuck” in a merging process. The most striking result of the next section is a theorem which states the equivalence between one of these classes and the class of graphs in which any



binary watershed is thin. In section 3.4, we will see that there exists a class of graphs in which extension-by-floodings are W-thinnings, a result which is non trivial.

### 3.3 Fusion and perfect fusion graphs

We begin with the definition of two classes of graphs adapted for merging.

**Definition 21.** We say that a graph  $(E, \Gamma)$  is a fusion graph if for any  $X \subseteq E$  such that  $|\mathcal{C}(\overline{X})| \geq 2$ , each  $A \in \mathcal{C}(\overline{X})$  can be merged for  $X$ .

Let  $X \subset E$ , and let  $A$  and  $B \in \mathcal{C}(\overline{X})$ ,  $A \neq B$ . We set  $\Gamma^*(A, B) = \Gamma^*(A) \cap \Gamma^*(B)$ . We say that  $A$  and  $B$  are neighbors if  $\Gamma^*(A, B) \neq \emptyset$ .

**Definition 22.** We say that the graph  $(E, \Gamma)$  is a perfect fusion graph if, for any  $X \subseteq E$ , any  $A$  and  $B \in \mathcal{C}(\overline{X})$  which are neighbors can be merged through  $\Gamma^*(A, B)$ .

Basic examples and counter-examples of fusion and perfect fusion graphs are given in Fig. 11.

These classes are linked by inclusion relations. The following property clarifies these links.

**Property 23** ([8]). Any perfect fusion graph is a fusion graph. The converse is not true.

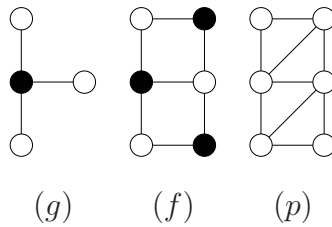


Fig. 11. Examples and counter-examples for different classes of graphs.  $(g)$ : A graph which is not a fusion graph,  $(f)$ : a fusion graph which is not a perfect fusion graph,  $(p)$ : a perfect fusion graph. In the graphs  $(g, f)$ , the black vertices constitute a set  $X$  which serves to prove that the graph does not belong to the pre-cited class.

Now, we present the main theorem of this section, which establishes that the class of graphs for which any watershed is thin is precisely the class of fusion graphs. As an immediate consequence of this theorem and Prop. 23, we see that all watersheds in perfect fusion graphs are thin.

**Theorem 24** ([8]). A graph  $G$  is a fusion graph if and only if any non-trivial watershed in  $G$  is thin.

Let us look at some examples to illustrate this property. The graphs of Fig. 9c, Fig. 9d and Fig. 9e are not fusion graphs; we see that they may indeed contain a non-thin watershed. Fig. 9d shows that the usual 4-adjacency relation is not a fusion graph. On the other hand, we have proved that the usual 8-adjacency is a fusion graph, and thus any watershed is thin on an 8-adjacency graph.

We give below three necessary and sufficient conditions for characterizing perfect fusion graphs. Recall that in perfect fusion graphs, any two components  $A, B$  of  $\mathcal{C}(\overline{X})$  which are neighbors can be merged through  $\Gamma^*(A) \cap \Gamma^*(B)$ . Thus, perfect fusion graphs constitute an ideal framework for region merging methods. In the sequel, we will use the symbol  $G^\blacktriangle$  to denote the graph of Fig. 12.

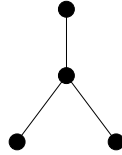


Fig. 12. Graph  $G^\blacktriangle$  used for a characterization of perfect fusion graphs (Th. 25).

**Theorem 25** ([8]). *Let  $(E, \Gamma)$  be a graph.*

*The four following statements are equivalent:*

- i)  $(E, \Gamma)$  is a perfect fusion graph;*
- ii) for any non-trivial watershed  $Y$  in  $E$ , each point  $x$  in  $Y$  is  $F$ -simple;*
- iii) for any connected subset  $A$  of  $E$ , the subgraph of  $(E, \Gamma)$  induced by  $A$  is a fusion graph;*
- iv) the graph  $G^\blacktriangle$  is not a subgraph of  $(E, \Gamma)$ ;*

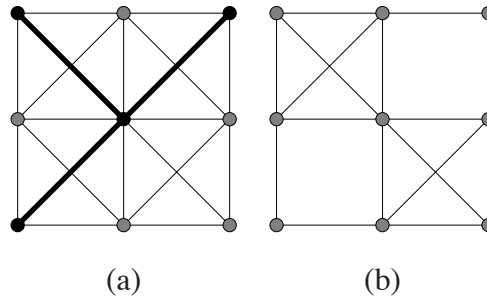


Fig. 13. Illustration of theorem 25. (a):  $G^\blacktriangle$  (in bold) is a subgraph of the 8-adjacency relation. (b):  $G^\blacktriangle$  is not a subgraph of this perfect fusion graph.

As an illustration of theorem 25, we can see on figure 13 that the graph  $G^\blacktriangle$  is a subgraph of the usual 8-adjacency relation, and thus graphs induced by the 8-adjacency relation are not perfect fusion graphs.

We conclude this section with a nice property of perfect fusion graphs, which can be useful to design hierarchical segmentation methods based on watersheds and region splitting. Consider the example of Fig. 14a, where a watershed  $X$  (black points) in the graph  $G$  separates  $\overline{X}$  into two components. Consider now the set  $Y$

(gray points) which is a watershed in the subgraph of  $G$  induced by one of these components. We can see that the union of the watersheds,  $X \cup Y$ , is not a watershed, since the point  $x$  is W-simple for  $X \cup Y$ . Prop. 26 shows that this problem cannot occur in any perfect fusion graph.

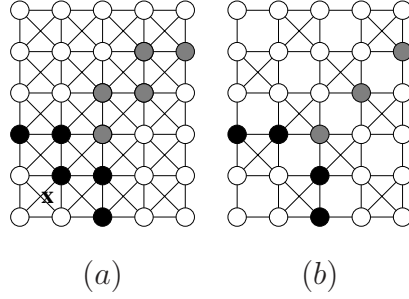


Fig. 14. Illustrations for Prop. 26. (a): The graph (8-adjacency relation) is not a perfect fusion graph, and the union of the watersheds is not a watershed. (b): The graph is a perfect fusion graph, the property holds.

**Property 26** ([8]). *Let  $G = (E, \Gamma)$  be a graph. If  $G$  is a perfect fusion graph, then for any watershed  $X \subset E$  in  $G$  and for any watershed  $Y \subset A$  in  $G_A$ , where  $A \in \mathcal{C}(\overline{X})$  and  $G_A$  is the subgraph of  $G$  induced by  $A$ , the set  $X \cup Y$  is a watershed in  $G$ .*

### 3.4 Topological watersheds and perfect fusion graphs

In the previous section, we have seen that an interesting framework for region merging is the class of perfect fusion graphs. For morphological merging schemes [3, 17], we still have to preserve the connection values. One of the consequences is that the divide of a topological watershed can be thick, even on fusion graphs. The following theorem show that this is not the case anymore on perfect fusion graphs. Furthermore, we can compute a W-thinning using the flooding algorithm.

**Theorem 27** ([12]). *Let  $G = (E, \Gamma)$  be a perfect fusion graph and let  $F \in \mathcal{F}(E)$ . The divide of any topological watershed of  $F$  is thin. Let  $X$  be an extension-by-flooding of  $F$ . Then the mosaic  $F_X$  associated with  $X$  is a W-thinning of  $F$ . Furthermore  $\overline{X}$  is a thin watershed.*

Note that the previous statement is false on fusion graphs which are not perfect fusion graphs (see counter-examples in [12]).

Another property states [12] that on perfect fusion graphs, the flooding algorithm is monotone, in the sense of remark 7: points of  $E$  are processed according to increasing altitude.

### 3.5 Perfect fusion grids

We conclude this section by intuitively introducing the perfect fusion grid [8]: this is a grid for structuring  $n$ -dimensional digital images that is a perfect fusion graph, whatever the dimension  $n$ . It is depicted in 2D on Fig. 14b and Fig. 15(b,c).

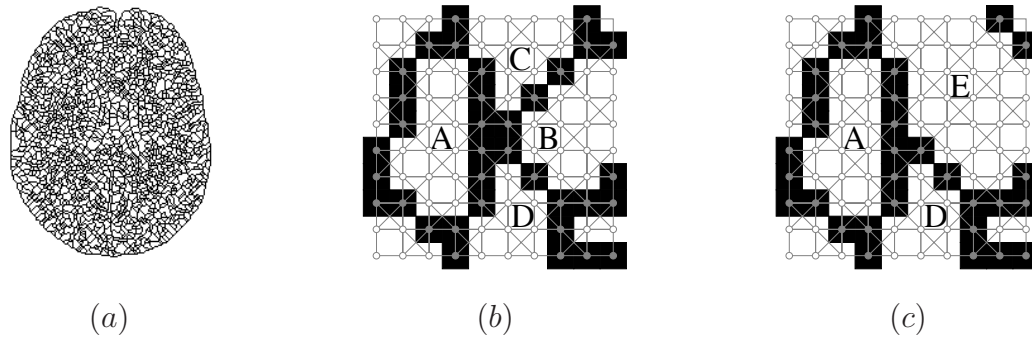


Fig. 15. (a): A watershed of Fig. 1a obtained on the perfect fusion grid; (b): a zoom of a part of (a) where the regions  $A$ ,  $B$ ,  $C$  and  $D$  correspond to the regions shown in Fig. 1c; in gray, the corresponding perfect fusion grid is superimposed; (c): same as (b) after having merged  $B$  and  $C$  to form a new region, called  $E$ .

It does thus constitute a structure on which neighboring regions can always be merged through their common neighborhood without problem with other regions. Fig. 15a shows a watershed of Fig. 1a obtained on this grid. Remark that the problems pointed out in the introduction do not exist in this case. The watershed does not contain any inner point. Any pair of neighboring regions can be merged by simply removing from the watershed the points which are adjacent to both regions (Fig. 15b,c). Furthermore, the resulting set is still a watershed. Observe that this grid is “between” the usual grids. In a forthcoming paper [7], we intend to prove that this is the unique such graph.

## 4 Conclusion

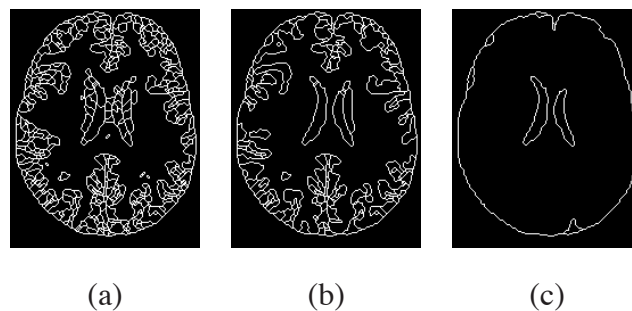


Fig. 16. Region merging on perfect fusion grid. (a,b,c) Several steps of merging starting from Fig. 15a.

In this paper, we have presented a framework that allows in particular the analysis of watershed algorithms. We have also described some classes of graphs adapted for region merging schemes. An important goal of this framework is to help the design of such schemes (see Fig. 16), and in particular morphological hierarchical merging schemes known under the name of *geodesic saliency of watershed contours* [17] schemes.

In a forthcoming paper [10], we also study definitions and properties of watersheds in the framework of edge-weighted graphs, a subclass of perfect fusion graphs. An important result is that in this framework, watersheds can be defined following the intuitive idea of flowing drops of water. Moreover, we have established the consistency of these watersheds with respect to definition in terms of catchment basins, and proved their optimality in terms of minimum spanning forests.

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