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Abstract

A point of a discrete object is called simple if it can be deleted from this object without altering topology. In this article, we present new characterizations of simple points which hold in dimensions 2, 3 and 4, and which lead to efficient algorithms for detecting such points. In order to prove these characterizations, we establish two confluence properties of the collapse operation which hold in the neighborhood of a point in spaces of low dimension. This work is settled in the framework of cubical complexes, which provides a sound topological basis for image analysis, and allows to retrieve the main notions and results of digital topology, in particular the notion of simple point.

Key Words: Cubical complex, topology preservation, collapse, simple point, confluence, 4D space.

Introduction

Topology-preserving operators, like homotopic skeletonization, are used in many applications of image analysis to transform an object while leaving unchanged its topological characteristics. Applications in 2D and 3D are already widely spread, and with the emergence of fast 3D image acquisition devices, such as medical X-ray and MRI scanners, there is a growing interest in considering a time sequence of 3D objects as a coherent 4D structure. For example, the segmentation of a moving heart muscle can be facilitated in this way [12].

In discrete grids ($\mathbb{Z}^2$, $\mathbb{Z}^3$, $\mathbb{Z}^4$), a topology-preserving transformation can be defined thanks to the notion of simple point [20]: intuitively, a point of an object is called simple if it can be deleted from this object without altering topology. This notion, pionneered by Duda, Hart, Munson [14], Golay [17] and Rosenfeld [27], has since been the subject of an abundant literature. In particular, local characterizations of simple points have been proposed, which allow efficient implementation of thinning procedures.

Let us illustrate informally the notion of simple point through some examples, first in 2D, then in 3D. In Fig. 1, the points (or pixels) x, y, z, t are not simple: the removal of x from the set X of pixels would create a new connected component of the complement $\overline{X}$ of X ; the removal of y would merge two connected components of $\overline{X}$; the removal of z would split a connected component of X ; and the removal of t would delete a connected component of X. On the other hand, the pixels a, b and c are simple pixels. We see that, in 2D, the notion of connectedness (for both X and $\overline{X}$) suffices to characterize simple pixels.

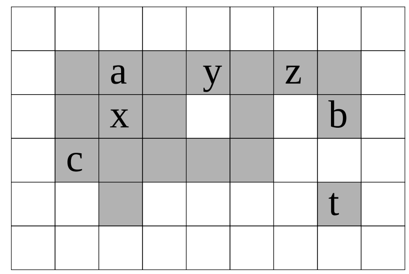


Figure 1. Illustration of 2D simple pixels. The set X is made of the pixels in gray, a, b, c are simple while x, y, z, t are not simple.

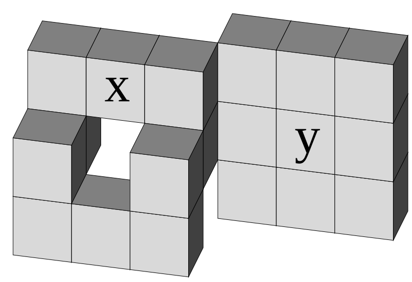


Figure 2. A set X of voxels. The voxels x and y are not simple.

Things are more difficult in 3D. Consider the example of the set X depicted in Fig. 2, removing the voxel x or the voxel y from X would not split, merge, create or suppress

any component of X nor any component of $\overline{X}$. However neither x nor y is simple, for the deletion of x (resp. y) causes the suppression (resp. creation) of a tunnel. Surprisingly, it is still possible to characterize 3D simple points by local conditions which are only based on connectedness (see [1, 9, 28]), but this is no longer true in 4D.

In this article, we use a definition of simple points ([4], see also [8]) based on the collapse operation. Collapse is an elementary topology-preserving transformation which has been introduced by Whitehead [29] and plays an important role in combinatorial topology, it can be seen as a discrete analogue of a continuous deformation (a homotopy). Notice that this definition of simple points makes sense in any dimension.

We present new characterizations of 2D, 3D and 4D simple points based on the collapse (Th. 13, Th. 14), which lead to simple, greedy linear-time algorithms for simplicity checking. We also retrieve in our framework, a characterization of 4D simple points established by T.Y. Kong [19], and some previously proposed characterizations of 3D simple points [19, 1, 9, 28].

In order to prove these characterizations, we establish some *confluence properties* of the collapse (Th. 11, Th. 12). These properties do not hold in general due to the existence of "topological monsters" such as the Bing's house ([10], see also [25]) and the dunce hat [30]; we show that they are indeed true in the neighborhood of a point, when the dimension of the space is such that this neighborhood is not large enough to contain such counter-examples.

This work is settled in the framework of cubical complexes. Abstract (cubical) complexes have been promoted in particular by V. Kovalevsky [22] in order to provide a sound topological basis for image analysis. For instance, in this framework, we retrieve the main notions and results of digital topology, such as the notion of simple point.

1 Cubical Complexes

Intuitively, a cubical complex may be thought of as a set of elements having various dimensions (*e.g.* cubes, squares, edges, vertices) glued together according to certain rules. In this section, we recall briefly some basic definitions on complexes, see also [7, 5, 6] for more details. We consider here $d-$dimensional complexes, mainly with $0 \leq d \leq 4$.

Let S be a set. If T is a subset of S, we write $T \subseteq S$. We denote by $|S|$ the number of elements of S.

Let $\mathbb{Z}$ be the set of integers. We consider the families of sets $\mathbb{F}^1_0$, $\mathbb{F}^1_1$, such that $\mathbb{F}^1_0 = \{\{a\} \mid a \in \mathbb{Z}\}$, $\mathbb{F}^1_1 = \{\{a, a+1\} \mid a \in \mathbb{Z}\}$. A subset f of $\mathbb{Z}^d$, $d \geq 2$, which is the Cartesian product of exactly m elements of $\mathbb{F}^1_1$ and $(d-m)$ elements of $\mathbb{F}^1_0$ is called a *face* or an $m-$*face* of $\mathbb{Z}^d$, m is the *dimension of* f, we write $\dim(f) = m$.

Observe that any non-empty intersection of faces is a face. For example, the intersection of two $2-$faces A and B may be either a $2-$face (if $A = B$), a $1-$face, a $0-$face, or the empty set.

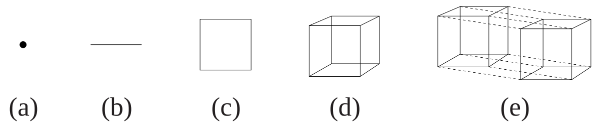


Figure 3. Graphical representations of: (a) a $0-$face, (b) a $1-$face, (c) a $2-$face, (d) a $3-$face, (e) a $4-$face.

We denote by $\mathbb{F}^d$ the set composed of all $m-$faces of $\mathbb{Z}^d$, with $0 \leq m \leq d$. An $m-$face of $\mathbb{Z}^d$ is called a *point* if $m = 0$, a *(unit) edge* if $m = 1$, a *(unit) square* if $m = 2$, a *(unit) cube* if $m = 3$, a *(unit) hypercube* if $m = 4$ (see Fig. 3).

Let f be a face in $\mathbb{F}^d$. We set $\hat{f} = \{g \in \mathbb{F}^d \mid g \subseteq f\}$ and $\hat{f}^* = \hat{f} \setminus \{f\}$.
Any $g \in \hat{f}$ is a *face of* f, and any $g \in \hat{f}^*$ is a *proper face of* f.
If X is a finite set of faces in $\mathbb{F}^d$, we write $X^- = \cup\{\hat{f} \mid f \in X\}$, X^- is the *closure of* X (see Fig. 4).

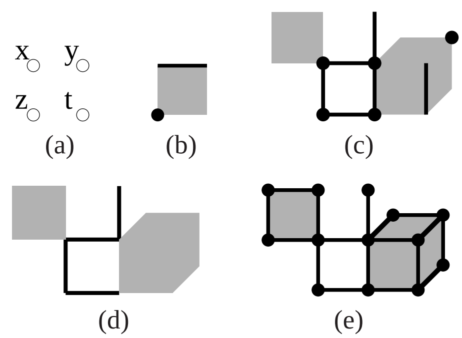


Figure 4. (a): Four points x, y, z, t. (b): A graphical representation of the set of faces $\{\{x, y, z, t\}, \{x, y\}, \{z\}\}$. (c): A set of faces X, which is not a complex. (d): The set X^+, composed by all facets of X. (e): The set X^-, *i.e.* the closure of X, which is a complex.

A set X of faces in $\mathbb{F}^d$ is a *cell* or an $m-$*cell* if there exists an $m-$face $f \in X$, such that $X = \hat{f}$. The *boundary of a cell* $\hat{f}$ is the set $\hat{f}^*$. For example, a $3-$cell is composed of 27 faces: a cube, six squares, twelve edges and eight points. Its boundary is composed of all these faces but the cube.

A finite set X of faces in $\mathbb{F}^d$ is a *complex (in* $\mathbb{F}^d$*)* if $X = X^-$. Any subset Y of a complex X which is also a complex is a *subcomplex of* X. If Y is a subcomplex of X, we write $Y \preceq X$. If X is a complex in $\mathbb{F}^d$, we also write $X \preceq \mathbb{F}^d$.

See in Fig. 4e an example of a complex, and in Fig. 4b,c,d examples of sets of faces which are not complexes. Also in Fig. 2 and Fig. 5, some complexes are represented. Notice that any cell is a complex.

Let $X \subseteq \mathbb{F}^d$, let $f \in X$ and $m = \dim(f)$. We say that f is a *facet of* X or an $m-$*facet of* X if there is no $g \in X$ such that $f \in \hat{g}^*$. We denote by X^+ the set composed of all facets of X (see Fig. 4).
If X is a complex, observe that in general, X^+ is not a complex, and that $[X^+]^- = X$.

Let $X \preceq \mathbb{F}^d$, $X \neq \emptyset$, the number $\dim(X) = \max\{\dim(f) \mid f \in X^+\}$ is the *dimension of* X. We say that X is an $m-$complex if $\dim(X) = m$.
We say that X is *pure* if, for each $f \in X^+$, we have $\dim(f) = \dim(X)$.
In Fig. 5, the complexes (a) and (f) are pure, while (b,c,d,e) are not.

Let $X \subseteq \mathbb{F}^d$ be a set of faces. A sequence $\pi = \langle f_0, \ldots, f_\ell \rangle$ of faces of X is a *path in* X *(from* f_0 *to* f_ℓ*)* if either f_i is a face of f_{i+1} or f_{i+1} is a face of f_i, for each $i \in \{0, \ldots, \ell - 1\}$.

Let $X \subseteq \mathbb{F}^d$. We say that X *is connected* if, for any two faces f, g in X, there is a path from f to g in X; otherwise we say that X *is disconnected*. We say that Y is a *(connected) component of* X if $Y \neq \emptyset$, $Y \subseteq X$, Y is connected and if Y is maximal for these properties (*i.e.*, we have $Z = Y$ whenever $Y \subseteq Z \subseteq X$ and Z is connected). Notice that the empty set is connected but has no connected component.

If X is an $m-$complex with $m \leq 1$, then X is also called a *graph* (see [16]). Examples of graphs can be seen in Fig. 12 and Fig. 13. Let X be a graph, and let $\pi = \langle f_0, \ldots, f_\ell \rangle$ be a path in X such that $\dim(f_0) = \dim(f_\ell) = 0$. The path π is said to be *closed* whenever $f_0 = f_\ell$, it is a *trivial path* whenever $\ell = 0$, it is said to be *elementary* if its faces are all distinct except that possibly $f_0 = f_\ell$. A graph which is constituted by the faces of a non-trivial elementary closed path is called a *cycle*. The graph X is *acyclic* if none of its subcomplexes is a cycle. A connected and acyclic graph is *a tree*.

2 Collapse and simple sets

Intuitively a subcomplex of a complex X is simple if its removal from X "does not change the topology of X". In this section we recall a definition of a simple subcomplex based on the operation of collapse [29, 16], which is a discrete analogue of a continuous deformation (a homotopy).

Let X be a complex in $\mathbb{F}^d$ and let $f \in X$. If there exists one face $g \in \hat{f}^*$ such that f is the only face of X which strictly includes g, then g is said to be *free for* X and the pair (f, g) is said to be a *free pair for* X. The complex, which is the closure of the set of all free faces for X, is called the *boundary of* X and is denoted by $Bd(X)$. Notice that, if (f, g) is a free pair, then we have necessarily $f \in X^+$ and $\dim(g) = \dim(f) - 1$.

Let X be a complex, and let (f, g) be a free pair for X. Let $m = \dim(f)$. The complex $X \setminus \{f, g\}$ is an *elementary collapse of* X, or an *elementary* $m-$*collapse of* X.
Let X, Y be two complexes. We say that X *collapses onto* Y if $Y = X$ or if there exists a *collapse sequence from* X *to* Y, *i.e.*, a sequence of complexes $\langle X_0, \ldots, X_\ell \rangle$ such that $X_0 = X$, $X_\ell = Y$, and X_i is an elementary collapse of X_{i-1}, for each $i \in \{1, \ldots, \ell\}$. If X collapses onto Y and Y is a complex made of a single point, we say that X *is collapsible*.

Fig. 5 illustrates a collapse sequence. Observe that, if X is a cell of any dimension, then X is collapsible. Also, a graph is a tree if and only if it is collapsible ([16]). Furthermore, it may easily be seen that the collapse operation preserves the number of connected components.

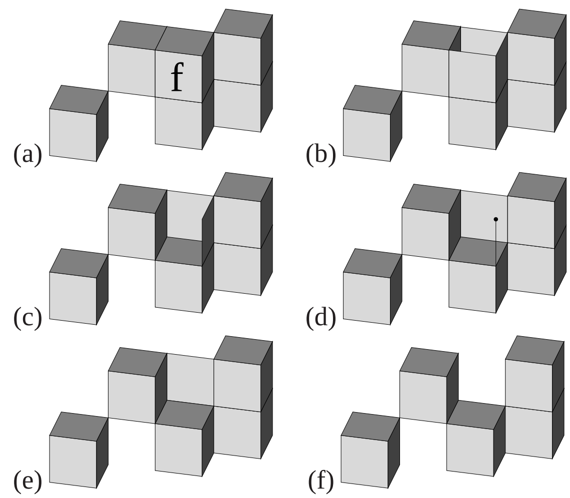


Figure 5. (a): a pure $3-$complex $X \preceq \mathbb{F}^3$, and a $3-$face $f \in X^+$. (f): a complex Y which is the detachment of $\hat{f}$ from X. (a-f): a collapse sequence from X to Y.

We say that the collapse sequence $\langle X_0, \ldots, X_\ell \rangle$ is *decreasing* if for any $i \in \{1, \ldots, \ell - 1\}$, we have $m \geq m'$ whenever X_i is an elementary $m-$collapse of X_{i-1} and X_{i+1} is an elementary $m'-$collapse of X_i. For example in Fig. 5, the collapse sequence $\langle a, b, c, d, e \rangle$ is decreasing, but $\langle a, b, c, d, e, f \rangle$ is not decreasing.

Let $\langle X_0, \ldots, X_\ell \rangle$ be a collapse sequence. If there exists $i \in \{1, \ldots, \ell - 1\}$ such that X_i is an elementary $m-$collapse of X_{i-1} and X_{i+1} is an elementary $m'-$collapse of X_i, with $m' > m$, then it may be seen that the sequence obtained by exchanging these two elementary collapse operations is still a collapse sequence from X_0 to X_ℓ. By induction, this proves the following property, which will be used later.

Proposition 1. *Let X,Y be two complexes. If X collapses onto Y, then there exists a decreasing collapse sequence from X to Y.*

Let X,Y be two complexes. Let Z be such that $X \cap Y \preceq Z \preceq Y$, and let $f,g \in Z \setminus X$. The pair (f,g) is a free pair for $X \cup Z$ if and only if (f,g) is a free pair for Z. Thus, by induction, we have the following property.

Proposition 2 ([3, 4])**.** *Let $X,Y \preceq \mathbb{F}^d$. The complex $X \cup Y$ collapses onto X if and only if Y collapses onto $X \cap Y$.*

The operation of detachment allows to remove a subset from a complex, while guaranteeing that the result is still a complex.

Definition 3 ([3, 4])**.** *Let $Y \subseteq X \preceq \mathbb{F}^d$. We set $X \oslash Y = (X^+ \setminus Y^+)^-$. The set $X \oslash Y$ is a complex which is the* detachment *of Y from X.*

In the following, we will be interested in the case where Y is a single cell. For example in Fig. 5, we see a complex X (a) containing a $3-$cell $\hat{f}$, and $X \oslash \hat{f}$ is depicted in (f).

Let us now recall here a definition of simplicity based on the collapse operation, which can be seen as a discrete counterpart of the one given by T.Y. Kong [19].

Definition 4 ([3, 4])**.** *Let $Y \subseteq X$; we say that Y is* simple for *X if X collapses onto $X \oslash Y$.*

The collapse sequence displayed in Fig. 5 (a-f) shows that the cell $\hat{f}$ (and the face f) is simple for the complex depicted in (a).

The notion of attachment, as introduced by T.Y. Kong [18, 19], leads to a local characterization of simple sets, which follows easily from Prop. 2.

Let $Y \preceq X \preceq \mathbb{F}^d$. The *attachment* of Y for X is the complex defined by $Att(Y,X) = Y \cap (X \oslash Y)$.

Proposition 5 ([3, 4])**.** *Let $Y \preceq X \preceq \mathbb{F}^d$. The complex Y is simple for X if and only if Y collapses onto $Att(Y,X)$.*

Fig. 6 shows the attachments of simple pixels a,b,c and non-simple pixels x,y,z,t of Fig. 1. We invite the reader to use these examples to illustrate Prop. 5.

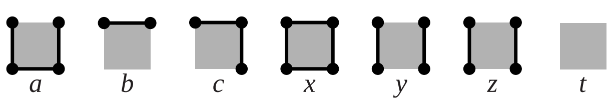


Figure 6. Attachments (in black) of simple pixels a,b,c and non-simple pixels x,y,z,t of Fig. 1.

Let us introduce informally the *Schlegel diagrams* as a graphical representation for visualizing the attachment of a cell. In Fig. 7a, the boundary of a $3-$cell $\hat{f}$ and its Schlegel diagram are depicted. The interest of this representation lies in the fact that a structure like $\hat{f}^*$ lying in the 3D space may be represented in the 2D plane. Notice that one $2-$face of the boundary, here the square $\{e,f,h,g\}$, is not represented like the other ones in the schlegel diagram, but we may consider that it is represented by the ouside space.

As an illustration of Prop. 5, Fig. 7b shows (both directly and by its Schlegel diagram) the attachment of $\hat{f}$ for the complex X of Fig. 5a, and we can easily verify that $\hat{f}$ collapses onto $Att(\hat{f},X)$. Also, Fig. 7c shows $Att(\hat{x},X)$ (see Fig. 2) and we can verify by Prop. 5 that x is not simple.

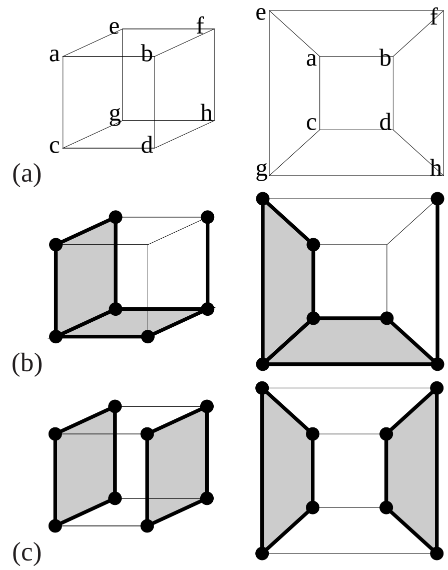


Figure 7. (a): The boundary of a $3-$cell and its Schlegel diagram. (b): The attachment of $\hat{f}$ for X (see Fig. 5a). (c): The attachment of $\hat{x}$ for X (see Fig. 2).

Representing 4D objects is not easy. To start with, let us consider Fig. 8a where a representation of the 3D complex X of Fig. 5a is given under the form of two horizontal cross-sections, each black dot representing a $3-$cell.

In a similar way, we may represent a 4D object by its "3D sections", as the object Y in Fig. 8b. Such an object may be thought of as a "time series of 3D objects". In Fig. 8b, each black dot represents a $4-$cell of the whole 4D complex Y.

Schlegel diagrams are particularly useful for representing the attachment of a 4D cell $\hat{f}$, whenever this attachment if not equal to $\hat{f}^*$. Fig. 9a shows the Schlegel diagram of the boundary of a $4-$cell (see Fig. 3e), where one of the eight $3-$faces is represented by the ouside space. Fig. 9b shows the Schlegel diagram of the attachment of the $4-$cell g in Y (see Fig. 8b). For example, the $3-$cell H represented in the center of the diagram is the intersection between the $4-$cell g and the $4-$cell h. Also, the $2-$cell I (resp. the $1-$cell

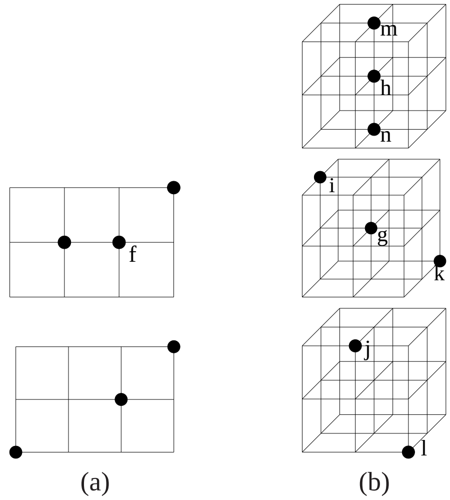


Figure 8. (a): An alternative representation of the 3D complex X of Fig. 5a. (b): A similar representation of a 4D complex Y.

J, the $1-$cell K, the $0-$cell L) is $g \cap i$ (resp. $g \cap j$, $g \cap k$, $g \cap l$). The two $2-$cells which are the intersections of g with, respectively, m and n, are both included in the $3-$cell H. Observe that the cell g is not simple (its attachment is not connected).

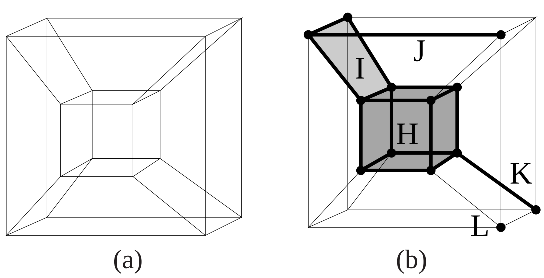


Figure 9. (a): The Schlegel diagram of the boundary of a $4-$cell. (b): The Schlegel diagram of the attachment of the $4-$cell g of Fig. 8b, which is not simple.

The following property easily follows from the definition of the boundary of a face, and may be checked on Fig. 9a.

Proposition 6. *Let f be a $4-$face. Then,*
i) any $2-$face of $\hat{f}^*$ is included in exactly two $3-$faces of $\hat{f}^*$; and
ii) any $1-$face of $\hat{f}^*$ is included in exactly three $3-$faces of $\hat{f}^*$; and
iii) any $1-$face of $\hat{f}^*$ is included in exactly three $2-$faces of $\hat{f}^*$.

3 Confluences

Let $X \preceq \mathbb{F}^d$. If f is a facet of X, then by Def. 4, $\hat{f}$ is simple if and only if X collapses onto $X \oslash \hat{f}$. From Prop. 5, we see that checking the simplicity of a cell $\hat{f}$ reduces to the search for a collapse sequence from $\hat{f}$ to $Att(\hat{f}, X)$. We will show in Sec. 4 that the huge number (especially in 4D) of possible such collapse sequences need not be exhaustively explored, thanks to the confluence properties (Th. 11 and Th. 12) introduced in this section.

Consider three complexes A, B, C. If A collapses onto C and A collapses onto B, then we know that A, B and C "have the same topology". If furthermore we have $C \preceq B \preceq A$, it is tempting to conjecture that B collapses onto C.

In the two-dimensional discrete plane $\mathbb{F}^2$, the above conjecture is true, for example any complex obtained by a collapse sequence from a full rectangle, collapses onto a point. We call it a confluence property. But quite surprisingly it does not hold in $\mathbb{F}^3$ (more generally in $\mathbb{F}^d, d \geq 3$), and this fact constitutes indeed one of the principal difficulties when dealing with certain global topological properties, such as the Poincaré conjecture. Classical counter-examples to this assertion are the Bing's house ([10], see also [25]) and the dunce hat ([30]).

In Fig. 10a, we see a classical (informal) representation of the Bing's house. The house has two rooms separated by a floor ; one can enter the lower room of the house by the chimney passing through the upper room, and vice-versa.

In Fig. 10b, we depict a Bing's house B which is a $2-$complex. For readability of the figure, only some of the $1-$faces and $2-$faces are displayed. This $2-$complex may be obtained by collapse from the $3-$complex depicted in Fig. 10c, which is composed of twenty-four $3-$cells. The dotted arrow suggests one half of a possible sequence of collapse operations, the other half being symmetrical to this one. The $2-$complex B contains no free face: we can verify that each $1-$face is contained in two or three $2-$faces.

From any $2-$complex, we may extract the graph composed by all the $1-$cells which are included in three or more $2-$cells. We call this graph the *signature of the* $2-$*complex*. In Fig. 10b, the signature of the Bing's house B is highlighted by a bold black line: it is composed of three connected cycles.

Fig. 11a depicts a triangulation of the dunce hat. Notice that the three sides of the biggest triangle (in bold) are identified, and that the different occurences of the point a are indeed representations of the same point (this remark also holds for points b and c). Notice also that only segments ab, bc and ca are included in three triangles, furthermore they form a cycle, which is the signature of the dunce hat.

In Fig. 11b, we show a realization of the dunce hat as a $2-$complex, which is very likely to be the smallest one which may be built in $\mathbb{F}^3$. For readability, only some

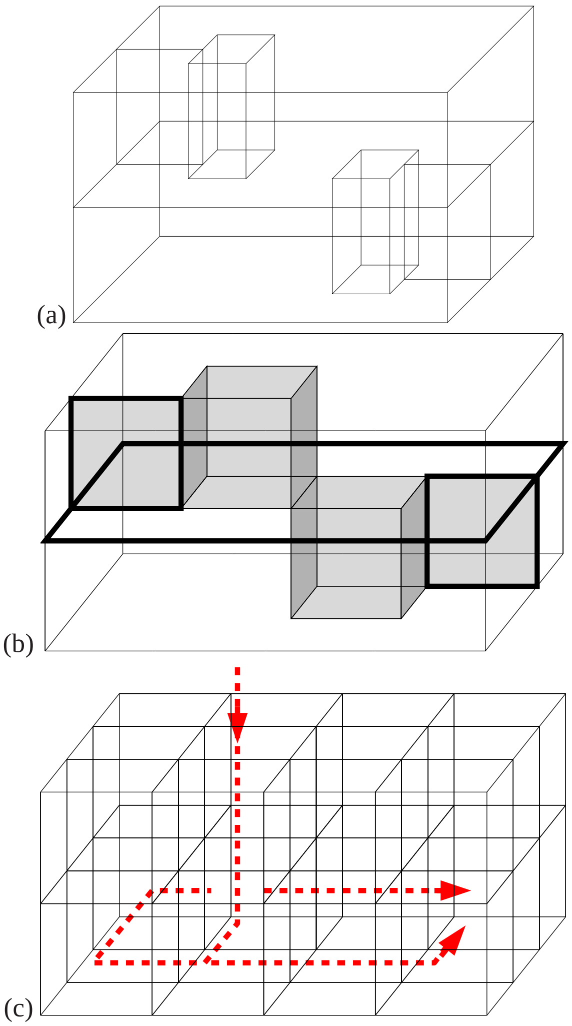


Figure 10. (a) The Bing's house with two rooms (classical representation). (b) A realization of a Bing's house as a 2−complex B. (c) A 3−complex made of 24 cubes. The arrows symbolize the order in which 3−collapse operations can be made in order to "carve" the lower room of the house. By performing a symmetrical operation for the upper room, we obtain the 2−complex in (b).

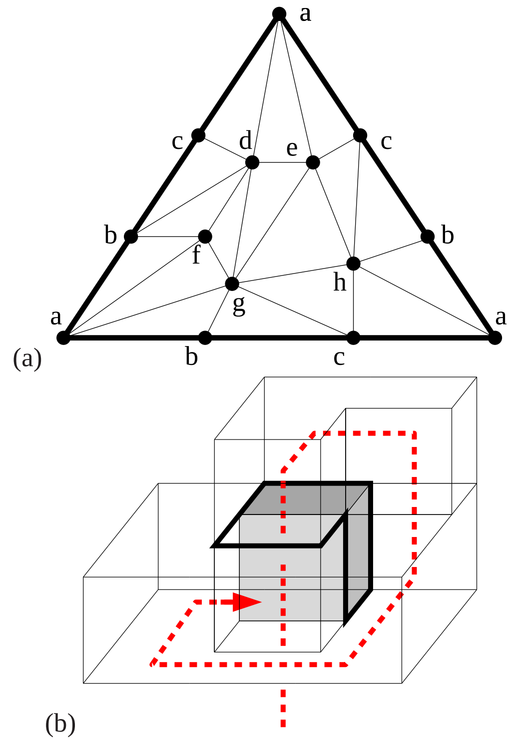


Figure 11. Dunce hats (see text).

1−faces and three 2−faces are displayed. The collapse sequence from a 3−complex to this 2−complex, composed of 12 elementary 3−collapse operations, is suggested by the dotted arrow. We may verify that the signature of this 2−complex is composed of the 1−cells highlighted by a bold black line: it is a cycle.

In this section we show that, in the boundary of a d−face with $d \leq 4$, there is "not enough room" to build such counter-examples, and thus some kinds of confluence properties hold.

We emphasize that for our purpose, it is sufficient to make a combinatorial proof for only one lemma (Lemma 7). Due to the high number of cases in dimension 4, we used a computer program for this proof. Notice that, however, it would not be possible to establish direcly, by exhaustive exploration of all possible configurations, the main properties proved in this paper (confluence properties and simple point characterizations): the number of possible configurations in the boundary of a four-dimensional face is 2^{80}.

Lemma 7. *Let f be a d−face with $d \in \{3,4\}$, and let X be a non-empty subcomplex of $\hat{f}^*$. Let us denote by $\overline{X}$ the complementary of X in $\hat{f}^*$. Suppose that* $\dim(X) = d-2$ *and that $\overline{X}$ is connected, then the two following statements hold:*
i) The complex X has at least one free $(d-3)$−face.
ii) If $d = 4$ and if X is pure, then the graph $Bd(X)$ is not

acyclic.

Proof. With the help of a computer program, we generated all the possible such subcomplexes of $\hat{f}^*$, and checked the property exhaustively. In the case $d = 4$, notice that $0-$ and $1-$facets of X play no role in the connectedness of $\overline{X}$, thus without loss of generality for proving statement i), we can suppose, as for ii), that X is a pure $2-$complex. The number of such complexes is 2^{24}. $\square$

Suppose that f is a $4-$face, then $\hat{f}^*$ is a $3-$complex. We observe that statements i) and ii) of Lemma 7 do not hold if, instead of being a subcomplex of the $3-$complex $\hat{f}^*$, X is a subcomplex of $\mathbb{F}^3$, due to the existence of counter-examples such as the Bing's house. Let B be a Bing's house which is a pure $2-$complex, we can see that B has no free $1-$face and $\overline{B}$ is connected, furthermore since $Bd(B) = \emptyset$, the graph $Bd(B)$ is acyclic.

We will also need the following result for the proofs of Prop. 9, Lemma 21 and Th. 15. We prove it here for the case of the boundary of a cell, but a more general property could be established in the framework of discrete manifolds (see [13]).

Proposition 8. *Let f be a $d-$face with $d \in \{2,3,4\}$, and let $Y \preceq X \preceq \hat{f}^*$ such that X collapses onto Y. Then, the sets $\overline{X} = \hat{f}^* \setminus X$ and $\overline{Y} = \hat{f}^* \setminus Y$ have the same number of connected components.*

Proof. It is sufficient to prove the proposition whenever $Y = X \setminus \{h,g\}$, with (h,g) being a free pair for X. We make the proof for $d = 4$, the other cases are similar and simpler. Let us call an $m-$path, a path in which each face has a dimension greater or equal to m. It may be seen that a subset Z of $\overline{X}$ is connected if and only if any two $3-$faces of Z are linked by a $2-$path. Let us denote by $|\mathcal{C}(Z)|$ the number of connected components of Z, thus we have $|\mathcal{C}(\overline{Y})| \neq |\mathcal{C}(\overline{X})|$ only if either h or g is a $2-$face.
Case 1: $\dim(g) = 2$. Hence, $\dim(h) = 3$. Since (h,g) is a free pair for X, hence $h \in X$, from Prop. 6i we deduce that g is included in exactly one $3-$face of $\overline{X}$, thus $|\mathcal{C}(\overline{Y})| = |\mathcal{C}(\overline{X})|$.
Case 2: $\dim(h) = 2$. Hence, $\dim(g) = 1$. Let A,B,C be the three $3-$faces of $\hat{f}^*$ which contain g (see Prop. 6ii), with $A \cap B = h$. Since g is free, these $3-$faces all belong to $\overline{X}$. Furthermore A and B are connected by the $2-$path $\langle A, A\cap C, C, C \cap B, B\rangle$ in $\overline{X}$. Thus, A and B are in the same connected component of $\overline{X}$, and $|\mathcal{C}(\overline{Y})| = |\mathcal{C}(\overline{X})|$. $\square$

We are now ready to introduce the confluence properties.

Proposition 9 (Downstream confluence)**.** *Let f be a $d-$face with $d \in \{2,3,4\}$, and let $A,B \preceq \hat{f}^*$ such that $B \preceq A$, A collapses onto B, and A is collapsible. Then, B is collapsible.*

Proof. We make the proof for $d = 4$, the other cases are similar and simpler. We only have to prove that B either is a point, or has a free face. If the latter is true, then by collapsing this face we obtain a subcomplex B' of A strictly included in B, which is such that A collapses onto B' (by transitivity). The result follows by induction on the size of B.
Let us consider the following (mutually exclusive) cases.
- $\dim(B) = 3$: Since A is collapsible, we have $A \neq \hat{f}^*$ and $B \neq \hat{f}^*$. Since B has at least one $3-$face, it can be easily seen that there exists a $2-$face of B which is a free face: since there are only eight $3-$faces in $\hat{f}^*$, this fact may be checked by enumeration (this property may also be derived from general properties of manifolds, see [13]).
- $\dim(B) = 2$: From Prop. 8 and our hypotheses, $\overline{B}$ is connected, thus by Lemma 7i, B has at least one free $1-$face.
- $\dim(B) = 1$: In other words, B is a graph. The hypotheses imply that B is indeed a connected and acyclic graph, *i.e.*, a tree. Since $\dim(B) = 1$, B cannot be a point, then it has at least one free $0-$face ([16]).
- $\dim(B) = 0$: In other words, B is a set of points. The hypotheses, and the fact that collapse preserves the number of connected components, imply that B is indeed a single point. $\square$

Prop. 20, Lemma 21 and Lemma 22, which may be found in the appendix, are needed in addition to Prop. 9 for the proof of Prop. 10.

Proposition 10 (Upstream confluence)**.** *Let f be a $d-$face with $d \in \{2,3,4\}$, and let $A,B \preceq \hat{f}^*$ such that $B \preceq A$, A is collapsible, and B is collapsible. Then, A collapses onto B.*

Proof. Let $k = |A|$, the property is trivially true when $k = 1$. Suppose now that $k > 1$, and suppose that the property holds for any complexes A', B' verifying the hypotheses of the theorem, whenever $k' < k$ (with $k' = |A'|$). From Lemma 21 and Lemma 22, there exists a pair of faces (h,g) such that (h,g) is free for A and either (h,g) is free for B or $\{h,g\} \cap B = \emptyset$.
Case 1: $\{h,g\} \cap B = \emptyset$. We set $A' = A \setminus \{h,g\}$, we have obviously $B \preceq A'$. By Prop. 9, A' is collapsible, furthermore $k' < k$. By the recurrence hypothesis, we deduce that A' collapses onto B, thus A collapses onto B.
Case 2: (h,g) is free for B. We set $A' = A \setminus \{h,g\}$, and $B' = B \setminus \{h,g\}$, we have obviously $B' \preceq A'$. By Prop. 9, both A' and B' are collapsible, furthermore $k' < k$. By the recurrence hypothesis, we deduce that A' collapses onto B'. Furthermore, it can easily be seen that any collapse sequence from A' to B' induces a collapse sequence from A to B (by removing the same pairs in the same order). $\square$

Th. 11 summarizes Prop. 9 and Prop. 10.

Theorem 11. *Let f be a $d-$face with $d \in \{2,3,4\}$, let $A,B \preceq \hat{f}^*$ such that $B \preceq A$, and A is collapsible. Then, B is collapsible if and only if A collapses onto B.*

The following theorem may be easily derived from Th. 11 and the fact that $\hat{f}$ is collapsible, its proof is left

to the reader.

Theorem 12. *Let f be a $d-$face with $d \in \{2,3,4\}$, and let $C,D \preceq \hat{f}^*$ such that $D \preceq C$, and $\hat{f}$ collapses onto D. Then, $\hat{f}$ collapses onto C if and only if C collapses onto D.*

4 New characterizations of simple cells

In the image processing literature, a (binary) digital image is often considered as a set of pixels in 2D or voxels in 3D. A pixel is an elementary square and a voxel is an elementary cube, thus an easy correspondance can be made between this classical view and the framework of cubical complexes.

If $X \preceq \mathbb{F}^d$ and if X is a pure $d-$complex, then we write $X \sqsubseteq \mathbb{F}^d$. In other words, $X \sqsubseteq \mathbb{F}^d$ means that X^+ is a set composed of $d-$faces (*e.g.*, pixels in 2D or voxels in 3D).

Notice that, if $X \sqsubseteq \mathbb{F}^d$ and if $\hat{f}$ is a $d-$cell of X, then $X \oslash \hat{f} \sqsubseteq \mathbb{F}^d$. There is indeed an equivalence between the operation on complexes which consists of removing (by detachment) a simple $d-$cell, and the removal of a 8-simple (resp. 26-simple, 80-simple) point in the framework of 2D (resp. 3D, 4D) digital topology (see [18, 19, 7, 5]).

From Prop. 5 and Th. 12, we have the following characterization of a simple cell, which does only depend on the status of the faces which are in the cell.

Theorem 13. *Let $X \sqsubseteq \mathbb{F}^d$, with $d \in \{2,3,4\}$. Let f be a facet of X, and let $A = Att(\hat{f},X)$. The two following statements hold:*
i) The cell $\hat{f}$ is simple for X if and only if $\hat{f}$ collapses onto A.
ii) If there exists a complex Z such that $A \preceq Z \preceq \hat{f}$, $\hat{f}$ collapses onto Z and Z does not collapse onto A, then $\hat{f}$ is not simple for X.

Now, thanks to Th. 13, if we want to check whether a cell $\hat{f}$ is simple or not, it is sufficient to apply the following greedy algorithm.

Algorithm $\mathcal{A}_1$: *Set $Z = \hat{f}$; Do*
Select any free pair (h,g) in $Z \setminus A$; set Z to $Z \setminus \{h,g\}$;
Continue until either $Z = A$ (answer yes) or no such pair is found (answer no).

If this algorithm returns "yes", then obviously $\hat{f}$ collapses onto A and by Th. 13i, $\hat{f}$ is simple. In the other case, by Th. 13ii, $\hat{f}$ is not simple.

By Th. 13 and Th. 11, we derive a second characterization which leads straightforwardly to a second greedy algorithm $\mathcal{A}_2$ for checking simplicity.

Theorem 14. *Let $X \sqsubseteq \mathbb{F}^d$, with $d \in \{2,3,4\}$. Let f be a facet of X, and let $A = Att(\hat{f},X)$. The two following statements hold:*
i) The cell $\hat{f}$ is simple for X if and only if A is collapsible.
ii) If there exists a complex Z such that A collapses onto Z and Z is not collapsible, then $\hat{f}$ is not simple for X.

Both algorithms may be implemented to run in linear time with respect to the number of elements in the attachment of a cell (Remark 16 will give some elements which support this claim).

Thanks to Th. 14 and the previous properties, we can also retrieve a characterization of simple cells proved by T.Y. Kong in [19], where arguments based on the continuous framework and several combinatorial lemmas were used. In contrast, our new proof is purely discrete and its combinatorial part is reduced to Lemma 7.

Let X be a complex in $\mathbb{F}^4$, and let us denote by n_i the number of $i-$faces of X, $i = 0,\ldots,4$. The *Euler characteristic* of X, written $\chi(X)$, is defined by $\chi(X) = n_0 - n_1 + n_2 - n_3 + n_4$. The Euler characteristic is a well-known topological invariant; in particular, it can be easily seen that collapse preserves it.

Theorem 15 (adapted from [19], theorem 9)**.** *Let $X \sqsubseteq \mathbb{F}^d$, with $d \in \{2,3,4\}$, let f be a facet of X, and let $A = Att(\hat{f},X)$. The facet f is simple for X if and only if the three following statements are true:*
i) A has exactly one connected component, and
ii) $\hat{f}^* \setminus A$ has exactly one connected component, and
iii) $\chi(A) = 1$.

Proof. Suppose that f is simple for X. By Th. 14, A is collapsible. Since collapse preserves the number of connected components we deduce i), and by Prop. 8 we deduce ii). Furthermore the Euler characteristic of a point is equal to 1, and collapse preserves the Euler characteristic, hence iii).
Conversely, suppose that f verifies i), ii) and iii). One and only one among the following cases occurs.
• $\dim(A) \leq 1$: In other words, A is a graph. From i) and iii), we deduce that A is a connected and acyclic graph, *i.e.*, a tree, and thus A is collapsible ([16]).
• $\dim(A) = 2$ and $d = 4$: If $d = 4$, by Lemma 7i, condition ii) implies that A has at least one free pair (h,g) and thus A collapses onto $A' = A \setminus \{h,g\}$. From the properties of collapse, we see that A' also verifies i), ii) and iii). If $\dim(A') < 2$, we deduce the result from the preceding case, otherwise the result comes by induction on the number of $2-$faces.
• $\dim(A) = 3$ and $d = 4$ (resp. $\dim(A) = 2$ and $d = 3$) : We know from ii) that $A \neq \hat{f}^*$. Since A has at least one $3-$face (resp. $2-$face), it can be easily seen that A has at least a free $2-$face (resp. $1-$face), see the proof of Prop. 9. Thus, similarly to the previous case, the result follows by induction. □

Remark 16. *This characterization also induces a linear-time algorithm for simplicity checking. Nevertheless, observe that this algorithm (let us call it $\mathcal{B}$) is composed of*

three steps: one for computing the Euler characteristic of the attachment, and two for extracting connected components. To extract connected components in linear time, one may classically apply a breadth-first exploration strategy. The same strategy may also be used to implement algorithms $\mathcal{A}_1$ and $\mathcal{A}_2$, thus in terms of number of operations, both $\mathcal{A}_1$ and $\mathcal{A}_2$ are comparable to one of the steps of $\mathcal{B}$.

Let us also mention another definition of simple points based on homology ([24], see also [21]). In this context, checking whether a point p is simple or not amounts to verify that all the homology groups of the neighborhood (or attachment) of p are trivial. However, computing homology groups requires a computational effort which is much greater than the one needed by algorithms $\mathcal{A}_1$ and $\mathcal{A}_2$.

In the case $d = 3$, we retrieve well-known characterizations of simple points (see [18, 1, 9, 28]), using only two conditions among the three ones of Th. 15. Of course, these characterizations also hold for dimension 2.

Theorem 17. *Let $X \sqsubseteq \mathbb{F}^3$, let f be a facet of X, and let $A = Att(\hat{f}, X)$. The facet f is simple for X if and only if statement i) and either statement ii) or statement iii) of Th. 15 hold.*

Proof. If i) and iii) hold, then since ii) is not used in the proof of Th. 15 for the 3D case, we are done. Suppose now that i) and ii) hold. The case $\dim(A) = 2$ is treated in the proof of Th. 15, suppose that $\dim(A) = 1$. From ii) and Lemma 7i, we deduce that A has at least a free pair (h, g). Let $A' = A \setminus \{h, g\}$, we can see that $\hat{f}^* \setminus A'$ is also connected. Thus by induction on the number of $1-$faces, A collapses onto a $0-$complex. By i), this $0-$complex is necessarily reduced to a single point. $\square$

5 Higher dimensions

Indeed, the results of this paper hold for any dimension strictly lower than a certain dimension D, which is the lowest dimension such that a counter-example like the Bing's house or the dunce hat may be built inside the boundary of a $D-$face. From Th. 11 and Th. 12, we know that $D > 4$. The notion of lump defined below helps us to formalize the problem that we study in this section.

Definition 18. *Let f be a $d-$face, with $d \in \mathbb{N}$, and let $X \preceq \hat{f}$. The complex X is a* lump (by collapse) *if $\hat{f}$ collapses onto X and X is not collapsible.*
We say that f is lump-free *if no subcomplex of $\hat{f}$ is a lump.*

Realizations of the Bing's house or the dunce hat as $2-$complexes (see Fig. 10b and Fig. 11b) are examples of complexes which are not collapsible and which may be obtained by collapse from a cuboid in $\mathbb{F}^3$, thus the existence of lumps in a face of dimension 4 and higher may be conjectured. On the other hand, from Prop. 9, we know that $2-$faces, $3-$faces and $4-$faces are lump-free.

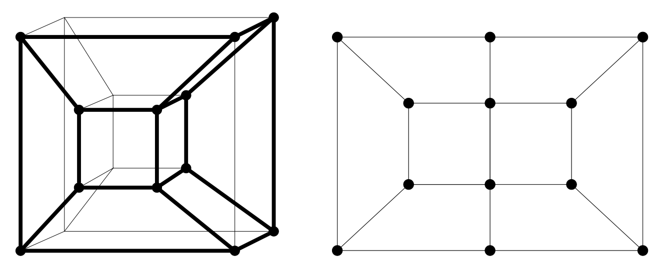
If a face of dimension D is not lump-free, it may be seen that the main theorems of this paper cannot be extended to dimension D. Let us consider for example the case of Th. 13, and take $X \sqsubseteq \mathbb{F}^D$ and a simple $D-$face x of X such that $Att(x, X)$ is a point. The existence of a lump contradicts the extension of Th. 13ii. Consider now the case of Th. 14, and take $X \sqsubseteq \mathbb{F}^D$ and a simple $D-$face x of X such that $Att(x, X)$ is a lump. By definition, the face x is simple but its attachment is not collapsible, a contradiction with the extension of Th. 14i.

The aim of this section is to answer the question: what is the highest dimension d such that a $d-$face is lump-free ?

Dimensions 6 and higher

It is in fact possible to build a Bing's house in $\hat{f}^*$, with f being a $6-$face (or a face of higher dimension). We give an informal description of this construction.

Let us consider the $1-$subcomplex of the boundary of a $4-$face, which is depicted in Fig. 12a.



(a) (b)

Figure 12. (a): A $1-$subcomplex of the boundary of a $4-$face. (b): Another view of this complex.

A $(d+1)-$face is obtained by the product of a $d-$face and a $1-$face (an operation on complexes directly derived from the Cartesian product operation). Let f be a $d-$face, let g be a $(d+1)-$face and let h be a $(d+2)-$face, if X is a subcomplex of $\hat{f}$ then in $\hat{g}$ we can embed two "independent copies" of X, and in $\hat{h}$ we can embed four independent copies of X (see Fig. 13 an example with $d = 2$).

Starting from the $4-$face of Fig. 12a, we can thus obtain by two product operations a $6-$face containing four independent copies of the $1-$complex depicted in Fig. 12b. Keeping only three of these copies, we can add them $2-$faces in order to obtain the $2-$complex sketched in Fig. 14 (a Bing's house).

Dimension 5

Such a construction is not feasible in 5D, thus we tried another strategy in order to find out whether there exists a

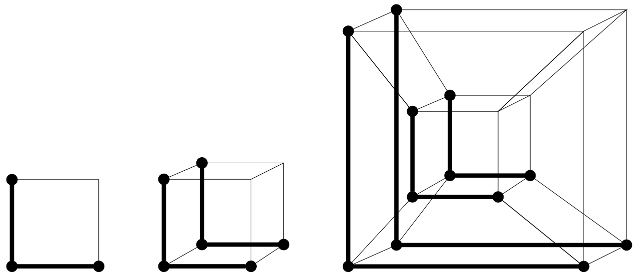


Figure 13. Illustration of the product operation.

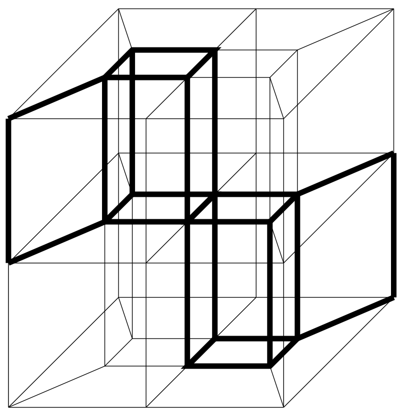


Figure 14. Sketch of a Bing's house in the boundary of a $6-$face (the two chimneys and the two walls are highlighted).

lump or not in the boundary of a $5-$face f.

We made a computer program which generates random collapse sequences starting from $\hat{f}$ and ending when no free face can be found, with the hope that one of these sequences will eventually terminate with a complex which is not reduced to a point. Such a complex must be a lump.

Surprisingly, this happens rather often (about one time every 50,000 trials, to compare with the gigantic number of possible collapse sequences, which is far beyond the possibility of an exhaustive exploration).

The shortest such collapse sequence that we found is made of 43 elementary collapse operations, and results in a pure $2-$complex having 47 facets (squares). This collapse sequence has then been checked "by hand".

The smallest lump that we found by this way is a pure $2-$complex X_{105} having 29 squares, 52 edges and 24 points. Unfortunately, it is very difficult to visualize such a complex object which lies in a $5-$dimensional space. Nevertheless, we can easily visualize its signature, which is depicted in Fig. 15a. Remarkably, the signature of X_{105} has the same structure (a cycle connected to a $1-$cell) as the signature of a variant of the dunce hat, displayed in Fig. 15b. It may be seen that there exists a sequence of one inverse elementary collapse and three elementary collapses from this variant to the dunce hat (Fig. 11a): $\langle +(daef, dae), -(daef, fde), -(daf, df), -(eaf, ef) \rangle$.

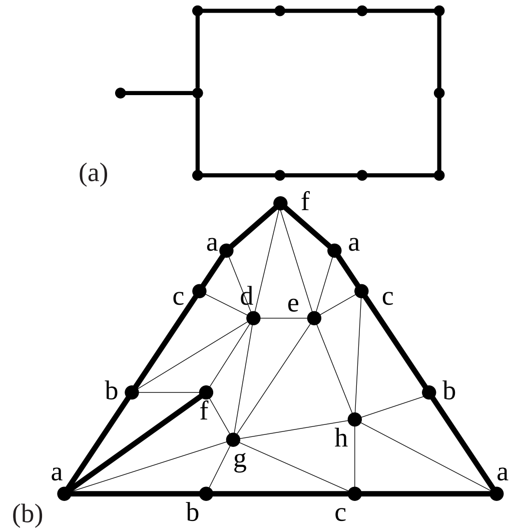


Figure 15. (a): The signature of X_{105}. (b): A variant of the dunce hat (triangulated).

Thanks to Th. 11 and from the preceding observations, we can conclude this section by the following theorem.

Theorem 19. *A face if lump-free if and only if its dimension is not strictly greater than* 4.

Conclusion

The new characterizations of simple points that we proved in this paper lead to simple and efficient algorithms for checking simplicity. In 2D and 3D, configurations of simple and non-simple points may be stored in a look-up table, but in 4D this is clearly impossible (there are 2^{80} possible configurations), thus such algorithms may be of practical interest. On the theoretical point of view, we proved these characterizations on the basis of new confluence properties, which turn out to be also keystones of a set of new results linking minimal non-simple sets [26], P-simple points [2] and critical kernels [3, 4], to appear in another article [11]. We also proved (Th. 19) that these characterizations and confluence properties do not hold beyond dimension 4.

Appendix

Proposition 20. *Let f be a $4-$face. If X is a pure $3-$dimensional subcomplex of $\hat{f}^*$, then the complex $Bd(X)$ has no free $1-$face.*

Proof. Let $k = |X^+|$, if $k = 1$ then the property is obvious. Suppose now that $k > 1$, and that the property holds for any $3-$subcomplex Y of $\hat{f}^*$ such that $|Y^+| < k$. Let $x \in |X^+|$, and let $Y = X \oslash x$. By the recurrence hypothesis, $Bd(Y)$ has no free $1-$face. If $\dim(Y \cap \hat{x}) < 2$ then it may be easily seen that $Bd(X)$ has no free $1-$face. Suppose now that $\dim(Y \cap \hat{x}) = 2$ and let h be a $2-$face in $Y \cap \hat{x}$. From Prop. 6i, we can see that h is free for Y. We also see that h is not free for X since it belongs to two $3-$cells of X, namely $\hat{x}$ and a $3-$cell $\hat{y}$ in Y. Any $1-$face of $Bd(Y)$ which is not in $\hat{h}$ is obviously not free for $Bd(X)$, let us consider a $1-$face g in $\hat{h}$. From Prop. 6ii and Prop. 6iii, g belongs to $\hat{x}$, $\hat{y}$ and $\hat{z}$ where z is a $3-$face of $\hat{f}^*$ distinct from x and y, and g also belongs to $\hat{h} = \hat{x} \cap \hat{y}$, $\hat{h'} = \hat{y} \cap \hat{z}$, and $\hat{h''} = \hat{z} \cap \hat{x}$. If $z \notin X$ then both h' and h'' are free for X, and if $z \in X$ then neither h, h' nor h'' is free for X, thus in all cases, g is not free for $Bd(X)$. $\square$

Lemma 21. *Let f be a $d-$face with $d \in \{2,3,4\}$, and let $A, B \preceq \hat{f}^*$ such that $B \preceq A$, B is collapsible, A is collapsible and* $\dim(B) < \dim(A)$. *Then, there exists $h, g \in A \setminus B$ such that* $\dim(h) = \dim(A)$ *and (h,g) is free for A.*

Proof. We make the proof for $d = 4$, the other cases are similar and simpler. Let $m = \dim(A)$, we have $m < d$. If $\dim(B) < m - 1$ then by Prop. 1 the proof is immediate, suppose from now that $\dim(B) = m - 1$. The case $m = 1$ is trivial.
Case $m = 2$: hence $\dim(B) = 1$, which means that B is a graph. The hypotheses imply that B is indeed a connected and acyclic graph, *i.e.*, a tree. Let A_2 be the subcomplex of A such that A_2^+ is the set of all the $2-$faces of A. Obviously A_2 is a pure $2-$dimensional subcomplex of $\hat{f}^*$, and since A is collapsible, $\overline{A}$ is connected (by Prop. 8), hence $\overline{A_2}$ is connected. From Lemma 7ii, we deduce that $Bd(A_2)$ is not acyclic. Thus, since B is a tree, B cannot contain $Bd(A_2)$, and there must exist a $1-$face g in $Bd(A_2) \setminus B$ and a $2-$face h in A (and not in B, since $\dim(B) < 2$) such that (h,g) is free for A.
Case $m = 3$. Let A_3 be the subcomplex of A such that A_3^+ is the set of all the $3-$faces of A. From Prop. 20 and Lemma 7i, we deduce that $\overline{Bd(A_3)}$ is disconnected. Thus, since B is collapsible, $\overline{B}$ is connected (by Prop. 8), and B cannot contain $Bd(A_3)$ (because $\dim(B) = 2$ and the number of connected components of $\overline{Bd(A_3)}$ does not change if $k-$faces (with $k \leq 2$) are added to $Bd(A_3)$). We conclude that there must exist a $2-$face g in $Bd(A_3) \setminus B$ and a $3-$face h in A such that (h,g) is free for A. $\square$

Lemma 22. *Let f be a $d-$face with $d \in \{2,3,4\}$, and let $A, B \preceq \hat{f}^*$ such that $B \preceq A$, B is collapsible, A is collapsible and* $\dim(B) = \dim(A)$. *Then, there exists h, g in A such that (h,g) is free for A, and either (h,g) is free for B or $\{h,g\} \cap B = \emptyset$.*

Proof. Let $m = \dim(B) = \dim(A)$. Since B is collapsible, by Prop. 1 we can deduce that B collapses onto a complex B', where $\dim(B') = m - 1$, B' contains all the $(m-1)-$facets of B, and B' is collapsible. Knowing that $B' \preceq A$, B' is collapsible, A is collapsible and $\dim(B') < \dim(A)$, by Lemma 21 we deduce that A has a free pair (h,g) such that $h \notin B'$, $g \notin B'$ and $\dim(h) = \dim(A)$. Since $g \notin B'$, g is not a $(m-1)-$facet of B. If $h \in B$ (hence $g \in B$) then, since (h,g) is free for A, we can see that (h,g) is also free for B, and we are done. Now if $h \notin B$, since h is the only $m-$face of A which strictly includes g, we see that if $g \in B$ then g would be a $(m-1)-$facet of B: a contradiction. Hence, $\{h,g\} \cap B = \emptyset$. $\square$

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