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Beta-expansions for cubic Pisot numbers

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Abstract. Real numbers can be represented in an arbitrary base $\beta > 1$ using the transformation $T_\beta : x \to \beta x \pmod{1}$ of the unit interval; any real number $x \in [0, 1]$ is then expanded into $d_\beta(x) = (x_i)_{i \geq 1}$ where $x_i = [\beta T_\beta^{i-1}(x)]$. The closure of the set of the expansions of real numbers of $[0, 1]$ is a subshift of $\{a \in \mathbb{N} \mid a < \beta\}$, called the beta-shift. This dynamical system is characterized by the beta-expansion of 1; in particular, it is of finite type if and only if $d_\beta(1)$ is finite; $\beta$ is then called a simple beta-number.

We first compute the beta-expansion of 1 for any cubic Pisot number. Next we show that cubic simple beta-numbers are Pisot numbers.

Introduction

Representations of real numbers in an arbitrary base $\beta > 1$, called beta-expansions, have been introduced by Rényi ([14]). They arise from the orbits of the piecewise-monotone transformation of the unit interval : $T_\beta : x \to \beta x \pmod{1}$. Such transformations were extensively studied in ergodic theory ([13]).

More precisely, any real number $x \in [0, 1]$ is expanded into $d_\beta(x) = (x_i)_{i \geq 1}$ where $x_i = [\beta T_\beta^{i-1}(x)]$. The nonnegative integers $d_i$ are elements of the digit alphabet $A = \{a \in \mathbb{N} \mid a < \beta\}$. These representations generalize standard representations in an integral base to a real base; indeed the beta-expansion of any real number of $[0, 1]$ can equivalently be obtained by the greedy algorithm. Only the beta-expansion of 1 differs.

Properties of beta-expansions are strongly related to symbolic dynamics ([4]). The closure of the set of infinite sequences, appearing as beta-expansions of numbers of the interval $[0, 1]$, is a dynamical system, that is, a closed shift-invariant subset of $A^\mathbb{N}$, called the beta-shift.

An important property of the beta-shift is that its nature is entirely determined, in a combinatorial manner, by the beta-expansion of 1: the beta-shift is sofic, that is to say the set of its finite factors is recognized by a finite automaton, if and only the beta-expansion of 1 is eventually periodic ([3]); it is of finite type, that is to say the set of its finite factors is defined by forbidding a finite set of words, if and only if the beta-expansion of 1 is finite ([12]).

When the beta-expansion of 1 is eventually periodic, $\beta$ is called a beta-number and when the beta-expansion of 1 is finite, $\beta$ is said to be a simple beta-number.
The eventually periodic beta-expansions were extensively studied by Bertrand ([3]) and by Schmidt ([15]). In particular, it is known that Pisot numbers are beta-numbers. Concerning Salem numbers, we only know that if $\beta$ is a Salem number of degree 4, then the beta-expansion of 1 is eventually periodic ([5]). It is conjectured that Salem numbers of degree 6 are still beta-numbers, but not all Salem numbers of degree 8 ([7]).

The domain of the Galois conjugates of all beta-numbers was also investigated independently by Solomyak ([16]) and by Flatto, Lagarias and Poonen ([8]).

For a general presentation of the beta-shift one can refer to [9].

In the following, we summarize properties of beta-numbers. We compute the beta-expansion of 1 for any cubic Pisot number and we establish a characterization of cubic simple beta-numbers, showing that they are Pisot numbers.

A very close problem, seen from the point of view of numeration systems, was studied by Akiyama ([1]). He showed that in the cubic case, the real numbers of the set $\mathbb{N}[\beta^{-1}]$ have a finite beta-expansion if and only $\beta$ is a Pisot unit and 1 has a finite beta-expansion. This finiteness problem is equivalent to a problem of fractal tiling generated by Pisot numbers.

1 Beta-numbers

Real numbers can be represented in an arbitrary base $\beta > 1$ using the transformation $T_\beta : x \to \beta x \pmod{1}$ of the unit interval; any real number $x \in [0, 1]$ is then expanded into $d_\beta(x) = \{x_i\}_{i \geq 1}$ where $x_i = |\beta T_\beta^{i-1}(x)|$. When a beta-expansion is of the form $w\ell^2$, the expansion is said to be eventually periodic. If a representation ends with infinitely many zeroes, like $w0^\infty$, it is said to be finite and the ending zeroes are omitted.

Let us denote by $S_\beta$ the closure of all beta-expansions of real numbers of $[0, 1]$ and by $\sigma$ the (one-sided) shift defined by $\sigma((x_i)_{i \geq 1}) = (x_{i+1})_{i \geq 1}$. The set $S_\beta$ endowed with the shift is called the beta-shift, it is a subshift of $A^\mathbb{N}$, $A$ being the digit set, i.e., $A = \{a \in \mathbb{N} \mid a < \beta\}$.

An important property ([13]) of the beta-shift $S_\beta$ is that its nature is entirely determined by $d_\beta(1)$ the beta-expansion of 1. Indeed, setting $d^*(1) = d_\beta(1)$ if $d_\beta(1)$ is infinite and $d^*(1) = (d_1 d_2 \cdots d_{n-1} d_n - 1)^\omega$ if $d_\beta(1) = d_1 d_2 \cdots d_{n-1} d_n$, a sequence $x$ of nonnegative integers belongs to $S_\beta$ if and only if it satisfies the following lexicographical order conditions: $\forall p \geq 0, \quad |\sigma^p(x) - d^*(1)| < d^*(1)$.

Recall that the beta-expansion of 1 also can be characterized ([13]) by lexicographical order conditions: let $d = (d_i)_{i \geq 1}$ be a sequence of nonnegative integers different from $10^\infty$, such that $\sum_{i \geq 1} d_i \beta^{-i} = 1$, with $d_1 \geq 1$ and for $i \geq 2$, $d_i \leq d_1$, then $d$ is the beta-expansion of 1 if and only if for all $p \geq 1, \sigma^p(d) < d$.

We recall that an algebraic integer $\beta$ strictly greater than 1 is called a Perron number if all its Galois conjugates have modulus strictly less than $\beta$, a Pisot number if all its Galois conjugates have modulus strictly less than 1, and a Salem number if all its conjugates are less than 1 in modulus and at least one conjugate has modulus 1.
Let $\beta$ be a beta-number. Denote by $d_\beta(1) = d_1 \ldots d_n(d_{n+1} \ldots d_{n+p})^n$, where $n$ and $p$ are chosen minimal, the beta-expansion of 1. Then the adjacency matrix $M_\beta$ of the finite automaton recognizing the set of its finite factors (Fig.1) is a primitive (i.e., its associated graph is strongly connected and the lengths of its cycles are relatively prime) nonnegative integral matrix whose spectral radius is $\beta$; so, from the Perron-Frobenius theorem, $\beta$ is a Perron number.

![Diagram](image)

Fig. 1. Automaton recognizing the set of the finite factors of $S_\beta$

The characteristic polynomial of $M_\beta$

$$P(X) = X^{n+p} - \sum_{i=1}^{n+p} d_i X^{n+p-i} - X^n + \sum_{i=1}^n d_i X^{n-i}$$

is called, following the terminology introduced by Hollander ([11]), the associated beta-polynomial.

As $P$ is a multiple of the minimal polynomial $M_\beta$ of $\beta$, $P(0) = d_{n+p} - d_n$ is a multiple of $|M_\beta(0)| = \prod |\beta_i|$ where $\beta_i$ runs over the set of algebraic conjugates of $\beta$. So, we get that $|\prod |\beta_i|$ has to be smaller than $|\beta|$.

As a consequence, in the quadratic case, the only beta-numbers are the Pisot numbers. Conversely, it is known that if $\beta$ is a Pisot number then $\beta$ is a beta-number ([2]). An important gap remains between Pisot and Perron numbers.

Example 1. The quadratic number $\beta = (1 + \sqrt{3})/2$ is not a beta-number since $M_\beta(X) = X^2 - X - 3$ and $M_\beta(0) > |\beta|$. 
Let \( \beta \) be the Pisot number \((3 + \sqrt{5})/2\), then \( \beta \) is a beta-number and \( d_\beta = 21^\omega \).

Let \( \beta \) be the golden ratio \((1 + \sqrt{5})/2\), then \( \beta \) is a simple beta-number and \( d_\beta(1) = 11 \).

On the other hand, the domain of the Galois conjugates of beta-numbers was studied by Solomyak ([16]) and independently by Flatto, Lagarias and Poonen ([8]). They showed in particular that if the beta-expansion of 1 is eventually periodic then the Galois conjugates of \( \beta \) have modulus less than the golden ratio \((1 + \sqrt{5})/2\). It was already known (see [9]) that \( \beta \) cannot have a Galois conjugate greater than 1.

Solomyak ([16]) proved that the topological closure of conjugates of beta-numbers and the one of conjugates of simple beta-numbers are the same. However, there is an important difference between the conjugates of beta-numbers and the ones of simple beta numbers: if \( \beta \) is a simple beta-number then \( \beta \) has no algebraic conjugate that is a nonnegative real number.

Indeed, let \( \beta \) be a simple beta-number and set \( d_\beta(1) = d_1 \ldots d_n \). Consider

\[
0, 1, \ldots, d_1 - 1
\]

0, 1, \ldots, \( d_1 \)-1

\[
1 \rightarrow 2 \rightarrow 3 \rightarrow \cdots \rightarrow n - 1 \rightarrow n
\]

\[
d_1 \rightarrow d_2 \rightarrow \cdots \rightarrow d_{n-1} \rightarrow d_n
\]

\[
0, \ldots, d_{n-1} - 1
\]

\[
0, \ldots, d_n - 1
\]

\[
\text{Fig. 2. Automaton recognizing the set of the finite factors of } S_\beta
\]

the finite automaton recognizing the set of the finite factors of the associated beta-shift (Fig. 2). Let \( M_\beta \) be the transition matrix of this automaton. The characteristic polynomial of \( M_\beta \), which is called the associated \textit{beta-polynomial},

\[
P(X) = X^n - \sum_{i=1}^{n} d_i X^{n-i}
\]

has only one positive real root.
Example 2. Salem numbers are roots of reciprocal polynomials. Thus if $\beta$ is a Salem number, $1/\beta > 0$ is a Galois conjugate of $\beta$, and so $\beta$ is not a simple beta-number.

The previous conditions are sufficient for a quadratic algebraic integer to be a simple beta-number.

Proposition 1. [10] The simple beta-numbers of degree 2 are exactly the quadratic Pisot numbers without a positive real Galois conjugate. They are the positive roots of the polynomials

$$X^2 - aX - b \quad \text{with} \quad a \geq b \geq 1.$$ 

The beta-expansion of 1 is then $d_\beta(1) = ab$.

Example 3. The minimal polynomial of $(1 + \sqrt{5})/2$ is $X^2 - X - 1$, $(1 + \sqrt{5})/2$ is a simple beta-number and $d_\beta(1) = 11$.

The minimal polynomial of $(3 + \sqrt{5})/2$ is $X^2 - 3X + 1$, therefore $(3 + \sqrt{5})/2$ is not a simple beta-number.

2 Beta-expansions of 1 for cubic Pisot numbers

Let us recall the characterization of cubic Pisot numbers due to Akiyama [1]

Theorem 1 (Akiyama [1]). Let $\beta > 1$ be a cubic number and let

$$M_\beta(x) = X^3 - aX^2 - bX - c$$

be its minimal polynomial.

Then $\beta$ is a Pisot number if and only if the inequalities

$$|b - 1| < a + c \quad \text{and} \quad (c^2 - b) < \text{sgn}(c)(1 + ac)$$

hold.

Remark 1. Note that $a$ must be a nonnegative integer.

The following theorem gives the $\beta$-expansion of 1 for any cubic Pisot number.

Theorem 2. Let $\beta$ be a cubic Pisot number and let

$$M_\beta(x) = X^3 - aX^2 - bX - c$$

be its minimal polynomial. Then the beta-expansion of 1 is

- Case 1: When $b \geq a$, then $d_\beta(1) = (a + 1)(b - 1 - a)(a + c - b)(b - c)c$.
- Case 2: When $0 \leq b \leq a$, if $c > 0$, $d_\beta(1) = abc$, otherwise,

$$d_\beta(1) = a[(b - 1)(c + a)]^{b'}. $$
\begin{itemize}
  \item Case 3: When \( -a < b < 0 \), if \( b + c \geq 0 \), then \( d_3(1) = (a - 1)(a + b)(b + c)c \), otherwise \( d_3(1) = (a - 1)(a - 1)(a + b - 1)(a + b - 1)c \).
  \item Case 4: When \( b \leq -a \), let \( k \) be the integer of \( \{ 2, 3, \ldots , a - 2 \} \) such that, denoting \( e_k = 1 - a + (a - 2) / k \), \( e_k \leq b < c < e_{k-1} \).
  \begin{itemize}
    \item If \( b(k - 1) + c(k - 2) \leq (k - 2) - (k - 1)a \), \( d_3(1) = d_1 \ldots d_{2k+2} \) with \( \begin{align*}
      d_1 &= a - 2, \\
      d_{k+2-i} &= -(k + 3 - i) + a(k + 2 - i) + b(k + 1 - i) + c(k - i), 3 \leq i \leq k \\
      d_k &= -k + ak + b(k - 1) + c(k - 2) \quad \text{and} \quad d_{2k+2} = c.
    \end{align*} \)
    \item If \( b(k - 1) + c(k - 2) > (k - 2) - (k - 1)a \), let \( m \) be the integer defined by \( m = \min \{ i \in \mathbb{N} \mid i + 1 \mid b + ic > i - (i + 1)a \} \).
  \end{itemize}
\end{itemize}

When \( m > 1 \), \( d_3(1) = (a - 2)(2a + b - 2)(2a + 2b + c - 2)(2a + 2b + 2c - 2) \).

When \( m = 1 \), \( d_3(1) = d_1 d_2 \ldots d_{m+3} d_{m+4} \), with \( \begin{align*}
      d_1 &= a - 2, \\
      d_2 &= 2a + b - 3, \\
      d_{m+3-i} &= 2a + b - 3 + (m + 1 - i)(a + b + c - 1) \quad m \geq 3, 3 \leq i \leq m, \\
      d_{m+1} &= 2a + b - 2 + (m - 1)(a + b + c - 1), \\
      d_{m+2} &= a + b - 1 + m(a + b + c - 1), \\
      d_{m+3} &= (m + 1)(a + b + c - 1).
    \end{align*} \)

\textbf{Example 4.} When \( a \geq b \geq 0 \) and \( c > 0 \), we obtain the only beta-expansion of 1 of length 3.

The smallest Pisot number has \( M_3 = X^3 - X - 1 \) as minimal polynomial, it is a simple beta-number and \( d_3(1) = 10001 \).

The positive root \( \beta \) of \( M_3 = X^3 - 3X^2 + 2X - 2 \) is a simple beta-number and \( d_3(1) = 2102 \).

The case where \( b \leq -a \) shows that from a cubic simple beta-number, we can obtain an arbitrary long beta-expansion of 1. For any integer \( k \) greater than or equal to 2, the real root \( \beta \) of the irreducible polynomial \( X^3 - (k + 2)X^2 + 2kX - k \), is a simple beta number whose integer part is equal to \( k \), and the beta-expansion of 1 has length \( 2k + 2 \). For \( k = 2 \), we get \( d_3(1) = 221002 \); for \( k = 3 \), we get \( d_3(1) = 31310203 \).

\textbf{Example 5.} The greatest positive root \( \beta \) of \( M_3 = X^3 - 2X^2 - X + 1 \) is a beta-number and \( d_3(1) = 2(01)^{\omega} \).

If \( \beta \) is the positive root of \( X^3 - 5X^2 + 3X - 2 \), then \( d_3(1) = 413^{\omega} \). When \( \beta \) is the greatest positive root of \( X^3 - 5X^2 + 2X + 2 \), then \( d_3(1) = 431^{\omega} \).

For any integer \( k \) greater than or equal to 3, the real root \( \beta \) of the irreducible polynomial \( X^3 - (k + 2)X^2 + (2k - 1)X - (k - 1) \), is a beta number whose integer part is equal to \( k \), and the beta-expansion of 1 is eventually periodic of period
1, the length of its preperiod $k$. For $k = 3$, we get $d_3(1) = 33024^2$; for $k = 4$, we get $d_4(1) = 42403^2$.

**Proof.** It is known that Pisot numbers are beta-numbers, thus, for any cubic Pisot number $\beta$, the beta-expansion of 1 is finite or eventually periodic. In any case, we first compute the associated beta-polynomial $P$. Next we prove that the sequence $d = (d_i)_{i \geq 1}$ of nonnegative integers obtained from the beta-polynomial satisfy lexicographical order conditions: for all $p \geq 1$, $\sigma^p(d) < d$.

First of all, we recall that, from Theorem 1, a cubic number $\beta$, greater than 1 and having

$$M_\beta(X) = X^3 - aX^2 - bX - c$$

as minimal polynomial, is a cubic Pisot number if and only if it both

$$|b - 1| < a + c \quad \text{and} \quad (c^2 - b) < sgn(c)(1 + ac)$$

hold.

Denote by $Q$ the complementary factor of the beta-polynomial $P$ defined by

$$P(X) = M_\beta(X)|Q(X).$$

As we shall see in what follows, the value of $Q$ depends upon the value of the coefficients of $M_\beta$.

**Case 1:** When $b > a$, as $\beta$ is a Pisot number, from Theorem 1, $c$ is a positive integer. In this case, the complementary factor is

$$Q(X) = X^2 - X + 1 \quad \text{and} \quad d_\beta(1) = (a + 1)(b - 1 - a)(a + c - b)(b - c).$$

Indeed, as $(c^2 - b) < sgn(c)(1 + ac)$ and $c > 0$, we get $c \leq a + 1$. As $|b - 1| < a + c$, we get $b - 1 < a$ and $0 \leq b - a + c$. From $b > a$, we get that $0 \leq b - a - 1$ and, as $c \leq a + 1$, that $a - b + c \leq a$. Finally as $0 \leq b - a + c$, we obtain $0 \leq b - c \leq a$.

**Case 2:** When $0 \leq b \leq a$, the complementary factor is then $Q(X) = 1$ and the associated beta-polynomial is equal to the minimal polynomial.

If $c > 0$, then $d_\beta(1) = abc$. Indeed, as $(c^2 - b) < sgn(c)(1 + ac)$, we get $c \leq a$.

If $c < 0$, then $d_\beta(1) = a(b - 1)(a + c)^{|c|}$. As $|b - 1| < a + c$, we get $b - 1 < a - 2$. As $(c^2 - b) < sgn(c)(1 + ac)$, we get that $c \geq -a$ and, consequently, $0 \leq c + a \leq a - 1$.

**Case 3:** When $-a < b < 0$, if $b + c > 0$ then the complementary factor is $Q(X) = X + 1$ and $d_\beta(1) = (a - 1)(a + b)(b + c)$. Indeed, as $-a < b < 0$, we obtain $1 \leq a + b \leq a - 1$. Since $b + c > 0$, $c$ is a positive integer. From $(c^2 - b) < sgn(c)(1 + ac)$, we get that $c \leq a - 1$ and $b + c \leq a - 2$.

If $b + c < 0$, then $Q(X) = 1$ and $d_\beta(1) = (a - 1)(a + b - 1)(a + b + c - 1)^{|c|}$.

As $-a < b < 0$, we get $0 \leq a + b - 1 \leq a - 2$. From $|b - 1| < a + c$, we get that $1 \leq a + b + c - 1$ and as $b + c < 0$, we obtain $a + b + c - 1 \leq a - 2$.

**Case 4:** First of all, since $|b - 1| < a + c$, we get $-a + 2 \leq b + c$. Moreover as $b \leq -a$, we get $c \geq 2$ and as $(c^2 - b) < sgn(c)(1 + ac)$, we obtain $c \leq a - 2$, thus $b + c \leq -2$. So, there exists an integer $k$ in $\{2, 3, \ldots, a - 2\}$, such that, denoting

$$e_k = 1 - a + (a - 2)/k, \quad e_k \leq b + c < e_{k-1}.$$  

When $b(k - 1) + c(k - 2) \leq (k - 2) - (k - 1)a$, the complementary factor is

$$Q(X) = \frac{(X^k - 1)(X^{k+1} - 1)}{(X - 1)^2}.$$
and $d_3(1) = d_1 \ldots d_{2k+2}$ with

$$d_1 = a - 2,$$

$$d_{k+i} = -(k + 3 - i) + a(k + 2 - i) + b(k + 1 - i) + c(k - i), k \geq 3, 3 \leq i \leq k$$

$$d_k = -k + ak + b(k - 1) + c(k - 2)$$

$$d_{k+1} = -(k - 1) + ak + bk + c(k - 1)$$

$$d_{k+i} = -(k - 2) + a(k - 1) + bk + c(k - 2) + c$$

$$d_{2k+2-i} = -(i - 2) + a(i - 1) + bi + c(i + 1), k \geq 3, 2 \leq i \leq (k - 1)$$

$$d_{2k+1} = b + 2c$$

$$d_{2k+2} = c.$$

We now verify that the lexicographical order conditions on $d_3(1)$ are satisfied.

As $2 \leq c \leq a - 2$ and $b + c \leq -2$, we get $d_{2k+1} \leq a - 4$. From $e_k \leq b + c$ and $b(k - 1) + c(k - 2) \leq (k - 2) - (k - 1)a$, we get $d_{2k+1} \geq 0$.

For $k \leq 3$ and $2 \leq i \leq k - 1$, $d_{2k+2-i} = -(i - 2) + a(i - 1) + bi + c(i + 1)$. As $b + c < e_i$, we get $d_{2k+2-i} < c$. As $-a + 2 \leq b + c$ and $b + 2c \geq 0$, we get $d_{2k+2-i} \geq i$.

As $e_k \leq b + c$, we obtain $d_{k+2} \geq 0$. Since $c \leq a - 2$, $d_{k+1} > d_{k+2}$ and since $b + c \leq -2$, $d_k > d_{k+1}$. Moreover from $b(k - 1) + c(k - 2) \leq (k - 2) - (k - 1)a$, we get $d_k \leq a - 2$.

For $k \leq 3$, if $b - 1 \leq a + c$, we obtain $d_2 < \ldots < d_{k+1}$. As $b + c < e_{k+1}$ and $b + 2c \leq 0$, we get $d_{k+1} < a - 2$. Moreover from $c \leq a - 2$ and $a + b + c - 1 > 0$, we get that $d_2 = 2a + b - 3$ is nonnegative.

All $d_i$'s are smaller than $d_1$, only $d_{2k+2}$ and $d_k$ can be equal to $d_1$. Therefore we have to verify that $d_k \geq d_{k+1}$ when $k \geq 3$ (otherwise $d_k = d_k$ and $d_k > d_{k+1}$).

If $d_k = a - 2$, then $b + c = e_k$, and $d_{k+1} = a - c - 1$. As $a + b + c - 1 > 0$, we obtain $d_{k+1} \leq d_2$. In case of equality, if $k = 3$, then $d_3 = d_k$ and $d_k > d_{k+2}$, otherwise $d_3 > d_2$ and $d_{k+1} > d_{k+2}$, therefore $d_3 > d_{k+2}$.

So lexicographical order conditions are satisfied and $d_1 \ldots d_{2k+2}$ is the beta-expansion of 1.

When $b(k - 1) + c(k - 2) > (k - 2) - (k - 1)a$, as $b \leq -a$, we get $k \geq 3$. Let $m$ be the integer defined by $m = \min \{i \in \mathbb{N} \mid (i + 1)b + ic > i - (i + 1)a\}$. Note that by definition of $m$, $m \leq k - 2$ and since $b \leq -a$, $m \geq 1$. In this case, the complementary factor is

$$Q(X) = \sum_{i=0}^{m} X^i.$$

The beta-expansion of 1 is then eventually periodic with period 1, the length of the preperiod is $m + 2$.

When $m = 1$, $P(X) = X^4 - (a - 1)X^3 - (a + b)X^2 - (b + c)X - c$ and

$$d_3(1) = (a - 2)(2a + b - 2)(2a + 2b + c - 2)(2a + 2b + 2c - 2)^m.$$

Here $d_3 = \delta_{m+1} = a + b - 1 + m(a + b + c - 1)$ and $d_4 = \delta_{m+2} = (m + 1)(a + b + c - 1)$.

When $m > 1$,

$$P(X) = X^{m+3} - (a - 1)X^{m+2} - (a + b - 1)X^{m+1} - \sum_{i=3}^{m} (a + b + c - 1)X^i - (a + b + c)X^2 - (b + c)X - c$$
and $d_3(1) = d_1d_2 \ldots d_{m+2}d_{m+3}$, with
\[d_1 = a - 2, \quad d_2 = 2a + b - 3,
\]
\[d_{m+3-i} = 2a + b - 3 + (m+1-i)(a+b+c-1) \quad m \geq 3, 3 \leq i \leq m,
\]
\[d_{m+1} = 2a + b - 2 + (m-1)(a+b+c-1),
\]
\[d_{m+2} = a + b - 1 + m(a+b+c-1),
\]
\[d_{m+3} = (m+1)(a+b+c-1).
\]

In both cases, $d_1 = a - 2$. Since $b(k-1) + c(k-2) > (k-2) - (k-1)a$ and $c \leq a - 2$, we get $2a + 3 \leq b$. Moreover as $b \leq -a$, $1 \leq d_2 \leq a - 2$ when $m = 1$, and $0 \leq d_2 \leq a - 3$ otherwise. By definition of $m$, $(m+1)b + mc > m - (m+1)a$, thus $d_{m+2} \geq 0$ and $d_{m+3} \geq c$. Since $e_k \leq b + c < e_{k-1}$ and $m \leq k - 2$, we obtain $d_{m+3} \leq a - 3$ and $d_{m+2} \leq a - c - 3$.

When $m > 1$, since $mb + (m-1)c \leq (m-1) - ma$, we get $d_{m+1} \leq a - 2$. As $0 \leq 2a + b - 2$ and $a + b + c - 1 > 0$, $d_{m+1} > 0$. Moreover as $a + b + c - 1 > 0$, one has $d_2 < d_3 < \ldots < d_{m+1}$. Note that, when $m \geq 3, d_1 \neq a - 2$.

We now study the cases where $d_i$ is not strictly smaller than $d_1$, when $m = 1$, only $d_2$ may be equal to $a - 2$, then $b = -a$ and $d_3 = c - 2$, thus $d_3 < d_2$. When $m > 1$, only $d_{m+1}$ may be equal to $a - 2$, then $mb = -ma - (m-1)c + (m-1)$, and thus $d_2 - d_{m+2} = a - 1 - c$ is a positive integer.

We have proved that the lexicographical order conditions on $d_3(1)$:
\[d_1d_2 \ldots d_{m+3} >_{lex} d_1d_{i+1} \ldots d_{m+3} \quad \text{for } 2 \leq i \leq m + 3,
\]
are satisfied, showing in this way that the announced beta-expansions of 1 are right.

**Remark 2.** The polynomials $Q$ that appear in the cubic case are cyclotomic. In the general case, $Q$ can be noncyclotomic and even nonreciprocal ([B]).

## 3 Cubic simple beta-numbers

In the following, we establish that cubic simple beta-numbers are Pisot numbers. Next we give necessary and sufficient conditions on the coefficients of the minimal polynomial of $\beta$ for $\beta$ to be a simple beta-number.

**Theorem 3.** If $\beta$ is a cubic simple beta-number then $\beta$ is a Pisot number.

**Remark 3.** This is no longer true for simple beta-numbers of degree 4. For example, the positive root of $X^4 - 3X^3 - 2X^2 - 3$ is a simple beta-number, but is not a Pisot number.

**Proof.** Let $\beta$ be a cubic simple beta-number and let
\[M_{\beta}(X) = X^3 - aX^2 - bX - c
\]
be its minimal polynomial. Then $\beta$ has no positive real algebraic conjugate and $c$ is a positive integer smaller than $|\beta|$. 
The condition on the product $c$ of the roots of the polynomial $M_\beta$, i.e., $|c| \leq |\beta|$, directly implies, when the Galois conjugates of $\beta$ are not real numbers, that $\beta$ is a Pisot number.

The only other case is the case where both Galois conjugates $\gamma_1$ and $\gamma_2$ of $\beta$ are negative real numbers. We then assume that $\beta$ is a cubic simple beta-number that is not a Pisot number, and show that these hypotheses are contradictory. Let $\gamma_1$ and $\gamma_2$ be the Galois conjugates of $\beta$. As $0 < c \leq |\beta|$, if one of the $\gamma_i$’s is smaller than $-1$ the other one is greater than $-1$. Moreover, as the modulus of a Galois conjugate of a beta-number is smaller than the golden ratio, one can suppose, for example, that

$$-\frac{1 + \sqrt{5}}{2} < \gamma_2 < -1 < \gamma_1 < 0 < \beta$$

Consequently, $M_\beta(-1) > 0$, in other words, $b > a + c + 1$. Note that here $a \in \{[\beta] - 2, [\beta] - 1\}$.

As $\beta$ is a simple beta-number, $d_\beta(1) = d_1d_2 \ldots d_n$. Denote by $P$ the associated $\beta$-polynomial:

$$P(X) = X^n - \sum_{i=1}^{n} d_i X^{n-i}$$

and denote by $Q = \sum_{i=0} q_i X^i$ the quotient of the division upon the increasing powers of $P$ by $M_\beta$. In other words,

$$P(X) = M_\beta(X)Q(X)$$

We shall show, by induction, that $q_0 \geq 1$, and that for all $i \geq 0$, $|q_{i+1}| > |q_i|$ with $\text{sgn}(q_{i+1}) = -\text{sgn}(q_i)$. We shall conclude from the growth of the moduli of its coefficients that $Q$ is an infinite series, and thus that $d_\beta(1)$ is not finite.

In what follows, we mainly use the fact that the $d_i$’s are nonnegative integers smaller than $|\beta|$ and the inequality $b \geq a + c + 2$.

First of all, as $d_n = q_0c$ and $d_n$ and $c$ are positive integers, $q_0 \geq 1$. Since $d_{n-1} = q_0b + q_1c$ and $q_0 \geq 1$, $d_{n-1} \geq q_0a + 2q_0 + (q_0 + q_1)c$. When $a = [\beta] - 1$, we directly get from $d_{n-1} \leq [\beta]$, that $q_1 < -q_0$. When $a = [\beta] - 2$, the lexicographical order conditions on $d_\beta(1)$ imply that

$$d_{n-1}d_n < d_1d_2 \ldots d_n.$$ 

By definition of beta-expansions, $d_1 = [\beta]$ and here $d_2 < d_n$. Indeed as

$$\gamma_2 = \frac{1}{2} \left( a - \beta + \sqrt{(a - \beta)^2 - 4c} \right),$$

and $\gamma_2 > -(1 + \sqrt{5})/2$, we get that

$$c > \frac{\sqrt{5} - 1}{2} \beta + \frac{1 + \sqrt{5}}{2} \beta \{\beta\},$$
and in particular, that $c > \beta/2$, consequently $d_n = c$ and that $\beta \{ \beta \} < c$. Thus $d_2 = [\beta \{ \beta \}]$ is strictly smaller than $d_n$. Therefore the previous lexicographical order condition implies that $d_{n-1} < [\beta \{ \beta \}]$. So as $d_{n-1} \geq [\beta \{ \beta \}] + (q_0 + q_1)c$, $q_1 < -q_0$.

As $d_{n-2} = q_0a + q_1 b + q_2 c$ and $q_1 < -q_0 < 0, d_{n-2} \leq (q_1 + q_0)a + 2q_1 + (q_1 + q_2)c$, that is $d_{n-2} < -[\beta \{ \beta \}] + (q_1 + q_2)c$, so $q_2 > -q_1$.

For all positive integers $i$, $d_{n-(2i+1)} = -q_3 + + q_2 + q_2i + q_2i+1c$. From $q_{2i} > 0$, we get $d_{n-(2i+1)} \geq q_2i - q_2i + q_2i + q_2i - q_2i + q_2i+1c$. From $(q_2i - q_2i) \geq 1$, $q_2i > 2i$ and $(q_2i - q_2i) > 1$, we obtain $d_{n-(2i+1)} > [\beta \{ \beta \}] + (q_2i + q_2i+1)c$, and thus $q_2i+1 < -q_{2i}$.

For all positive integers $i$, $d_{n-(2i+2)} = -q_2i - 1 + q_2i + q_2i + 1b + q_2i+2c$. From $q_{2i+1} \leq 0$, we get $d_{n-(2i+2)} \leq (q_2i + q_2i + 1 + q_2i + 1 - q_2i + 1 + q_2i+1 + q_2i+2)$. As $(q_2i + q_2i + 1) \leq -1, q_{2i+1} < -2i + 1$ and $(q_2i + q_2i + 1) < -1$, we get $d_{n-(2i+2)} < -[\beta \{ \beta \} + (q_2i + q_2i + 2)c$, thus $q_2i+2 > -q_{2i+2}$.

So $Q$ is an infinite series; consequently if $\beta$ is not a Pisot number, $d_\beta(1)$ is not finite.

As a consequence of Theorems 2 and 3, we obtain the above characterization of cubic simple beta-numbers.

**Proposition 2.** Let $\beta$ be a cubic Pisot number and let 

$$M_\beta(x) = x^3 - ax^2 - bx - c$$

be its minimal polynomial.

Then $\beta$ is a simple beta-number if and only it satisfies one of the following conditions:

- **Case 1:** $b \geq 0$ and $c > 0$
- **Case 2:** $-a < b < 0$ and $b + c \geq 0$
- **Case 3:** $b \leq -a$ and $b(k - 1) + c(k - 2) \leq (k - 2)(k - 1)a$, where $k$ is the integer in $\{2, 3, \ldots, a - 2\}$ such that, denoting $e_k = 1 - a + (a - 2)/k$, $e_k \leq b + c < e_{k-1}$.

The problem of finding such a characterization remains open for simple beta-numbers of higher degree.

**References**


