β -expansions for cubic Pisot numbers

Frédérique Bassino

► To cite this version:

Frédérique Bassino. β -expansions for cubic Pisot numbers. 5th Latin American Theoretical Informatics (LATIN'2002), 2002, United States. pp.141-152. hal-00619858

HAL Id: hal-00619858

<https://hal.science/hal-00619858>

Submitted on 6 Oct 2011

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Beta-expansions for cubic Pisot numbers

Frédérique Bassino

I.G.M., Université de Marne La Vallée
77454 Marne-la-Vallée Cedex 2. France
e-mail: bassino@univ-mlv.fr

Abstract. Real numbers can be represented in an arbitrary base $\beta > 1$ using the transformation $T_\beta : x \rightarrow \beta x \pmod{1}$ of the unit interval; any real number $x \in [0, 1]$ is then expanded into $d_\beta(x) = (x_i)_{i \geq 1}$ where $x_i = \lfloor \beta T_\beta^{i-1}(x) \rfloor$.

The closure of the set of the expansions of real numbers of $[0, 1[$ is a subshift of $\{a \in \mathbb{N} \mid a < \beta\}^{\mathbb{N}}$, called the beta-shift. This dynamical system is characterized by the beta-expansion of 1; in particular, it is of finite type if and only if $d_\beta(1)$ is finite; β is then called a simple beta-number.

We first compute the beta-expansion of 1 for any cubic Pisot number. Next we show that cubic simple beta-numbers are Pisot numbers.

Introduction

Representations of real numbers in an arbitrary base $\beta > 1$, called beta-expansions, have been introduced by Rényi ([14]). They arise from the orbits of the piecewise-monotone transformation of the unit interval : $T_\beta : x \rightarrow \beta x \pmod{1}$. Such transformations were extensively studied in ergodic theory ([13]).

More precisely, any real number $x \in [0, 1]$ is expanded into $d_\beta(x) = (x_i)_{i \geq 1}$ where $x_i = \lfloor \beta T_\beta^{i-1}(x) \rfloor$. The nonnegative integers d_i are elements of the digit alphabet $A = \{a \in \mathbb{N} \mid a < \beta\}$. These representations generalize standard representations in an integral base to a real base; indeed the beta-expansion of any real number of $[0, 1[$ can equivalently be obtained by the greedy algorithm. Only the beta-expansion of 1 differs.

Properties of beta-expansions are strongly related to symbolic dynamics ([4]). The closure of the set of infinite sequences, appearing as beta-expansions of numbers of the interval $[0, 1[$, is a dynamical system, that is, a closed shift-invariant subset of $A^{\mathbb{N}}$, called the *beta-shift*.

An important property of the beta-shift is that its nature is entirely determined, in a combinatorial manner, by the beta-expansion of 1: the beta-shift is sofic, that is to say the set of its finite factors is recognized by a finite automaton, if and only the beta-expansion of 1 is eventually periodic ([3]); it is of finite type, that is to say the set of its finite factors is defined by forbidding a finite set of words, if and only if the beta-expansion of 1 is finite ([12]).

When the beta-expansion of 1 is eventually periodic, β is called a *beta-number* and when the beta-expansion of 1 is finite, β is said to be a *simple beta-number*.

The eventually periodic beta-expansions were extensively studied by Bertrand ([3]) and by Schmidt ([15]). In particular, it is known that Pisot numbers are beta-numbers. Concerning Salem numbers, we only know that if β is a Salem number of degree 4, then the beta-expansion of 1 is eventually periodic ([5]). It is conjectured that Salem numbers of degree 6 are still beta-numbers, but not all Salem numbers of degree 8 ([7]).

The domain of the Galois conjugates of all beta-numbers was also investigated independently by Solomyak ([16]) and by Flatto, Lagarias and Poonen ([8]).

For a general presentation of the beta-shift one can refer to [9].

In the following, we summarize properties of beta-numbers. We compute the beta-expansion of 1 for any cubic Pisot number and we establish a characterization of cubic simple beta-numbers, showing that they are Pisot numbers.

A very close problem, seen from the point of view of numeration systems, was studied by Akiyama ([1]). He showed that in the cubic case, the real numbers of the set $\mathbb{N}[\beta^{-1}]$ have a finite beta-expansion if and only if β is a Pisot unit and 1 has a finite beta-expansion. This finiteness problem is equivalent to a problem of fractal tiling generated by Pisot numbers.

1 Beta-numbers

Real numbers can be represented in an arbitrary base $\beta > 1$ using the transformation $T_\beta : x \rightarrow \beta x \pmod{1}$ of the unit interval; any real number $x \in [0, 1]$ is then expanded into $d_\beta(x) = (x_i)_{i \geq 1}$ where $x_i = \lfloor \beta T_\beta^{i-1}(x) \rfloor$. When a beta-expansion is of the form uv^ω , the expansion is said to be *eventually periodic*. If a representation ends with infinitely many zeros, like $u0^\omega$, it is said to be *finite* and the ending zeros are omitted.

Let us denote by S_β the closure of all beta-expansions of real numbers of $[0, 1]$ and by σ the (one-sided) shift defined by $\sigma((x_i)_{i \geq 1}) = (x_{i+1})_{i \geq 1}$. The set S_β endowed with the shift is called the beta-shift, it is a subshift of $A^\mathbb{N}$, A being the digit set, *i.e.*, $A = \{a \in \mathbb{N} \mid a < \beta\}$.

An important property ([13]) of the beta-shift S_β is that its nature is entirely determined by $d_\beta(1)$ the beta-expansion of 1. Indeed, setting $d^*(1) = d_\beta(1)$ if $d_\beta(1)$ is infinite and $d^*(1) = (d_1 d_2 \dots d_{n-1} (d_n - 1))^\omega$ if $d_\beta(1) = d_1 d_2 \dots d_{n-1} d_n$, a sequence x of nonnegative integers belongs to S_β if and only if it satisfies the following lexicographical order conditions: $\forall p \geq 0, \quad \sigma^p(x) \leq d^*(1)$.

Recall that the beta-expansion of 1 also can be characterized ([13]) by lexicographical order conditions: let $d = (d_i)_{i \geq 1}$ be a sequence of nonnegative integers different from 10^ω , such that $\sum_{i \geq 1} d_i \beta^{-i} = 1$, with $d_1 \geq 1$ and for $i \geq 2$, $d_i \leq d_1$, then d is the beta-expansion of 1 if and only if for all $p \geq 1$, $\sigma^p(d) < d$.

We recall that an algebraic integer β strictly greater than 1 is called a *Perron number* if all its Galois conjugates have modulus strictly less than β , a *Pisot number* if all its Galois conjugates have modulus strictly less than 1, and a *Salem number* if all its conjugates are less than 1 in modulus and at least one conjugate has modulus 1.

Let β be a beta-number. Denote by $d_\beta(1) = d_1 \dots d_n(d_{n+1} \dots d_{n+p})^\omega$, where n and p are chosen minimal, the beta-expansion of 1. Then the adjacency matrix $\mathcal{M}_\beta$ of the finite automaton recognizing the set of its finite factors (Fig.1) is a *primitive* (i.e., its associated graph is strongly connected and the lengths of its cycles are relatively prime) nonnegative integral matrix whose spectral radius is β ; so, from the Perron-Frobenius theorem, β is a Perron number.

Fig. 1. Automaton recognizing the set of the finite factors of S_β

The characteristic polynomial of $\mathcal{M}_\beta$

$$P(X) = X^{n+p} - \sum_{i=1}^{n+p} d_i X^{n+p-i} - X^n + \sum_{i=1}^n d_i X^{n-i}$$

is called, following the terminology introduced by Hollander ([11]), the associated *beta-polynomial*.

As P is a multiple of the minimal polynomial M_β of β , $P(0) = d_{n+p} - d_n$ is a multiple of $|M_\beta(0)| = |\prod \beta_i|$, where β_i runs over the set of algebraic conjugates of β . So, we get that $|\prod \beta_i|$ has to be smaller than $\lfloor \beta \rfloor$.

As a consequence, in the quadratic case, the only beta-numbers are the Pisot numbers. Conversely, it is known that if β is a Pisot number then β is a beta-number ([2]). An important gap remains between Pisot and Perron numbers.

Example 1. The quadratic number $\beta = (1 + \sqrt{13})/2$ is not a beta-number since $M_\beta(X) = X^2 - X - 3$ and $M_\beta(0) = 3 > \lfloor \beta \rfloor$.

Let β be the Pisot number $(3+\sqrt{5})/2$, then β is a beta-number and $d_\beta = 21^\omega$.

Let β be the golden ratio $(1+\sqrt{5})/2$, then β is a simple beta-number and $d_\beta(1) = 11$.

On the other hand, the domain of the Galois conjugates of beta-numbers was studied by Solomyak ([16]) and independently by Flatto, Lagarias and Poonen ([8]). They showed in particular that if the beta-expansion of 1 is eventually periodic then the Galois conjugates of β have modulus less than the golden ratio $(1+\sqrt{5})/2$. It was already known (see [9]) that β cannot have a Galois conjugate greater than 1.

Solomyak ([16]) proved that the topological closure of conjugates of beta-numbers and the one of conjugates of simple beta-numbers are the same. However, there is an important difference between the conjugates of beta-numbers and the ones of simple beta numbers: if β is a simple beta-number then β has no algebraic conjugate that is a nonnegative real number.

Indeed, let β be a simple beta-number and set $d_\beta(1) = d_1 \dots d_n$. Consider

Fig. 2. Automaton recognizing the set of the finite factors of S_β

the finite automaton recognizing the set of the finite factors of the associated beta-shift (Fig. 2). Let $\mathcal{M}_\beta$ be the transition matrix of this automaton. The characteristic polynomial of $\mathcal{M}_\beta$, which is called the associated *beta-polynomial*,

$$P(X) = X^n - \sum_{i=1}^n d_i X^{n-i}$$

has only one positive real root.

Example 2. Salem numbers are roots of reciprocal polynomials. Thus if β is a Salem number, $1/\beta > 0$ is a Galois conjugate of β , and so β is not a simple beta-number.

The previous conditions are sufficient for a quadratic algebraic integer to be a simple beta-number.

Proposition 1. [10] *The simple beta-numbers of degree 2 are exactly the quadratic Pisot numbers without a positive real Galois conjugate. They are the positive roots of the polynomials*

$$X^2 - aX - b \quad \text{with} \quad a \geq b \geq 1,$$

The beta-expansion of 1 is then $d_\beta(1) = ab$.

Example 3. The minimal polynomial of $(1 + \sqrt{5})/2$ is $X^2 - X - 1$, $(1 + \sqrt{5})/2$ is a simple beta-number and $d_\beta(1) = 11$.

The minimal polynomial of $(3 + \sqrt{5})/2$ is $X^2 - 3X + 1$, therefore $(3 + \sqrt{5})/2$ is not a simple beta-number.

2 Beta-expansions of 1 for cubic Pisot numbers

Let us recall the characterization of cubic Pisot numbers due to Akiyama ([1])

Theorem 1 (Akiyama [1]). *Let $\beta > 1$ be a cubic number and let*

$$M_\beta(x) = X^3 - aX^2 - bX - c$$

be its minimal polynomial.

Then β is a Pisot number if and only if it both inequalities

$$|b - 1| < a + c \quad \text{and} \quad (c^2 - b) < \text{sgn}(c)(1 + ac)$$

hold.

Remark 1. Note that a must be a nonnegative integer.

The following theorem gives the β -expansion of 1 for any cubic Pisot number.

Theorem 2. *Let β be a cubic Pisot number and let*

$$M_\beta(x) = X^3 - aX^2 - bX - c$$

be its minimal polynomial. Then the beta-expansion of 1 is

- *Case 1 :* When $b \geq a$, then $d_\beta(1) = (a + 1)(b - 1 - a)(a + c - b)(b - c)c$.
- *Case 2:* When $0 \leq b \leq a$, if $c > 0$, $d_\beta(1) = abc$, otherwise,

$$d_\beta(1) = a[(b - 1)(c + a)]^\omega.$$

- Case 3: When $-a < b < 0$, if $b + c \geq 0$, then $d_\beta(1) = (a-1)(a+b)(b+c)c$, otherwise $d_\beta(1) = (a-1)(a+b-1)(a+b+c-1)^\omega$
- Case 4: When $b \leq -a$, let k be the integer of $\{2, 3, \dots, a-2\}$ such that, denoting $e_k = 1 - a + (a-2)/k$, $e_k \leq b + c < e_{k-1}$.
 - If $b(k-1) + c(k-2) \leq (k-2) - (k-1)a$, $d_\beta(1) = d_1 \dots d_{2k+2}$ with

$$\begin{aligned}
 d_1 &= a-2, \\
 d_{k+2-i} &= -(k+3-i) + a(k+2-i) + b(k+1-i) + c(k-i), 3 \leq i \leq k \\
 d_k &= -k + ak + b(k-1) + c(k-2) \\
 d_{k+1} &= -(k-1) + ak + bk + c(k-1) \\
 d_{k+2} &= -(k-2) + a(k-1) + bk + ck \\
 d_{2k+2-i} &= -(i-2) + a(i-1) + bi + c(i+1) \quad k \geq 3, 2 \leq i \leq (k-1) \\
 d_{2k+1} &= b+2c \quad \text{and} \quad d_{2k+2} = c.
 \end{aligned}$$

- If $b(k-1) + c(k-2) > (k-2) - (k-1)a$, let m be the integer defined by $m = \min\{i \in \mathbb{N} \text{ such that } (i+1)b + ic > i - (i+1)a\}$.

$$\text{When } m = 1, d_\beta(1) = (a-2)(2a+b-2)(2a+2b+c-2)(2a+2b+2c-2)^\omega.$$

$$\text{When } m > 1, d_\beta(1) = d_1 d_2 \dots d_{m+2} d_{m+3}^\omega, \text{ with}$$

$$\begin{aligned}
 d_1 &= a-2, \quad d_2 = 2a+b-3, \\
 d_{m+3-i} &= 2a+b-3 + (m+1-i)(a+b+c-1) \quad m \geq 3, 3 \leq i \leq m, \\
 d_{m+1} &= 2a+b-2 + (m-1)(a+b+c-1), \\
 d_{m+2} &= a+b-1 + m(a+b+c-1), \\
 d_{m+3} &= (m+1)(a+b+c-1).
 \end{aligned}$$

Example 4. When $a \geq b \geq 0$ and $c > 0$, we obtain the only beta-expansion of 1 of length 3.

The smallest Pisot number has $M_\beta = X^3 - X - 1$ as minimal polynomial, it is a simple beta-number and $d_\beta(1) = 10001$.

The positive root β of $M_\beta = X^3 - 3X^2 + 2X - 2$ is a simple beta-number and $d_\beta(1) = 2102$.

The case where $b \leq -a$ shows that from a cubic simple beta-number, we can obtain an arbitrary long beta-expansion of 1. For any integer k greater than or equal to 2, the real root β of the irreducible polynomial $X^3 - (k+2)X^2 + 2kX - k$, is a simple beta number whose integer part is equal to k , and the beta-expansion of 1 has length $2k+2$. For $k=2$, we get $d_\beta(1) = 221002$; for $k=3$, we get $d_\beta(1) = 31310203$.

Example 5. The greatest positive root β of $M_\beta = X^3 - 2X^2 - X + 1$ is a beta-number and $d_\beta(1) = 2(01)^\omega$.

If β is the positive root of $X^3 - 5X^2 + 3X - 2$, then $d_\beta(1) = 413^\omega$. When β is the greatest positive root of $X^3 - 5X^2 + X + 2$, then $d_\beta(1) = 431^\omega$.

For any integer k greater than or equal to 3, the real root β of the irreducible polynomial $X^3 - (k+2)X^2 + (2k-1)X - (k-1)$, is a beta number whose integer part is equal to k , and the beta-expansion of 1 is eventually periodic of period

1, the length of its preperiod k . For $k = 3$, we get $d_\beta(1) = 3302^\omega$; for $k = 4$, we get $d_\beta(1) = 42403^\omega$.

Proof. It is known that Pisot numbers are beta-numbers, thus, for any cubic Pisot number β , the beta-expansion of 1 is finite or eventually periodic. In any case, we first compute the associated beta-polynomial P . Next we prove that the sequence $d = (d_i)_{i \geq 1}$ of nonnegative integers obtained from the beta-polynomial satisfy lexicographical order conditions: for all $p \geq 1$, $\sigma^p(d) < d$.

First of all, we recall that, from Theorem 1, a cubic number β , greater than 1 and having

$$M_\beta(X) = X^3 - aX^2 - bX - c$$

as minimal polynomial, is a cubic Pisot number if and only if it both

$$|b - 1| < a + c \quad \text{and} \quad (c^2 - b) < \text{sgn}(c)(1 + ac)$$

hold.

Denote by Q the *complementary factor* of the beta-polynomial P defined by $P(X) = M_\beta(X)Q(X)$. As we shall see in what follows, the value of Q depends upon the value of the coefficients of M_β .

Case 1: When $b > a$, as β is a Pisot number, from Theorem 1, c is a positive integer. In this case, the complementary factor is $Q(X) = X^2 - X + 1$ and $d_\beta(1) = (a + 1)(b - 1 - a)(a + c - b)(b - c)c$.

Indeed, as $(c^2 - b) < \text{sgn}(c)(1 + ac)$ and $c > 0$, we get $c \leq a + 1$. As $|b - 1| < a + c$, we get $b - 1 - a \leq a$ and $0 \leq a - b + c$. From $b > a$, we get that $0 \leq b - a - 1$ and, as $c \leq a + 1$, that $a - b + c \leq a$. Finally as $0 \leq a - b + c \leq a$, we obtain $0 \leq b - c \leq a$.

Case 2: When $0 \leq b \leq a$, the complementary factor is then $Q(X) = 1$ and the associated beta-polynomial is equal to the minimal polynomial.

If $c > 0$, then $d_\beta(1) = abc$. Indeed, as $(c^2 - b) < \text{sgn}(c)(1 + ac)$, we get $c \leq a$.

If $c < 0$, then $d_\beta(1) = a[(b - 1)(a + c)]^\omega$. As $|b - 1| < a + c$, we get $b - 1 \leq a - 2$. As $(c^2 - b) < \text{sgn}(c)(1 + ac)$, we get that $c \geq -a$ and, consequently, $0 \leq c + a \leq a - 1$.

Case 3: When $-a < b < 0$, if $b + c \geq 0$ then the complementary factor is $Q(X) = X + 1$ and $d_\beta(1) = (a - 1)(a + b)(b + c)c$. Indeed, as $-a < b < 0$, we obtain $1 \leq a + b \leq a - 1$. Since $b + c \geq 0$, c is a positive integer. From $(c^2 - b) < \text{sgn}(c)(1 + ac)$, we get that $c \leq a - 1$ and $b + c \leq a - 2$.

If $b + c < 0$, then $Q(X) = 1$ and $d_\beta(1) = (a - 1)(a + b - 1)(a + b + c - 1)^\omega$. As $-a < b < 0$, we get $0 \leq a + b - 1 \leq a - 2$. From $|b - 1| < a + c$, we get that $1 \leq a + b + c - 1$ and as $b + c < 0$, we obtain $a + b + c - 1 \leq a - 2$.

Case 4: First of all, since $|b - 1| < a + c$, we get $-a + 2 \leq b + c$. Moreover as $b \leq -a$, we get $c \geq 2$ and as $(c^2 - b) < \text{sgn}(c)(1 + ac)$, we obtain $c \leq a - 2$, thus $b + c \leq -2$. So, there exists an integer k in $\{2, 3, \dots, a - 2\}$, such that, denoting $e_k = 1 - a + (a - 2)/k$, $e_k \leq b + c < e_{k-1}$.

When $b(k - 1) + c(k - 2) \leq (k - 2) - (k - 1)a$, the complementary factor is

$$Q(X) = \frac{(X^k - 1)(X^{k+1} - 1)}{(X - 1)^2}$$

and $d_\beta(1) = d_1 \dots d_{2k+2}$ with

$$\begin{aligned} d_1 &= a - 2, \\ d_{k+2-i} &= -(k+3-i) + a(k+2-i) + b(k+1-i) + c(k-i), k \geq 3, 3 \leq i \leq k \\ d_k &= -k + ak + b(k-1) + c(k-2) \\ d_{k+1} &= -(k-1) + ak + bk + c(k-1) \\ d_{k+2} &= -(k-2) + a(k-1) + bk + ck \\ d_{2k+2-i} &= -(i-2) + a(i-1) + bi + c(i+1) \quad k \geq 3, 2 \leq i \leq (k-1) \\ d_{2k+1} &= b + 2c \quad \text{and} \quad d_{2k+2} = c. \end{aligned}$$

We now verify that the lexicographical order conditions on $d_\beta(1)$ are satisfied.

As $2 \leq c \leq a-2$ and $b+c \leq -2$, we get $d_{2k+1} \leq a-4$. From $e_k \leq b+c$ and $b(k-1) + c(k-2) \leq (k-2) - (k-1)a$, we get $d_{2k+1} \geq 0$.

For $k \leq 3$ and $2 \leq i \leq k-1$, $d_{2k+2-i} = -(i-2) + a(i-1) + bi + c(i+1)$. As $b+c < e_i$, we get $d_{2k+2-i} < c$. As $-a+2 \leq b+c$ and $b+2c \geq 0$, we get $d_{2k+2-i} \geq i$.

As $e_k \leq b+c$, we obtain $d_{k+2} \geq 0$. Since $c \leq a-2$, $d_{k+1} > d_{k+2}$ and since $b+c \leq -2$, $d_k > d_{k+1}$. Moreover from $b(k-1) + c(k-2) \leq (k-2) - (k-1)a$, we get $d_k \leq a-2$.

For $k \leq 3$, as $|b-1| < a+c$, we obtain $d_2 < \dots < d_{k-1}$. As $b+c < e_{k-1}$ and $b+2c \leq 0$, we get $d_{k-1} < a-2$. Moreover from $c \leq a-2$ and $a+b+c-1 > 0$, we get that $d_2 = 2a+b-3$ is nonnegative.

All d_i 's are smaller than d_1 , only d_{2k+2} and d_k can be equal to d_1 . Therefore we have to verify that $d_2 \geq d_{k+1}$ when $k \geq 3$ (otherwise $d_2 = d_k$ and $d_k > d_{k+1}$). If $d_k = a-2$, then $b+c = e_k$, and $d_{k+1} = a-c-1$. As $a+b+c-1 > 0$, we obtain $d_{k+1} \leq d_2$. In case of equality, if $k=3$, then $d_3 = d_k$ and $d_k > d_{k+2}$, otherwise $d_3 > d_2$ and $d_{k+1} > d_{k+2}$, therefore $d_3 > d_{k+2}$.

So lexicographical order conditions are satisfied and $d_1 \dots d_{2k+2}$ is the beta-expansion of 1.

When $b(k-1) + c(k-2) > (k-2) - (k-1)a$, as $b \leq -a$, we get $k \geq 3$. Let m be the integer defined by $m = \min\{i \in \mathbb{N} \text{ such that } (i+1)b + ic > i - (i+1)a\}$. Note that by definition of m , $m \leq k-2$ and since $b \leq -a$, $m \geq 1$. In this case, the complementary factor is

$$Q(X) = \sum_{i=0}^m X^i.$$

The beta-expansion of 1 is then eventually periodic with period 1, the length of the preperiod is $m+2$.

When $m=1$, $P(X) = X^4 - (a-1)X^3 - (a+b)X^2 - (b+c)X - c$ and

$$d_\beta(1) = (a-2)(2a+b-2)(2a+2b+c-2)(2a+2b+2c-2)^\omega.$$

Here $d_3 = d_{m+2} = a+b-1+m(a+b+c-1)$ and $d_4 = d_{m+3} = (m+1)(a+b+c-1)$.

When $m > 1$,

$$\begin{aligned} P(X) &= X^{m+3} - (a-1)X^{m+2} - (a+b-1)X^{m+1} - \sum_{i=3}^m (a+b+c-1)X^i \\ &\quad - (a+b+c)X^2 - (b+c)X - c \end{aligned}$$

and $d_\beta(1) = d_1 d_2 \dots d_{m+2} d_{m+3}^\omega$, with

$$\begin{aligned} d_1 &= a - 2, & d_2 &= 2a + b - 3, \\ d_{m+3-i} &= 2a + b - 3 + (m+1-i)(a+b+c-1) & m \geq 3, 3 \leq i \leq m, \\ d_{m+1} &= 2a + b - 2 + (m-1)(a+b+c-1), \\ d_{m+2} &= a + b - 1 + m(a+b+c-1), \\ d_{m+3} &= (m+1)(a+b+c-1). \end{aligned}$$

In both cases, $d_1 = a - 2$. Since $b(k-1) + c(k-2) > (k-2) - (k-1)a$ and $c \leq a - 2$, we get $-2a + 3 \leq b$. Moreover as $b \leq -a$, $1 \leq d_2 \leq a - 2$ when $m = 1$, and $0 \leq d_2 \leq a - 3$ otherwise. By definition of m , $(m+1)b + mc > m - (m+1)a$, thus $d_{m+2} \geq 0$ and $d_{m+3} \geq c$. Since $e_k \leq b + c < e_{k-1}$ and $m \leq k - 2$, we obtain $d_{m+3} \leq a - 3$ and $d_{m+2} \leq a - c - 3$.

When $m > 1$, since $mb + (m-1)c \leq (m-1) - ma$, we get $d_{m+1} \leq a - 2$. As $0 \leq 2a + b - 2$ and $a + b + c - 1 > 0$, $d_{m+1} > 0$. Moreover as $a + b + c - 1 > 0$, one has $d_2 < d_3 < \dots < d_{m+1}$. Note that, when $m \geq 3$, $d_2 \neq a - 2$.

We now study the cases where d_i is not strictly smaller than d_1 . When $m = 1$, only d_2 may be equal to $a - 2$, then $b = -a$ and $d_3 = c - 2$, thus $d_3 < d_2$. When $m > 1$, only d_{m+1} may be equal to $a - 2$, then $mb = -ma - (m-1)c + (m-1)$, and thus $d_2 - d_{m+2} = a - 1 - c$ is a positive integer.

We have proved that the lexicographical order conditions on $d_\beta(1)$:

$$d_1 d_2 \dots d_{m+3}^\omega >_{lex} d_i d_{i+1} \dots d_{m+3}^\omega \quad \text{for } 2 \leq i \leq m+3,$$

are satisfied, showing in this way that the announced beta-expansions of 1 are right.

Remark 2. The polynomials Q that appear in the cubic case are cyclotomic. In the general case, Q can be noncyclotomic and even nonreciprocal ([6]).

3 Cubic simple beta-numbers

In the following, we establish that cubic simple beta-numbers are Pisot numbers. Next we give necessary and sufficient conditions on the coefficients of the minimal polynomial of β for β to be a simple beta-number.

Theorem 3. *If β is a cubic simple beta-number then β is a Pisot number.*

Remark 3. This is no longer true for simple beta-numbers of degree 4. For example, the positive root of $X^4 - 3X^3 - 2X^2 - 3$ is a simple beta-number, but is not a Pisot number.

Proof. Let β be a cubic simple beta-number and let

$$M_\beta(X) = X^3 - aX^2 - bX - c$$

be its minimal polynomial. Then β has no positive real algebraic conjugate and c is a positive integer smaller than $\lfloor \beta \rfloor$.

The condition on the product c of the roots of the polynomial M_β , *i.e.*, $|c| \leq \lfloor \beta \rfloor$, directly implies, when the Galois conjugates of β are not real numbers, that β is a Pisot number.

The only other case is the case where both Galois conjugates γ_1 and γ_2 of β are negative real numbers. We then assume that β is a cubic simple beta-number that is not a Pisot number, and show that these hypotheses are contradictory. Let γ_1 and γ_2 be the Galois conjugates of β . As $0 < c \leq \lfloor \beta \rfloor$, if one of the γ_i 's is smaller than -1 the other one is greater than -1 . Moreover, as the modulus of a Galois conjugate of a beta-number is smaller than the golden ratio, one can suppose, for example, that

$$-\frac{1+\sqrt{5}}{2} < \gamma_2 < -1 < \gamma_1 < 0 < \beta$$

Consequently, $M_\beta(-1) > 0$, in other words, $b > a + c + 1$. Note that here $a \in \{\lfloor \beta \rfloor - 2, \lfloor \beta \rfloor - 1\}$.

As β is a simple beta-number, $d_\beta(1) = d_1 d_2 \dots d_n$. Denote by P the associated β -polynomial:

$$P(X) = X^n - \sum_{i=1}^n d_i X^{n-i}$$

and denote by $Q = \sum_{i \geq 0} q_i X^i$ the quotient of the division upon the increasing powers of P by M_β . In other words,

$$P(X) = M_\beta(X)Q(X)$$

We shall show, by induction, that $q_0 \geq 1$, and that for all $i \geq 0$, $|q_{i+1}| > |q_i|$ with $\text{sgn}(q_{i+1}) = -\text{sgn}(q_i)$. We shall conclude from the growth of the moduli of its coefficients that Q is an infinite series, and thus that $d_\beta(1)$ is not finite.

In what follows, we mainly use the fact that the d_i 's are nonnegative integers smaller than $\lfloor \beta \rfloor$ and the inequality $b \geq a + c + 2$.

First of all, as $d_n = q_0 c$ and d_n and c are positive integers, $q_0 \geq 1$. Since $d_{n-1} = q_0 b + q_1 c$ and $q_0 \geq 1$, $d_{n-1} \geq q_0 a + 2q_0 + (q_0 + q_1)c$. When $a = \lfloor \beta \rfloor - 1$, we directly get from $d_{n-1} \leq \lfloor \beta \rfloor$, that $q_1 < -q_0$. When $a = \lfloor \beta \rfloor - 2$, the lexicographical order conditions on $d_\beta(1)$ imply that

$$d_{n-1} d_n < d_1 d_2 \dots d_n.$$

By definition of beta-expansions, $d_1 = \lfloor \beta \rfloor$ and here $d_2 < d_n$. Indeed as

$$\gamma_2 = \frac{1}{2} \left(a - \beta + \sqrt{(a - \beta)^2 - \frac{4c}{\beta}} \right),$$

and $\gamma_2 > -(1 + \sqrt{5})/2$, we get that

$$c > \frac{\sqrt{5}-1}{2} \beta + \frac{1+\sqrt{5}}{2} \beta \{\beta\},$$

and in particular, that $c > \beta/2$, consequently $d_n = c$ and that $\beta\{\beta\} < c$. Thus $d_2 = \lfloor \beta\{\beta\} \rfloor$ is strictly smaller than d_n . Therefore the previous lexicographical order condition implies that $d_{n-1} < \lfloor \beta \rfloor$. So, as $d_{n-1} \geq \lfloor \beta \rfloor + (q_0 + q_1)c$, $q_1 < -q_0$.

As $d_{n-2} = q_0a + q_1b + q_2c$ and $q_1 < -q_0 < 0$, $d_{n-2} \leq (q_1 + q_0)a + 2q_1 + (q_1 + q_2)c$, that is $d_{n-2} < -\lfloor \beta \rfloor + (q_1 + q_2)c$, so $q_2 > -q_1$.

For all positive integers i , $d_{n-(2i+1)} = -q_{2i-2} + q_{2i-1}a + q_{2i}b + q_{2i+1}c$. From $q_{2i} > 0$, we get $d_{n-(2i+1)} \geq (q_{2i-1} + q_{2i})a + q_{2i} + (q_{2i} - q_{2i-2}) + (q_{2i} + q_{2i+1})c$. From $(q_{2i-1} + q_{2i}) \geq 1$, $q_{2i} > 2i$ and $(q_{2i} - q_{2i-2}) > 1$, we obtain $d_{n-(2i+1)} > \lfloor \beta \rfloor + (q_{2i} + q_{2i+1})c$, and thus $q_{2i+1} < -q_{2i}$.

For all positive integers i , $d_{n-(2i+2)} = -q_{2i-1} + q_{2i}a + q_{2i+1}b + q_{2i+2}c$. From $q_{2i+1} < 0$, we get $d_{n-(2i+2)} \leq (q_{2i} + q_{2i+1})a + q_{2i+1} + (q_{2i+1} - q_{2i-1}) + (q_{2i+1} + q_{2i+2})c$. As $(q_{2i} + q_{2i+1}) \leq -1$, $q_{2i+1} < -(2i + 1)$ and $(q_{2i+1} - q_{2i-1}) < -1$, we get $d_{n-(2i+2)} < -\lfloor \beta \rfloor + (q_{2i+1} + q_{2i+2})c$, thus $q_{2i+2} > -q_{2i+1}$.

So Q is an infinite series; consequently if β is not a Pisot number, $d_\beta(1)$ is not finite.

As a consequence of Theorems 2 and 3, we obtain the above characterization of cubic simple beta-numbers.

Proposition 2. *Let β be a cubic Pisot number and let*

$$M_\beta(x) = X^3 - aX^2 - bX - c$$

be its minimal polynomial.

Then β is a simple beta-number if and only if it satisfies one of the following conditions:

- *Case 1: $b \geq 0$ and $c > 0$*
- *Case 2: $-a < b < 0$ and $b + c \geq 0$*
- *Case 3: $b \leq -a$ and $b(k - 1) + c(k - 2) \leq (k - 2) - (k - 1)a$, where k is the integer in $\{2, 3, \dots, a - 2\}$ such that, denoting $e_k = 1 - a + (a - 2)/k$, $e_k \leq b + c < e_{k-1}$.*

The problem of finding such a characterization remains open for simple beta-numbers of higher degree.

References

- [1] S. Akiyama. Cubic Pisot units with finite beta expansions. In F. Halter-Koch and R.F. Tichy, editors, *Algebraic Number Theory and Diophantine Analysis*, 11–26. de Gruyter, 2000.
- [2] A. Bertrand. Développements en base de Pisot et répartition modulo 1. *C. R. Acad. Sci. Paris*, 285:419–421, 1977.
- [3] A. Bertrand-Mathis. Développement en base θ , répartition modulo 1 de la suite $(x\theta^n)_{n \geq 0}$, langages codés et θ -shift. *Bull. Soc. Math. France*, 114:271–323, 1986.
- [4] F. Blanchard. β -expansions and symbolic dynamics. *Theor. Comput. Sci.*, 65:131–141, 1989.

- [5] D. W. Boyd. Salem numbers of degree four have periodic expansions. In *Number theory*, pages 57–64. de Gruyter, 1989.
- [6] D. W. Boyd. On beta expansions for Pisot numbers. *Mathematics of Computation*, 65(214):841–860, 1996.
- [7] D. W. Boyd. On the beta expansion for Salem numbers of degree 6. *Mathematics of Computation*, 65(214):861–875, 1996.
- [8] L. Flatto, J. Lagarias, and B. Poonen. The zeta function of the beta transformation. *Ergodic Theory Dynamical Systems*, 14:237–266, 1994.
- [9] C. Frougny. Numeration Systems, chapter 7, in M. Lothaire, *Algebraic Combinatorics on Words*. Cambridge University Press, to appear, available at <http://www-igm.univ-mlv.fr/~berstel/Lothaire/>.
- [10] C. Frougny and B. Solomyak. Finite β -expansions. *Ergodic Theory Dynamical Systems*, 12:713–723, 1992.
- [11] M. Hollander. Greedy numeration systems and regularity. *Theory of Computing Systems*, 31:111–133, 1998.
- [12] S. Ito and Y. Takahashi. Markov subshifts and realization of β -expansions. *J. Math. Soc. Japan*, 26:33–55, 1974.
- [13] W. Parry. On the beta expansions of real numbers. *Acta Math. Acad. Sci. Hung.*, 11:401–416, 1960.
- [14] A. Rényi. Representations for real numbers and their ergodic properties. *Acta Math. Acad. Sci. Hung.*, 8:477–493, 1957.
- [15] K. Schmidt. On periodic expansions of Pisot numbers and Salem numbers. *Bull. London Math. Soc.*, 12:269–278, 1980.
- [16] B. Solomyak. Conjugates of beta-numbers and the zero-free domain for a class of analytic functions. *Proc. London Math. Soc.*, 68(3):477–498, 1994.