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Beta-expansions for cubic Pisot numbers

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Abstract. Real numbers can be represented in an arbitrary base $\beta > 1$ using the transformation $T_\beta : x \rightarrow \beta x \pmod{1}$ of the unit interval; any real number $x \in [0, 1]$ is then expanded into $d_\beta(x) = (x_i)_{i \geq 1}$ where $x_i = \lfloor \beta T_{\beta^{-1}}(x) \rfloor$.

The closure of the set of the expansions of real numbers of $[0, 1]$ is a subshift of $\{a \in \mathbb{N} \mid a < \beta\}^\mathbb{N}$, called the beta-shift. This dynamical system is characterized by the beta-expansion of 1; in particular, it is of finite type if and only if $d_\beta(1)$ is finite; $\beta$ is then called a simple beta-number.

We first compute the beta-expansion of 1 for any cubic Pisot number. Next we show that cubic simple beta-numbers are Pisot numbers.

Introduction

Representations of real numbers in an arbitrary base $\beta > 1$, called beta-expansions, have been introduced by Rényi ([14]). They arise from the orbits of the piecewise-monotone transformation of the unit interval : $T_\beta : x \rightarrow \beta x \pmod{1}$. Such transformations were extensively studied in ergodic theory ([13]).

More precisely, any real number $x \in [0, 1]$ is expanded into $d_\beta(x) = (x_i)_{i \geq 1}$ where $x_i = \lfloor \beta T_{\beta^{-1}}(x) \rfloor$. The nonnegative integers $d_i$ are elements of the digit alphabet $A = \{a \in \mathbb{N} \mid a < \beta\}$. These representations generalize standard representations in an integral base to a real base; indeed the beta-expansion of any real number of $[0, 1]$ can equivalently be obtained by the greedy algorithm. Only the beta-expansion of 1 differs.

Properties of beta-expansions are strongly related to symbolic dynamics ([4]). The closure of the set of infinite sequences, appearing as beta-expansions of numbers of the interval $[0, 1]$, is a dynamical system, that is, a closed shift-invariant subset of $A^\mathbb{N}$, called the beta-shift.

An important property of the beta-shift is that its nature is entirely determined, in a combinatorial manner, by the beta-expansion of 1: the beta-shift is sofic, that is to say the set of its finite factors is recognized by a finite automaton, if and only the beta-expansion of 1 is eventually periodic ([3]); it is of finite type, that is to say the set of its finite factors is defined by forbidding a finite set of words, if and only if the beta-expansion of 1 is finite ([12]).

When the beta-expansion of 1 is eventually periodic, $\beta$ is called a beta-number and when the beta-expansion of 1 is finite, $\beta$ is said to be a simple beta-number.
The eventually periodic beta-expansions were extensively studied by Bertrand ([3]) and by Schmidt ([15]). In particular, it is known that Pisot numbers are beta-numbers. Concerning Salem numbers, we only know that if $\beta$ is a Salem number of degree 4, then the beta-expansion of 1 is eventually periodic ([5]). It is conjectured that Salem numbers of degree 6 are still beta-numbers, but not all Salem numbers of degree 8 ([7]).

The domain of the Galois conjugates of all beta-numbers was also investigated independently by Solomyak ([16]) and by Flatto, Lagarias and Poonen ([8]).

For a general presentation of the beta-shift one can refer to [9].

In the following, we summarize properties of beta-numbers. We compute the beta-expansion of 1 for any cubic Pisot number and we establish a characterization of cubic simple beta-numbers, showing that they are Pisot numbers.

A very close problem, seen from the point of view of numeration systems, was studied by Akiyama ([1]). He showed that in the cubic case, the real numbers of the set $\mathbb{N}[\beta^{-1}]$ have a finite beta-expansion if and only $\beta$ is a Pisot unit and 1 has a finite beta-expansion. This finiteness problem is equivalent to a problem of fractal tiling generated by Pisot numbers.

1 Beta-numbers

Real numbers can be represented in an arbitrary base $\beta > 1$ using the transformation $T_\beta : x \rightarrow \beta x \mod 1$ of the unit interval; any real number $x \in [0, 1]$ is then expanded into $d_\beta(x) = (x_i)_{i \geq 1}$ where $x_i = \lfloor \beta T_\beta^{i-1}(x) \rfloor$. When a beta-expansion is of the form $u\omega^r$, the expansion is said to be eventually periodic. If a representation ends with infinitely many zeros, like $u0^\omega$, its is said to be finite and the ending zeros are omitted.

Let us denote by $S_\beta$ the closure of all beta-expansions of real numbers of $[0, 1]$ and by $\sigma$ the (one-sided) shift defined by $\sigma((x_i)_{i \geq 1}) = (x_{i+1})_{i \geq 1}$. The set $S_\beta$ endowed with the shift is called the beta-shift, it is a subshift of $A^\mathbb{N}$, $A$ being the digit set, i.e., $A = \{a \in \mathbb{N} \mid a < \beta\}$.

An important property ([13]) of the beta-shift $S_\beta$ is that its nature is entirely determined by $d_\beta(1)$ the beta-expansion of 1. Indeed, setting $d^*(1) = d_\beta(1)$ if $d_\beta(1)$ is infinite and $d^*(1) = (d_1, d_2, \ldots, d_{n-1}, d_n)$ if $d_\beta(1) = d_1 d_2 \ldots d_{n-1} d_n$, a sequence $x$ of nonnegative integers belongs to $S_\beta$ if and only if it satisfies the following lexicographical order conditions: $\forall p \geq 0, \quad \sigma^p(x) \leq d^*(1)$.

Recall that the beta-expansion of 1 also can be characterized ([13]) by lexicographical order conditions: let $d = (d_i)_{i \geq 1}$ be a sequence of nonnegative integers different from $0^\omega$, such that $\sum_{i \geq 1} d_i \beta^{-i} = 1$, with $d_1 \geq 1$ and for $i \geq 2, d_i \leq d_1$, then $d$ is the beta-expansion of 1 if and only if for all $p \geq 1, \sigma^p(d) < d$.

We recall that an algebraic integer $\beta$ strictly greater than 1 is called a Perron number if all its Galois conjugates have modulus strictly less than $\beta$, a Pisot number if all its Galois conjugates have modulus strictly less than 1, and a Salem number if all its conjugates are less than 1 in modulus and at least one conjugate has modulus 1.
Let $\beta$ be a beta-number. Denote by $d_\beta(1) = d_1 \ldots d_n (d_{n+1} \ldots d_{n+p})^m$, where $n$ and $p$ are chosen minimal, the beta-expansion of 1. Then the adjacency matrix $M_\beta$ of the finite automaton recognizing the set of its finite factors (Fig. 1) is a primitive (i.e., its associated graph is strongly connected and the lengths of its cycles are relatively prime) nonnegative integral matrix whose spectral radius is $\beta$; so, from the Perron-Frobenius theorem, $\beta$ is a Perron number.

![Diagram](image)

**Fig. 1.** Automaton recognizing the set of the finite factors of $S_\beta$

The characteristic polynomial of $M_\beta$

$$P(X) = X^{n+p} - \sum_{i=1}^{n+p} d_i X^{n+p-i} - X^n + \sum_{i=1}^{n} d_i X^{n-i}$$

is called, following the terminology introduced by Hollander ([11]), the associated beta-polynomial.

As $P$ is a multiple of the minimal polynomial $M_\beta$ of $\beta$, $P(0) = d_{n+p} - d_n$ is a multiple of $|M_\beta(0)| = \prod |\beta_i|$ where $\beta_i$ runs over the set of algebraic conjugates of $\beta$. So, we get that $|\prod |\beta_i|$ has to be smaller than $|\beta|$.

As a consequence, in the quadratic case, the only beta-numbers are the Pisot numbers. Conversely, it is known that if $\beta$ is a Pisot number then $\beta$ is a beta-number ([12]). An important gap remains between Pisot and Perron numbers.

**Example 1.** The quadratic number $\beta = (1 + \sqrt{3})/2$ is not a beta-number since $M_\beta(X) = X^2 - X - 3$ and $M_\beta(0) > |\beta|$. 
Let $\beta$ be the Pisot number $(3+\sqrt{5})/2$, then $\beta$ is a beta-number and $d_{\beta} = 21^\omega$.
Let $\beta$ be the golden ratio $(1 + \sqrt{5})/2$, then $\beta$ is a simple beta-number and $d_{\beta}(1) = 11$.

On the other hand, the domain of the Galois conjugates of beta-numbers was studied by Solomyak ([16]) and independently by Flatto, Lagarias and Poonen ([8]). They showed in particular that if the beta-expansion of 1 is eventually periodic then the Galois conjugates of $\beta$ have modulus less than the golden ratio $(1 + \sqrt{5})/2$. It was already known (see [9]) that $\beta$ cannot have a Galois conjugate greater than 1.

Solomyak ([16]) proved that the topological closure of conjugates of beta-numbers and the one of conjugates of simple beta-numbers are the same. However, there is an important difference between the conjugates of beta-numbers and the ones of simple beta numbers: if $\beta$ is a simple beta-number then $\beta$ has no algebraic conjugate that is a nonnegative real number.

Indeed, let $\beta$ be a simple beta-number and set $d_{\beta}(1) = d_1 \ldots d_n$. Consider

the finite automaton recognizing the set of the finite factors of $S_{\beta}$

![Image of the automaton recognizing the set of the finite factors of $S_{\beta}$]

0, 1, ..., $d_1 - 1$

$0, 1, ..., d_2 - 1$

$0, 1, ..., d_3 - 1$

$0, 1, ..., d_{n-1} - 1$

$0, 1, ..., d_n - 1$

**Fig. 2.** Automaton recognizing the set of the finite factors of $S_{\beta}$

the finite automaton recognizing the set of the finite factors of the associated beta-shift (Fig. 2). Let $M_{\beta}$ be the transition matrix of this automaton. The characteristic polynomial of $M_{\beta}$, which is called the associated beta-polynomial,

$$P(X) = X^n - \sum_{i=1}^{n} d_i X^{n-i}$$

has only one positive real root.
Example 2. Salem numbers are roots of reciprocal polynomials. Thus if $\beta$ is a Salem number, $1/\beta > 0$ is a Galois conjugate of $\beta$, and so $\beta$ is not a simple beta-number.

The previous conditions are sufficient for a quadratic algebraic integer to be a simple beta-number.

**Proposition 1.** [10] The simple beta-numbers of degree 2 are exactly the quadratic Pisot numbers without a positive real Galois conjugate. They are the positive roots of the polynomials

$$X^2 - aX - b \quad \text{with} \quad a \geq b \geq 1.$$ 

The beta-expansion of 1 is then $d_\beta(1) = ab$.

Example 3. The minimal polynomial of $(1 + \sqrt{5})/2$ is $X^2 - X - 1$, $(1 + \sqrt{5})/2$ is a simple beta-number and $d_\beta(1) = 11$.

The minimal polynomial of $(3 + \sqrt{5})/2$ is $X^2 - 3X + 1$, therefore $(3 + \sqrt{5})/2$ is not a simple beta-number.

2 Beta-expansions of 1 for cubic Pisot numbers

Let us recall the characterization of cubic Pisot numbers due to Akiyama [1]

**Theorem 1 (Akiyama [1]).** Let $\beta > 1$ be a cubic number and let

$$M_\beta(x) = X^3 - aX^2 - bX - c$$

be its minimal polynomial.

Then $\beta$ is a Pisot number if and only if the both inequalities

$$|b - 1| < a + c \quad \text{and} \quad (c^2 - b) < \text{sgn}(c)(1 + ac)$$

hold.

**Remark 1.** Note that $a$ must be a nonnegative integer.

The following theorem gives the $\beta$-expansion of 1 for any cubic Pisot number.

**Theorem 2.** Let $\beta$ be a cubic Pisot number and let

$$M_\beta(x) = X^3 - aX^2 - bX - c$$

be its minimal polynomial. Then the beta-expansion of 1 is

- Case 1: When $b \geq a$, then $d_\beta(1) = (a + 1)(b - 1 - a)(a + c - b)(b - c)c$.
- Case 2: When $0 \leq b \leq a$, if $c > 0$, $d_\beta(1) = abc$, otherwise,

$$d_\beta(1) = a[(b - 1)(c + a)]^\circ.$$
– Case 3: When $-a < b < 0$, if $b + c \geq 0$, then $d_3(1) = (a - 1)(a + b)(b + c)c$, otherwise $d_3(1) = (a - 1)(a + b - 1)(a + b + c - 1)^c$.

– Case 4: When $b \leq -a$, let $k$ be the integer of $\{2, 3, \ldots, a - 2\}$ such that, denoting $e_k = 1 - a + (a - 2)/k$, $e_k \leq b + c < e_{k-1}$.

  - If $b(k - 1) + c(k - 2) \leq (k - 2) - (k - 1)a$, $d_3(1) = d_1 \ldots d_{2k+2}$ with
    \[
    d_1 = a - 2, \\
    d_{k+2-i} = -(k + 3 - i) + a(k + 2 - i) + b(k + 1 - i) + c(k - i), 3 \leq i \leq k \\
    d_k = -k + ak + b(k - 1) + c(k - 2) \\
    d_{k+1} = -(k - 1) + ak + bk + c(k - 1) \\
    d_{k+2} = -(k - 2) + a(k - 1) + bk + ck \\
    d_{2k+2-i} = -(i - 2) + a(i - 1) + bi + c(i + 1) \quad k \geq 3, 2 \leq i \leq (k - 1) \\
    d_{2k+1} = b + 2c \\ and \quad d_{2k+2} = c.
    \]

  - If $b(k - 1) + c(k - 2) > (k - 2) - (k - 1)a$, let $m$ be the integer defined by $m = \min \{i \in \mathbb{N} \text{ such that } (i + 1)b + ic > i - (i + 1)a\}$.

  When $m = 1$, $d_3(1) = (a - 2)(2a + b - 2)(2a + 2b + c - 2)(2a + 2b + 2c - 2)^c$.

  When $m > 1$, $d_3(1) = d_1 d_2 \ldots d_{m-2} d_{m-3}^c$, with

\[
\begin{align*}
    d_1 &= a - 2, \\
    d_2 &= 2a + b - 3, \\
    d_{m+3-i} &= 2a + b - 3 + (m + 1 - i)(a + b + c - 1) \quad m \geq 3, 3 \leq i \leq m, \\
    d_{m+1} &= 2a + b - 2 + (m - 1)(a + b + c - 1), \\
    d_{m+2} &= a + b - 1 + m(a + b + c - 1), \\
    d_{m+3} &= (m + 1)(a + b + c - 1).
\end{align*}
\]

Example 4. When $a \geq b \geq 0$ and $c > 0$, we obtain the only beta-expansion of 1 of length 3.

The smallest Pisot number has $M_3 = X^3 - X - 1$ as minimal polynomial, it is a simple beta-number and $d_3(1) = 10001$.

The positive root $\beta$ of $M_3 = X^3 - 3X^2 + 2X - 2$ is a simple beta-number and $d_3(1) = 2102$.

The case where $b < -a$ shows that from a cubic simple beta-number, we can obtain an arbitrary long beta-expansion of 1. For any integer $k$ greater than or equal to 2, the real root $\beta$ of the irreducible polynomial $X^3 - (k^2 + 2)X^2 + 2kX - k,$ is a simple beta number whose integer part is equal to $k$, and the beta-expansion of 1 has length $2k + 2$. For $k = 2$, we get $d_3(1) = 221002$; for $k = 3$, we get $d_3(1) = 31310203$.

Example 5. The greatest positive root $\beta$ of $M_3 = X^3 - 2X^2 - X + 1$ is a beta-number and $d_3(1) = 2(01)^c$.

If $\beta$ is the positive root of $X^3 - 5X^2 + 3X - 2$, then $d_3(1) = 413^c$. When $\beta$ is the greatest positive root of $X^3 - 5X^2 + X + 2$, then $d_3(1) = 431^c$.

For any integer $k$ greater than or equal to 3, the real root $\beta$ of the irreducible polynomial $X^3 - (k + 2)X^2 + (2k - 1)X - (k - 1)$, is a beta number whose integer part is equal to $k$, and the beta-expansion of 1 is eventually periodic of period
1, the length of its preperiod \( k \). For \( k = 3 \), we get \( d_\beta(1) = 3302^{2^k} \); for \( k = 4 \), we get \( d_\beta(1) = 42403^{2^k} \).

**Proof.** It is known that Pisot numbers are beta-numbers, thus, for any cubic Pisot number \( \beta \), the beta-expansion of 1 is finite or eventually periodic. In any case, we first compute the associated beta-polynomial \( P \). Next we prove that the sequence \( d = (d_i)_{i \geq 1} \) of nonnegative integers obtained from the beta-polynomial satisfy lexicographical order conditions: for all \( p \geq 1 \), \( \sigma^p(d) < d \).

First of all, we recall that, from Theorem 1, a cubic number \( \beta \), greater than 1 and having

\[
M_\beta(X) = X^3 - aX^2 - bX - c
\]
as minimal polynomial, is a cubic Pisot number if and only if it both

\[
|b - 1| < a + c \quad \text{and} \quad (c^2 - b) \leq \text{sgn}(c)(1 + ac)
\]
hold.

Denote by \( Q \) the complementary factor of the beta-polynomial \( P \) defined by

\[
P(X) = M_\beta(X)Q(X).
\]
As we shall see in what follows, the value of \( Q \) depends upon the value of the coefficients of \( M_\beta \).

**Case 1:** When \( b > a \), as \( \beta \) is a Pisot number, from Theorem 1, \( c \) is a positive integer. In this case, the complementary factor is \( Q(X) = X^2 - X + 1 \) and \( d_\beta(1) \equiv (a + 1)(b - 1 - a)(a + c - b)(b - c) \).

Indeed, as \( (c^2 - b) < \text{sgn}(c)(1 + ac) \) and \( c > 0 \), we get \( c \leq a + 1 \). As \( |b - 1| < a + c \), we get \( b - 1 - a \leq a \) and \( 0 \leq a - b + c \). From \( b > a \), we get that \( 0 \leq b - a - 1 \) and, as \( c \leq a + 1 \), that \( a - b + c \leq a \). Finally as \( 0 \leq a - b + c \leq a \), we obtain \( 0 \leq b - c \leq a \).

**Case 2:** When \( 0 \leq b \leq a \), the complementary factor is then \( Q(X) = 1 \) and the associated beta-polynomial is equal to the minimal polynomial.

If \( c > 0 \), then \( d_\beta(1) = abc \). Indeed, as \( (c^2 - b) < \text{sgn}(c)(1 + ac) \), we get \( c \leq a \).

If \( c < 0 \), then \( d_\beta(1) = a(b - 1)(a + c)^{|c|} \). As \( |b - 1| < a + c \), we get \( b - 1 \leq a - 2 \). As \( (c^2 - b) < \text{sgn}(c)(1 + ac) \), we get that \( c \geq -a \) and, consequently, \( 0 \leq c + a \leq a - 1 \).

**Case 3:** When \( -a < b < 0 \), if \( b + c \geq 0 \) then the complementary factor is \( Q(X) = X + 1 \) and \( d_\beta(1) \equiv (a - 1)(a + b)(b + c) \). Indeed, as \( -a < b < 0 \), we obtain \( 1 \leq a + b \leq a - 1 \). Since \( b + c \geq 0 \), \( c \) is a positive integer. From \( (c^2 - b) < \text{sgn}(c)(1 + ac) \), we get that \( c \leq a - 1 \) and \( b + c \leq a - 2 \).

If \( b + c < 0 \), then \( Q(X) = 1 \) and \( d_\beta(1) \equiv (a - 1)(a + b - 1)(a + b + c - 1)^{|c|} \).

As \( -a < b < 0 \), we get \( 0 \leq a + b - 1 \leq a - 2 \). From \( |b - 1| < a + c \), we get that \( 1 \leq a + b - c - 1 \) and as \( b + c < 0 \), we obtain \( a + b + c - 1 \leq a - 2 \).

**Case 4:** First of all, since \( |b - 1| < a + c \), we get \(-a + 2 \leq b + c \). Moreover as \( b \leq -a \), we get \( c \geq 2 \) and as \( (c^2 - b) < \text{sgn}(c)(1 + ac) \), we obtain \( c \leq a - 2 \), thus \( b + c \leq -2 \). So, there exists an integer \( k \) in \( \{2, 3, \ldots, a - 2\} \), such that, denoting \( e_k = 1 - a + (a - 2)/k \), \( e_k \leq b + c < e_{k-1} \).

When \( b(k - 1) + c(k - 2) \leq (k - 2) - (k - 1)a \), the complementary factor is

\[
Q(X) = \frac{(X^k - 1)(X^{k+1} - 1)}{(X - 1)^2}
\]
and $d_3(1) = d_1 \ldots d_{2k+2}$ with

\[
d_1 = a - 2, \\
d_{2k+2-i} = -(k + 3 - i) + a(k + 2 - i) + b(k + 1 - i) + c(k - i), k \geq 3, 3 \leq i \leq k \\
d_k = -k + ak + b(k - 1) + c(k - 2) \\
d_{k+1} = -(k - 1) + ak + bk + c(k - 1) \\
d_{k+2} = -(k - 2) + a(k - 1) + bk + c(k - 1)
\]

As $b + c < e_i$, we get $d_{2k+2-i} < c$. As $-a + 2 \leq b + c$ and $b + 2c \geq 0$, we get $d_{2k+2-i} \geq i$.

As $e_k \leq b + c$, we obtain $d_{k+2} \geq 0$. Since $c \leq a - 2$, $d_{k+1} > d_{k+2}$ and since $b + c \leq -2$, $d_k > d_{k+1}$. Moreover from $b(k - 1) + c(k - 2) \leq (k - 2) - (k - 1)a$, we get $d_k \leq a - 2$.

For $k \leq 3$, as $b - 1 \leq a + c$, we obtain $d_{2} < \ldots < d_{k+1}$. As $b + c < e_{k+1}$ and $b + 2c \leq 0$, we get $d_{k+1} \leq a - 2$. Moreover from $c \leq a - 2$ and $a + b + c - 1 > 0$, we get that $d_k = 2a + b - 3$ is nonnegative.

All $d_i$'s are smaller than $d_1$, only $d_{2k+2}$ and $d_k$ can be equal to $d_1$. Therefore we have to verify that $d_2 \geq d_{k+1}$ when $k \geq 3$ (otherwise $d_2 = d_k$ and $d_{k+1} > d_{k+2}$).

If $d_k = a - 2$, then $b + c = e_k$, and $d_{k+1} = a - c - 1$. As $a + b + c - 1 > 0$, we obtain $d_{k+1} \leq d_2$. In case of equality, if $k = 3$, then $d_3 = d_k$ and $d_{k+1} > d_{k+2}$, otherwise $d_3 > d_2$ and $d_{k+1} > d_{k+2}$, therefore $d_3 > d_{k+2}$.

So lexicographical order conditions are satisfied and $d_1 \ldots d_{2k+2}$ is the beta-expansion of 1.

When $b(k - 1) + c(k - 2) > (k - 2) - (k - 1)a$, as $b \leq -a$, we get $k \geq 3$. Let $m$ be the integer defined by $m = \min\{i \in \mathbb{N} \mid (i + 1)b + ic > i - (i + 1)a\}$. Note that by definition of $m$, $m \leq k - 2$ and since $b \leq -a$, $m \geq 1$. In this case, the complementary factor is

\[
Q(X) = \sum_{i=0}^{m} X^i.
\]

The beta-expansion of 1 is then eventually periodic with period 1, the length of the preperiod is $m + 2$.

When $m = 1$, $P(X) = X^4 - (a - 1)X^3 - (a + b)X^2 - (b + c)X - c$ and

\[
d_3(1) = (a - 2)(2a + b - 2)(2a + 2b + c - 2)(2a + 2b + 2c - 2)\omega.
\]

Here $d_3 = c_{m+2} = a + b - 1 + m(a + b + c - 1)$ and $d_k = d_{m+1} = (m+1)(a+b+c-1)$.

When $m > 1$,

\[
P(X) = X^{m+3} - (a - 1)X^{m+2} - (a + b - 1)X^{m+1} - \sum_{i=3}^{m}(a + b + c - 1)X^i
\]

\[-(a + b + c)X^2 - (b + c)X - c\]
and $d_3(1) = d_1 d_2 \ldots d_{m+2} d_{m+3}$, with

$$
\begin{align*}
d_1 &= a - 2, & d_2 &= 2a + b - 3, \\
d_{m+3-i} &= a + b - 3 + (m + 1 - i)(a + b + c - 1) & m &\geq 3, 3 \leq i \leq m, \\
d_{m+1} &= 2a + b - 2 + (m - 1)(a + b + c - 1), \\
d_{m+2} &= a + b - 1 + m(a + b + c - 1), \\
d_{m+3} &= (m + 1)(a + b + c - 1).
\end{align*}
$$

In both cases, $d_1 = a - 2$. Since $b(k - 1) + c(k - 2) > (k - 2) - (k - 1)a$ and $c \leq a - 2$, we get $-2a + 3 \leq b$. Moreover as $b \leq -a$, $1 \leq d_2 \leq a - 2$ when $m = 1$, and $0 \leq d_2 \leq a - 3$ otherwise. By definition of $m$, $(m + 1)b + mc > m - (m + 1)a$, thus $d_{m+2} \geq 0$ and $d_{m+3} \geq c$. Since $e_k \leq b + c < e_{k-1}$ and $m \leq k - 2$, we obtain $d_{m+3} \leq a - 3$ and $d_{m+2} \leq a - c - 3$.

When $m > 1$, since $mb + (m - 1)c \leq (m - 1) - ma$, we get $d_{m+1} \leq a - 2$. As $0 \leq 2a + b - 2$ and $a + b + c - 1 > 0$, $d_{m+1} > 0$. Moreover as $a + b + c - 1 > 0$, one has $d_2 < d_3 < \ldots < d_{m+1}$. Note that, when $m \geq 3$, $d_2 \neq a - 2$.

We now study the cases where $d_i$ is not strictly smaller than $d_1$. When $m = 1$, only $d_2$ may be equal to $a - 2$, then $b = -a$ and $d_3 = c - 2$, thus $d_3 < d_2$. When $m > 1$, only $d_{m+1}$ may be equal to $a - 2$, then $mb = -ma - (m - 1)c + (m - 1)$, and thus $d_2 - d_{m+2} = a - 1 - c$ is a positive integer.

We have proved that the lexicographical order conditions on $d_3(1)$:

$$
d_1 d_2 \ldots d_{m+3} >_{lex} d_1 d_{i+1} \ldots d_{m+3} \quad \text{for } 2 \leq i \leq m + 3,
$$

are satisfied, showing in this way that the announced beta-expansions of 1 are right.

Remark 2. The polynomials $Q$ that appear in the cubic case are cyclotomic. In the general case, $Q$ can be noncyclotomic and even nonreciprocal ([8]).

3 Cubic simple beta-numbers

In the following, we establish that cubic simple beta-numbers are Pisot numbers. Next we give necessary and sufficient conditions on the coefficients of the minimal polynomial of $\beta$ for $\beta$ to be a simple beta-number.

Theorem 3. If $\beta$ is a cubic simple beta-number then $\beta$ is a Pisot number.

Remark 3. This is no longer true for simple beta-numbers of degree 4. For example, the positive root of $X^4 - 3X^3 - 2X^2 - 3$ is a simple beta-number, but is not a Pisot number.

Proof. Let $\beta$ be a cubic simple beta-number and let

$$
M_\beta(X) = X^3 - aX^2 - bX - c
$$

be its minimal polynomial. Then $\beta$ has no positive real algebraic conjugate and $c$ is a positive integer smaller than $|\beta|$.
The condition on the product $c$ of the roots of the polynomial $M_\beta$, i.e., $|c| \leq |\beta|$, directly implies, when the Galois conjugates of $\beta$ are not real numbers, that $\beta$ is a Pisot number.

The only other case is the case where both Galois conjugates $\gamma_1$ and $\gamma_2$ of $\beta$ are negative real numbers. We then assume that $\beta$ is a cubic simple beta-number that is not a Pisot number, and show that these hypotheses are contradictory. Let $\gamma_1$ and $\gamma_2$ be the Galois conjugates of $\beta$. As $0 < c \leq |\beta|$, if one of the $\gamma_i$’s is smaller than $-1$ the other one is greater than $-1$. Moreover, as the modulus of a Galois conjugate of a beta-number is smaller than the golden ratio, one can suppose, for example, that

$$\frac{-1 + \sqrt{5}}{2} < \gamma_2 < -1 < \gamma_1 < 0 < \beta$$

Consequently, $M_\beta(-1) > 0$, in other words, $b > a + c + 1$. Note that here $a \in \{[\beta] - 2, [\beta] - 1\}$.

As $\beta$ is a simple beta-number, $d_\beta(1) = d_1 d_2 \ldots d_n$. Denote by $P$ the associated $\beta$-polynomial:

$$P(X) = X^n - \sum_{i=1}^{n} d_i X^{n-i}$$

and denote by $Q = \sum_{i \geq 0} q_i X^i$ the quotient of the division upon the increasing powers of $P$ by $M_\beta$. In other words,

$$P(X) = M_\beta(X) Q(X)$$

We shall show, by induction, that $q_0 \geq 1$, and that for all $i \geq 0$, $|q_{i+1}| > |q_i|$ with $\text{sgn}(q_{i+1}) = -\text{sgn}(q_i)$. We shall conclude from the growth of the moduli of its coefficients that $Q$ is an infinite series, and thus that $d_\beta(1)$ is not finite.

In what follows, we mainly use the fact that the $d_i$’s are nonnegative integers smaller than $|\beta|$ and the inequality $b \geq a + c + 2$.

First of all, as $d_n = q_0 c$ and $d_n$ and $c$ are positive integers, $q_0 \geq 1$. Since $d_{n-1} = q_0 b + q_1 c$ and $q_0 \geq 1$, $d_{n-1} \geq q_0 a + 2q_0 + (q_0 + q_1)c$. When $a = |\beta| - 1$, we directly get from $d_{n-1} \leq |\beta|$, that $q_1 < -q_0$. When $a = |\beta| - 2$, the lexicographical order conditions on $d_\beta(1)$ imply that

$$d_{n-1} d_n < d_1 d_2 \ldots d_n.$$  

By definition of beta-expansions, $d_1 = |\beta|$ and here $d_2 < d_n$. Indeed as

$$\gamma_2 = \frac{1}{2} \left( a - \beta + \sqrt{(a - \beta)^2 - 4c} \right),$$

and $\gamma_2 > -(1 + \sqrt{5})/2$, we get that

$$c > \frac{\sqrt{5} - 1}{2} \beta + \frac{1 + \sqrt{5}}{2} \beta.$$
and in particular, that $c > \beta/2$, consequently $d_n = c$ and that $\beta \{ \beta \} < c$. Thus $d_2 = |\beta \{ \beta \}|$ is strictly smaller than $d_n$. Therefore the previous lexicographical order condition implies that $d_{n-1} < |\beta|$. So, as $d_{n-1} \geq |\beta| + (q_0 + q_1)c, q_1 < -q_0$.

As $d_{n-2} = q_0a + q_1b + q_2c$ and $q_1 < -q_0 < 0, d_{n-2} \leq (q_1 + q_0)a + 2q_1 + (q_1 + q_2)c$, that is $d_{n-2} < -|\beta| + (q_1 + q_2)c$, so $q_2 > -q_1$.

For all positive integers $i$, $d_{n-(2i+1)} = -q_{2i+2} + q_{2i+1}a + q_3b + q_{2i+1}c$. From $q_{2i+1} > 0$, we get $d_{n-(2i+1)} = (q_{2i+1} + q_3)a + q_{2i+1} + (q_{2i+1} + q_{2i+1})c$. From $(q_{2i+1} + q_{2i+1}) \geq 1, q_{2i+1} > 2i$ and $(q_{2i+1} + q_{2i+1}) > 1$, we obtain $d_{n-(2i+1)} > |\beta| + (q_{2i+1} + q_{2i+1})c$, and thus $q_{2i+1} < -q_{2i+1}$.

For all positive integers $i$, $d_{n-(2i+2)} = -q_{2i+1} + q_{2i+1}a + q_{2i+1} + q_{2i+1}c$. From $q_{2i+1} < 0$, we get $d_{n-(2i+2)} \leq (q_{2i+1} + q_{2i+1})a + q_{2i+1} + (q_{2i+1} + q_{2i+1})c$. As $(q_{2i+1} + q_{2i+1}) \leq -1, q_{2i+1} < -(2i+1)$ and $(q_{2i+1} + q_{2i+1}) \leq -1, we get $d_{n-(2i+2)} < -|\beta| + (q_{2i+1} + q_{2i+1})c$, thus $q_{2i+2} > -q_{2i+2}$.

So $Q$ is an infinite series; consequently if $\beta$ is not a Pisot number, $d_\beta(1)$ is not finite.

As a consequence of Theorems 2 and 3, we obtain the above characterization of cubic simple beta-numbers.

**Proposition 2.** Let $\beta$ be a cubic Pisot number and let

$$M_\beta(x) = X^3 - aX^2 - bX - c$$

be its minimal polynomial.

Then $\beta$ is a simple beta-number if and only if it satisfies one of the following conditions:

- Case 1: $b \geq 0$ and $c > 0$
- Case 2: $-a < b < 0$ and $b + c \geq 0$
- Case 3: $b \leq -a$ and $b(k + 1) - c(k - 2) \leq (k - 2) - (k - 2)a$, where $k$ is the integer in $\{2, 3, \ldots, a - 2\}$ such that, denoting $e_k = 1 - a + (a - 2)/k$, $e_k \leq b + c < e_{k-1}$.

The problem of finding such a characterization remains open for simple beta-numbers of higher degree.

**References**


