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Beta-expansions for cubic Pisot numbers

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Abstract. Real numbers can be represented in an arbitrary base $\beta > 1$ using the transformation $T_\beta : x \to \beta x \pmod{1}$ of the unit interval; any real number $x \in [0, 1]$ is then expanded into $d_\beta(x) = (x_i)_{i \geq 1}$ where $x_i = \lfloor \beta T_{\beta - 1}^{-i}(x) \rfloor$.

The closure of the set of the expansions of real numbers of $[0, 1]$ is a shift of $\{a \in \mathbb{N} \mid a < \beta\}^\mathbb{N}$, called the beta-shift. This dynamical system is characterized by the beta-expansion of 1; in particular, it is of finite type if and only if $d_\beta(1)$ is finite; $\beta$ is then called a simple beta-number.

We first compute the beta-expansion of 1 for any cubic Pisot number. Next we show that cubic simple beta-numbers are Pisot numbers.

Introduction

Representations of real numbers in an arbitrary base $\beta > 1$, called beta-expansions, have been introduced by Rényi ([14]). They arise from the orbits of the piecewise-monotone transformation of the unit interval $T_\beta : x \to \beta x \pmod{1}$. Such transformations were extensively studied in ergodic theory ([13]).

More precisely, any real number $x \in [0, 1]$ is expanded into $d_\beta(x) = (x_i)_{i \geq 1}$ where $x_i = \lfloor \beta T_{\beta - 1}^{-i}(x) \rfloor$. The nonnegative integers $d_i$ are elements of the digit alphabet $A = \{a \in \mathbb{N} \mid a < \beta\}$. These representations generalize standard representations in an integral base to a real base; indeed the beta-expansion of any real number of $[0, 1]$ can equivalently be obtained by the greedy algorithm. Only the beta-expansion of 1 differs.

Properties of beta-expansions are strongly related to symbolic dynamics ([4]). The closure of the set of infinite sequences, appearing as beta-expansions of numbers of the interval $[0, 1]$, is a dynamical system, that is, a closed shift-invariant subset of $A^\mathbb{N}$, called the beta-shift.

An important property of the beta-shift is that its nature is entirely determined, in a combinatorial manner, by the beta-expansion of 1: the beta-shift is sofic, that is to say the set of its finite factors is recognized by a finite automaton, if and only the beta-expansion of 1 is eventually periodic ([3]); it is of finite type, that is to say the set of its finite factors is defined by forbidding a finite set of words, if and only if the beta-expansion of 1 is finite ([12]).

When the beta-expansion of 1 is eventually periodic, $\beta$ is called a beta-number and when the beta-expansion of 1 is finite, $\beta$ is said to be a simple beta-number.
The eventually periodic beta-expansions were extensively studied by Bertrand ([3]) and by Schmidt ([15]). In particular, it is known that Pisot numbers are beta-numbers. Concerning Salem numbers, we only know that if $\beta$ is a Salem number of degree 4, then the beta-expansion of 1 is eventually periodic ([5]). It is conjectured that Salem numbers of degree 6 are still beta-numbers, but not all Salem numbers of degree 8 ([7]).

The domain of the Galois conjugates of all beta-numbers was also investigated independently by Solomyak ([16]) and by Flatto, Lagarias and Poonen ([8]).

For a general presentation of the beta-shift one can refer to [9].

In the following, we summarize properties of beta-numbers. We compute the beta-expansion of 1 for any cubic Pisot number and we establish a characterization of cubic simple beta-numbers, showing that they are Pisot numbers.

A very close problem, seen from the point of view of numeration systems, was studied by Akiyama ([1]). He showed that in the cubic case, the real numbers of the set $\mathbb{N}[\beta^{-1}]$ have a finite beta-expansion if and only $\beta$ is a Pisot unit and 1 has a finite beta-expansion. This finiteness problem is equivalent to a problem of fractal tiling generated by Pisot numbers.

1 Beta-numbers

Real numbers can be represented in an arbitrary base $\beta > 1$ using the transformation $T_{\beta} : x \to \beta x \mod 1$ of the unit interval; any real number $x \in [0, 1]$ is then expanded into $d_{\beta}(x) = (x_i)_{i \geq 1}$ where $x_i = \lfloor \beta T_{\beta}^{i-1}(x) \rfloor$. When a beta-expansion is of the form $u\bar{w}^\omega$, the expansion is said to be eventually periodic. If a representation ends with infinitely many zeroes, like $u\bar{0}^\omega$, it is said to be finite and the ending zeroes are omitted.

Let us denote by $S_{\beta}$ the closure of all beta-expansions of real numbers of $[0, 1]$ and by $\sigma$ the (one-sided) shift defined by $\sigma((x_i)_{i \geq 1}) = (x_{i+1})_{i \geq 1}$. The set $S_{\beta}$ endowed with the shift is called the beta-shift, it is a subshift of $A^\mathbb{N}$, $A$ being the digit set, i.e., $A = \{a \in \mathbb{N} | a < \beta\}$.

An important property ([13]) of the beta-shift $S_{\beta}$ is that its nature is entirely determined by $d_{\beta}(1)$ the beta-expansion of 1. Indeed, setting $d^*(1) = d_{\beta}(1)$ if $d_{\beta}(1)$ is infinite and $d^*(1) = (d_1 d_2 \ldots d_{n-1} d_n)$ if $d_{\beta}(1) = d_1 d_2 \ldots d_{n-1} d_n$, a sequence $x$ of nonnegative integers belongs to $S_{\beta}$ if and only if it satisfies the following lexicographical order conditions: $\forall p \geq 0, \quad \sigma^p(x) \leq d^*(1)$.

Recall that the beta-expansion of 1 also can be characterized ([13]) by lexicographical order conditions: let $d = (d_i)_{i \geq 1}$ be a sequence of nonnegative integers different from $10^\omega$, such that $\sum_{i \geq 1} d_i \beta^{-i} = 1$, with $d_1 \geq 1$ and for $i \geq 2$, $d_i \leq d_1$, then $d$ is the beta-expansion of 1 if and only if for all $p \geq 1, \sigma^p(d) < d$.

We recall that an algebraic integer $\beta$ strictly greater than 1 is called a Perron number if all its Galois conjugates have modulus strictly less than $\beta$, a Pisot number if all its Galois conjugates have modulus strictly less than 1, and a Salem number if all its conjugates are less than 1 in modulus and at least one conjugate has modulus 1.
Let \( \beta \) be a beta-number. Denote by \( d_\beta(1) = d_1 \ldots d_n d_{n+1} \ldots d_{n+p} \), where \( n \) and \( p \) are chosen minimal, the beta-expansion of 1. Then the adjacency matrix \( M_\beta \) of the finite automaton recognizing the set of its finite factors (Fig.1) is a primitive (i.e., its associated graph is strongly connected and the lengths of its cycles are relatively prime) nonnegative integral matrix whose spectral radius is \( \beta \); so, from the Perron-Frobenius theorem, \( \beta \) is a Perron number.

![Diagram of automaton](image)

**Fig. 1.** Automaton recognizing the set of the finite factors of \( S_\beta \)

The characteristic polynomial of \( M_\beta \)

\[
P(X) = X^{n+p} - \sum_{i=1}^{n+p} d_i X^{n+p-i} - X^n + \sum_{i=1}^{n} d_i X^{n-i}
\]

is called, following the terminology introduced by Holländer ([11]), the associated beta-polynomial.

As \( P \) is a multiple of the minimal polynomial \( M_\beta \) of \( \beta \), \( P(0) = d_{n+p} - d_n \) is a multiple of \( |M_\beta(0)| = \prod \beta_i \) where \( \beta_i \) runs over the set of algebraic conjugates of \( \beta \). So, we get that \( \prod \beta_i \) has to be smaller than \( |\beta| \).

As a consequence, in the quadratic case, the only beta-numbers are the Pisot numbers. Conversely, it is known that if \( \beta \) is a Pisot number then \( \beta \) is a beta-number ([2]). An important gap remains between Pisot and Perron numbers.

**Example 1.** The quadratic number \( \beta = (1 + \sqrt{3})/2 \) is not a beta-number since \( M_\beta(X) = X^2 - X - 3 \) and \( M_\beta(0) > |\beta| \).
Let $\beta$ be the Pisot number $(3 + \sqrt{5})/2$, then $\beta$ is a beta-number and $d_\beta = 21^\omega$.

Let $\beta$ be the golden ratio $(1 + \sqrt{5})/2$, then $\beta$ is a simple beta-number and $d_\beta(1) = 11$.

On the other hand, the domain of the Galois conjugates of beta-numbers was studied by Solomyak ([16]) and independently by Flatto, Lagarias and Poonen ([8]). They showed in particular that if the beta-expansion of 1 is eventually periodic then the Galois conjugates of $\beta$ have modulus less than the golden ratio $(1 + \sqrt{5})/2$. It was already known (see [9]) that $\beta$ cannot have a Galois conjugate greater than 1.

Solomyak ([16]) proved that the topological closure of conjugates of beta-numbers and the one of conjugates of simple beta-numbers are the same. However, there is an important difference between the conjugates of beta-numbers and the ones of simple beta numbers: if $\beta$ is a simple beta-number then $\beta$ has no algebraic conjugate that is a nonnegative real number.

Indeed, let $\beta$ be a simple beta-number and set $d_\beta(1) = d_1 \ldots d_n$. Consider

\[ 0, 1, \ldots, d_1 - 1 \]

\[ \begin{array}{c}
1 \\
\downarrow d_1 \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
2 \\
\downarrow d_2 \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
3 \\
\downarrow \cdots \\
n-1 \\
\downarrow d_{n-1} \\
n
\end{array} \]

\[ \begin{array}{c}
0, \ldots, d_2 - 1 \\
0, \ldots, d_3 - 1 \\
0, \ldots, d_{n-1} - 1 \end{array} \]

\[ \begin{array}{c}
0, \ldots, d_n - 1 \\
0, \ldots, d_1 - 1 \\
0, 1, \ldots, d_1 - 1 \\
\cdots \\
0, \ldots, d_2 - 1 \\
0, \ldots, d_3 - 1 \\
0, \ldots, d_{n-1} - 1 \\
0, \ldots, d_n - 1 \\
\cdots \\
0, \ldots, d_2 - 1 \\
0, \ldots, d_3 - 1 \\
0, \ldots, d_{n-1} - 1 \\
0, \ldots, d_n - 1 \\
\cdots \\
0, \ldots, d_2 - 1 \\
0, \ldots, d_3 - 1 \\
0, \ldots, d_{n-1} - 1 \\
0, \ldots, d_n - 1 \\
\cdots \\
0, \ldots, d_2 - 1 \\
0, \ldots, d_3 - 1 \\
0, \ldots, d_{n-1} - 1 \\
0, \ldots, d_n - 1 \\
\cdots \\
0, \ldots, d_2 - 1 \\
0, \ldots, d_3 - 1 \\
0, \ldots, d_{n-1} - 1 \\
0, \ldots, d_n - 1 \\
\end{array} \]

**Fig. 2.** Automaton recognizing the set of the finite factors of $S_\beta$

the finite automaton recognizing the set of the finite factors of the associated beta-shift (Fig. 2). Let $M_\beta$ be the transition matrix of this automaton. The characteristic polynomial of $M_\beta$, which is called the associated beta-polynomial,

\[ P(X) = X^n - \sum_{i=1}^{n} d_i X^{n-i} \]

has only one positive real root.
Example 2. Salem numbers are roots of reciprocal polynomials. Thus if \( \beta \) is a Salem number, \( 1/\beta > 0 \) is a Galois conjugate of \( \beta \), and so \( \beta \) is not a simple beta-number.

The previous conditions are sufficient for a quadratic algebraic integer to be a simple beta-number.

Proposition 1. [10] The simple beta-numbers of degree 2 are exactly the quadratic Pisot numbers without a positive real Galois conjugate. They are the positive roots of the polynomials

\[ X^2 - aX - b \quad \text{with} \quad a \geq b \geq 1. \]

The beta-expansion of 1 is then \( d_\beta(1) = ab \).

Example 3. The minimal polynomial of \( (1 + \sqrt{5})/2 \) is \( X^2 - X - 1 \), \( (1 + \sqrt{5})/2 \) is a simple beta-number and \( d_\beta(1) = 11 \).

The minimal polynomial of \( (3 + \sqrt{5})/2 \) is \( X^2 - 3X + 1 \), therefore \( (3 + \sqrt{5})/2 \) is not a simple beta-number.

2 Beta-expansions of 1 for cubic Pisot numbers

Let us recall the characterization of cubic Pisot numbers due to Akiyama ([1])

Theorem 1 (Akiyama [1]). Let \( \beta > 1 \) be a cubic number and let

\[ M_\beta(x) = X^3 - aX^2 - bX - c \]

be its minimal polynomial.

Then \( \beta \) is a Pisot number if and only if it both inequalities

\[ |b - 1| < a + c \quad \text{and} \quad (c^2 - b) < \text{sgn}(c)(1 + ac) \]

hold.

Remark 1. Note that \( a \) must be a nonnegative integer.

The following theorem gives the \( \beta \)-expansion of 1 for any cubic Pisot number.

Theorem 2. Let \( \beta \) be a cubic Pisot number and let

\[ M_\beta(x) = X^3 - aX^2 - bX - c \]

be its minimal polynomial. Then the beta-expansion of 1 is

- Case 1: When \( b \geq a \), then \( d_\beta(1) = (a + 1)(b - 1 - a)(a + c - b)(b - c)c \).
- Case 2: When \( 0 \leq b \leq a \), if \( c > 0 \), \( d_\beta(1) = abc \), otherwise,

\[ d_\beta(1) = a[(b - 1)(c + a)]^2. \]
Case 3: When $-a < b < 0$, if $b + c \geq 0$, then $d_3(1) = (a - 1)(a + b)(b + c)c$, otherwise $d_3(1) = (a - 1)(a + b - 1)(a + b - c - 1)c$.

Case 4: When $b \leq -a$, let $k$ be the integer of $\{2, 3, \ldots, a - 2\}$ such that, denoting $e_k = 1 - a + (a - 2)/k, e_k \leq b + c < e_{k-1}$.

- If $b(k - 1) + c(k - 2) \leq (k - 2) - (k - 1)a$, $d_3(1) = d_1 \ldots d_{2k+2}$ with
  
  \[ d_1 = a - 2, \]
  \[ d_{k+2-i} = -(k + 3 - i) + a(k + 2 - i) + b(k + 1 - i) + c(k - i), 3 \leq i \leq k \]
  \[ d_k = -(k + a) + b(k - 1) + c(k - 2) \]
  \[ d_{k+1} = -(k - 1) + a(k + 1) + bk + c(k - 1) \]
  \[ d_{k+2} = -(k - 2) + a(k - 1) + bk + ck \]
  \[ d_{2k+2-i} = -(i - 2) + a(i - 1) + bi + c(i + 1) \quad k \geq 3, 2 \leq i \leq (k - 1) \]
  \[ d_{2k+1} = b + 2c \quad \text{and} \quad d_{2k+2} = c. \]

- If $b(k - 1) + c(k - 2) > (k - 2) - (k - 1)a$, let $m$ be the integer defined by $m = \min\{i \in \mathbb{N} \text{ such that } (i + 1)b + ic > i - (i + 1)a\}$. When $m > 1$, $d_3(1) = (a - 2)(2a + b - 2)(2a + 2b + c - 2)(2a + 2b + 2c - 2)c$.

When $m > 1$, $d_3(1) = d_1d_2 \ldots d_{m+3}d_{m+3}$, with

\[ d_1 = a - 2, \]
\[ d_2 = 2a + b - 3, \]
\[ d_{m+3-i} = 2a + b - 3 + (m + 1 - i)(a + b + c - 1) \quad m \geq 3, 3 \leq i \leq m, \]
\[ d_{m+1} = 2a + b - 2 + (m - 1)(a + b + c - 1), \]
\[ d_{m+2} = a + b - 1 + m(a + b + c - 1), \]
\[ d_{m+3} = (m + 1)(a + b + c - 1). \]

Example 4. When $a \geq b \geq 0$ and $c > 0$, we obtain the only beta-expansion of 1 of length 3.

The smallest Pisot number has $M_3 = X^3 - X - 1$ as minimal polynomial, it is a simple beta-number and $d_3(1) = 10001$.

The positive root $\beta$ of $M_3 = X^3 - 3X^2 + 2X - 2$ is a simple beta-number and $d_3(1) = 2102$.

The case where $b \leq -a$ shows that from a cubic simple beta-number, we can obtain an arbitrary long beta-expansion of 1. For any integer $k$ greater than or equal to 2, the real root $\beta$ of the irreducible polynomial $X^3 -(k+2)X^2 + 2kX - k$, is a simple beta number whose integer part is equal to $k$, and the beta-expansion of 1 has length $2k + 2$. For $k = 2$, we get $d_{3}(1) = 221002$; for $k = 3$, we get $d_{3}(1) = 31310203$.

Example 5. The greatest positive root $\beta$ of $M_3 = X^3 - 2X^2 - X + 1$ is a beta-number and $d_{3}(1) = 20101c$.

If $\beta$ is the positive root of $X^3 - 5X^2 + 3X - 2$, then $d_{3}(1) = 413c$. When $\beta$ is the greatest positive root of $X^3 - 5X^2 + 2X + 2$, then $d_{3}(1) = 431c$.

For any integer $k$ greater than or equal to 3, the real root $\beta$ of the irreducible polynomial $X^3 -(k+2)X^2 + (2k-1)X -(k-1)$, is a beta number whose integer part is equal to $k$, and the beta-expansion of 1 is eventually periodic of period
1, the length of its preperiod \( k \). For \( k = 3 \), we get \( d_B(1) = 3302^{2^k} \); for \( k = 4 \), we get \( d_B(1) = 42403^{2^k} \).

**Proof.** It is known that Pisot numbers are beta-numbers, thus, for any cubic Pisot number \( \beta \), the beta-expansion of 1 is finite or eventually periodic. In any case, we first compute the associated beta-polynomial \( P \). Next we prove that the sequence \( d = (d_i)_{i \geq 1} \) of nonnegative integers obtained from the beta-polynomial satisfy lexicographical order conditions: for all \( p \geq 1 \), \( \sigma^p(d) < d \).

First of all, we recall that, from Theorem 1, a cubic number \( \beta \), greater than 1 and having
\[
M_\beta(X) = X^3 - aX^2 - bX - c
\]
as minimal polynomial, is a cubic Pisot number if and only if it both
\[
|b - 1| < a + c \quad \text{and} \quad (c^2 - b) < \text{sgn}(c)(1 + ac)
\]
hold.

Denote by \( Q \) the **complementary factor** of the beta-polynomial \( P \) defined by
\[
P(X) = M_\beta(X)Q(X).
\]
As we shall see in what follows, the value of \( Q \) depends upon the value of the coefficients of \( M_\beta \).

**Case 1:** When \( b > a \), as \( \beta \) is a Pisot number, from Theorem 1, \( a \) is a positive integer. In this case, the complementary factor is \( Q(X) = X^2 - X + 1 \) and \( d_B(1) = (a + 1)(b - 1 - a)(a + c - b)(b - c) \).

Indeed, as \( (c^2 - b) < \text{sgn}(c)(1 + ac) \) and \( c > 0 \), we get \( c \leq a + 1 \). As \( |b - 1| < a + c \) and \( b - 1 - a \leq a \) and \( 0 \leq a - b + c \). From \( b > a \), we get that \( 0 \leq b - a - 1 \) and, as \( c \leq a + 1 \), that \( a - b + c \leq a \). Finally as \( 0 \leq a - b + c \leq b \), we obtain \( 0 \leq b - c \leq a \).

**Case 2:** When \( 0 \leq b \leq a \), the complementary factor is then \( Q(X) = 1 \) and the associated beta-polynomial is equal to the minimal polynomial.

If \( c > 0 \), then \( d_B(1) = abc \). Indeed, as \( (c^2 - b) < \text{sgn}(c)(1 + ac) \), we get \( c \leq a \).

If \( c < 0 \), then \( d_B(1) = a(b - 1)(a + c)^{|c|} \). As \( |b - 1| < a + c \) and \( b - 1 \leq a - 1 \), we get \( c \geq -a \) and, consequently, \( 0 \leq c + a \leq a - 1 \).

**Case 3:** When \( -a < b < 0 \), if \( b + c \geq 0 \) then the complementary factor is \( Q(X) = X + 1 \) and \( d_B(1) = (a - 1)(a + b)(b + c) \). Indeed, as \( -a < b < 0 \), we obtain \( 1 \leq a + b \leq a - 1 \). Since \( b + c \geq 0 \), \( c \) is a positive integer. From \( c^2 - b < \text{sgn}(c)(1 + ac) \), we get that \( c \leq a - 1 \) and \( b + c \leq a - 2 \).

If \( b + c < 0 \), then \( Q(X) = 1 \) and \( d_B(1) = (a - 1)(a + b - 1)(a + b + c - 1)^{|c|} \). As \( -a < b < 0 \), we get \( 0 \leq a + b - 1 \leq a - 2 \). From \( |b - 1| < a + c \), we get that \( 1 \leq a + b + c - 1 \) and \( b + c < 0 \), we obtain \( a + b + c - 1 \leq a - 2 \).

**Case 4:** First of all, since \( |b - 1| < a + c \), we get \( -a + 2 \leq b + c \). Moreover as \( b \leq -a \), we get \( c \geq 2 \) and as \( (c^2 - b) < \text{sgn}(c)(1 + ac) \), we obtain \( c \leq a + 2 \), thus \( b + c \leq -2 \). So, there exists an integer \( k \) in \( \{2, 3, \ldots, a - 2\} \), such that, denoting \( e_k = 1 - a + (a - 2)/k \), \( c \leq b + c < e_k \).

When \( b(k - 1) + c(k - 2) \leq (k - 2) - (k - 1)a \), the complementary factor is
\[
Q(X) = \frac{(X^k - 1)(X^{k+1} - 1)}{(X - 1)^2}
\]
and \( d_3(1) = d_1 \ldots d_{2k+2} \) with
\[
d_1 = a - 2, \\
d_{k+2,i} = -(k + 3 - i) + a(k + 2 - i) + b(k + 1 - i) + c(k - i), k \geq 3, 3 \leq i \leq k \\
d_k = -k + ak + b(k - 1) + c(k - 2) \\
d_{k+1} = -(k - 1) + ak + bk + c(k - 1) \\
d_{k+2} = -(k - 2) + a(k - 1) + bk +ck \\
d_{k+2,i} = -(i - 2) + a(i - 1) + bi + c(i + 1), k \geq 3, 2 \leq i \leq (k - 1) \\
d_{2k+1} = b + 2c \quad \text{and} \quad d_{2k+2} = c.
\]

We now verify that the lexicographical order conditions on \( d_3(1) \) are satisfied.

As \( 2 \leq c \leq a - 2 \) and \( b + c \leq -2 \), we get \( d_{2k+1} \leq a - 4 \). From \( e_k \leq b + c \) and 
\( b(k - 1) + c(k - 2) \leq (k - 2) - (k - 1)a \), we get \( d_{2k+1} \geq 0 \).

For \( k \leq 3 \) and \( 2 \leq i \leq k - 1 \), \( d_{k+2,i} = -(i - 2) + a(i - 1) + bi + c(i + 1). \)
As \( b + c < e_i \), we get \( d_{k+2,i} < c \). As \( -a + 2 \leq b + c \) and \( b + 2c \geq 0 \), we get 
\( d_{2k+1} \leq a - 2 \).

For \( k \leq 3 \), as \( b[k - 1] \leq a + c \), we obtain \( d_2 < \cdots < d_{k-1} \). As \( b + c < e_{k-1} \) and \( b + 2c \leq 0 \), we get \( d_{k-1} \leq a - 2 \). Moreover from \( b(k - 1) + c(k - 2) \leq (k - 2) - (k - 1)a \),
we get \( d_{k-1} \leq a - 2 \).

All \( d_i \)'s are smaller than \( d_1 \), only \( d_{2k+2} \) and \( d_k \) can be equal to \( d_1 \). Therefore we have to verify that \( d_3 \geq d_{k+1} \) when \( k \geq 3 \) (otherwise \( d_3 = d_k \) and \( d_k > d_{k+1} \)).

If \( d_k = a - 2 \), then \( b + c = e_k \), and \( d_{k+1} = a - c - 1 \). As \( a + b + c - 1 > 0 \), we obtain \( d_{k+1} \leq d_2 \). In case of equality, if \( k = 3 \), then \( d_3 = d_k \) and \( d_k > d_{k+1} \), otherwise \( d_3 > d_2 \) and \( d_{k+1} > d_{k+2} \), therefore \( d_3 > d_{k+2} \).

So lexicographical order conditions are satisfied and \( d_1 \ldots d_{2k+2} \) is the beta-
expansion of \( 1 \).

When \( b(k - 1) + c(k - 2) > (k - 2) - (k - 1)a \), as \( b \leq -a \), we get \( k \geq 3 \). Let \( m \) 
be the integer defined by \( m = \min \{i \in \mathbb{N} : (i+1)b + ic > i - (i+1)a \} \).
Note that by definition of \( m \), \( m \leq k - 2 \) and since \( b \leq -a \), \( m \geq 1 \). In this case, 
the complementary factor is 
\[
Q(X) = \sum_{i=0}^{m} X^i.
\]

The beta-expansion of \( 1 \) is then eventually periodic with period \( 1 \), the length of the preperiod is \( m + 2 \).

When \( m = 1 \), \( P(X) = X^4 - (a - 1)X^3 - (a + b)X^2 - (b + c)X - c \) and 
\[
d_3(1) = (a - 2)(2a + b - 2)(2a + 2b + c - 2)(2a + 2b + 2c - 2)^m.
\]
Here \( d_3 = d_{m+2} = a + b - 1 + m(a + b + c - 1) \) and \( d_4 = d_{m+3} = (m+1)(a + b + c - 1) \).

When \( m > 1 \), 
\[
P(X) = X^{m+3} - (a - 1)X^{m+2} - (a + b - 1)X^{m+1} - \sum_{i=3}^{m} (a + b + c - 1)X^i \\
-(a + b + c)X^2 - (b + c)X - c
\]
and $d_3(1) = d_1d_2 \cdots d_{m+2}d_{m+3}$, with

$$
\begin{align*}
  d_1 &= a - 2, \\
  d_2 &= 2a + b - 3, \\
  d_{m+3-i} &= 2a + b - 3 + (m + 1 - i)(a + b + c - 1) \quad m \geq 3, 3 \leq i \leq m, \\
  d_{m+1} &= 2a + b - 2 + (m - 1)(a + b + c - 1), \\
  d_{m+2} &= a + b - 1 + m(a + b + c - 1), \\
  d_{m+3} &= (m + 1)(a + b + c - 1).
\end{align*}
$$

In both cases, $d_1 = a - 2$. Since $b(k - 1) + c(k - 2) > (k - 2) - (k - 1)a$ and $c \leq a - 2$, we get $-2a + 3 \leq b$. Moreover as $b \leq -a$, $1 \leq d_2 \leq a - 2$ when $m = 1$, and $0 \leq d_2 \leq a - 3$ otherwise. By definition of $m$, $(m + 1)b + mc > m - (m + 1)a$, thus $d_{m+2} \geq 0$ and $d_{m+3} \geq c$. Since $e_3 \leq b + c < e_{k-1}$ and $m \leq k - 2$, we obtain $d_{m+3} \leq a - 3$ and $d_{m+2} \leq a - c - 3$.

When $m > 1$, since $mb + (m - 1)c \leq (m - 1) - ma$, we get $d_{m+1} \leq a - 2$. As $0 \leq 2a + b - 2$ and $a + b + c - 1 > 0$, $d_{m+1} > 0$. Moreover as $a + b + c - 1 > 0$, one has $d_2 < d_3 \cdots < d_{m+1}$. Note that, when $m \geq 3$, $d_2 \neq a - 2$.

We now study the cases where $d_i$ is not strictly smaller than $d_1$. When $m = 1$, only $d_2$ may be equal to $a - 2$, then $b = -a$ and $d_3 = c - 2$, thus $d_3 < d_2$. When $m > 1$, only $d_{m+1}$ may be equal to $a - 2$, then $mb = -ma - (m - 1)c + (m - 1)$, and thus $d_2 - d_{m+2} = a - 1 - c$ is a positive integer.

We have proved that the lexicographical order conditions on $d_3(1)$:

$$
\begin{align*}
  d_1d_2 \cdots d_{m+3} >_{lex} &d_id_{i+1} \cdots d_{m+3} \quad \text{for } 2 \leq i \leq m + 3,
\end{align*}
$$

are satisfied, showing in this way that the announced beta-expansions of 1 are right.

Remark 2. The polynomials $Q$ that appear in the cubic case are cyclotomic. In the general case, $Q$ can be noncyclotomic and even nonreciprocal ($\not\mid 0$).

## 3 Cubic simple beta-numbers

In the following, we establish that cubic simple beta-numbers are Pisot numbers. Next we give necessary and sufficient conditions on the coefficients of the minimal polynomial of $\beta$ for $\beta$ to be a simple beta-number.

**Theorem 3.** If $\beta$ is a cubic simple beta-number then $\beta$ is a Pisot number.

Remark 3. This is no longer true for simple beta-numbers of degree 4. For example, the positive root of $X^4 - 3X^3 - 2X^2 - 3$ is a simple beta-number, but is not a Pisot number.

**Proof.** Let $\beta$ be a cubic simple beta-number and let

$$
M_\beta(X) = X^3 - aX^2 - bX - c
$$

be its minimal polynomial. Then $\beta$ has no positive real algebraic conjugate and $c$ is a positive integer smaller than $|\beta|$.
The condition on the product $c$ of the roots of the polynomial $M_\beta$, i.e., $|c| \leq |\beta|$, directly implies, when the Galois conjugates of $\beta$ are not real numbers, that $\beta$ is a Pisot number.

The only other case is the case where both Galois conjugates $\gamma_1$ and $\gamma_2$ of $\beta$ are negative real numbers. We then assume that $\beta$ is a cubic simple beta-number that is not a Pisot number, and show that these hypotheses are contradictory. Let $\gamma_1$ and $\gamma_2$ be the Galois conjugates of $\beta$. As $0 < c \leq |\beta|$, if one of the $\gamma_i$’s is smaller than $-1$ the other one is greater than $-1$. Moreover, as the modulus of a Galois conjugate of a beta-number is smaller than the golden ratio, one can suppose, for example, that

$$- \frac{1 + \sqrt{5}}{2} < \gamma_2 < -1 < \gamma_1 < 0 < \beta$$

Consequently, $M_\beta(-1) > 0$, in other words, $b > a + c + 1$. Note that here $a \in \{|\beta| - 2, |\beta| - 1\}$.

As $\beta$ is a simple beta-number, $d_\beta(1) = d_1 d_2 \ldots d_n$. Denote by $P$ the associated $\beta$-polynomial:

$$P(X) = X^n - \sum_{i=1}^n d_i X^{n-i}$$

and denote by $Q = \sum_{i \geq 0} q_i X^i$ the quotient of the division upon the increasing powers of $P$ by $M_\beta$. In other words,

$$P(X) = M_\beta(X) Q(X)$$

We shall show, by induction, that $q_0 \geq 1$, and that for all $i \geq 0$, $|q_{i+1}| > |q_i|$ with $\text{sgn}(q_{i+1}) = -\text{sgn}(q_i)$. We shall conclude from the growth of the moduli of its coefficients that $Q$ is an infinite series, and thus that $d_\beta(1)$ is not finite.

In what follows, we mainly use the fact that the $d_i$’s are nonnegative integers smaller than $|\beta|$ and the inequality $b \geq a + c + 2$.

First of all, as $d_n = q_0 c$ and $d_n$ and $c$ are positive integers, $q_0 \geq 1$. Since $d_{n-1} = q_0 b + q_1 c$ and $q_0 \geq 1$, $d_{n-1} \geq q_0 a + 2q_0 + (q_0 + q_1)c$. When $a = |\beta| - 1$, we directly get from $d_{n-1} \leq |\beta|$, that $q_1 < -q_0$. When $a = |\beta| - 2$, the lexicographical order conditions on $d_\beta(1)$ imply that

$$d_{n-1} d_n < d_1 d_2 \ldots d_n.$$

By definition of beta-expansions, $d_1 = |\beta|$ and here $d_2 < d_n$. Indeed as

$$\gamma_2 = \frac{1}{2} (a - \beta + \sqrt{(a - \beta)^2 - 4c}),$$

and $\gamma_2 > -(1 + \sqrt{5})/2$, we get that

$$c > \frac{\sqrt{5} - 1}{2} \beta + \frac{1 + \sqrt{5}}{2} \beta \{\beta\},$$
and in particular, that $c > \beta/2$, consequently $d_n = c$ and that $\beta \{\beta\} < c$. Thus $d_2 = \lfloor \beta \{\beta\} \rfloor$ is strictly smaller than $d_n$. Therefore the previous lexicographical order condition implies that $d_{n-1} < \lfloor \beta \rfloor$. So, as $d_{n-1} \geq \lfloor \beta \rfloor + (q_0 + q_1)c, q_1 < -q_0$.

As $d_{n-2} = q_0 a + q_1 b + q_2 c$ and $q_1 < -q_0 < 0, d_{n-2} \leq (q_1 + q_0)a + 2q_1 + (q_1 + q_2)c$, that is $d_{n-2} < -\lfloor \beta \rfloor + (q_1 + q_2)c$, so $q_2 > -q_1$.

For all positive integers $i$, $d_{n-(2i+1)} = -q_{2i+1} + q_{2i-1}a + q_{2i}b + q_{2i+1}c$. From $q_{2i} > 0$, we get $d_{n-(2i+1)} \geq (q_{2i-1} + q_{2i})a + q_{2i}b + (q_{2i+1} - q_{2i-2}) + (q_{2i} + q_{2i+1})c$. From $(q_{2i-1} + q_{2i}) \geq 1, q_{2i} > 2i$ and $(q_{2i} - q_{2i-2}) > 1$, we obtain $d_{n-(2i+1)} > [\beta] + (q_{2i} + q_{2i+1})c$, and thus $q_{2i+1} < -q_{2i}$.

For all positive integers $i$, $d_{n-(2i+2)} = -q_{2i+1} + q_{2i}a + q_{2i+1}b + q_{2i+2}c$. From $q_{2i+1} < 0$, we get $d_{n-(2i+1)} \leq (q_{2i} + q_{2i+1})a + q_{2i+1}b + q_{2i+2}c$ and $q_{2i+1} < -(2i + 1)$ and $(q_{2i+1} - q_{2i-2}) < -1$, we get $d_{n-(2i+2)} < -\lfloor \beta \rfloor + (q_{2i+1} + q_{2i+2})c$, thus $q_{2i+2} > -q_{2i+1}$.

So $Q$ is an infinite series; consequently if $\beta$ is not a Pisot number, $d_\beta(1)$ is not finite.

As a consequence of Theorems 2 and 3, we obtain the above characterization of cubic simple beta-numbers.

**Proposition 2.** Let $\beta$ be a cubic Pisot number and let

$$M_\beta(x) = X^3 - aX^2 - bX - c$$

be its minimal polynomial.

Then $\beta$ is a simple beta-number if and only it satisfies one of the following conditions:

- Case 1: $b \geq 0$ and $c > 0$
- Case 2: $-a < b < 0$ and $b + c \geq 0$
- Case 3: $b \leq -a$ and $b(k - 1) + c(k - 2) \leq (k - 2) - (k - 1)a$, where $k$ is the integer in $\{2, 3, \ldots, a - 2\}$ such that, denoting $e_k = 1 - a + (a - 2)/k, e_k \leq b + c < e_{k-1}$.

The problem of finding such a characterization remains open for simple beta-numbers of higher degree.

**References**


